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RESEARCH EXPOSITION

The Conley Conjecture and Beyond

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Abstract This is (mainly) a survey of recent results on the problem of the existence of infinitely many periodic orbits for Hamiltonian diffeomorphisms and Reeb flows. We focus on the Conley conjecture, proved for a broad class of closed symplectic manifolds, asserting that under some natural conditions on the manifold every Hamiltonian diffeomorphism has infinitely many (simple) periodic orbits. We discuss in detail the established cases of the conjecture and related results including an analog of the conjecture for Reeb flows, the cases where the conjecture is known to fail, the question of the generic existence of infinitely many periodic orbits, and local geometrical conditions that force the existence of infinitely many periodic orbits. We also show how a recently established variant of the Conley conjecture for Reeb flows can be applied to prove the existence of infinitely many periodic orbits of a low-energy charge in a non-vanishing magnetic field on a surface other than a sphere.

Keywords Periodic orbits · Hamiltonian diffeomorphisms and Reeb flows · Conley conjecture · Floer and contact homology · Twisted geodesic or magnetic flows

Mathematics Subject Classification 53D40 · 37J10 · 37J45 · 37J55

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1 Introduction

Hamiltonian systems tend to have infinitely many periodic orbits. For many phase spaces, every system, without any restrictions, has infinitely many simple periodic orbits. Moreover, even if not holding unconditionally, this is still a C^∞-generic property of Hamiltonian systems for the majority of phase spaces. Finally, for some phase spaces, a system has infinitely many simple periodic orbits when certain natural local conditions are met.

This paper is mainly a survey focusing on this phenomenon for Hamiltonian diffeomorphisms and Reeb flows. The central theme of the paper is the so-called Conley conjecture, proved for a broad class of closed symplectic manifolds and asserting that under some natural conditions on the manifold every Hamiltonian diffeomorphism has infinitely many (simple) periodic orbits. We discuss in detail the established cases of the conjecture and related results, including an analog of the conjecture for Reeb flows, and also the manifolds for which the conjecture is known to fail. In particular, we investigate local geometrical conditions that force globally the existence of infinitely many periodic orbits and consider the question of the generic existence of infinitely many periodic orbits.

We also briefly touch upon the applications to dynamical systems of physical origin. For instance, we show how a recently established variant of the Conley conjecture for Reeb flows can be used to prove the existence of infinitely many simple periodic orbits of a low-energy charge in a non-vanishing magnetic field on a surface other than a sphere.

Our perspective on the problem and the methods we use here are mainly Morse theoretic, broadly understood, and homological. In this framework, the reasons for the existence of periodic orbits lie at the interplay between local dynamical features of the system and the global behavior of the homology "counting" the periodic orbits, e.g., Floer, symplectic or contact homology.

This is not the only perspective on the subject. For instance, in dimensions two and three, one can alternatively use exceptionally powerful methods of low-dimensional dynamics (see, e.g., Franks 1992, 1996; Franks and Handel 2003; Le Calvez 2006) and holomorphic curves (see, e.g., Bramham and Hofer 2012; Hofer et al. 1998, 2003). In higher dimensions, however, most of the results on this class of problems rely on homological methods.

It is important to note that for Hamiltonian diffeomorphisms, in contrast with the classical setting of geodesic flows on a majority of manifolds as in Gromoll and Meyer (1969), the existence of infinitely many simple periodic orbits is not forced by the homological growth. Likewise, the local dynamical features we consider here are usually of different flavor from, say, homoclinic intersections or elliptic fixed points often used in dynamics to infer under favorable circumstances the existence of infinitely many periodic orbits. There is no known single unifying explanation for the ubiquity of Hamiltonian systems with infinitely many periodic orbits. Even in the cases where the Conley conjecture holds, this is usually a result of several disparate phenomena.

The survey can be read at three levels. First of all, we give a broad picture, explain the main ideas, results, and conjectures in a non-technical way, paying attention not only to what has been proved but also to what is not known. This side of the survey requires very little background in symplectic and contact topology and dynamics from the reader. However, we also give the necessary technical details and conditions when stating the most important results. Although we recall the relevant definitions in due course, this level of the survey is intended for a more expert reader. Finally, in several instances, we attempt to explain the main ideas of the proofs or even to sketch the arguments. In particular, in Sect. 3 we outline the proof of the Conley conjecture; here we freely use Floer homology and some other, not entirely standard, symplectic topological tools.

The survey is organized as follows. In Sect. 2, we discuss the Conley conjecture (its history, background, and the state of the art) and the generic existence results, and also set the conventions and notation used throughout the paper. A detailed outline of the proof is, as has been mentioned above, given in Sect. 3. The rest of the paper is virtually independent of this section. We discuss the Conley conjecture and other related phenomena for Reeb flows in Sect. 4 and applications of the contact Conley conjecture to twisted geodesic flows, which govern the motion of a charge in a magnetic field, in Sect. 5. Finally, in Sect. 6, we turn to the manifolds for which the Conley conjecture fails and, taking the celebrated Frank's theorem (see Franks 1992, 1996) as a starting point, show how certain local geometrical features of a system can force the existence of infinitely many periodic orbits. Here we also briefly touch upon the problem of the existence of infinitely many simple periodic orbits for symplectomorphisms and for some other types of "Hamiltonian"systems.

2 Conley Conjecture

2.1 History and Background

As has been pointed out in the introduction, for many closed symplectic manifolds, every Hamiltonian diffeomorphism has infinitely many simple periodic orbits and, in fact, simple periodic orbits of arbitrarily large period whenever the fixed points are isolated. This unconditional existence of infinitely many periodic orbits is often referred to as the Conley conjecture. The conjecture was, indeed, formulated by Conley (1984) for tori, and since then it has been a subject of active research focusing on establishing the existence of infinitely many periodic orbits for broader and broader classes of symplectic manifolds or Hamiltonian diffeomorphisms.

