

# Abundance of 3-Planes on Real Projective Hypersurfaces

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**Abstract** We show that a generic real projective  $n$ -dimensional hypersurface of odd degree  $d$ , such that  $4(n - 2) = \binom{d+3}{3}$ , contains “many” real 3-planes, namely, in the logarithmic scale their number has the same rate of growth,  $d^3 \log d$ , as the number of complex 3-planes. This estimate is based on the interpretation of a suitable signed count of the 3-planes as the Euler number of an appropriate bundle.

Everything you can imagine is real.  
Pablo Picasso

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## 1 Introduction

### 1.1 Phenomenon of Abundance

Up to our knowledge, the phenomenon of abundance of real solutions in certain real enumerative problems was observed for the first time in [Itenberg et al. \(2003\)](#), where it

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was shown that the number of real rational curves of degree  $d$  interpolating a generic collection of  $3d - 1$  real points in the real projective plane grows, in the logarithmic scale, as fast as the number of complex curves. Since then, similar abundance phenomena were observed in various other real enumerative problems (cf., [Itenberg et al. 2013](#); [Georgieva and Zinger 2013](#); [Kharlamov and Rasdeaconu 2014](#)). In particular, in our previous paper [Finashin and Kharlamov \(2013\)](#) we proved that a generic real hypersurface of degree  $2n - 1$  in a real projective space of dimension  $n + 1$  contains at least  $(2n - 1)!!$  real lines, which is approximately the square root of the number of complex lines [the same bound was obtained by [Okonek and Teleman \(2014\)](#)]. This estimate was based on the signed counting of the real lines by means of the Euler number of an appropriate vector bundle, and as a result gave the following relations

$$n_d^{\mathbb{C}} \geq n_d^{\mathbb{R}} \geq n_d^{\mathbb{R}, \min} \geq n_d^e, \quad \log n_d^e \sim \frac{1}{2} \log n_d^{\mathbb{C}},$$

where  $n_d^{\mathbb{C}}$  and  $n_d^{\mathbb{R}}$  denote respectively the numbers of complex and real lines on a generic real hypersurface of odd degree  $d$  in a projective space of dimension  $\frac{d+3}{2}$ , the symbol  $n_d^{\mathbb{R}, \min}$  stands for the minimum of  $n_d^{\mathbb{R}}$  taken over all such generic hypersurfaces and  $n_d^e$  stands for the above mentioned signed count of real lines. (Here, the number  $n_d^{\mathbb{R}}$  depends on the choice of such a hypersurface, while  $n_d^e$ ,  $n_d^{\mathbb{R}, \min}$ , and  $n_d^{\mathbb{C}}$  depend only on  $d$ .)

The aim of the present paper is to show that a similar abundance phenomenon holds also as soon as we count the real 3-planes on generic real hypersurfaces of *odd* degree  $d$  in a real projective space of an appropriate dimension, which we will still denote by  $n + 1$ .

To achieve this goal we follow the same approach as in [Finashin and Kharlamov \(2013\)](#). Namely, the variety of (complex or real) 3-planes on a hypersurface that is defined by a homogeneous polynomial  $f$  of degree  $d$  in  $n + 2$  variables is viewed as the zero locus  $\{s_f = 0\}$  in the Grassmannian  $[G_4(\mathbb{C}^{n+2})$  or  $G_4(\mathbb{R}^{n+2})$ , respectively], where  $s_f$  is the determined by  $f$  section of the symmetric power  $\text{Sym}^d \tau_{4,n+2}^*$  of the dual to the tautological (complex or real) 4-dimensional vector bundle  $\tau_{4,n+2}$  on the corresponding Grassmannian. It is well-known [see, for example, [Debarre and Manivel \(1998\)](#), Theorem 1.2] that the section  $s_f$  is transversal to the zero section for a generic choice of  $f$ . Thus, if  $\dim\{s_f = 0\} = 4(n - 2) - \binom{d+3}{3}$  is zero, then, in the complex setting, the Chern number  $c_{4(n-2)}(\text{Sym}^d \tau_{4,n+2}^*)[G_4(\mathbb{C}^{n+2})]$  is equal to the number of complex 3-planes, while in the real setting, the Euler number of  $\text{Sym}^d \tau_{4,n+2}^*$  on  $G_4(\mathbb{R}^{n+2})$  counts the real 3-planes with signs.

The main feature of such a signed count is its invariance: dependence only on  $d$  and not on the choice of a hypersurface (in particular, independence of the topology of the hypersurface). This count is well defined, if  $d$  is odd (see Proposition 3.1.3), which is the only interesting case for invariant counting and lower bounds, since in the case of even  $d$  the real locus of the hypersurface can be empty. Note also that the count of 3-planes makes sense only if  $n = 2 + \frac{1}{4} \binom{d+3}{3}$  (in higher dimension the 3-planes come in families, and in lower dimension their number is zero).

In order to state the results, let us introduce the following notation: assuming that  $d$  is odd, let us denote by  $\mathcal{N}_d^{\mathbb{C}}$ ,  $\mathcal{N}_d^{\mathbb{R}}$  and  $\mathcal{N}_d^{\mathbb{R},\min}$  the number of complex 3-planes on a generic hypersurface of degree  $d$  in a complex projective space of dimension  $3 + \frac{1}{4}\binom{d+3}{3}$ , the number of real 3-planes on a generic real hypersurface of degree  $d$  in a real projective space of dimension  $3 + \frac{1}{4}\binom{d+3}{3}$ , and the minimum of  $\mathcal{N}_d^{\mathbb{R}}$  taken over all generic real hypersurfaces as above (the number  $\mathcal{N}_d^{\mathbb{C}}$  does not depend on the choice of a generic hypersurface). To avoid cumbersome discussions of explicit orientation conventions needed to fix the sign of the Euler number in question, we take into account only its absolute value and denote the latter by  $\mathcal{N}_d^e$ . Note that the numbers introduced are linked by the following trivial relations

$$\mathcal{N}_d^{\mathbb{C}} \geq \mathcal{N}_d^{\mathbb{R}} \geq \mathcal{N}_d^{\mathbb{R},\min} \geq \mathcal{N}_d^e \geq 0. \tag{1.1.1}$$

**Theorem 1.1.1** *The invariants  $\mathcal{N}_d^{\mathbb{C}}$ ,  $\mathcal{N}_d^e$  are positive for each odd  $d$  and satisfy the following asymptotic relations as (odd)  $d \rightarrow \infty$ :*

$$\log \mathcal{N}_d^e = \frac{1}{12}d^3 \log d + O(d^3) \leq \log \mathcal{N}_d^{\mathbb{C}} \leq \frac{1}{6}d^3 \log d + O(d^3).$$

Our conjecture is that, in fact,  $\log \mathcal{N}_d^{\mathbb{C}} \sim 2 \log \mathcal{N}_d^e$  which would imply that  $\log \mathcal{N}_d^{\mathbb{C}} \sim \frac{1}{6}d^3 \log d$ . It seems to us that even the positivity of  $\mathcal{N}_d^{\mathbb{R}}$  (which follows from  $\mathcal{N}_d^e \neq 0$ ) was not acknowledged in the literature before.

Amazingly, the answers that we obtain in the real setting look more simple than those in the complex setting. Similar phenomena are observed in other enumerative problems, see [Finashin and Kharlamov \(2013\)](#) and Sect. 5.

**Corollary 1.1.2** *As odd degree  $d$  increases, the invariants  $\mathcal{N}_d^{\mathbb{C}}$  and  $\mathcal{N}_d^{\mathbb{R},\min}$  have the same rate of growth in the logarithmic scale. More precisely,*

$$\frac{1}{12}d^3 \log d + O(d^3) \leq \log \mathcal{N}_d^{\mathbb{R},\min} \leq \log \mathcal{N}_d^{\mathbb{C}} \leq \frac{1}{6}d^3 \log d + O(d^3).$$

(For us the same rate of growth for two sequences  $f(n)$ ,  $g(n)$ , means existence of a constant  $C$  such that  $|f(n)| \leq C|g(n)|$  and  $|g(n)| \leq C|f(n)|$  for all sufficiently big  $n$ .)

### 1.2 Examples, Applications, and Related Results

In the case of even  $d$ , we still have  $\mathcal{N}_d^{\mathbb{C}} > 0$  [see, for example, [Debarre and Manivel \(1998\)](#), Theorem 2.1] as well as  $\log \mathcal{N}_d^{\mathbb{C}} \leq \frac{1}{6}d^3 \log d + O(d^3)$  (see Proposition 2.5.1). By contrary, if  $d$  is even, then  $\mathcal{N}_d^e$  either vanishes or is defined only modulo 2, see the explanation in Remark (1) after Proposition 3.1.3.

Note that the positivity  $\mathcal{N}_d^{\mathbb{R}} \geq \mathcal{N}_d^e > 0$  implies that a generic real projective  $n$ -dimensional hypersurface of odd degree  $d$  with  $4(n - 2) > \binom{d+3}{3}$  contains an infinite number of real 3-planes, and then due to [Debarre and Manivel \(1998\)](#), Theorem 2.1 the variety of these real 3-planes is of (pure) dimension  $4(n - 2) - \binom{d+3}{3}$  if  $n > 6$ . (In fact, Debarre and Manivel have proved in [\(2000\)](#) such positivity and pure dimension

results for the variety of real  $r$ -planes on odd degree real complete intersections, but only under the assumption that the dimension of the ambient projective space is large enough. Apparently, for hypersurfaces, due to this dimension assumption their result applies only to  $d \leq 3$ .)

The approach that we develop in this paper can be applied to counting real  $(2r - 1)$ -planes on real projective  $n$ -dimensional hypersurfaces of any odd degree  $d$ , under an appropriate dimension condition, that is  $2r(n - 2r + 2) = \binom{d+2r-1}{2r-1}$ . (Counting of even dimensional planes gives a trivial result, since the dimension of the corresponding vector bundle is odd, and the Euler count, whenever it gives an integer rather than a modulo 2 residue, would give zero.) The result of such counting still gives an invariant, and hence provides, like in the cases  $r = 1$  [considered in [Finashin and Kharlamov \(2013\)](#)] and  $r = 2$  (considered in this paper), an effective universal lower bound for the number of real  $(2r - 1)$ -planes on a hypersurface. We restricted ourselves here to the case of 3-planes, since for the moment in the higher dimensional cases we can not provide an explicit answer, but can only set up an upper bound (see Sect. 2.5), give an implicit formula in the form of multivariate Cauchy integral (see Theorem 5.3.1), and suggest a conjecture (see Conjecture 2.6).

### 1.3 The Content

In Sect. 2, we recall some facts from the complex Schubert calculus that are related to counting the number of projective subspaces in hypersurfaces. In Sect. 3, we discuss real Schur polynomials and the modifications required to make similar counting in the real setting. The techniques developed in these sections are applied in Sect. 4 to prove the main results. In Sect. 5 we discuss a few other real enumerative problems that can be solved by using the same methods.

### 1.4 Conventions

If in a homology or cohomology notation the coefficients are not specified, then they are supposed to be integer. The notation  $p_i$  for the Pontryagin classes may refer to  $p_i(\tau_{k,\infty}) \in H^*(G_k(\mathbb{R}^\infty))$  as well as for their pull-backs in  $H^*(G_k(\mathbb{R}^{k+n}))$ , in  $H^*(\tilde{G}_k(\mathbb{R}^\infty))$ , and in  $H^*(\tilde{G}_k(\mathbb{R}^{k+n}))$  ( $\tilde{G}$  stands for the Grassmannians of oriented subspaces). This ambiguity should be resolved by the context.

Our decision not to fix explicit orientations results in a number of identities valid up to sign, and we write  $x = \pm y$ , which means that  $x = y$ , or  $x = -y$ . The symbol  $\square$  marks the end of a proof, or signifies that the corresponding statement is either a citation or an immediate consequence of previous claims.

## 2 Elements of Complex Schubert Calculus

### 2.1 Schubert Basis

By a  $k$ -partition of  $n \in \mathbb{Z}_{\geq 0}$  we mean a decreasing integer sequence  $\alpha = (\alpha_1, \dots, \alpha_k)$ ,  $\alpha_1 \geq \dots \geq \alpha_k \geq 0$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_k = n$ . Graphically  $\alpha$  is presented as a Young–

*Ferrers diagram of size n.* For example, the constant  $k$ -partition  $\mathbf{m} = (m, \dots, m)$  is presented by the  $k \times m$  rectangle. In what follows we assume that some  $k > 0$  is fixed throughout the whole section, and omit sometimes the vanishing components of  $\alpha$ .

A filtration  $0 \subset \mathbb{C}^1 \subset \mathbb{C}^2 \subset \dots$  of  $\mathbb{C}^\infty$  yields a CW-decomposition of Grassmannian  $G_k(\mathbb{C}^\infty)$  into open *Schubert cells*  $C_\alpha$  indexed with  $k$ -partitions, namely, a  $k$ -subspace  $L \subset \mathbb{C}^\infty$  belongs to  $C_\alpha$  if and only if  $\alpha_{k+1-s} = \min\{j \mid \dim(L \cap \mathbb{C}^{j+s}) = s\}$ , for each  $1 \leq s \leq k$ .