The Conley conjecture was proved for the so-called weakly non-degenerate Hamiltonian diffeomorphisms in Salamon and Zehnder (1992) (see also Conley and Zehnder 1983b) and for all Hamiltonian diffeomorphisms of surfaces other than S^2 in Franks and Handel (2003) (see also Le Calvez 2006). In its original form for the tori, the conjecture was established in Hingston (2009) (see also Mazzucchelli 2013), and the case of an arbitrary closed, symplectically aspherical manifold was settled in Ginzburg (2010). The proof was extended to rational, closed symplectic manifolds M with $c_1(TM)|_{\pi_2(M)} = 0$ in Ginzburg and Gürel (2009b), and the rationality requirement was then eliminated in Hein (2012). In fact, after Salamon and Zehnder (1992), the main difficulty in establishing the Conley conjecture for more and more general manifolds with aspherical first Chern class, overcome in this series of works, lied in proving the conjecture for totally degenerate Hamiltonian diffeomorphisms not covered by Salamon and Zehnder (1992). (The internal logic in Franks and Handel (2003) and Le Calvez (2006), relying on methods from low-dimensional dynamics, was somewhat different.) Finally, in Ginzburg and Gürel (2012) and Chance et al. (2013), the Conley conjecture was proved for negative monotone symplectic manifolds. (The main new difficulty here was in the non-degenerate case.)

Two other variants of the Hamiltonian Conley conjecture have also been investigated. One of them is the existence of infinitely many periodic orbits for Hamiltonian diffeomorphisms with displaceable support; see, e.g., Frauenfelder and Schlenk (2007), Gürel (2008), Hofer and Zehnder (1994), Schwarz (2000) and Viterbo (1992). Here the form ω is usually required to be aspherical, but the manifold M is not necessarily closed. The second one is the Lagrangian Conley conjecture or, more generally, the Conley conjecture for Hamiltonians with controlled behavior at infinity on cotangent bundles (see Hein 2011; Long 2000; Lu 2009, 2011; Mazzucchelli 2011) or even some twisted cotangent bundles (see Frauenfelder et al. 2012). In this survey, however, we focus mainly on the case of closed manifolds.

The Conley conjecture looks deceptively similar to the well-known conjecture that every closed simply connected Riemannian manifold (e.g., S^n) carries infinitely many non-trivial closed geodesics. However, this appears to be a very different problem than the Conley conjecture, for the latter does not distinguish trivial and non-trivial orbits. For instance, the proof of the Lagrangian Conley conjecture for the pure kinetic energy Hamiltonian simply detects the constant geodesics. We will further discuss the connection between the two conjectures in Sects. 4 and 6.

What makes the Conley conjecture difficult and even counterintuitive from a homological perspective is that there seems to be no obvious homological reason for the existence of infinitely many simple periodic orbits. As we have already mentioned, there is no homological growth: the Floer homology of a Hamiltonian diffeomorphism does not change under iterations and remains isomorphic, up to a Novikov ring, to the homology of the manifold. (In that sense, the difficulty *is* similar to that in proving the existence of infinitely many non-trivial closed geodesics on, say, S^n where the rank of the homology of the free loop space remains bounded as a function of the degree.)

Ultimately, one can expect the Conley conjecture to hold for the majority of closed symplectic manifolds. There are, however, notable exceptions. The simplest one is S^2: an irrational rotation of S^2 about the z axis has only two periodic orbits, which are also the fixed points; these are the poles. In fact, any manifold that admits a Hamiltonian torus action with isolated fixed points also admits a Hamiltonian diffeomorphism with finitely many periodic orbits. For instance, such a diffeomorphism is generated by a generic element of the torus. In particular, flag manifolds (hence the complex projective spaces and the Grassmannians), and, more generally, most of the coadjoint orbits of compact Lie groups as well as symplectic toric manifolds all admit Hamiltonian diffeomorphisms with finitely many periodic orbits. In dimension two, there are also such examples with interesting dynamics. Namely, there exist area preserving diffeomorphisms of S^2 with exactly three ergodic measures: two fixed points and the area form; see Anosov and Katok (1970) and, e.g., Fayad and Katok (2004). These are the so-called pseudo-rotations. By taking direct products of pseudo-rotations, one obtains Hamiltonian diffeomorphisms of the products of several copies of S^2 with finite number of ergodic measures, and hence with finitely many periodic orbits. It would be extremely interesting to construct a Hamiltonian analog of pseudo-rotations for, say, $\mathbb{CP}^2$.

In all known examples of Hamiltonian diffeomorphisms with finitely many periodic orbits, all periodic orbits are fixed points, i.e., no new orbits are created by passing to the iterated diffeomorphisms, cf. Sect. 6. Furthermore, all such Hamiltonian diffeomorphisms are non-degenerate, and the number of fixed points is exactly equal to the sum of Betti numbers. Note also that Hamiltonian diffeomorphisms with finitely many periodic orbits are extremely non-generic; see Ginzburg and Gürel (2009c) and Sect. 2.2.

In any event, the class of manifolds admitting "counterexamples" to the Conley conjecture appears to be very narrow, which leads one to the question of finding further sufficient conditions for the Conley conjecture to hold. There are several hypothetical candidates. One of them, conjectured by the second author of this paper, is that the minimal Chern number N of M is sufficiently large, e.g., $N > \dim M$. (The condition $c_1(TM)|_{\pi_2(M)} = 0$ corresponds to $N = \infty$.) More generally, it might be sufficient to require the Gromov–Witten invariants of M to vanish, as suggested by Michael Chance and Dusa McDuff, or even the quantum product to be undeformed. No results in these directions have been proved to date. Note also that for all known "counterexamples" to the Conley conjecture $\mathrm{H}_*(M; \mathbb{Z})$ is concentrated in even degrees.

Another feature of Hamiltonian diffeomorphisms with finitely many periodic orbits is that, for many classes of manifolds, the actions or the actions and the mean indices of their simple periodic orbits must satisfy certain resonance relations of Floer homological nature; see Chance et al. (2013), Ginzburg and Gürel (2009b), Ginzburg and Kerman (2010) and Kerman (2012). (There are also analogs of such resonance relations for Reeb flows, which we will briefly touch upon in Sect. 4.) Although the very existence of homological resonance relations in the Hamiltonian setting is an interesting, new and unexpected phenomenon, and some of the results considered here do make use of these relations, their discussion is outside the scope of this paper.

2.2 Results: The State of the Art

In this section, we briefly introduce our basic conventions and notation and then state the most up-to-date results on the Conley conjecture and generic existence of infinitely many periodic orbits for Hamiltonian diffeomorphisms.