With any  $k$ -partition  $\alpha$  we associate a homology and a cohomology classes of Grassmannians as follows. The closure  $\text{Cl}(C_\alpha)$  is the so-called *Schubert variety*; being equipped with the complex orientation it yields the *Schubert class*  $[C_\alpha] \in H_{2n}(G_k(\mathbb{C}^\infty))$ , where  $n = |\alpha|$ . The cohomology class  $\sigma_\alpha \in H^{2n}(G_k(\mathbb{C}^\infty))$  associated to  $\alpha$  is characterized by

$$\sigma_\alpha([C_\beta]) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

In other words, the classes  $\sigma_\alpha$  taken over all  $k$ -partitions of size  $n$  form an additive basis in  $H^{2n}(G_k(\mathbb{C}^\infty))$  such that  $h = \sum_\alpha (h([C_\alpha])\sigma_\alpha$  for any  $h \in H^*(G_k(\mathbb{C}^\infty))$ .

Note that the Schubert cell  $C_\alpha \subset G_k(\mathbb{C}^\infty)$  is contained in a finite dimensional Grassmannian  $G_k(\mathbb{C}^{k+m})$  if and only if  $\alpha_1 \leq m$ , that is if the Young–Ferrers diagram of  $\alpha$  lies inside the  $(k \times m)$ -rectangle diagram. It follows that the additive bases of  $H_*(G_k(\mathbb{C}^{k+m}))$  and  $H^*(G_k(\mathbb{C}^{k+m}))$  are given respectively by  $[C_\alpha]$  and  $\sigma_\alpha$  such that  $\alpha_1 \leq m$  (here and below, to simplify notation, we denote by  $\sigma_\alpha$  not only the class in  $H^*(G_k(\mathbb{C}^\infty))$  but also its pull-back in  $H^*(G_k(\mathbb{C}^{k+m}))$ ).

We say that  $k$ -partitions  $\alpha$  and  $\beta$  are *m-complementary to each other* if  $\alpha_i + \beta_{k+1-i} = m$  for  $i = 1, \dots, k$  (so that, in particular,  $\alpha_1, \beta_1 \leq m$ ). It is well-known (and easy to check) that the Schubert cycles of  $m$ -complementary  $k$ -partitions are Poincare-dual in  $G_k(\mathbb{C}^{k+m})$ .

**Proposition 2.1.1** *Schubert classes  $[C_\alpha]$  and  $[C_\beta]$  have intersection index 1 in  $G_k(\mathbb{C}^{k+m})$  if  $k$ -partitions  $\alpha$  and  $\beta$  are  $m$ -complementary, and index 0 if not.  $\square$*

### 2.2 Schur Polynomials

Denote by  $U_k$  the unitary group and by  $U_1^k$  its maximal torus formed by the diagonal matrices. The inclusion map  $U_1^k \subset U_k$  induces a map of the classifying spaces

$$BU_1^k = (\mathbb{C}P^\infty)^k \xrightarrow{\phi} G_k(\mathbb{C}^\infty) = BU_k,$$

and the induced cohomology map  $\phi^* : H^*(G_k(\mathbb{C}^\infty)) \rightarrow H^*(\mathbb{C}P^\infty)^k \cong \mathbb{Z}[z_1, \dots, z_k]$  is independent of the choice of the maximal torus, since such tori form a single conjugacy class.

For the unitary groups, the so called *splitting principle* can be expressed formally as follows.

**Theorem 2.2.1** (e.g., Bott and Tu 1982) *The ring homomorphism  $\phi^*$  is monomorphic and its image coincides with the subring  $\mathbb{Z}^S[z] \subset \mathbb{Z}[z]$  formed by the symmetric polynomials. The Chern classes  $c_r(\tau_k^*) \in H^*(G_k(\mathbb{C}^\infty))$ ,  $1 \leq r \leq k$ , of the dual tautological vector bundle  $\tau_k^*$  over  $G_k(\mathbb{C}^\infty)$  are sent to the elementary symmetric polynomials*

$$\phi^*(c_r) = \varepsilon_r(z_1, \dots, z_k) = \sum_{i_1 < \dots < i_r} z_{i_1} \dots z_{i_r}.$$

These classes  $c_r = c_r(\tau_k^*)$  are multiplicative generators of the ring  $H^*(G_k(\mathbb{C}^\infty))$ . □

We call  $z_i$ ,  $1 \leq i \leq k$ , the *Chern roots* as they are the formal roots of  $t^k - c_1 t^{k-1} + \dots + (-1)^k c_k$ . For each  $h \in H^*(G_k(\mathbb{C}^\infty))$ , we say that  $\phi^*(h) \in \mathbb{Z}^S[z]$  is the *root polynomial* representing class  $h$ .

The root polynomials  $s_\alpha \in \mathbb{Z}^S[z]$  that represent classes  $\sigma_\alpha \in H^{2n}(G_k(\mathbb{C}^\infty))$ ,  $n = |\alpha|$ , are called *Schur polynomials*. The relation  $h = \sum_\alpha (h([C_\alpha])\sigma_\alpha$  implies

$$\phi^*(h) = \sum_\alpha \lambda_\alpha(h) s_\alpha,$$

where  $\lambda_\alpha(h) = h[C_\alpha] \in \mathbb{Z}$  will be called *the Schur coefficients* for  $h$  (or for  $\phi^*(h)$ ).

Recall that the Schur polynomials can be calculated using the following *generalized Vandermonde polynomial*

$$V_{\alpha+\delta}(z) = \sum_{\tau \in S_k} \text{sign}(\tau) z_{\tau(1)}^{\alpha_1+k-1} z_{\tau(2)}^{\alpha_2+k-2} \dots z_{\tau(k)}^{\alpha_k}$$

where  $\text{sign}(\tau)$  is the sign of a permutation  $\tau \in S_k$ , and  $\delta = (k - 1, k - 2, \dots, 1, 0)$ . Recall that the usual Vandermonde polynomial is

$$V_\delta(z) = \sum_{\tau \in S_k} \text{sign}(\tau) z_{\tau(1)}^{k-1} z_{\tau(2)}^{k-2} \dots z_{\tau(k-1)} = \prod_{1 \leq i < j \leq k} (z_i - z_j).$$

**Proposition 2.2.2** (e.g., Stanley 1999, Theorem 7.15.1) *For any  $k$ -partition  $\alpha$  we have*

$$s_\alpha = \frac{V_{\alpha+\delta}}{V_\delta}.$$

□

*Example (1)* The Schur polynomial  $s_{1,\dots,1}$  with  $r \leq k$  components “1” is the elementary symmetric polynomial

$$\varepsilon_r(z_1, \dots, z_k) = \sum_{i_1 < \dots < i_r} z_{i_1} \dots z_{i_r} = \phi^*(c_r),$$

it is the root polynomial of  $c_r = \sigma_{1,\dots,1}$  (cf., Theorem 2.2.1).

- (2) The Schur polynomial  $s_{m, \dots, m}$  with  $k$  components “ $m$ ” equals  $(z_1 \dots z_k)^m$ , it is the root polynomial of  $c_k^m$ .

### 2.3 Multivariate Cauchy Integral Formula for the Coefficients $\lambda_\alpha$

**Lemma 2.3.1** For any  $f \in \mathbb{Z}^S[z]$ ,  $z = (z_1, \dots, z_k)$ , and any  $k$ -partition  $\alpha$ , we have

$$\lambda_\alpha(f) = \frac{1}{k!(2\pi i)^k} \int_{T^k} f(z) \overline{s_\alpha(z)} V_\delta(z) \overline{V_\delta(z)} \frac{dz}{z},$$

where  $T^k = \{z \in \mathbb{C}^k : |z_1| = \dots = |z_k| = 1\}$  and  $\frac{dz}{z} = \frac{dz_1}{z_1} \dots \frac{dz_k}{z_k}$ .

*Proof* The monomials  $z^\alpha$  with  $\alpha = (\alpha_1, \dots, \alpha_k)$ ,  $\alpha_i \geq 0$ , form an orthonormal basis in  $\mathbb{C}[z]$  with respect to the inner product

$$\langle f_1, f_2 \rangle = \frac{1}{(2\pi i)^k} \int_{T^k} f_1(z) \overline{f_2(z)} \frac{dz}{z}.$$

It follows that for each pair of  $k$ -partitions  $\alpha$  and  $\beta$ ,  $\langle V_{\alpha+\delta}, V_{\beta+\delta} \rangle = k! \langle z^{\alpha+\delta}, z^{\beta+\delta} \rangle$ . Thus, according to Proposition 2.2.2, the Schur polynomials  $s_\alpha$  form an orthonormal basis in the vector space  $\mathbb{Z}^S[z] \otimes \mathbb{C}$ , with respect to the modified inner product

$$\langle f_1, f_2 \rangle_{\text{Sym}} = \frac{1}{k!(2\pi i)^k} \int_{T^k} f_1(z) \overline{f_2(z)} V_\delta(z) \overline{V_\delta(z)} \frac{dz}{z}, \tag{2.3.1}$$

namely,

$$\begin{aligned} \langle s_\alpha, s_\beta \rangle_{\text{Sym}} &= \left\langle \frac{V_{\alpha+\delta}}{V_\delta}, \frac{V_{\beta+\delta}}{V_\delta} \right\rangle_{\text{Sym}} = \frac{1}{k!(2\pi i)^k} \int_{T^k} V_{\alpha+\delta}(z) \overline{V_{\beta+\delta}(z)} \frac{dz}{z} \\ &= \frac{1}{k!} \langle V_{\alpha+\delta}, V_{\beta+\delta} \rangle = \langle z^{\alpha+\delta}, z^{\beta+\delta} \rangle. \end{aligned}$$

The claim of the Lemma follows now from  $\lambda_\alpha(f) = \langle f, s_\alpha \rangle_{\text{Sym}}$  and (2.3.1). □

**Corollary 2.3.2** For each  $h \in H^{2km}(G_k(\mathbb{C}^\infty))$  its value  $h([G_k(\mathbb{C}^{k+m})])$  on the fundamental class  $[G_k(\mathbb{C}^{k+m})] \in H_{2km}(G_k(\mathbb{C}^\infty))$  is given by

$$h([G_k(\mathbb{C}^{k+m})]) = \lambda_{m, \dots, m}(\phi^*(h)) = \frac{1}{k!(2\pi i)^k} \int_{T^k} \frac{\phi^*(h)(z)}{z^m} V_\delta(z) \overline{V_\delta(z)} \frac{dz}{z},$$

where  $\phi^*(h) \in \mathbb{Z}^S[z]$  is the root polynomial of  $h$  and  $z^m$  in the denominator stands for  $(z_1 \dots z_k)^m$ .

*Proof* We apply Lemma 2.3.1 to  $\alpha = (m, \dots, m)$  and use Example 2.2.2(2). □

### 2.4 Counting $(k - 1)$ -Planes on Projective Hypersurfaces

We denote by  $c_{\text{top}} \in H^*(G_k(\mathbb{C}^{k+m}))$  the top Chern class of the symmetric power  $\text{Sym}^d(\tau_{k,m}^*)$  of the dual to the tautological bundle  $\tau_{k,m}^*$  on  $G_k(\mathbb{C}^{k+m})$ , and by  $f_d(z) = \phi^*(c_{\text{top}}) \in \mathbb{Z}^S[z]$  the root polynomial of  $c_{\text{top}}$ . The number of  $(k - 1)$ -planes in a projective hypersurface can be evaluated as the following Chern number.

**Proposition 2.4.1** (e.g., [Debarre and Manivel 1998](#)) *Assume that  $X \subset P^{m+k-1}$  is a generic hypersurface of degree  $d \geq 1$ , and that  $mk = \binom{d+k-1}{k-1}$ . Then,  $X$  contains a finite number of projective  $(k - 1)$ -planes and this number is equal to  $c_{\text{top}}$  evaluated on the fundamental class  $[G_k(\mathbb{C}^{k+m})]$ .* □

Let us denote the above Chern number  $c_{\text{top}}([G_k(\mathbb{C}^{k+m})])$  by  $\mathcal{N}_{d,k}^{\mathbb{C}}$ . Proposition 2.4.1 together with Corollary 2.3.2 provides the following integral formula.

**Corollary 2.4.2** *If  $mk = \binom{d+k-1}{k-1}$ , then*

$$\mathcal{N}_{d,k}^{\mathbb{C}} = \lambda_{m,\dots,m}(f_d) = \frac{1}{k!(2\pi i)^k} \int_{T^k} \frac{f_d(z)}{z^{\mathbf{m}}} V_{\delta}(z) \overline{V_{\delta}}(z) \frac{dz}{z}.$$

□

To estimate this integral, we need the following well-known *root factorization formula* for  $f_d$ .

**Proposition 2.4.3** (e.g., [Debarre and Manivel 1998](#)) *For any  $d \geq 1$ ,*

$$f_d(z) = \prod_{\ell_1+\dots+\ell_k=d} (\ell_1 z_1 + \dots + \ell_k z_k). \tag{2.4.1}$$

□

We call the factors in the right hand side of (2.4.1) the *root factors*.

### 2.5 An Upper Bound

In this section we study the growth rate of the sequence  $\mathcal{N}_{d,k}^{\mathbb{C}}$  in the logarithmic scale.