2.2.1 Conventions and Notation

Let us first recall the relevant terminology, some of which have already been used in the previous section. A closed symplectic manifold (M^{2n}, ω) is said to be *monotone (negative monotone)* if $[\omega]|_{\pi_2(M)} = \lambda c_1(TM)|_{\pi_2(M)}$ for some non-negative (respectively, negative) constant λ and *rational* if $\langle[\omega], \pi_2(M)\rangle = \lambda_0\mathbb{Z}$, i.e., the integrals of ω over spheres in M form a discrete subgroup of $\mathbb{R}$. The positive generator N of the discrete subgroup $\langle c_1(TM), \pi_2(M)\rangle \subset \mathbb{R}$ is called the *minimal Chern number* of M. When this subgroup is zero, we set $N = \infty$. A manifold M is called *symplectic CY (Calabi–Yau)* if $c_1(M)|_{\pi_2(M)} = 0$ and *symplectically aspherical* if $c_1(TM)|_{\pi_2(M)} = 0 = [\omega]|_{\pi_2(M)}$. A symplectically aspherical manifold is monotone, and a monotone or negative monotone manifold is rational.

All Hamiltonians H considered in this paper are assumed to be k-periodic in time (i.e., H is a function $S^1_k \times M \to \mathbb{R}$, where $S^1_k = \mathbb{R}/k\mathbb{Z}$) and the period k is always a positive integer. When the period k is not specified, it is equal to one, and $S^1 = \mathbb{R}/\mathbb{Z}$. We set $H_t = H(t, \cdot)$ for $t \in S^1_k$. The (time-dependent) Hamiltonian vector field X_H of H is defined by $i_{X_H}\omega = -dH$. A *Hamiltonian diffeomorphism* is the time-one map, denoted by φ_H or just φ, of the time-dependent Hamiltonian flow (i.e., Hamiltonian isotopy) φ^t_H generated by X_H. It is preferable throughout this section to view φ as an element, determined by φ^t_H, of the universal covering of the group of Hamiltonian diffeomorphisms. A one-periodic Hamiltonian H can also be treated as k-periodic. In this case, we will use the notation $H^{\natural k}$ and, abusing terminology, call $H^{\natural k}$ the kth iteration of H.

In what follows, we identify the periodic orbits of H (i.e., of φ^t_H) with integer period k and periodic orbits of φ. A periodic orbit x of H is *non-degenerate* if the linearized return map $d\varphi|_x \colon T_{x(0)}M \to T_{x(0)}M$ has no eigenvalues equal to one. Following Salamon and Zehnder (1992), we call x *weakly non-degenerate* if at least one of the eigenvalues is different from one and *totally degenerate* otherwise. Finally, a periodic orbit is said to be *strongly non-degenerate* if no roots of unity are among the eigenvalues of $d\varphi|_x$. This terminology carries over to Hamiltonians H and Hamiltonian diffeomorphisms φ. For instance, φ is non-degenerate if all its one-periodic orbits are non-degenerate and strongly non-degenerate if all iterations φ^k are non-degenerate, etc.

2.2.2 Results

The following theorem is the most general variant of the Conley conjecture proved to date.

Theorem 2.1 (Conley Conjecture) *Assume that M is a closed symplectic manifold satisfying one of the following conditions*:

(CY) $c_1(TM)|_{\pi_2(M)} = 0$,
(NM) *M is negative monotone.*

Then every Hamiltonian diffeomorphism φ of M with finitely many fixed points has simple periodic orbits of arbitrarily large period.

As an immediate consequence, every Hamiltonian diffeomorphism of M, whether or not the fixed-point set is finite, satisfying either (CY) or (NM) has infinitely many simple periodic orbits. In fact, when the fixed points of φ are isolated, one can be even more specific: if (CY) holds, every sufficiently large prime p occurs as the period of a simple orbit and, moreover, one can show that there exists a sequence of integers $l_i \to \infty$ such that all p^{l_i} are periods of simple orbits. Consequently, the number of integers less than k that occur as periods of simple periodic orbits grows at least as fast as (in fact, faster than) $k/\ln k - C$ for some constant C. This lower growth bound is typical for the Conley conjecture type results; see also Ginzburg et al. (2014) and Sect. 4 for the case of Reeb flows, and Hingston (1993) for the growth of closed geodesics on S^2. (In dimension two, however, stronger growth results have been established in some cases; see, e.g., Le Calvez 2006; Viterbo 1992, Prop. 4.13 and also Bramham and Hofer 2012; Franks and Handel 2003; Kerman 2012.) When M is negative monotone, it is only known that there is a sequence of arbitrarily large primes occurring as simple periods at least when, in addition, φ is assumed to be weakly non-degenerate; see Sect. 3.1.1 for the definition.

The (CY) case of the theorem is proved in Hein (2012); see also Ginzburg (2010) and, respectively, Ginzburg and Gürel (2009b) for the proofs when M is symplectically aspherical, and when M is rational and (CY) holds. The negative monotone case is established in Chance et al. (2013) and Ginzburg and Gürel (2012). For both classes of the ambient manifolds, the proof of Theorem 2.1 amounts to analyzing two cases. When M is CY, the “non-degenerate case” of the Conley conjecture is based on the observation, going back to Salamon and Zehnder (1992), that unless φ has a fixed point of a particular type called a *symplectically degenerate maximum* or an *SDM*, new simple periodic orbits of high period must be created to generate the Floer homology in degree $n = \dim M/2$. (For negative monotone manifolds, the argument is more involved.) In the “degenerate case” one shows that the presence of an SDM fixed point implies the existence of infinitely many periodic orbits; see Hingston (2009) and also Ginzburg (2010). We outline the proof of Theorem 2.1 for rational CY manifolds in Sect. 3.

Among closed symplectic manifolds M with $c_1(TM)|_{\pi_2(M)} = 0$ are tori and Calabi–Yau manifolds. In fact, the manifolds meeting this requirement are more numerous than it might seem. As is proved in Fine and Panov (2013), for every finitely presented group G there exists a closed symplectic 6-manifold M with $\pi_1(M) = G$

and $c_1(TM) = 0$. A basic example of a negative monotone symplectic manifold is a smooth hypersurface of degree $d > n + 2$ in $\mathbb{CP}^{n+1}$. More generally, a transverse intersection M of m hypersurfaces of degrees $d_1, \ldots, d_m$ in $\mathbb{CP}^{n+m}$ is negative monotone iff $d_1 + \cdots + d_m > n + m + 1$; see Lawson and Michelsohn (1989, p. 88) and also, for $n = 4$, (McDuff and Salamon 2004, pp. 429–430). A complete intersection M is CY when $d_1 + \cdots + d_m = n + m + 1$ and (strictly) monotone when $d_1 + \cdots + d_m < n + m + 1$. Thus "almost all" complete intersections are negative monotone. Note also that the product of a symplectically aspherical manifold and a negative monotone manifold is again negative monotone.