**Proposition 2.5.1** *If we fix  $k \geq 1$  and vary  $d \geq 1$  so that  $\frac{1}{k} \binom{d+k-1}{k-1}$  is integer, then the invariant  $\mathcal{N}_{d,k}^{\mathbb{C}}$  defined in Sect. 2.4 has the following asymptotics as  $d \rightarrow \infty$ :*

$$\log(\mathcal{N}_{d,k}^{\mathbb{C}}) \leq \frac{1}{(k - 1)!} d^{k-1} \log d + O(d^{k-2} \log d).$$

*Proof* Since  $|\ell_1 x_1 + \dots + \ell_k x_k| \leq \ell_1 + \dots + \ell_k = d$  at each point of  $T^k$ , the integral formula given in Corollary 2.4.2 implies that there exists a constant  $C$  such that  $\mathcal{N}_{d,k}^{\mathbb{C}} \leq C d^{b(d)}$ , where  $b(d) = \binom{d+k-1}{k-1} = \frac{1}{(k-1)!} d^{k-1} + O(d^{k-2})$ . □



### 2.6 Conjecture

Some heuristic arguments make plausible to conjecture that the sequence  $\mathcal{N}_{d,k}^{\mathbb{C}}$  has the following logarithmic asymptotics:

$$\log(\mathcal{N}_{d,k}^{\mathbb{C}}) \sim \frac{1}{(k-1)!} d^{k-1} \log d \quad \text{for } k \text{ fixed and } d \rightarrow \infty.$$

## 3 Elements of Real Schubert Calculus

### 3.1 Orientability and the Euler Class for the Symmetric Powers

We denote the tangent bundle of a real Grassmannian  $G_k(\mathbb{R}^{k+m})$  by  $T_{k,m}$  and the tautological  $k$ -dimensional vector bundle over  $G_k(\mathbb{R}^{k+m})$  by  $\tau_{k,m}$ .

**Lemma 3.1.1** *For any  $k, m \geq 0$ , we have*

$$w_1(T_{k,m}) = (k+m)w_1(\tau_{k,m}).$$

*In particular,  $G_k(\mathbb{R}^{k+m})$  is orientable if and only if  $k+m$  is even.*

*Proof* Note that  $T_{k,m} = \text{Hom}(\tau_{k,m}, \tau_{k,m}^\perp)$ , where  $\tau_{k,m}^\perp$  is the  $m$ -dimensional vector bundle orthogonal to  $\tau_{k,m}$ , so that  $\tau_{k,m} + \tau_{k,m}^\perp$  is a trivial bundle and, in particular,  $w_1(\tau_{k,m}) = w_1(\tau_{k,m}^\perp)$ . Following the splitting principle, we write  $w_1(\tau_{k,m}) = \sum_{i=1}^k a_i$  and  $w_1(\tau_{k,m}^\perp) = \sum_{j=1}^m b_j$ , where  $a_i, b_j \in H^1(G_k(\mathbb{R}^{k+m}; \mathbb{Z}/2))$ , which gives

$$w_1(T_{k,m}) = \sum_{i,j} (a_i + b_j) = mw_1(\tau_{k,m}) + kw_1(\tau_{k,m}^\perp) = (k+m)w_1(\tau_{k,m}).$$

□

**Lemma 3.1.2** (1) *The vector bundle  $\text{Sym}^d(\tau_{k,m}^*)$  is orientable if and only if  $\binom{d+k-1}{k}$  is even.*

(2) *If  $\dim \text{Sym}^d(\tau_{k,m}^*) = \dim G_k(\mathbb{R}^{k+m})$ , then  $w_1(\text{Sym}^d(\tau_{k,m}^*)) = dm w_1(\tau_{k,m})$ . In particular, under the assumption that  $\dim \text{Sym}^d(\tau_{k,m}^*) = \dim G_k(\mathbb{R}^{k+m})$  the bundle  $\text{Sym}^d(\tau_{k,m}^*)$  is orientable if and only if  $dm$  is even.*

*Proof* For proving (1), we use the splitting principle and obtain the following expression for the total Stiefel-Whitney class,  $W_*$ , of the symmetric power  $\text{Sym}^d(\tau_{k,m})$ :

$$W_*(\text{Sym}^d(\tau_{k,m})) = \prod_{\ell_1+\dots+\ell_k=d} (1 + \ell_1 a_1 + \dots + \ell_k a_k)$$

with respect to  $w_1(\tau_{k,m}) = \sum_{i=1}^k a_i$ . Hence,

$$w_1(\text{Sym}^d(\tau_{k,m})) = \sum_{\ell_1+\dots+\ell_k=d} (\ell_1 a_1 + \dots + \ell_k a_k) = n(a_1 + \dots + a_k)$$

where

$$n = \frac{d}{k} \binom{d+k-1}{k-1} = \binom{d+k-1}{k},$$

from where  $w_1(\text{Sym}^d(\tau_{k,m})) = 0$  if and only if  $n = \binom{d+k-1}{k}$  is even.

To deduce (2), note that  $\binom{d+k-1}{k-1} = \dim \text{Sym}^d(\tau_{k,m}) = \dim G_k(\mathbb{R}^{k+m}) = km$  implies  $n = \frac{d}{k}km = dm$ . □

As an immediate consequence we obtain the following result.

**Proposition 3.1.3** *Assume that the dimension of the vector bundle  $\text{Sym}^d(\tau_{k,m}^*)$ , that is  $\binom{d+k-1}{k-1}$ , is equal to the dimension  $km$  of the Grassmannian  $G_k(\mathbb{R}^{k+m})$ . Assume also that  $k + m = dm \pmod 2$ . Then, the Euler number  $e(\text{Sym}^d(\tau_{k,m}^*))[G_k(\mathbb{R}^{k+m})]$  with respect to the local coefficient system twisted by  $w_1(G_k(\mathbb{R}^{k+m}))$  is an integer well-defined up to sign.* □

*Remarks* (1) The first assumption of Proposition 3.1.3 is always satisfied in what follows, since it simply signifies that the virtual dimension of the variety  $F_{k-1}(X)$  of  $(k - 1)$ -planes contained in a hypersurface  $X \subset P^{k+m-1}$  of degree  $d$  is 0. Note that for even  $d$  there exist real hypersurfaces  $X$  with  $X(\mathbb{R}) = \emptyset$ , for instance, the Fermat hypersurface, and thus, a signed count of the real  $(k - 1)$ -planes on such hypersurfaces (if invariant) gives 0. Note also that if  $km$  is odd, then even when the Euler number is well defined, it is equal to zero, as it happens for any real odd dimensional vector bundle.

For odd  $d$  the second assumption just means that  $k$  is even, so, the case of even  $k$  and odd  $d$  is the only one which makes sense to study. As the case  $k = 2$  was analyzed in [Finashin and Kharlamov \(2013\)](#), we are concerned in what follows with the case  $k = 4$ .

(2) Calculation of the Euler number  $e(\text{Sym}^d(\tau_{k,m}^*))[G_k(\mathbb{R}^{k+m})]$  can (and will) be done by passing to the Grassmannian  $\tilde{G}_k(\mathbb{R}^{k+m})$  of orientable  $k$ -planes and its (orientable) dual tautological bundle  $\tilde{\tau}_{k,m}^*$ , because of the relation

$$e(\text{Sym}^d(\tau_{k,m}^*))[G_k(\mathbb{R}^{k+m})] = \pm \frac{1}{2} e(\text{Sym}^d(\tilde{\tau}_{k,m}^*))[\tilde{G}_k(\mathbb{R}^{k+m})]. \tag{3.1.1}$$

### 3.2 Pontryagin Classes and the Real Root Polynomials

Let  $\tilde{G}_{2k}(\mathbb{R}^\infty)$  be the Grassmannian of oriented  $2k$ -planes in  $\mathbb{R}^\infty$ , and  $G_{2k}(\mathbb{R}^\infty)$  be that of non oriented  $2k$ -planes. Denote by  $\tilde{\vartheta} : \tilde{G}_{2k}(\mathbb{R}^\infty) \rightarrow G_{2k}(\mathbb{C}^\infty)$  the composition of the double covering  $\pi : \tilde{G}_{2k}(\mathbb{R}^\infty) \rightarrow G_{2k}(\mathbb{R}^\infty)$  and of the inclusion  $\vartheta : G_{2k}(\mathbb{R}^\infty) \subset G_{2k}(\mathbb{C}^\infty)$ . The following description of the integer cohomology ring  $H^*(\tilde{G}_{2k}(\mathbb{R}^\infty))/\text{Tors}$  is classical.

**Theorem 3.2.1** (e.g., [Brown 1982](#))

- (1) The ring  $H^*(G_{2k}(\mathbb{R}^\infty))/\text{Tors}$  is freely generated by the Pontryagin classes  $p_i = (-1)^i \vartheta^*(c_{2i}) \in H^{4i}(G_{2k}(\mathbb{R}^\infty))$ ,  $1 \leq i \leq k$ .
- (2) The ring  $H^*(\tilde{G}_{2k}(\mathbb{R}^\infty))/\text{Tors}$  is generated by the Pontryagin classes

$$\tilde{p}_i = \pi^*(p_i) = (-1)^i \tilde{\vartheta}^*(c_{2i}) \in H^{4i}(\tilde{G}_{2k}(\mathbb{R}^\infty)), \quad 1 \leq i \leq k$$

and the Euler class  $e_{2k} \in H^{2k}(\tilde{G}_{2k}(\mathbb{R}^\infty))$ , the only defining relation in these generators is  $\tilde{p}_k = e_{2k}^2$ . □

In the rest of the paper, we allow ambiguity and keep traditional notation  $p_i$  for the classes  $\tilde{p}_i$ , and moreover, use the same notation for the Pontryagin classes induced in the Grassmannians  $H^*(G_{2k}(\mathbb{R}^n))/\text{Tors}$  and  $H^*(\tilde{G}_{2k}(\mathbb{R}^n))/\text{Tors}$ .

The embedding  $SO_2^k \subset SO_{2k}$  given by  $2 \times 2$  special orthogonal matrices along the diagonal, gives a maximal torus in  $SO_{2k}$  and induces a map between the classifying spaces

$$(CP^\infty)^k = (BSO_2)^k \xrightarrow{\phi_{\mathbb{R}}} BSO_{2k} = \tilde{G}_{2k}(\mathbb{R}^\infty).$$

This map associates to a  $k$ -tuple  $(\xi_1, \dots, \xi_k)$  of  $SO_2$ -bundles their Whitney sum  $\xi_1 \oplus \dots \oplus \xi_k$ . We denote by  $x_1, \dots, x_k$  the standard (Euler class) generators of  $H^*((BSO_2)^k)$  and, for a given a class  $h \in H^*(\tilde{G}_{2k}(\mathbb{R}^\infty))$ , by a *real root polynomial* of  $h$  we mean the polynomial  $\phi_{\mathbb{R}}^*(h) \in \mathbb{Z}[x_1, \dots, x_k] = H^*((BSO_2)^k)$ .

**Lemma 3.2.2** Consider any class  $h \in H^*(G_{2k}(\mathbb{C}^\infty))$  and let  $h_{\mathbb{R}} = \tilde{\vartheta}^*(h) \in H^*(\tilde{G}_{2k}(\mathbb{R}^\infty))$  denote its pull-back. Then the real root polynomial  $\phi_{\mathbb{R}}^*(h_{\mathbb{R}})(x)$  is obtained from the complex one,  $\phi^*(h)(z)$ , by letting  $z_{2k-1} = -z_{2k} = x_k$ , that is

$$\phi_{\mathbb{R}}^*(h_{\mathbb{R}})(x_1, \dots, x_k) = \phi^*(h)(x_1, -x_1, \dots, x_k, -x_k).$$

*Proof* The tautological embedding  $SO_2 \rightarrow SU_2 \subset U_2$  is conjugate to the embedding  $SO_2 \rightarrow SU_2 \subset U_2$  defined as the composition of the *antidiagonal homomorphism*  $SO_2 \rightarrow SO_2 \times SO_2$ ,  $g \mapsto (g, g^{-1})$ , the isomorphism  $SO_2 \times SO_2 \cong U_1 \times U_1$  and the maximal torus inclusion  $U_1 \times U_1 \subset U_2$ . This leads to a commutative up to conjugation diagram

$$\begin{array}{ccc} (SO_2)^k & \longrightarrow & (U_1)^{2k} \\ \downarrow & & \downarrow \\ SO_{2k} & \longrightarrow & U_{2k} \end{array}$$

with the “vertical” block-diagonal inclusion maps, the  $k$ -th power of the antidiagonal homomorphism  $SO_2 \rightarrow SO_2 \times SO_2 \cong U_1 \times U_1$  at the “top”, and the tautological homomorphism at the “bottom”. Passing to the corresponding classifying spaces we obtain a commutative up to homotopy diagram

$$\begin{CD}
 (BSO_2)^k = (\mathbb{C}P^\infty)^k @>(\Delta_a)^k>> (\mathbb{C}P^\infty)^{2k} = (BU_1)^{2k} \\
 @V\phi_{\mathbb{R}}VV @VV\phi_{\mathbb{C}}V \\
 BSO_{2k} = \tilde{G}_{2k}(\mathbb{R}^\infty) @>\tilde{\vartheta}>> G_{2k}(\mathbb{C}^\infty) = BU_{2k}
 \end{CD}$$

where  $\Delta_a : \mathbb{C}P^\infty \rightarrow (\mathbb{C}P^\infty)^2$  is the *antidiagonal map*  $z \mapsto (z, \bar{z})$ . The induced map in cohomology yields a commutative diagram

$$\begin{CD}
 H^*(\mathbb{C}P^\infty)^k = \mathbb{Z}[x_1, \dots, x_k] @<(\Delta_a^*)^k<< \mathbb{Z}[z_1, \dots, z_{2k}] = H^*(\mathbb{C}P^\infty)^{2k} \\
 @A\phi_{\mathbb{R}}^*AA @AA\phi_{\mathbb{C}}^*A \\
 H^*(\tilde{G}_{2k}(\mathbb{R}^\infty)) @<\tilde{\vartheta}^*<< H^*(G_{2k}(\mathbb{C}^\infty))
 \end{CD}$$

where  $z_{2i-1} \mapsto x_i, z_{2i} \mapsto -x_i, i = 1, \dots, k$ . □

We consider two subrings of the ring  $\mathbb{Z}[x], x = (x_1, \dots, x_k)$ . The *Pontryagin ring*  $\mathbb{Z}^P[x] = \mathbb{Z}[x_1^2, \dots, x_k^2] \cap \mathbb{Z}^S[x]$  is formed by the symmetric polynomials in  $x_i^2, 1 \leq i \leq k$ , and the *Euler–Pontryagin ring*  $\mathbb{Z}^{EP}[x] = \mathbb{Z}[x_1^2, \dots, x_k^2, x_1 \dots x_k] \cap \mathbb{Z}^S[x]$ , where  $\mathbb{Z}[x_1^2, \dots, x_k^2, x_1 \dots x_k]$  is generated by the squares  $x_i^2$  and the product  $x_1 \dots x_k$ .