As has been pointed out in Sect. 2.1, we expect an analog of the theorem to hold when N is large. [In the (CY) case, $N = \infty$.] However, at this stage it is only known that the number of simple periodic orbits is bounded from below by $\lceil N/n \rceil$ when M^{2n} is rational and $2N > 3n$; see Ginzburg and Gürel (2009b, Thm. 1.3).

Let us now turn to the question of the generic existence of infinitely many simple periodic orbits. Conjecturally, for any closed symplectic manifold M, a C^∞-generic Hamiltonian diffeomorphism has infinitely many simple periodic orbits . This, however, is unknown (somewhat surprisingly) and appears to be a non-trivial problem. In all results to date, some assumptions on M are required for the proof.

Theorem 2.2 (Generic existence) *Assume that M^{2n} is a closed symplectic manifold with minimal Chern number N, meeting one of the following requirements*:

(i) $\mathrm{H}_{odd}(M; R) \neq 0$ *for some ring R,*
(ii) $N \geq n + 1$*,*
(iii) *M is $\mathbb{CP}^n$ or a complex Grassmannian or a product of one of these manifolds with a closed symplectically aspherical manifold.*

Then strongly non-degenerate Hamiltonian diffeomorphisms with infinitely many simple periodic orbits form a C^∞-residual set in the space of all C^∞-smooth Hamiltonian diffeomorphisms.

This theorem is proved in Ginzburg and Gürel (2009c). In (iii), instead of explicitly specifying M, we could have required that M is monotone and that there exists $u \in \mathrm{H}_{*<2n}(M)$ with $2n - \deg u < 2N$ and $w \in \mathrm{H}_{*<2n}(M)$ and α in the Novikov ring of M such that $[M] = (\alpha u) * w$ in the quantum homology. We refer the reader to Ginzburg and Gürel (2009c) for other examples when this condition is satisfied and a more detailed discussion.

The proof of the theorem when (i) holds is particularly simple. Namely, in this case, a non-degenerate Hamiltonian diffeomorphism φ with finitely many periodic orbits must have a non-hyperbolic periodic orbit. Indeed, it follows from Floer theory that φ has a non-hyperbolic fixed point or a hyperbolic fixed point with negative eigenvalues. When φ has finitely many periodic orbits, we can eliminate the latter case by passing to an iteration of φ. To finish the proof it suffices to apply the Birkhoff–Lewis–Moser theorem, Moser (1977). (This argument is reminiscent of the reasoning in, e.g., Markus and Meyer (1980) where the generic existence of solenoids for Hamiltonian flows is established.) The proofs of the remaining cases rely on the fact, already mentioned in Sect. 2.1, that under our assumptions on M the indices and/or actions of the periodic orbits of φ must satisfy certain resonance relations when φ has only finitely many periodic orbits; see Ginzburg and Gürel (2009b) and Ginzburg and Kerman (2010). These resonance relations can be easily broken by a C^∞-small perturbation of φ, and the theorem follows.

It is interesting to look at these results in the context of the closing lemma, which implies that the existence of a dense set of periodic orbits is C^1-generic for Hamiltonian diffeomorphisms; see Pugh and Robinson (1983). Thus, once the C^∞-topology is replaced by the C^1-topology a much stronger result than the generic existence of infinitely many periodic orbits holds—the generic dense existence. However, this is no longer true for the C^r-topology with $r > \dim M$ as the results of M. Herman show (see Herman 1991a, b), and the above conjecture on the C^∞-generic existence of infinitely many periodic orbits can be viewed as a hypothetical variant of a C^∞-closing lemma. (Note also that in the closing lemma one can require the perturbed diffeomorphism to be C^∞-smooth, but only C^1-close to the original one, as long as only finitely many periodic orbits are created. It is not clear to us whether one can produce infinitely many periodic orbits by a C^∞-smooth C^1-small perturbation.)

An interesting consequence of Theorem 2.2 pointed out in Polterovich and Shelukhin (2014) is that non-autonomous Hamiltonian diffeomorphisms (i.e., Hamiltonian diffeomorphisms that cannot be generated by autonomous Hamiltonians) on a manifold meeting the conditions of the theorem form a C^∞-residual subset in the space of all C^∞-smooth Hamiltonians. Indeed, when $k > 1$, simple k-periodic orbits of an autonomous Hamiltonian diffeomorphism are never isolated, and hence, in particular, never non-degenerate.

Remark 2.3 The proofs of Theorems 2.1 and 2.2 utilize Hamiltonian Floer theory. Hence, either M is required in addition to be weakly monotone [i.e., M is monotone or $N > n - 2$; see (Hofer and Salamon 1995; McDuff and Salamon 2004; Ono 1995; Salamon 1999) for more details] or the proofs ultimately, although not explicitly, must rely on the machinery of multi-valued perturbations and virtual cycles [see (Fukaya and Ono 1999; Fukaya et al. 2009; Liu and Tian 1998) or, for the polyfold approach, (Hofer et al. 2010, 2011) and references therein]. In the latter case, the ground field in the Floer homology must have zero characteristic.

3 Outline of the Proof of the Conley Conjecture

3.1 Preliminaries

In this section we recall, very briefly, several definitions and results needed for the proof of the (CY) case of Theorem 2.1 and also some terminology used throughout the paper.

3.1.1 The Mean Index and the Conley–Zehnder Index

To every continuous path $\Phi\colon [0,\ 1] \to \mathrm{Sp}(2n)$ starting at $\Phi(0) = I$ one can associate the *mean index* $\Delta(\Phi) \in \mathbb{R}$, a homotopy invariant of the path with fixed end-points; see Long (2002) and Salamon and Zehnder (1992). To give a formal definition, recall first that a map Δ from a Lie group to $\mathbb{R}$ is said to be a quasimorphism if it fails to be a homomorphism only up to a constant, i.e., $|\Delta(\Phi\Psi) - \Delta(\Phi) - \Delta(\Psi)| < const$, where the constant is independent of Φ and Ψ. One can prove that there is a unique quasimorphism $\Delta \colon \widetilde{\mathrm{Sp}}(2n) \to \mathbb{R}$ which is continuous and homogeneous [i.e., $\Delta(\Phi^k) = k\Delta(\Phi)$] and satisfies the normalization condition: $\Delta(\Phi_0) = 2$ for $\Phi_0(t) = e^{2\pi i t} \oplus I_{2n-2}$ with $t \in [0, 1]$, in the self-explanatory notation; see Barge and Ghys (1992). This quasimorphism is the mean index. (The continuity requirement holds automatically and is not necessary for the characterization of Δ, although this is not immediately obvious. Furthermore, Δ is also automatically conjugation invariant, as a consequence of the homogeneity.)

The mean index $\Delta(\Phi)$ measures the total rotation angle of certain unit eigenvalues of $\Phi(t)$ and can be explicitly defined as follows. For $A \in \mathrm{Sp}(2)$, set $\rho(A) = e^{i\lambda} \in S^1$ when A is conjugate to the rotation by λ counterclockwise, $\rho(A) = e^{-i\lambda} \in S^1$ when A is conjugate to the rotation by λ clockwise, and $\rho(A) = \pm 1$ when A is hyperbolic with the sign determined by the sign of the eigenvalues of A. Then $\rho \colon \mathrm{Sp}(2) \to S^1$ is a continuous (but not C^1) function, which is conjugation invariant and equal to det on $\mathrm{U}(1)$. A matrix $A \in \mathrm{Sp}(2n)$ with distinct eigenvalues, can be written as the direct sum of matrices $A_j \in \mathrm{Sp}(2)$ and a matrix with complex eigenvalues not lying on the unit circle. We set $\rho(A)$ to be the product of $\rho(A_j) \in S^1$. Again, ρ extends to a continuous function $\rho \colon \mathrm{Sp}(2n) \to S^1$, which is conjugation invariant (and hence $\rho(AB) = \rho(BA)$) and restricts to det on $\mathrm{U}(n)$; see, e.g., Salamon and Zehnder (1992). Finally, given a path $\Phi \colon [0, 1] \to \mathrm{Sp}(2n)$, there is a continuous function $\lambda(t)$ such that $\rho(\Phi(t)) = e^{i\lambda(t)}$, measuring the total rotation of the "preferred" eigenvalues on the unit circle, and we set $\Delta(\Phi) = (\lambda(1) - \lambda(0))/2$.

Assume now that the path Φ is *non-degenerate*, i.e., by definition, all eigenvalues of the end-point $\Phi(1)$ are different from one. We denote the set of such matrices in $\mathrm{Sp}(2n)$ by $\mathrm{Sp}^*(2n)$. It is not hard to see that $\Phi(1)$ can be connected to a hyperbolic symplectic transformation by a path Ψ lying entirely in $\mathrm{Sp}^*(2n)$. Concatenating this path with Φ, we obtain a new path Φ'. By definition, the *Conley–Zehnder index* $\mu_{\mathrm{CZ}}(\Phi) \in \mathbb{Z}$ of Φ is $\Delta(\Phi')$. One can show that $\mu_{\mathrm{CZ}}(\Phi)$ is well-defined, i.e., independent of Ψ. Furthermore, following Salamon and Zehnder (1992), let us call Φ *weakly non-degenerate* if at least one eigenvalue of $\Phi(1)$ is different from one and *totally degenerate* otherwise. The path is *strongly non-degenerate* if all its "iterations" Φ^k are non-degenerate, i.e., none of the eigenvalues of $\Phi(1)$ is a root of unity.

The indices Δ and μ_{CZ} have the following properties:

(CZ1) $|\Delta(\Phi) - \mu_{\mathrm{CZ}}(\tilde{\Phi})| \leq n$ for every sufficiently small non-degenerate perturbation $\tilde{\Phi}$ of Φ; moreover, the inequality is strict when Φ is weakly non-degenerate.

(CZ2) $\mu_{\mathrm{CZ}}(\Phi^k)/k \to \Delta(\Phi)$ as $k \to \infty$, when Φ is strongly non-degenerate; hence the name "mean index" for Δ.

Note that with our conventions the Conley–Zehnder index of a path parametrized by $[0, 1]$ and generated by a small negative definite quadratic Hamiltonian on $\mathbb{R}^{2n}$ is n.

Let now M^{2n} be a symplectic manifold and $x \colon S^1 \to M$ be a contractible loop. A *capping* of x is a map $u \colon D^2 \to M$ such that $u|_{S^1} = x$. Two cappings u and v of x are considered to be equivalent if the integrals of $c_1(TM)$ and ω over the sphere $u\#v$ obtained by attaching u to v are equal to zero. A capped closed curve $\bar{x} = (x, u)$ is, by definition, a closed curve x equipped with an equivalence class of a capping. In what follows, a capping is always indicated by the bar.

For a capped one-periodic (or k-periodic) orbit $\bar{x}$ of a Hamiltonian $H\colon S^1 \times M \to \mathbb{R}$, we can view the linearized flow $d\varphi^t_H|_x$ along x as a path in $\mathrm{Sp}(2n)$ by fixing a trivialization of u^*TM and restricting it to x. With this convention in mind, the above definitions and constructions apply to $\bar{x}$ and, in particular, we have the mean index $\Delta(\bar{x})$ and, when x is non-degenerate, the Conley–Zehnder index $\mu_{\mathrm{CZ}}(\bar{x})$ defined. These indices are independent of the trivialization of u^*TM, but may depend on the capping. Furthermore, (CZ1) and (CZ2) hold. The difference of the indices of (x, u) and (x, v) is equal to $2\,\langle c_1(TM), u\#v\rangle$. Hence, when M is a symplectic CY manifold, the indices are independent of the capping and thus assigned to x. The terminology we introduced for paths in $\mathrm{Sp}(2n)$ translates word-for-word to periodic orbits and Hamiltonian diffeomorphisms, cf. Sect. 2.2.1.

3.1.2 Floer Homology

In this section, we recall the construction and basic properties of global, filtered and local Hamiltonian Floer homology on a weakly monotone, rational symplectic manifold (M^{2n}, ω).