**Proposition 3.2.3** *For any  $k \geq 1$  and  $x = (x_1, \dots, x_k)$ :*

- (1) *the map  $\phi_{\mathbb{R}}^*$  yields a monomorphism  $H^*(\tilde{G}_{2k}(\mathbb{R}^\infty))/Tors \rightarrow \mathbb{Z}[x]$ ;*
- (2) *the image of  $\phi_{\mathbb{R}}^*$  is  $\mathbb{Z}^{EP}[x]$ ;*
- (3) *the image of  $\phi_{\mathbb{R}}^* \circ \tilde{\vartheta}^*$  is the image of  $\phi_{\mathbb{R}}^* \circ \pi^*$  and is  $\mathbb{Z}^P[x]$ ;*
- (4)  *$\phi_{\mathbb{R}}^*(p_i) = \varepsilon_i(x_1^2, \dots, x_k^2)$  and  $\phi_{\mathbb{R}}^*(e_{2k}) = x_1 \dots x_k$ .*

*Proof* Follows immediately from Lemma 3.2.2, Theorems 3.2.1 and 2.2.1. □

**Corollary 3.2.4** *The maps  $\phi_{\mathbb{R}}^* \circ \pi^*$  and  $\phi_{\mathbb{R}}^*$  identify the rings  $H^*(G_{2k}(\mathbb{R}^\infty))/Tors$  and  $H^*(\tilde{G}_{2k}(\mathbb{R}^\infty))/Tors$  with  $\mathbb{Z}^P[x]$  and  $\mathbb{Z}^{EP}[x]$ , respectively. The map*

$$\pi^* : H^*(G_{2k}(\mathbb{R}^\infty))/Tors \rightarrow H^*(\tilde{G}_{2k}(\mathbb{R}^\infty))/Tors$$

*induced by the projection  $\pi : \tilde{G}_{2k}(\mathbb{R}^\infty) \rightarrow G_{2k}(\mathbb{R}^\infty)$  is identified then with the inclusion homomorphism  $\mathbb{Z}^P[x] \rightarrow \mathbb{Z}^{EP}[x]$ .* □

### 3.3 Integral Schubert Cycles of Infinite Order in $G_{2k}(\mathbb{R}^\infty)$

As in the complex case, the  $k$ -partitions  $\alpha$  determine a CW-decomposition of  $G_k(\mathbb{R}^\infty)$  into *real Schubert cells*  $C_{\alpha, \mathbb{R}} = C_\alpha \cap G_k(\mathbb{R}^\infty)$  with the property  $C_{\alpha, \mathbb{R}} \subset G_k(\mathbb{R}^{k+m})$  if and only if  $\alpha_1 \leq m$ . The cells  $C_{\alpha, \mathbb{R}}$  lead in general only to  $\mathbb{Z}/2$ -homology classes

$$[C_{\alpha, \mathbb{R}}]_2 \in H_{|\alpha|}(G_k(\mathbb{R}^{k+m}); \mathbb{Z}/2) \rightarrow H_{|\alpha|}(G_k(\mathbb{R}^\infty); \mathbb{Z}/2), \quad \alpha_1 \leq m \leq \infty$$

(here and further on, the same notation is used for the classes in Grassmannians for different  $m$ , including  $m = \infty$ ).

For a  $k$ -partition  $\beta = (\beta_1, \dots, \beta_k)$ , denote by  $2\beta$  the  $k$ -partition  $(2\beta_1, \dots, 2\beta_k)$  and by  $\beta(2)$  the  $2k$ -partition  $(\beta_1, \beta_1, \dots, \beta_k, \beta_k)$ , where each of the  $\beta_i$  is repeated twice. We say that  $2k$ -partition  $\alpha$  is an *even partition* if it can be presented as  $\alpha = 2\beta(2)$ , and that  $\alpha$  is an *odd partition* if it has form  $\alpha = 2\beta(2) + \mathbf{1} = (2\beta_1 + 1, \dots, 2\beta_k + 1)$  for some  $k$ -partition  $\beta$ .

If  $\alpha$  is an even  $2k$ -partition, then the real Schubert cell  $C_{\alpha, \mathbb{R}}$  defines an integral homology class, which we denote

$$[C_{\alpha, \mathbb{R}}] \in H_{|\alpha|}(G_{2k}(\mathbb{R}^{2k+m}))/\text{Tors}, \quad \alpha_1 \leq m \leq \infty$$

(concerning the sign of  $[C_{\alpha, \mathbb{R}}]$ , in what follows we do not need to make any specific choice of the orientation).

**Theorem 3.3.1 (Pontryagin 1947)** *Assume that  $m \geq 0$  is even and  $r$  or  $2km - r$  is less than  $m$ . Then the Schubert classes  $[C_{\alpha, \mathbb{R}}]$  for all even  $2k$ -partitions  $\alpha$  of dimension  $|\alpha| = r$  form a basis in the group  $H_r(G_{2k}(\mathbb{R}^{2k+m}))/\text{Tors}$ .  $\square$*

*Remarks* Pontryagin’s claim [see Pontryagin (1947), Theorem 1] concerns the orientable Grassmannians, which is stronger than what we stated here. On the other hand, his claim covers only the case of  $2km - r < m$ . The case  $r < m$  can be derived from that one via Poincare duality, since the Poincare-dual Schubert cycles are represented by  $m$ -complementary even  $2k$ -partitions (for even  $m$ ). Indeed, for even  $m$  [in which case Grassmannian  $G_{2k}(\mathbb{R}^{2k+m})$  is orientable by Lemma 3.1.1], the intersection index of Schubert cycles  $[C_{\alpha, \mathbb{R}}] \circ [C_{\beta, \mathbb{R}}]$  in  $G_{2k}(\mathbb{R}^{2k+m})$  is  $\pm 1$  if  $\alpha$  and  $\beta$  are complementary even  $2k$ -partitions, and otherwise that index is 0.

### 3.4 Real Schur Polynomials

Letting  $m \rightarrow \infty$  in Theorem 3.3.1, we can conclude that the classes  $[C_{\alpha, \mathbb{R}}]$  for all even  $2k$ -partitions  $\alpha$  form an additive basis in  $H_*(G_{2k}(\mathbb{R}^\infty))/\text{Tors}$ . Let us introduce the dual additive basis,  $\{\sigma_{\alpha, \mathbb{R}}\}_{\text{even } \alpha}$ , in  $H^*(G_{2k}(\mathbb{R}^\infty))/\text{Tors}$ ; namely, let  $\sigma_{\alpha, \mathbb{R}}$  be the integral cohomology class taking value 1 on  $[C_{\alpha, \mathbb{R}}]$  and vanishing on the other integral Schubert classes. Then, we take the pull-back of  $\sigma_{\alpha, \mathbb{R}}$  by the double covering  $\pi : \tilde{G}_{2k}(\mathbb{R}^\infty) \rightarrow G_{2k}(\mathbb{R}^\infty)$ , that is  $\tilde{\sigma}_{\alpha, \mathbb{R}} = \pi^*(\sigma_{\alpha, \mathbb{R}}) \in H^*(\tilde{G}_{2k}(\mathbb{R}^\infty))$ , and define the *real Schur polynomial of an even  $2k$ -partition  $\alpha$*  as the root polynomial

$$s_{\alpha, \mathbb{R}} = \phi_{\mathbb{R}}^*(\tilde{\sigma}_{\alpha, \mathbb{R}}) \in \mathbb{Z}^P[x] \subset \mathbb{Z}^{EP}[x],$$

(cf., Corollary 3.2.4).

The relation between the real and the complex Schur polynomials can be described as follows. Consider the ring isomorphism  $T$  between the Chern ring

$$\mathbb{Z}^S[z_1, \dots, z_k] \cong H^*(G_k(\mathbb{C}^\infty))$$

and the Pontryagin ring

$$\mathbb{Z}^P[x_1, \dots, x_k] \cong H^*(G_{2k}(\mathbb{R}^\infty))/\text{Tors}$$

sending  $z_i$  to  $x_i^2$ , in terms of polynomial rings, and equivalently, the Chern classes  $c_i$  to the Pontryagin classes  $p_i, i = 1, \dots, k$ , in terms of cohomology rings.

**Proposition 3.4.1** (e.g., Fuks 1986, Section 2.3.E) *Let  $\beta$  be a  $k$ -partition and  $\alpha = 2\beta(2)$ . Then,  $c_1^{\gamma_1} \dots c_k^{\gamma_k}[C_\beta] = p_1^{\gamma_1} \dots p_k^{\gamma_k}[C_{\alpha, \mathbb{R}}]$  for any Chern class  $c_1^{\gamma_1} \dots c_k^{\gamma_k}$ , and the isomorphism  $T : \mathbb{Z}^S[z_1, \dots, z_k] \rightarrow \mathbb{Z}^P[x_1, \dots, x_k]$  sends the Schur polynomial  $s_\beta$  to the real Schur polynomial  $s_{\alpha, \mathbb{R}}$ .  $\square$*

Now we extend the definition of the real root polynomials to odd  $2k$ -partitions,  $\alpha' = \alpha + \mathbf{1}$  (where  $\alpha$  is an even partition) by letting  $\tilde{\sigma}_{\alpha', \mathbb{R}}$  be the product of  $\tilde{\sigma}_{\alpha, \mathbb{R}}$  by the Euler class  $e_{2k} \in H^{2k}(\tilde{G}_{2k}(\mathbb{R}^\infty))$  and define similarly the real Schur polynomial of  $\alpha'$  as the root polynomial

$$s_{\alpha', \mathbb{R}} = \phi_{\mathbb{R}}^*(\tilde{\sigma}_{\alpha', \mathbb{R}}) \in \mathbb{Z}^{EP}[z].$$

**Corollary 3.4.2** *For any even  $2k$ -partition  $\alpha = 2\beta(2)$ , and the odd partition  $\alpha' = \alpha + \mathbf{1}$  we have*

- (1)  $s_{\alpha, \mathbb{R}}(x_1, \dots, x_k) = s_\beta(x_1^2, \dots, x_k^2)$ ;
- (2)  $\tilde{s}_{\alpha', \mathbb{R}}(x_1, \dots, x_k) = x_1 \dots x_k s_\beta(x_1^2, \dots, x_k^2)$ ;
- (3) if  $\sigma_\beta \in H^{2n}(G_k(\mathbb{C}^\infty))$  is expressed as a polynomial  $\sigma_\beta = f(c_1, \dots, c_k)$ , then

$$\begin{aligned} \tilde{\sigma}_{\alpha, \mathbb{R}} &= f(p_1, \dots, p_k), \\ \tilde{\sigma}_{\alpha', \mathbb{R}} &= e_{2k} f(p_1, \dots, p_k). \end{aligned}$$

*Proof* A straightforward consequence of Propositions 3.4.1 and 3.2.3.  $\square$

**Corollary 3.4.3** *Assume that  $\alpha = \beta(2)$ ,  $\beta = 2\gamma + \mathbf{r}$ , where  $\gamma$  is any  $k$ -partition and  $\mathbf{r} = (r, \dots, r)$  is a  $k$ -partition with  $r$  equal to either 0 or 1. Then,*

$$s_{\alpha, \mathbb{R}}(x) = \frac{V_{\beta+2\delta}}{V_{2\delta}} = \frac{\begin{vmatrix} x_1^{\beta_1+2(k-1)} & x_1^{\beta_2+2(k-2)} & \dots & x_1^{\beta_k} \\ \dots & \dots & \dots & \dots \\ x_k^{\beta_1+2(k-1)} & x_k^{\beta_2+2(k-2)} & \dots & x_k^{\beta_k} \end{vmatrix}}{\begin{vmatrix} x_1^{2(k-1)} & x_1^{2(k-2)} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ x_k^{2(k-1)} & x_k^{2(k-2)} & \dots & 1 \end{vmatrix}}.$$

*Proof* It follows from Corollary 3.4.2 and Proposition 2.2.2, since  $V_{2\gamma}(x) = V_\gamma(x^2)$  and  $V_{2\gamma+\mathbf{r}}(x) = V_\gamma(x^2)(x_1 \dots x_k)^r$  for any  $k$ -partition  $\gamma$ .  $\square$