The *action* of a one-periodic Hamiltonian H on a capped loop $\bar{x} = (x, u)$ is, by definition,

$$\mathcal{A}_H(\bar{x}) = -\int_u \omega + \int_{S^1} H_t(x(t))\,dt.$$

The space of capped closed curves is a covering space of the space of contractible loops, and the critical points of $\mathcal{A}_H$ on this covering space are exactly capped one-periodic orbits of X_H. The *action spectrum* $\mathcal{S}(H)$ of H is the set of critical values of $\mathcal{A}_H$. This is a zero measure set; see, e.g., Hofer and Zehnder (1994). When M is rational, $\mathcal{S}(H)$ is a closed and hence nowhere dense set. [Otherwise, $\mathcal{S}(H)$ is dense.] These definitions extend to k-periodic orbits and Hamiltonians in the obvious way. Clearly, the action functional is homogeneous with respect to iteration:

$$\mathcal{A}_{H^{\natural k}}(\bar{x}^k) = k\mathcal{A}_H(\bar{x}).$$

Here $\bar{x}^k$ stands for the kth iteration of the capped orbit $\bar{x}$.

For a Hamiltonian $H\colon S^1 \times M \to \mathbb{R}$ and $\varphi = \varphi_H$, we denote by $\mathrm{HF}_*(\varphi)$ or, when the action filtration is essential, by $\mathrm{HF}^{(a,\,b)}_*(H)$ the *Floer homology* of H, where a and b are not in $\mathcal{S}(H)$. We refer the reader to, e.g., McDuff and Salamon (2004) and Salamon (1999) for a detailed construction of the Floer homology and to Ginzburg and Gürel (2009b) for a treatment particularly tailored for our purposes. Here we only mention that, when H is non-degenerate, $\mathrm{HF}^{(a,\,b)}_*(H)$ is the homology of a complex generated by the capped one-periodic orbits of H with action in the interval (a, b) and graded by the Conley–Zehnder index. Furthermore, $\mathrm{HF}_*(\varphi) \cong \mathrm{H}_{*+n}(M) \otimes \Lambda$, where Λ is a suitably defined Novikov ring. As a consequence, $\mathrm{HF}_n(\varphi) \neq 0$ when M is symplectic CY. (For our purposes it is sufficient to take $\mathbb{Z}_2$ as the ground field.)

When x is an isolated one-periodic orbit of H, one can associate to it the so-called *local Floer homology* $\mathrm{HF}_*(x)$ of x. This is the homology of a complex generated by the orbits x_i which x splits into under a C^∞-small non-degenerate perturbation. The differential ∂ is defined similarly to the standard Floer differential, and to show that $\partial^2 = 0$ it suffices to prove that the Floer trajectories connecting the orbits x_i cannot approach the boundary of an isolating neighborhood of x. This is an immediate consequence of Floer (1989, Thm. 3); see also McLean (2012) for a different proof. The resulting homology is well defined, i.e., independent of the perturbation. The local Floer homology $\mathrm{HF}_*(x)$ carries only a relative grading. To have a genuine $\mathbb{Z}$-grading it is enough to fix a trivialization of $TM|_x$. In what follows, such a trivialization will usually come from a capping of x, and we will then write $\mathrm{HF}_*(\bar{x})$. Clearly, the grading is independent of the capping when $c_1(TM)|_{\pi_2(M)} = 0$. Hence, in the symplectic (CY) case, the local Floer homology is associated to the orbit x itself. With relative grading, the local Floer homology is defined for the germ of a time-dependent Hamiltonian flow or, when x is treated as a fixed point, of a Hamiltonian diffeomorphism. The local Floer homology is invariant under deformations of H as long as x stays uniformly isolated.

Example 3.1 When x is non-degenerate, $\mathrm{HF}_*(\bar{x}) \cong \mathbb{Z}_2$ is concentrated in degree $\mu_{\mathrm{CZ}}(\bar{x})$. When x is an isolated critical point of an autonomous C^2-small Hamiltonian F (with trivial capping), the local Floer homology is isomorphic to the local Morse homology $\mathrm{HM}_{*+n}(F, x)$ of F at x (see Ginzburg 2010), also known as critical modules, which is in turn isomorphic to $\mathrm{H}_*(\{F < c\} \cup \{x\}, \{F < c\})$, where $F(x) = c$. The isomorphism $\mathrm{HF}_*(x) \cong \mathrm{HM}_{*+n}(F, x)$ is a local analog of the isomorphism between the Floer and Morse homology groups of a C^2-small Hamiltonian; see Salamon and Zehnder (1992) and references therein.

Let us now state three properties of local Floer homology, which are essential for what follows.

First of all, $\mathrm{HF}_*(\bar{x})$ is supported in the interval $[\Delta(\bar{x}) - n, \Delta(\bar{x}) + n]$:

$$\operatorname{supp} \mathrm{HF}_*(\bar{x}) \subset [\Delta(\bar{x}) - n, \Delta(\bar{x}) + n], \tag{3.1}$$

i.e., the homology vanishes in the degrees outside this interval. Moreover, when x is weakly non-degenerate, the support lies in the open interval. These facts readily follow from (CZ1) and the continuity of the mean index.

Secondly, the local Floer homology groups are building blocks for the ordinary Floer homology. Namely, assume that for $c \in \mathcal{S}(H)$ there are only finitely many one-periodic orbits $\bar{x}_i$ with $\mathcal{A}_H(\bar{x}_i) = c$. Then all these orbits are isolated and

$$\mathrm{HF}_*^{(c-\epsilon,\, c+\epsilon)}(H) = \bigoplus \mathrm{HF}_*(\bar{x}_i),$$

when M is rational and $\epsilon > 0$ is sufficiently small. Furthermore, it is easy to see that, even without the rationality condition, $\mathrm{HF}_l(\varphi) = 0$ when all one-periodic orbits of H are isolated and have local Floer homology vanishing in degree l.