**Lemma 3.4.4** *The classes  $\tilde{\sigma}_{\alpha, \mathbb{R}}$  for all even and odd  $2k$ -partitions  $\alpha$  form an additive basis of the group  $H^*(\tilde{G}_{2k}(\mathbb{R}^\infty))/\text{Tors}$ . The corresponding real Schur polynomials  $\sigma_{\alpha, \mathbb{R}}$  form an additive basis in  $\mathbb{Z}^{EP}[x]$ .*

*Proof* Classes  $\tilde{\sigma}_{\alpha, \mathbb{R}}$  for all even  $2k$ -partitions  $\alpha$ , being dual to a basis in  $H_*(G_{2k}(\mathbb{R}^\infty))$ , give an additive basis in  $\pi^*(H^*(G_{2k}(\mathbb{R}^\infty)/\text{Tors})) \subset H^*(\tilde{G}_{2k}(\mathbb{R}^\infty))/\text{Tors}$ . This subgroup is generated by products of the Pontryagin classes  $p_i$ , by Theorem 3.2.1. By the same theorem,  $H^*(\tilde{G}_{2k}(\mathbb{R}^\infty))/\text{Tors}$  is generated by 1 and the Euler class  $e_{2k}$  as a module over  $H^*(G_{2k}(\mathbb{R}^\infty)/\text{Tors})$ , so, the classes  $\tilde{\sigma}_{\alpha, \mathbb{R}}$  for all odd  $2k$ -partitions  $\alpha$ , together with  $\tilde{\sigma}_{\alpha, \mathbb{R}}$  for even  $2k$ -partitions  $\alpha$  form a basis in  $H^*(\tilde{G}_{2k}(\mathbb{R}^\infty))/\text{Tors}$ .  $\square$

### 3.5 The Duality Between the Classes $[\tilde{C}_{\alpha, \mathbb{R}}]$ and $\tilde{\sigma}_{\alpha, \mathbb{R}}$ in $\tilde{G}_{2k}(\mathbb{R}^\infty)$

In  $\tilde{G}_{2k}(\mathbb{R}^\infty)$ , we have a pair of cells,  $C_{\alpha, \mathbb{R}}^\pm$ , that form the pull-back of  $C_{\alpha, \mathbb{R}}$  with respect to the double covering  $\pi : \tilde{G}_{2k}(\mathbb{R}^\infty) \rightarrow G_{2k}(\mathbb{R}^\infty)$ . The closure of the union  $\tilde{C}_{\alpha, \mathbb{R}} = C_{\alpha, \mathbb{R}}^+ \cup C_{\alpha, \mathbb{R}}^-$  yields an integral class of infinite order  $[\tilde{C}_{\alpha, \mathbb{R}}] \in H_{|\alpha|}(\tilde{G}_{2k}(\mathbb{R}^\infty))/\text{Tors}$  if  $\alpha$  is an even or an odd  $2k$ -partition. Here, the orientations of  $C_{\alpha, \mathbb{R}}^\pm$  are chosen to be respected by the covering  $\pi$  (and, therefore, by the deck transformation), and for even  $2k$ -partitions  $\alpha$  we just take the sum of cells  $C_{\alpha, \mathbb{R}}^\pm$ , that is the pull-back  $[\tilde{C}_{\alpha, \mathbb{R}}] = \pi^!([C_{\alpha, \mathbb{R}}])$ , while for odd  $2k$ -partitions  $\alpha$  we take the difference  $[\tilde{C}_{\alpha, \mathbb{R}}] = [C_{\alpha, \mathbb{R}}^+ - C_{\alpha, \mathbb{R}}^-]$ , where the sign “-” signifies reversion of the orientation of  $C_{\alpha, \mathbb{R}}^-$ .

*Remarks* These classes  $[\tilde{C}_{\alpha, \mathbb{R}}]$  form a basis in  $H_*(\tilde{G}_{2k}(\mathbb{R}^\infty); \mathbb{Q})$ . Furthermore, they are divisible by 2 in the integral homology, and their halves form an additive basis in  $H_*(\tilde{G}_{2k}(\mathbb{R}^\infty))/\text{Tors}$ . This can be derived from Theorem 1 in Pontryagin (1947), which treats indeed a more subtle case of finite Grassmannians  $\tilde{G}_k(\mathbb{R}^{k+m})$ .

**Lemma 3.5.1** *Consider a Schubert class  $[\tilde{C}_{\alpha, \mathbb{R}}] \in H_{|\alpha|}(\tilde{G}_{2k}(\mathbb{R}^\infty))/\text{Tors}$ , where  $\alpha = (\alpha_1, \dots, \alpha_{2k})$  is an even or an odd  $2k$ -partition. Then the class  $[\tilde{C}_{\alpha, \mathbb{R}}] \cap e_{2k} \in H_{|\alpha|-2k}(\tilde{G}_{2k}(\mathbb{R}^\infty))$  is represented, up to sign and torsion homology elements, by the Schubert class  $\tilde{C}_{\alpha', \mathbb{R}}$ , where  $\alpha' = \alpha - \mathbf{1} = (\alpha_1 - 1, \dots, \alpha_{2k} - 1)$ , if  $\alpha_{2k} \geq 1$ . If  $\alpha_{2k} = 0$ , then  $[\tilde{C}_{\alpha, \mathbb{R}}] \cap e_{2k} = 0$ .*

*Proof* The Euler class  $e_{2k}$  restricted to  $\tilde{G}_{2k}(\mathbb{R}^{2k+m})$  is known to be dual to a fundamental class  $[\tilde{G}_{2k}(\mathbb{R}^{2k+m-1})]$  of  $\tilde{G}_{2k}(\mathbb{R}^{2k+m-1})$ . Therefore, the cap product with  $e_{2k}$  is realized by the intersection of  $[\tilde{C}_{\alpha, \mathbb{R}}]$  with  $[\tilde{G}_{2k}(\mathbb{R}^{2k+m-1})]$  for  $m$  sufficiently big with respect to  $|\alpha|$ , which corresponds to subtraction  $\mathbf{1}$  from  $\alpha$ .  $\square$

**Proposition 3.5.2** *Assume that  $\alpha$  and  $\beta$  are  $2k$ -partitions, each one is either even or odd, and  $|\alpha| = |\beta| = n$ . Then  $[\tilde{C}_{\alpha, \mathbb{R}}] \cap \tilde{\sigma}_{\beta, \mathbb{R}}$  is equal to  $\pm 2$  if  $\alpha = \beta$ , and 0 if  $\alpha \neq \beta$ .*

*Proof* For even  $\alpha$ , both  $[\tilde{C}_{\alpha, \mathbb{R}}]$  and  $\tilde{\sigma}_{\alpha, \mathbb{R}}$  are in the invariant subspaces of the covering involution, acting in the integral homology and cohomology, respectively. For odd  $\alpha$  these classes are both in the skew-invariant subspaces. This implies that  $[\tilde{C}_{\alpha, \mathbb{R}}] \cap \tilde{\sigma}_{\beta, \mathbb{R}} = 0$  if  $\alpha$  and  $\beta$  have different parity.

If both  $\alpha$  and  $\beta$  are even, then  $[\tilde{C}_{\alpha, \mathbb{R}}] = \pi^!([C_{\alpha, \mathbb{R}}])$  and  $\tilde{\sigma}_{\beta, \mathbb{R}} = \pi^! \sigma_{\beta, \mathbb{R}}$ . Thus,  $[\tilde{C}_{\alpha, \mathbb{R}}] \cap \tilde{\sigma}_{\beta, \mathbb{R}} = 2([C_{\alpha, \mathbb{R}}] \cap \sigma_{\beta, \mathbb{R}}) = \pm 2$  if  $\alpha = \beta$  and 0 otherwise, since  $\sigma_{\alpha, \mathbb{R}}$  was defined as the dual to  $[C_{\alpha, \mathbb{R}}]$  up to the choice of orientation of the latter.

If both  $\alpha$  and  $\beta$  are odd, then  $\alpha = \alpha' + \mathbf{1}$ ,  $\beta = \beta' + \mathbf{1}$ , where  $\alpha'$  and  $\beta'$  are even. It follows that

$$[\tilde{C}_{\alpha, \mathbb{R}}] \cap \tilde{\sigma}_{\beta, \mathbb{R}} = [\tilde{C}_{\alpha, \mathbb{R}}] \cap (e_{2k} \cup \tilde{\sigma}_{\beta', \mathbb{R}}) = ([\tilde{C}_{\alpha, \mathbb{R}}] \cap e_{2k}) \cap \tilde{\sigma}_{\beta', \mathbb{R}} = \tilde{C}_{\alpha', \mathbb{R}} \cap \tilde{\sigma}_{\beta', \mathbb{R}}$$

where the last identity is due to Lemma 3.5.1. So, this case is reduced to the previous one. □

### 3.6 Calculation of the Real Schur Coefficients

By Lemma 3.4.4, each class  $h \in H^*(\tilde{G}_{2k}(\mathbb{R}^\infty))/\text{Tors}$  can be decomposed into an integer linear combination

$$h = \sum_{\text{even and odd } 2k\text{-partitions } \alpha} \lambda_\alpha \tilde{\sigma}_{\alpha, \mathbb{R}}.$$

This gives the corresponding decomposition of the root polynomial  $f = \phi_{\mathbb{R}}^* h \in \mathbb{Z}^{EP}[z]$ :

$$f(x) = \sum_{\text{even and odd } 2k\text{-partitions } \alpha} \lambda_\alpha s_{\alpha, \mathbb{R}}.$$

The coefficients  $\lambda_\alpha = \lambda_\alpha(h) \in \mathbb{Z}$  will be called *real Schur coefficients* of  $h$ , and of its root polynomial  $f$ . If  $h \in H^*(\tilde{G}_{2k}(\mathbb{R}^\infty))$ , we put  $\lambda_\alpha(h) = \lambda_\alpha(h/\text{Tors})$ .

**Lemma 3.6.1** *Assume that  $\alpha$  is an even or odd  $2k$ -partition, and  $h \in H^n(\tilde{G}_{2k}(\mathbb{R}^\infty))$ . Then  $\lambda_\alpha(h) = \pm \frac{1}{2} [\tilde{C}_{\alpha, \mathbb{R}}] \cap h$  (and in particular,  $\lambda_\alpha(h) = 0$  if  $|\alpha| \neq n$ ).*

*Proof* By Proposition 3.5.2,  $\{\tilde{\sigma}_{\alpha, \mathbb{R}}\}$  and  $\{\frac{1}{2}[\tilde{C}_{\alpha, \mathbb{R}}]\}$  form dual, up to sign, bases in  $H_*(\tilde{G}_{2k}(\mathbb{R}^\infty); \mathbb{Q})$ . □

**Corollary 3.6.2** *For any class  $h \in H^*(\tilde{G}_{2k}(\mathbb{R}^\infty))$ , we have*

$$\frac{1}{2} h[\tilde{G}_{2k}(\mathbb{R}^{m+2k})] \in \mathbb{Z} = \pm \lambda_{\mathbf{m}},$$

where  $\mathbf{m}$  is the constant  $2k$ -partition  $(m, \dots, m)$ .

*Proof* We apply Lemma 3.6.1 to the Grassmannian  $\tilde{G}_{2k}(\mathbb{R}^{m+2k})$ , which is a Schubert variety for  $2k$ -partition  $\mathbf{m}$ . □

Like in Lemma 2.3.1, we can find the coefficients  $\lambda_\alpha$  using a residue formula.

**Lemma 3.6.3** *For any  $f \in \mathbb{Z}^{EP}[x]$ ,  $x = (x_1, \dots, x_k)$ , and an even or odd  $2k$ -partition  $\alpha$ , we have*

$$\lambda_\alpha(f) = \pm \frac{1}{k!(2\pi i)^k} \int_{T^k} f(x) \overline{s_{\alpha, \mathbb{R}}(x)} V_{2\delta}(x) \overline{V_{2\delta}(x)} \frac{dx}{x}$$

where  $T^k = \{x \in \mathbb{C}^k : |x_1| = \dots = |x_k| = 1\}$ .



*Proof* The proof is the same as that of Lemma 2.3.1, except that we use an expression of the real Schur polynomials in Corollary 3.4.3 instead of the complex one, which was given in Proposition 2.2.2.  $\square$

In the case of  $\lambda_m$  that we are interested in, we obtain a formula analogous to the one in Corollary 2.3.2.

**Corollary 3.6.4** *For any class  $h \in H^{2km}(\tilde{G}_{2k}(\mathbb{R}^\infty))$  its value on the fundamental class  $[\tilde{G}_{2k}(\mathbb{R}^{m+2k})] \in H_{2km}(\tilde{G}_{2k}(\mathbb{R}^\infty))$  is determined by the following integral*

$$\frac{1}{2}h \cap [\tilde{G}_{2k}(\mathbb{R}^{m+2k})] = \pm \lambda_m(\phi_{\mathbb{R}}^*(h)) = \frac{1}{k!(2\pi i)^k} \int_{T^k} \frac{\phi_{\mathbb{R}}^*(h)(x)}{x^m} V_{2\delta}(x) \overline{V_{2\delta}(x)} \frac{dx}{x},$$

where  $\phi_{\mathbb{R}}^*(h) \in \mathbb{Z}^{EP}[x]$  is the real root polynomial of  $h$ .  $\square$

### 4 Proof of Theorem 1.1.1

#### 4.1 Multivariate Integral Formula for the Number of Real 3-Planes on Hypersurfaces

Throughout this section  $d \geq 1$  is odd, as it is in Theorem 1.1.1. We start with a real analogue of the root factorization formula (2.4.1) that takes the following form, in which we call it the *real root factorization formula*.

**Proposition 4.1.1** *Let  $f_d(x)$  denote the real root polynomial of the Euler class  $e_N = e_N(\text{Sym}^d(\tilde{\tau}_{2k,\infty}^*)) \in H^N(\tilde{G}_{2k}(\mathbb{R}^\infty))$ ,  $N = \binom{d+2k-1}{2k-1}$ . Then*

$$(-1)^{\frac{N}{2}} f_d^2(x) = \prod_{\ell_1 + \ell_{\bar{1}} + \dots + \ell_k + \ell_{\bar{k}} = d, \ell_i, \ell_{\bar{i}} \geq 0} ((\ell_1 - \ell_{\bar{1}})x_1 + \dots + (\ell_k - \ell_{\bar{k}})x_k).$$

The factors in the above product formula will be called *real root factors*.

*Remarks* Since we did not fix an orientation of  $\text{Sym}^d(\tilde{\tau}_{2k,\infty}^*)$ , the polynomial  $f_d$  is well-defined only up to sign. In what follows [see formula (4.2.2)] we will determine the sign so that the corresponding Euler number, which we denote  $\mathcal{N}_d^e$ , becomes positive.

*Proof* The operation of taking  $d$ -th symmetric power defines the maps  $\phi_d^{\mathbb{R}} : \tilde{G}_{2k}(\mathbb{R}^\infty) \rightarrow \tilde{G}_N(\mathbb{R}^\infty)$  and  $\phi_d^{\mathbb{C}} : G_{2k}(\mathbb{C}^\infty) \rightarrow G_N(\mathbb{C}^\infty)$  commuting (up to homotopy) with the tautological embedding maps  $\tilde{\vartheta}_{2k}$  and  $\tilde{\vartheta}_N$

$$\begin{array}{ccc} \tilde{G}_{2k}(\mathbb{R}^\infty) & \xrightarrow{\phi_d^{\mathbb{R}}} & \tilde{G}_N(\mathbb{R}^\infty) \\ \tilde{\vartheta}_{2k} \downarrow & & \downarrow \tilde{\vartheta}_N \\ G_{2k}(\mathbb{C}^\infty) & \xrightarrow{\phi_d^{\mathbb{C}}} & G_N(\mathbb{C}^\infty). \end{array}$$

Then, the result follows from Lemma 3.2.2 and Proposition 2.4.3, since  $e^2(\tilde{\tau}_{N,\infty}^*) = p_{\frac{N}{2}}(\tilde{\tau}_{N,\infty}^*) = (-1)^{\frac{N}{2}} \tilde{\theta}_N^*(c_N(\tau_{N,\infty}^*))$ . □

Theorem 1.1.1 concerns the case  $k = 2$ , and in this case Proposition 4.1.1 reads as follows

$$f_d^2(x) = \prod_{\ell_1+\ell_1+\ell_2+\ell_2=d, \ell_i, \ell_i \geq 0} ((\ell_1 - \ell_1)x_1 + (\ell_2 - \ell_2)x_2). \tag{4.1.1}$$

**Theorem 4.1.2** *Assume that  $X \subset P^{m+3}$  is a generic real hypersurface of odd degree  $d$ , where  $\binom{d+3}{3} = 4m$ . Then the number of real 3-planes in  $X$  is finite and bounded from below by*

$$\mathcal{N}_d^e = \pm \frac{1}{2(2\pi i)^2} \int_{T^2} \frac{f_d(x)}{x^m} V_{2\delta}(x) \overline{V_{2\delta}(x)} \frac{dx}{x},$$

where  $T^2 = \{x \in \mathbb{C}^2 : |x_1| = |x_2| = 1\}$  and  $f_d(x)$  is the polynomial determined by the formula (4.1.1).

*Proof* It follows from Corollary 3.6.4 and Proposition 4.1.1. □

### 4.2 Factorization of $f_d$ in the Case of 3-Planes

The polynomial  $g_d(x) = f_d^2(x) \in \mathbb{Z}^{EP}[x]$  can be written as

$$g_d = g_{d,0} g_{d,1} \cdots g_{d, \frac{d-1}{2}},$$

where  $g_{d,i}, 0 \leq i \leq \frac{d-1}{2}$ , is the product of those real root factors that give  $|\ell_1 - \ell_1| + |\ell_2 - \ell_2| = d - 2i$  (or, equivalently,  $\min(\ell_1, \ell_1) + \min(\ell_2, \ell_2) = i$ ).

**Lemma 4.2.1** *We have  $g_{d,i} = g_{d-2i,0}^{i+1}$ , and thus  $g_d = g_{d,0} g_{d-2,0}^2 g_{d-4,0}^3 \cdots$*

*Proof* The real root factors in  $g_{d,i}$  are the same as in  $g_{d-2i,0}$ , but appear as many times as there are partitions  $i = s_1 + s_2$ , with  $s_j = \min\{\ell_j, \ell_j\} \geq 0, j = 1, 2$ . □

It is convenient to group the real root factors of  $g_d$  by letting

$$h_d(x_1, x_2) = \prod_{\ell_1, \ell_2 \geq 1, \ell_1 + \ell_2 = d} (\ell_1 x_1 + \ell_2 x_2)$$

and

$$\tilde{h}_d(x_1, x_2) = \prod_{\ell_1, \ell_2 \geq 0, \ell_1 + \ell_2 = d} (\ell_1 x_1 + \ell_2 x_2) = d^2 x_1 x_2 h_d(x_1, x_2).$$

**Lemma 4.2.2** *The following identity holds for any odd  $d \geq 1$ :*

$$g_{d,0} = [d^2 x_1 x_2 h_d(x_1, x_2) h_d(x_1, -x_2)]^2 = \left[ \frac{1}{d^2 x_1 x_2} \tilde{h}_d(x_1, x_2) \tilde{h}_d(x_1, -x_2) \right]^2.$$

*Proof* The real root factors of  $g_{d,0}$  that are taken from  $h_d(\pm x_1, \pm x_2)$  go in groups of four elements, respectively to the four combinations of signs before the coefficients, if the both coefficients do not vanish [which corresponds in (4.1.1) to the cases when both  $|\ell_1 - \ell_{\bar{1}}|$  and  $|\ell_2 - \ell_{\bar{2}}|$  are non-zero and  $\ell_1 + \ell_{\bar{1}} + \ell_2 + \ell_{\bar{2}} = d$ ]. Their product converts then to  $[h_d(x_1, x_2) h_d(x_1, -x_2)]^2$ . The additional term  $d^2 x_1 x_2$  is involved because if just one of the coefficients  $l_1, \bar{l}_1, l_2, \bar{l}_2$  is non-zero (and hence, is equal to  $d$ ), then only two combinations of the signs give root factors contributing to  $g_{d,0}$ .  $\square$

**Corollary 4.2.3** *The polynomial  $g_d$  is a complete square of the polynomial*

$$\begin{aligned} f_d(x) &= \pm [d^2 x_1 x_2 h_d(x_1, x_2) h_d(x_1, -x_2)] \\ &\quad [(d - 2)^2 x_1 x_2 h_{d-2}(x_1, x_2) h_{d-2}(x_1, -x_2)]^2 \dots \\ &= \pm [d!!(d - 2)!! \dots]^2 (x_1 x_2)^{\frac{d^2-1}{8}} \prod_{i=0}^{\frac{d-1}{2}} [h_{d-2i}(x_1, x_2) h_{d-2i}(x_1, -x_2)]^{i+1} \end{aligned}$$

$\square$

*Example 4.2.4* If  $d = 1$ , then  $h_1 = 1$  (there are no suitable real root factors) and  $g_1 = g_{1,0} = x_1^2 x_2^2$ . In the first nontrivial case,  $d = 3$ , we have

$$\begin{aligned} h_3(x_1, \pm x_2) &= (2x_1 \pm x_2)(x_1 \pm 2x_2) = (\pm 5x_1 x_2 + 2(x_1^2 + x_2^2)), \\ g_{3,0}(x_1, x_2) &= [(9x_1 x_2)(5x_1 x_2 + 2(x_1^2 + x_2^2))(-5x_1 x_2 + 2(x_1^2 + x_2^2))]^2, \\ g_3(x_1, x_2) &= g_{3,0} g_{1,0}^2 = [9x_1^3 x_2^3 (4(x_1^2 + x_2^2)^2 - 25x_1^2 x_2^2)]^2, \\ \pm f_3(x_1, x_2) &= 9x_1^3 x_2^3 (4(x_1^2 + x_2^2)^2 - 25x_1^2 x_2^2). \end{aligned}$$

Applying Theorem 4.1.2 and using that  $V_{2\delta} \bar{V}_{2\delta} = -\frac{(x_1^2 - x_2^2)^2}{x_1^2 x_2^2}$ , we conclude that it is the coefficient at  $(x_1 x_2)^7$  in  $-\frac{1}{2} f_3(x_1^2 - x_2^2)^2$  that gives us the signed count of real 3-planes on a generic real 7-dimensional cubic hypersurface. This coefficient is equal to  $-\frac{1}{2}(9 \times 42) = -189$ , which gives us  $\mathcal{N}_3^e = 189$ . An alternative way to get this coefficient is to apply Corollary 3.4.3 to get the following expressions for the real Schur polynomials,  $s_{(5,5,5,5),\mathbb{R}} = (x_1 x_2)^5$  and

$$s_{(7,7,3,3),\mathbb{R}} = \frac{V_{(7,3)+(2,0)}}{V_{(2,0)}} = \frac{\begin{vmatrix} x_1^9 & x_1^3 \\ x_2^9 & x_2^3 \end{vmatrix}}{\begin{vmatrix} x_1^2 & 1 \\ x_2^2 & 1 \end{vmatrix}} = (x_1 x_2)^3 (x_1^4 + x_1^2 x_2^2 + x_2^4),$$

which allows us to get the decomposition  $f_3(x_1, x_2) = 9[4s_{(7,7,3,3),\mathbb{R}} - 21s_{(5,5,5,5),\mathbb{R}}]$  with  $\lambda_{5,5,5,5} = -189$  and gives us the same result  $\mathcal{N}_3^e = 189$ .

Finally, we can write  $\mathcal{N}_3^{\mathbb{R}} \geq \mathcal{N}_3^{\mathbb{R}, \min} \geq \mathcal{N}_3^e = 189$ , which implies that a real 7-dimensional cubic has generically at least 189 real 3-planes. [See also [Finashin and Kharlamov \(2013\)](#), Subsection 6.2, for one more method to find  $\mathcal{N}_3^e = 189$  by a direct calculation of the characteristic number  $e_{20}[\tilde{G}_4(\mathbb{R}^9)]$ , where  $e_{20} = e_4^3(4p_1^2 - 25e_4^2)$  is represented by the root polynomial  $f_3(x)$ .] To compare with, one can find  $\mathcal{N}_3^{\mathbb{C}} = 321489$ .

Further computations (using the program Macoley2) show that  $\mathcal{N}_5^e = 37655727525$ , whereas  $\mathcal{N}_5^{\mathbb{C}} = 64127725294951805931404297113125$ .

As in Lemma 4.2.2, the root factors  $(\ell_1x_1 + \ell_2x_2)(\ell_2x_1 + \ell_1x_2)$  in  $h_{d-2i}(x_1, x_2)$  can be grouped together with the factors  $(\ell_1x_1 - \ell_2x_2)(\ell_2x_1 - \ell_1x_2)$  in  $h_{d-2i}(x_1, -x_2)$  to give us the product

$$\begin{aligned}
 (\ell_1^2x_1^2 - \ell_2^2x_2^2)(\ell_2^2x_1^2 - \ell_1^2x_2^2) &= \ell_1^2\ell_2^2(x_1^2 + x_2^2)^2 - (\ell_1^2 + \ell_2^2)^2x_1^2x_2^2 \\
 &= (\ell_1\ell_2x_1x_2)^2 \left[ \left(\frac{x_1}{x_2} + \frac{x_2}{x_1}\right)^2 - \left(\frac{\ell_1}{\ell_2} + \frac{\ell_2}{\ell_1}\right)^2 \right] \tag{4.2.1}
 \end{aligned}$$

This product enters in  $f_d$  in the power  $(i + 1)$ . Therefore, we can put

$$\begin{aligned}
 f_d(x) &= [d!!(d - 2)!! \dots]^2 (x_1x_2)^{\frac{d^2-1}{8}} \prod_{\ell \in \mathcal{L}_d} \\
 &\quad \left[ \ell_1^2\ell_2^2x_1^2x_2^2 \left( -\frac{(x_1^2 + x_2^2)^2}{x_1^2x_2^2} + \frac{(\ell_1^2 + \ell_2^2)^2}{\ell_1^2\ell_2^2} \right) \right]^{i+1} \tag{4.2.2}
 \end{aligned}$$

where

$$\mathcal{L}_d = \left\{ (\ell_1, \ell_2, i) \mid \ell_1 + \ell_2 = d - 2i, \ell_1, \ell_2 \geq 1, 0 \leq i \leq \frac{d-1}{2} \right\}.$$

Note that in the initial definition of  $f_d$  it was defined only up to sign (see the remark after Proposition 4.1.1), and from now on we eliminate this ambiguity by prescribing to  $f_d$  the sign given by formula (4.2.2).