Finally, the local Floer homology enjoys a certain periodicity property as a function of the iteration order. To be more specific, let us call a positive integer k an *admissible* *iteration* of x if the multiplicity of the generalized eigenvalue one for the iterated linearized Poincaré return map $d\varphi^k|_x$ is equal to its multiplicity for $d\varphi|_x$. In other words, k is admissible if and only if it is not divisible by the degree of any root of unity different from one among the eigenvalues of $d\varphi|_x$. For instance, when x is totally degenerate (the only eigenvalue is one) or strongly non-degenerate (no roots of unity among the eigenvalues), all $k \in \mathbb{N}$ are admissible. For any x, all sufficiently large primes are admissible. We have

Theorem 3.2 (Ginzburg and Gürel 2010) *Let $\bar{x}$ be a capped isolated one-periodic orbit of a Hamiltonian $H: S^1 \times M \to \mathbb{R}$. Then x^k is also an isolated one-periodic orbit of $H^{\natural k}$ for all admissible k, and the local Floer homology groups of $\bar{x}$ and $\bar{x}^k$ coincide up to a shift of degree*:

$$\mathrm{HF}_*(\bar{x}^k) = \mathrm{HF}_{*+s_k}(\bar{x}) \quad \textit{for some } s_k.$$

Furthermore, $\lim_{k\to\infty} s_k/k = \Delta(\bar{x})$ *and* $s_k = k\Delta(\bar{x})$ *for all k when x is totally degenerate. Moreover, when* $\mathrm{HF}_{n+\Delta(\bar{x})}(\bar{x}) \neq 0$*, the orbit x is totally degenerate.*

The first part of this theorem is an analog of the result from Gromoll and Meyer (1969) for Hamiltonian diffeomorphisms. One can replace a capping of x by a trivialization of $TM|_x$ with the grading and indices now associated with that trivialization. The theorem is not obvious, although not particularly difficult. First, by using a variant of the Künneth formula and some simple tricks, one can reduce the problem to the case where x is a totally degenerate constant orbit with trivial capping. [Hence, in particular, $\Delta(\bar{x}) = 0$]. Then we have the isomorphisms $\mathrm{HF}_*(\bar{x}) = \mathrm{HM}_{*+n}(F, x)$, where $F: M \to \mathbb{R}$, near x, is the generating function of φ, and $\mathrm{HF}_*(\bar{x}^k) = \mathrm{HM}_{*+n}(kF, x) = \mathrm{HM}_{*+n}(F, x)$. Thus, in the totally degenerate case, $s_k = 0$, and the theorem follows; see Ginzburg and Gürel (2010) for a complete proof. [The fact that x^k is automatically isolated when k is admissible, reproved in Ginzburg and Gürel (2010), has been known for some time; see Chow et al. (1981).]

As a consequence of Theorem 3.2 or of Chow et al. (1981), the iterated orbit x^k is automatically isolated for all k if it is isolated for some finite collection of iterations k (depending on the degrees of the roots of unity among the eigenvalues). Furthermore, it is easy to see that then the map $k \mapsto \mathrm{HF}_*(\bar{x}^k)$ is periodic up to a shift of grading, and hence the function $k \mapsto \dim \mathrm{HF}_*(\bar{x}^k)$ is bounded.

An isolated orbit x is said to be *homologically non-trivial* if $\mathrm{HF}_*(x) \neq 0$. (The choice of trivialization along the orbit is clearly immaterial here.) These are the orbits detected by the filtered Floer homology. For instance, a non-degenerate orbit is homologically non-trivial. By Theorem 3.2, an admissible iteration of a homologically non-trivial orbit is again homologically non-trivial. It is not known if, in general, an iteration of a homologically non-trivial orbit can become homologically trivial while remaining isolated.

We refer the reader to Ginzburg (2010) and Ginzburg and Gürel (2009b, 2010) for a further discussion of local Floer homology.

As we noted in Sect. 2.2, the proof of the general case of the Conley conjecture for symplectic CY manifolds hinges on the fact that the presence of an orbit of a particular type, a *symplectically degenerate maximum* or an *SDM*, automatically implies the existence of infinitely many simple periodic orbits. To be more precise, an isolated periodic orbit x is said to be a symplectically degenerate maximum if $\mathrm{HF}_n(x) \neq 0$ and $\Delta(x) = 0$ for some trivialization. This definition makes sense even for the germs of Hamiltonian flows or Hamiltonian diffeomorphisms. An SDM orbit is necessarily totally degenerate by the "moreover" part of (3.1) or Theorem 3.2. Sometimes it is also convenient to say that an orbit is an SDM with respect to a particular capping. For instance, a capped orbit $\bar{x}$ is an SDM if it is an SDM for the trivialization associated with the capping, i.e., $\mathrm{HF}_n(\bar{x}) \neq 0$ and $\Delta(\bar{x}) = 0$.

Example 3.3 Let $H: \mathbb{R}^{2n} \to \mathbb{R}$ be an autonomous Hamiltonian with an isolated critical point at $x = 0$. Assume furthermore that x is a local maximum and that all eigenvalues (in the sense of, e.g., Arnold 1989, App. 6) of the Hessian $d^2H(x)$ are equal to zero. Then x (with constant trivialization or, equivalently, trivial capping) is an SDM of H. For instance, the origin in $\mathbb{R}^2$ is an SDM for $H(p, q) = p^4 + q^4$ or $H(p, q) = p^2 + q^4$, but not for $H(p, q) = ap^2 + bq^2$ for any $a \neq 0$ and $b \neq 0$.

Remark 3.4 There are several other ways to define an SDM. The following conditions are equivalent (see Ginzburg and Gürel 2010, Prop. 5.1):

- the orbit $\bar{x}$ is a symplectically degenerate maximum of H;
- $\mathrm{HF}_n(\bar{x}^{k_i}) \neq 0$ for some sequence of admissible iterations $k_i \to \infty$;
- the orbit x is totally degenerate, $\mathrm{HF}_n(\bar{x}) \neq 0$ and $\mathrm{HF}_n(\bar{x}^k) \neq 0$ for at least one admissible iteration $k \geq n + 1$.

3.2 The Non-Degenerate Case of the Conley Conjecture

The following proposition settling, in particular, the non-degenerate case of the Conley conjecture for symplectic CY manifolds is a refinement of the main result from Salamon and Zehnder (1992). It is proved in Ginzburg (2010) and Ginzburg and Gürel (2009b), although the argument given below is somewhat different from the original proof.

Proposition 3.5 *Assume that $c_1(TM)|_{\pi_2(M)} = 0$ and that $\varphi = \varphi_H$ has finitely many fixed points and none of these points is an SDM.* (*This is the case when, e.g., φ is weakly non-degenerate.*) *Then φ has a simple periodic orbit of period k for every sufficiently large prime k.*

The key to the proof is the fact that $\mathrm{HF}_n(\varphi^k) \neq 0$ for all k and that, even when $\omega|_{\pi_2(M)} \neq 0$, the condition $c_1(TM)|_{\pi_2(M)} = 0$ guarantees that all recappings of every orbit have the same mean index and the same (graded) local Floer homology.

Proof First, note that when k is prime, every k-periodic orbit is either simple or the k-th iteration of a fixed point. For every isolated fixed point x, we have three mutually exclusive possibilities:

- $\Delta(x) \neq 0$,
- $\Delta(x) = 0$ and $\mathrm{HF}_n(x) = 0$,
- $\Delta(x) = 0$ but $\mathrm{HF}_n(x) \neq 0$.