We can also summarize the above formulae as follows:

$$\begin{aligned}
 &h_d(x_1, x_2)h_d(x_1, -x_2) \\
 &= \pm \prod_{\ell_1 + \ell_2 = d, \ell_1, \ell_2 \geq 1} (x_1x_2)^2 (\ell_1\ell_2)^2 \left[ -\frac{(x_1^2 + x_2^2)^2}{x_1^2x_2^2} + \frac{(\ell_1^2 + \ell_2^2)^2}{\ell_1^2\ell_2^2} \right], \\
 f_d(x) &= C_d x_1^N x_2^N \prod_{(\ell_1, \ell_2, i) \in \mathcal{L}_d} (\ell_1\ell_2)^{2(i+1)} \left[ -\frac{(x_1^2 + x_2^2)^2}{x_1^2x_2^2} + \frac{(\ell_1^2 + \ell_2^2)^2}{\ell_1^2\ell_2^2} \right]^{i+1}, \tag{4.2.3}
 \end{aligned}$$

where  $C_d = [d!!(d - 2)!! \dots]^2$  and  $N = \frac{1}{2} \binom{d+3}{3}$ .

### 4.3 Positivity and the Maximum

**Proposition 4.3.1** *The function  $F_d(x) = \frac{f_d(x)}{(x_1x_2)^m}$ , where  $m = \frac{1}{4} \binom{d+3}{3}$  (cf. Theorem 4.1.2) takes positive real values (and, in particular, does not vanish) at each point of  $T^2$ . The maximal value of  $F_d(x)$  on  $T^2$  is achieved along the two-component curve  $x_2 = \pm ix_1$ , and this value equals*

$$M_d = C_d \prod_{(\ell_1, \ell_2, i) \in \mathcal{L}_d} (\ell_1^2 + \ell_2^2)^{2(i+1)} = \prod_{i=0}^{\frac{d-1}{2}} \prod_{\ell=0}^{\frac{d-1}{2}-i} (\ell^2 + (d - 2i - \ell)^2)^{2(i+1)}. \tag{4.3.1}$$

*Proof* According to (4.2.3), the factors of  $(x_1x_2)^{2(1-d)}h_d(x_1, x_2)h_d(x_1, -x_2)$  are equal to

$$\ell_1^2 \ell_2^2 [-(t + t^{-1})^2 + (k + k^{-1})^2], \quad \text{where } t = \frac{x_1}{x_2}, |t| = 1, \text{ and } k = \frac{\ell_1}{\ell_2}, k > 0.$$

Since  $t + t^{-1}$  is real for  $|t| = 1$ , these factors are real, and thus  $F_d(x)$  is real for all  $x \in T^2$ . Since  $0 \leq (t + t^{-1})^2 \leq 4 \leq (k + k^{-1})^2$ ,  $F_d(x)$  can only vanish for  $k + k^{-1} = 2$ , that is for  $\ell_1 = \ell_2$ , which is impossible under our assumption that  $d$  is odd. The maximal value  $(k + k^{-1})^2$  of the factor  $\ell_1^2 \ell_2^2 |-(t + t^{-1})^2 + (k + k^{-1})^2|$  is achieved as  $(t + t^{-1})^2$  takes its minimal value (equal to 0), that is along the circles  $x_2 = \pm ix_1$ .

Since these circles are common for all the partitions  $(\ell_1, \ell_2)$  involved in the formula for  $F_d$ , the product of all the factors achieves its maximum value along the same circles, and this value is as indicated in the statement. □

**Proposition 4.3.2** *The sequence  $M_d$  given by (4.3.1) has the following asymptotic growth in the log-scale:*

$$\log M_d = \frac{1}{12}d^3 \log d + O(d^3).$$

*Proof* According to the first order Euler–Maclaurin formula, the following relations hold for any function  $f \in C^1[0, d]$  and any  $r \in \mathbb{N}$ :

$$\sum_{\ell=0}^r f(\ell) = \int_0^r f(t) dt + \frac{1}{2}[f(0) + f(r)] + \rho_1(r),$$

where  $\rho_1(r) = \int_0^r (x - [x] - \frac{1}{2})|f'(t)| dt$  and, hence,

$$|\rho_1(r)| \leq \frac{1}{2} \int_0^r |f'(t)| dt.$$

We apply the latter bound to  $f(t) = \log(t^2 + (d - t)^2)$  and  $r = \frac{d-1}{2}$ . Since  $f' < 0$  on  $[0, r]$ , this gives

$$|\rho_1(r)| \leq \frac{1}{2} \left( f(0) - f\left(\frac{d-1}{2}\right) \right) < \log d.$$

The logarithm of the product  $J_d = \log\left(\prod_{\ell=0}^{\frac{d-1}{2}} (\ell^2 + (d - \ell)^2)\right)$  can be written in the form

$$J_d = \sum_{\ell=0}^{\frac{d-1}{2}} \log(\ell^2 + (d - \ell)^2) = \int_0^{\frac{d-1}{2}} \log(t^2 + (d - t)^2) dt + \log d + \frac{1}{2} \log\left(\frac{d^2 + 1}{2}\right) + \rho_1(r)$$

where in its turn

$$\begin{aligned} \int_0^{\frac{d-1}{2}} \log(t^2 + (d - t)^2) dt &= t \log(t^2 + (d - t)^2) \Big|_0^{\frac{d-1}{2}} - \int_0^{\frac{d-1}{2}} t \frac{2t + 2(t - d)}{t^2 + (d - t)^2} dt \\ &= \frac{d - 1}{2} \log\left(\frac{d^2 + 1}{2}\right) - \tau(d). \end{aligned}$$

By  $\tau(d)$  we denoted here the integral that can be evaluated as follows:

$$\begin{aligned} \tau(d) &= \int_0^{\frac{d-1}{2}} t \frac{2t + 2(t - d)}{t^2 + (d - t)^2} dt = \int_0^{\frac{d-1}{2}} \left[ 2 + \frac{\frac{d}{2}(4t - 2d)}{2t^2 - 2dt + d^2} - \frac{d^2}{2t^2 - 2dt + d^2} \right] dt \\ &= (d - 1) + \frac{d}{2} \log(t^2 + (d - t)^2) \Big|_0^{\frac{d-1}{2}} - d \arctan\left(\frac{2t}{d} - 1\right) \Big|_0^{\frac{d-1}{2}} \\ &= (d - 1) + \frac{d}{2} \left[ \log\left(\frac{d^2 + 1}{2}\right) - 2 \log d \right] + d \left( \arctan \frac{1}{d} - \frac{\pi}{4} \right). \end{aligned}$$

This implies a uniform estimate  $|\tau(d)| \leq md + M$  with the constants  $m, M$  independent of  $d$ . Thus, we can write  $J_d = d \log d + O(d)$ ,  $O(d) \leq m'd + M$ , with a constant  $m'$  independent on  $d$  and make an estimate

$$J_{d-2i} = (d - 2i) \log(d - 2i) + O(d - 2i), \quad O(d - 2i) \leq m'd + M.$$

Then

$$\begin{aligned} 2(i + 1)J_{d-2i} &= \log\left(\prod_{\ell=0}^{\frac{d-1}{2}-i} (\ell^2 + (d - 2i - \ell)^2)^{2(i+1)}\right) \\ &= 2(i + 1)(d - 2i) \log(d - 2i) + (i + 1)O(d), \end{aligned}$$

and a bound  $(i + 1)O(d) \leq O(d^2)$  gives

$$\begin{aligned} \log M_d &= 2 \sum_{i=0}^{\frac{d-1}{2}} [(i + 1)(d - 2i) \log(d - 2i) + O(d^2)] \\ &= 2 \int_0^{\frac{d-1}{2}} (t + 1)(d - 2t) \log(d - 2t) dt \\ &\quad + O(d^3) = 2[-\frac{2}{3}t^3 + \frac{1}{2}(d - 2)t^2 + dt] \log(d - 2t) \Big|_0^{\frac{d-1}{2}} + K_d + O(d^3) \\ &= K_d + O(d^3) \end{aligned}$$

where

$$\begin{aligned} K_d &= 4 \int_0^{\frac{d-1}{2}} \frac{\frac{2}{3}t^3 - \frac{1}{2}(d - 2)t^2 - dt}{2t - d} dt \\ &= \frac{2}{3} \int_0^{\frac{d-1}{2}} \left( 2t^2 - \frac{d - 6}{2}t - \frac{d(d+6)}{4} \right) dt \\ &\quad - \frac{2}{3} \frac{d^2(d + 6)}{4} \int_0^{\frac{d-1}{2}} \frac{dt}{2t - d} = O(d^3) - \frac{1}{12}d^2(d + 6) \log(d - 2t) \Big|_0^{\frac{d-1}{2}} \end{aligned}$$

and finally

$$\log M_d = \frac{1}{12}d^3 \log d + O(d^3). \quad \square$$

### 4.4 Asymptotics

Consider a sequence of functions

$$F_d(x) = \prod_{(\ell,r) \in \mathcal{L}_d} \phi_{\ell,r}(x), \quad x \in T^2, \quad \text{where } d \in \mathbb{N}.$$

Assume that  $\phi_{\ell,r}(x) > 0$  for all  $x \in T^2$ ,  $d$  and  $(\ell, r)$ , and that the maximum of  $\phi_{\ell,r}$  over  $T^2$  is equal to  $M_{\ell,r}$  and is achieved on a submanifold  $L \subset T^2$ , which is common for all  $d$  and  $\ell$ . This is the case in our setting, where  $\ell = (\ell_1, \ell_2)$  and

$$\begin{aligned} \phi_{\ell,r}(x_1, x_2) &= \left[ (\ell_1^2 + \ell_2^2)^2 - \ell_1^2 \ell_2^2 \frac{(x_1^2 + x_2^2)^2}{x_1^2 x_2^2} \right]^{r+1} \\ &= (\ell_1^2 + \ell_2^2)^{2(r+1)} \left[ 1 - \frac{4\ell_1^2 \ell_2^2}{(\ell_1^2 + \ell_2^2)^2} \sin^2 \phi \right]^{r+1} \end{aligned}$$

and  $M_{\ell,r} = (\ell_1^2 + \ell_2^2)^{r+1}$ . Namely, if  $x_j = e^{i\phi_j}$ ,  $\phi_j \in [0, 2\pi]$ ,  $j = 1, 2$ , then

$$\frac{(x_1^2+x_2^2)^2}{x_1^2x_2^2} = \left(\frac{x_1}{x_2} + \frac{x_1}{x_2}\right)^2 = (e^{i(\phi_1-\phi_2)} + e^{-i(\phi_1-\phi_2)})^2 = 4 \cos^2(\phi_1 - \phi_2) = 4 \sin^2 \phi$$

where  $\phi = \frac{\pi}{2} - \phi_1 + \phi_2$ . Thus,  $(x_1, x_2) = (e^{i\phi_1}, e^{i\phi_2})$  is a point of maximum if and only if  $\phi = 0$  or  $\phi = \pi$ , so that in our case  $L$  consists of two disjoint circles  $\phi_1 - \phi_2 = \pm \frac{\pi}{2}$ .

We are concerned about the asymptotics, in the logarithmic scale, of the integral

$$I_d = \frac{1}{8(\pi i)^2} \int_{T^2} F_d(x)W(x) \frac{dx}{x}$$

for a certain function  $W(x) \geq 0$  on  $T^2$ , namely, for  $W(x) = V_{2\delta}(x)\overline{V}_{2\delta}(x)$

$$\begin{aligned} W(x) &= \left| \begin{matrix} 1 & x_1^2 \\ 1 & x_2^2 \end{matrix} \right| \left| \begin{matrix} 1 & x_1^{-2} \\ 1 & x_2^{-2} \end{matrix} \right| = -\frac{(x_1^2 - x_2^2)^2}{x_1^2x_2^2} = -[e^{2i(\phi_1-\phi_2)} + e^{-2i(\phi_1-\phi_2)} - 2] \\ &= 2(1 - \cos 2(\phi_1 - \phi_2)) = 2(1 - \sin 2\phi), \quad \phi = \frac{\pi}{2} - \phi_1 + \phi_2. \end{aligned}$$

**Proposition 4.4.1** *Under the above choice of functions  $F_d$  the following log-scale asymptotic development holds:*

$$\log I_d = \frac{1}{12}d^3 \log d + O(d^3).$$

*Proof* An upper bound

$$\log I_d \leq \frac{1}{12}d^3 \log d + O(d^3)$$

follows from  $\log M_d = \frac{1}{12}d^3 \log d + O(d^3)$  (see Proposition 4.3.2).

By the localization principle, the lower bound follows from positivity of  $F_d$  and  $W$ . Indeed, let us consider the tubular neighborhood  $U$  of  $L$  defined by  $|\sin 2\phi| \leq \frac{1}{2}$ , that is  $|\phi| \leq \phi_0 = \frac{\pi}{12}$ . Then  $1 \leq W(x) \leq 3$  for  $x \in U$ , and

$$\begin{aligned} \log I_d + \log 8\pi^2 &\geq \log \int_U F_d(x)W(x) \frac{dx}{x} \\ &\geq \log[\text{Area}(U) \min_U F_d(x) \min_U W(x)] \\ &\geq \log\left(\frac{4}{3}\pi^2\right) + \log\left(\prod_{(\ell_1, \ell_2, r) \in \mathcal{L}_d} (\ell_1^2 + \ell_2^2)^{2r} \left[1 - \frac{4\ell_1^2\ell_2^2}{(\ell_1^2 + \ell_2^2)^2} \sin^2 \phi_0\right]^r\right) \\ &= \log\left(\frac{4}{3}\pi^2\right) + \log M_d + \mathcal{R} \end{aligned}$$



where

$$\mathcal{R} = \log \prod \left[ 1 - \frac{4\ell_1^2 \ell_2^2}{(\ell_1^2 + \ell_2^2)^2} \sin^2 \phi_0 \right]^r.$$

Finally, note that

$$|\mathcal{R}| \leq c \sum_{(\ell_1, \ell_2, r) \in \mathcal{L}_d} r \frac{4\ell_1^2 \ell_2^2}{(\ell_1^2 + \ell_2^2)^2} \sin^2 \phi_0 \leq cd|\mathcal{L}_d| = O(d^3),$$

where  $|\mathcal{L}_d| = \binom{d+2}{2} = O(d^2)$  is the cardinality of  $\mathcal{L}_d$  and  $c > 0$  is a constant independent of  $d$ . □

### 4.5 Proof of Theorem 1.1.1

The asymptotic upper bound for  $\mathcal{N}_d^{\mathbb{C}}$  is established in Proposition 2.5.1. The positivity of  $\mathcal{N}_d^e$  follows from Theorem 4.1.2 and the positivity of  $F_d$  (see Proposition 4.3.1). The asymptotic expression for  $\mathcal{N}_d^e$  follows from Theorem 4.1.2 and Proposition 4.4.1.

## 5 Concluding Remarks

### 5.1 Counting 3-Planes in the Complete Intersections

The above approach can be also applied to counting 3-planes in the complete intersections, except that the formulas become more cumbersome and the asymptotics is difficult to disclose. For simplicity, let us restrict ourselves to the intersections of cubic hypersurfaces. Recall that for one cubic hypersurface (see Example 4.2.4) the Euler class is given by

$$e_{20} = e(\text{Sym}^3(\tilde{\tau}_{4,\infty}^*)) = \pm 9e^3(25e^2 - 4p_1^2) \in H^{20}(\tilde{G}_4(\mathbb{R}^\infty)).$$

An intersection  $X$  of  $r$  generic real cubic hypersurfaces in  $P^{m+3}$  contains a finite number of 3-planes if the dimension  $20r$  of  $e_{20}^r$  is equal to  $4m = \dim \tilde{G}_4(\mathbb{R}^{m+4})$  [see, for example, Debarre and Manivel (2000)], that is if  $m = 5r$ . The signed count of real 3-planes in  $X$  gives a number  $\mathcal{N}_{3,\dots,3}^e$ , which is the half of the corresponding count of oriented 3-planes that is  $e_{20}^r[\tilde{G}_4(\mathbb{R}^{m+4})]$ , so we obtain

$$2\mathcal{N}_{3,\dots,3}^e = \pm 9^r e^{3r} (25e^2 - 4p_1^2)^r [\tilde{G}_4(\mathbb{R}^{m+4})].$$

Due to the multiplicative structure of the Euler–Pontryagin ring, the right-hand side is equal to the result of substitution of the Catalan numbers  $C_k$  instead of  $t^k$  in the polynomial  $9^r (25 - 4t)^r$ . Standard manipulation with generating functions and radii of convergence shows that the rate of growth of this sequence is linear in the logarithmic scale:

$$\log \mathcal{N}_{3,\dots,3}^e \sim 4r \log 3.$$

To count the number  $\mathcal{N}_{3,\dots,3}^{\mathbb{C}}$  of complex 3-planes on a generic intersection of cubic hypersurfaces we can use the multivariate integral Cauchy formula (cf., Corollary 2.4.2):

$$\frac{1}{4!(2\pi i)^4} \int_{T^4} \frac{\prod_{\ell_1+\dots+\ell_4=3} (\ell_1 z_1 + \dots + \ell_4 z_4)^r}{z^{\mathbf{m}}} V_{\delta}(z) \overline{V_{\delta}(z)} \frac{dz}{z}.$$

Applying to this integral the saddle point version of the Laplace method, namely, by deforming the integration cycle locally keeping the points of the maximal absolute value of the product (this value is equal to  $3^{20}$ ) but making the values of the product real at each point of a small neighborhood of the locus of maxima, we obtain

$$\log \mathcal{N}_{3,\dots,3}^{\mathbb{C}} \sim 20r \log 3 \sim 5 \log \mathcal{N}_{3,\dots,3}^e.$$

### 5.2 Another Enumerative Problem with the 3-Planes

Note that the Schubert cell  $C_{2,2} \subset G_4(\mathbb{C}^{2n+4})$  is formed by the projective 3-planes in  $P^{2n+3}$  which intersect  $P^{2n-1} \subset P^{2n+3}$  along a line. Therefore, if we choose a generic set  $S = \{S_1, \dots, S_{2n}\}$  of projective  $(2n - 1)$ -dimensional subspaces  $S_i \subset P^{2n+3}$ , then the number  $\mathcal{N}_{\boxplus 2n}^{\mathbb{C}}$  of 3-planes  $L \subset P^{2n+3}$  such that the intersections  $L \cap S_i, 1 \leq i \leq 2n$ , are lines, can be found by evaluation of the power of the Schubert class  $\sigma_{2,2}$ :

$$\mathcal{N}_{\boxplus 2n}^{\mathbb{C}} = \sigma_{2,2}^{2n}[G_4(\mathbb{C}^{2n+4})].$$

In the real setting, we define the number  $\mathcal{N}_{\boxplus 2n}^{\mathbb{R}}$  similarly, by counting the real 3-planes intersecting a generic set  $S$  of real subspaces  $S_i \subset P^{2n+3}$  along lines. This number depends on the choice of a generic set  $S$ , and we denote by  $\mathcal{N}_{\boxplus 2n}^{\min}$  the minimum of  $\mathcal{N}_{\boxplus 2n}^{\mathbb{R}}$  with respect to all generic choices of such  $S$ . The number  $\sigma_{2,2,\mathbb{R}}^{2n}[G_4(\mathbb{R}^{2n+4})]$  can be interpreted as the signed count of the real 3-planes, and thus its absolute value, which we denote by  $\mathcal{N}_{\boxplus 2n}^e$ , provides an estimate

$$\mathcal{N}_{\boxplus 2n}^e \leq \mathcal{N}_{\boxplus 2n}^{\min} \leq \mathcal{N}_{\boxplus 2n}^{\mathbb{R}} \leq \mathcal{N}_{\boxplus 2n}^{\mathbb{C}}.$$

Here,  $\sigma_{2,2,\mathbb{R}} \in H^4(G_4(\mathbb{R}^{2n+4}))$  is nothing but the Pontryagin class  $p_1$ . The following result shows that for this enumerative problem we get once more a fixed proportion in the logarithmic scale between the number of real solutions and the number of complex ones.

**Theorem 5.2.1** *Numbers  $\mathcal{N}_{\boxplus 2n}^{\mathbb{C}}$  and  $\mathcal{N}_{\boxplus 2n}^e$  have the logarithmic asymptotics*

$$\begin{aligned} \log \mathcal{N}_{\boxplus 2n}^{\mathbb{C}} &= 2n \log 20 + o(n), \\ \log \mathcal{N}_{\boxplus 2n}^e &= 2n \log 2 + o(n). \end{aligned}$$

In particular,

$$\log \mathcal{N}_{\boxplus^{2n}}^e \sim \frac{1}{\log_2 20} \log \mathcal{N}_{\boxplus^{2n}}^{\mathbb{C}}.$$

**Proposition 5.2.2** *The signed count of the real 3-planes intersecting each of the given 2n generic (2n - 1)-planes in  $\mathbb{R}P^{2n+3}$  along a line gives the following answer:*

$$p_1^{2n}(\tau_{4,2n}^*)[G_4(\mathbb{R}^{2n+4})] = \frac{1}{n+1} \binom{2n}{n}.$$

*Proof* In the Pontryagin ring  $H^*(G_4(\mathbb{R}^{2n+4}))/\text{Tors}$  we have (cf., Proposition 3.4.1)

$$p_2^n(\tau_{4,2n}^*)[G_4(\mathbb{R}^{2n+4})] = \sigma_{2,2,2,2,\mathbb{R}}^n[G_4(\mathbb{R}^{2n+4})] = \sigma_{2n,2n,2n,2n,\mathbb{R}}[G_4(\mathbb{R}^{2n+4})] = 1.$$

Therefore, our statement would follow from  $p_1^{2n} = \frac{1}{n+1} \binom{2n}{n} p_2^n$ .

On the other hand, there is a similar well known identity  $c_1^{2n} = \frac{1}{n+1} \binom{2n}{n} c_2^n$  in  $H^*(G_2(\mathbb{C}^{n+2}))$  which follows easily from the Pieri rule. So, it is left to refer to Proposition 3.4.1, or just to use the ring isomorphism between  $H^*(G_4(\mathbb{R}^{2n+4}))/\text{Tors}$  and

$$H^*(G_2(\mathbb{C}^{n+2})) = \mathbb{Z}[c_1, c_2, \tilde{c}_1, \dots, \tilde{c}_n] / \{(1 + c_1 + c_2)(1 + \tilde{c}_1 + \dots + \tilde{c}_n) = 1\}$$

that puts in correspondence  $c_i$  and  $p_i, i = 1, 2$ . □

**Corollary 5.2.3** *Given a generic set  $\{S_1, \dots, S_{2n}\}$  of  $(2n - 1)$ -planes in  $\mathbb{R}P^{2n+3}$ , there exist at least  $\frac{1}{n+1} \binom{2n}{n}$  real 3-planes that intersect each of the  $S_i$ 's along a line.*

Now, let us look for the asymptotical behavior of the number of complex solutions in the same Schubert problem as that in Proposition 5.2.2.

**Proposition 5.2.4** *The following asymptotic approximation holds:*

$$\log \sigma_{2,2}^{2n}[G_4(\mathbb{C}^{2n+4})] = 2n \log 20 + o(n).$$

*Proof* Applying Corollary 2.3.2 to  $h = \sigma_{2,2}^{2n}$  we obtain

$$\sigma_{2,2}^{2n}[G_4(\mathbb{C}^{2n+4})] = \frac{1}{24(2\pi i)^4} \int_{T^4} \frac{s_{2,2}^{2n}}{x^{2n}} \frac{V^2(x)}{x^3} \frac{dx}{x} = \frac{1}{24(2\pi i)^4} \int_{T^4} f(x)g^n(x) \frac{dx}{x}$$

where  $f(x) = \frac{V^2(x)}{x^3} = V(x)\overline{V(x)}$  is real and non-negative, as well as

$$g(x) = \frac{s_{2,2}^2}{x^2} = s_{2,2}\overline{s_{2,2}} = |s_{2,2}|^2.$$

Thus, we deal with a kind of Laplace integrals and can use the following result, which follows for instance from Maslov and Fedoryuk (1989), Theorem 2.1 (where the asymptotic is given under the conditions much weaker than the analyticity that we require below).

**Lemma 5.2.5** *Assume that  $f$  and  $g$  are real analytic functions taking non-negative values on a compact domain  $R \subset \mathbb{R}^d$ . Let  $M$  be the maximal value of  $g$  on  $R$ . Assume in addition that  $f$  and  $g$  do not vanish identically. Then,*

$$F(n) = \int_R f(x)g^n(x)dx$$

has the following logarithmic asymptotics

$$\lim_{n \rightarrow \infty} \frac{\log F(n)}{n} = \log M. \quad \square$$

Passing to the polar coordinates,  $x = \exp(i\theta)$ , we obtain

$$\frac{1}{24(2\pi i)^4} \int_R f(\exp(i\theta))g(\exp(i\theta))^n d\theta,$$

where  $R = [0, 2\pi]^4$ , and the maximal value of  $g(x) = |s_{2,2}(x)|^2$  is  $20^2$ , since the sum of the coefficients of  $s_{2,2} = x_1^2x_2^2 + \dots + x_1^2x_3x_4 + \dots + 2x_1x_2x_3x_4$  is 20.  $\square$

*Proof of Theorem 5.2.1* It follows from Proposition 5.2.4 combined with Stirling formula applied to Proposition 5.2.2.  $\square$

### 5.3 Multivariate Integral Formula for the Number of Real $(2k - 1)$ -Planes on Hypersurfaces

Theorem 4.1.2 and its proof extends easily to the case of counting  $2k - 1$ -planes with any  $k \in \mathbb{N}$ .

**Theorem 5.3.1** *Assume that  $X \subset P^{m+3}$  is a generic real hypersurface of odd degree  $d$  and that  $\binom{d+2k-1}{2k-1} = 2km$ . Then the number,  $\mathcal{N}_d^{\mathbb{R}}$ , of real  $(2k - 1)$ -subspaces in  $X$  is finite and bounded from below by the number  $\mathcal{N}_d^e \geq 0$  that is given by the multivariate integral formula*

$$\mathcal{N}_d^e = \pm \frac{1}{k!(2\pi i)^k} \int_{T^k} \frac{f_d(x)}{x^{\mathbf{m}}} V_{2\delta}(x) \overline{V_{2\delta}(x)} \frac{dx}{x},$$

where  $f_d(x)$  is the polynomial satisfying the formula of Proposition 4.1.1.

Some heuristic arguments applied to the above integral formula suggest a conjecture that  $\mathcal{N}_d^e$  is non zero for each odd  $d$  and any  $k$ , and that, for  $k$  fixed and odd  $d$  growing to infinity, the following asymptotic relation holds (cf., Conjecture 2.6)

$$\log \mathcal{N}_d^e \sim \frac{1}{2(2k - 1)!} d^{2k-1} \log d.$$

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