Here we are using the fact that M is CY, and hence the indices are independent of the capping. The last case, where x is an SDM, is ruled out by the assumptions of the proposition.

In the first case, $\mathrm{HF}_n(x^k) = 0$ when $k|\Delta(x)| > 2n$, and hence x cannot contribute to $\mathrm{HF}_n(\varphi^k)$ when k is large. In the second case, $\mathrm{HF}_n(x^k) = \mathrm{HF}_n(x) = 0$ for all admissible iterations by Theorem 3.2. In particular, x again cannot contribute to $\mathrm{HF}_n(\varphi^k)$ for all large primes k. It follows that, under the assumptions of the proposition, $\mathrm{HF}_n(\varphi^k) = 0$ for all large primes k unless φ has a simple periodic orbit of period k. □

Remark 3.6 Although this argument relies on Theorem 3.2 which is not entirely trivial, a slightly different logical organization of the proof would enable one to utilize a much simpler of version of the theorem; see Ginzburg (2010) and Ginzburg and Gürel (2009b).

With Proposition 3.5 established, it remains to deal with the degenerate case of the Conley conjecture, i.e., the case where φ has an SDM. We do this in the next section; see Theorem 3.8.

Remark 3.7 When M is negative monotone and φ has an SDM fixed point, the degenerate case of Theorem 2.1 follows from Theorem 3.8, just as for CY manifolds. However, the non-degenerate case requires a totally new proof. The argument relies on the subadditivity property of spectral invariants; see Chance et al. (2013) and Ginzburg and Gürel (2012) for more details.

3.3 Symplectically Degenerate Maxima

In this section, we show that a Hamiltonian diffeomorphism with an SDM fixed point has infinitely many simple periodic orbits. We assume that M is rational as in Ginzburg and Gürel (2009b). The case of irrational CY manifolds is treated in Hein (2012).

Theorem 3.8 (Ginzburg and Gürel 2009b) *Let $\varphi = \varphi_H$ be a Hamiltonian diffeomorphism of a closed rational symplectic manifold M, generated by a one-periodic Hamiltonian H. Assume that some iteration φ^{k_0} has finitely many k_0-periodic orbits and one of them, $\bar{x}$, is an SDM.*

(i) *Then φ has infinitely many simple periodic orbits.*
(ii) *Moreover, φ has simple periodic orbits of arbitrarily large prime period if, in addition, $k_0 = 1$ and $\omega|_{\pi_2(M)} = 0$ or $c_1(M)|_{\pi_2(M)} = 0$.*

This theorem is in turn a consequence of the following result.

Theorem 3.9 (Ginzburg and Gürel 2009b) *Assume that (M^{2n}, ω) is closed and rational, and let $\bar{x}$ be an SDM of H. Set $c = \mathcal{A}_H(\bar{x})$. Then for every sufficiently small $\epsilon > 0$ there exists k_ϵ such that*

$$\mathrm{HF}_{n+1}^{(kc+\delta_k,\, kc+\epsilon)}\left(H^{\natural k}\right) \neq 0 \textit{ for all } k > k_\epsilon \textit{ and some } \delta_k \textit{ with } 0 < \delta_k < \epsilon. \quad (3.2)$$

For instance, to prove case (ii) of Theorem 3.8 when M is CY it suffices to observe that no kth iteration of a fixed point can contribute to the Floer homology in degree $n+1$ for any action interval when k is a sufficiently large prime and φ has finitely many fixed points. When $\omega|_{\pi_2(M)} = 0$, the argument is similar, but now the action filtration is used in place of the degree. The proof of case (i) is more involved; see Ginzburg and Gürel (2009b, Sect. 3) where some more general results are also established.

It is worth pointing out that although Theorem 3.9 guarantees the existence of capped simple k-periodic orbits $\bar{y}$ with action close to $kc = \mathcal{A}_{H^{\natural k}}(\bar{x}^k)$, we do not claim that the orbits y are close to x^k or even intersect a neighborhood of x. In general, essentially nothing is known about the location of these orbits and hypothetically a small neighborhood of x may contain no periodic orbits at all. However, as is proved in Yan (2014), the orbit x is in a certain sense an accumulation point for periodic orbits when, e.g., $M = \mathbb{T}^2$.

Outline of the proof of Theorem 3.9 Composing if necessary φ_H^t with a loop of Hamiltonian diffeomorphisms, we can easily reduce the problem to the case where $\bar{x}$ is a constant one-periodic orbit with trivial capping; see Ginzburg and Gürel (2009b, Prop. 2.9 and 2.10). Henceforth, we write x rather than $\bar{x}$ and assume that $dH_t(x) = 0$ for all t.

The key to the proof is the following geometrical characterization of SDMs:

Lemma 3.10 (Ginzburg 2010; Hingston 2009) *Let x be an isolated constant one-periodic orbit for a germ of a time-dependent Hamiltonian flow φ_H^t. Assume that x (with constant trivialization) is an SDM. Then there exists a germ of a time-dependent Hamiltonian flow φ_K^t near x such that the two flows generate the same time-one map, i.e., $\varphi_K = \varphi_H$, and K_t has a strict local maximum at x for every t. Furthermore, one can ensure that the Hessian $d^2K_t(x)$ is arbitrarily small. In other words, for every $\eta > 0$ one can find such a Hamiltonian K_t with $\|d^2K_t(x)\| < \eta$.*

Remark 3.11 Strictly speaking, contrary to what is stated in Ginzburg and Gürel (2009b, Rmk. 5.9, 2010, Prop. 5.2), this lemma is not quite a characterization of SDMs in the sense that it is not clear if every x for which such Hamiltonians K_t exist is necessarily an SDM. However, in fact, K_t can be taken to meet an additional requirement ensuring, in essence, that the t-dependence of K_t is minor. With this condition, introduced in Hingston (2009, Lemma 4) as that K is relatively autonomous (see also Ginzburg 2010, Sect. 5 and 6), the lemma gives a necessary and sufficient condition for an SDM.

Outline of the proof of Lemma 3.10 The proof of Lemma 3.10 is rather technical, but the idea of the proof is quite simple. Set $\varphi = \varphi_H$. First, note that all eigenvalues of $d\varphi|_x : T_xM \to T_xM$ are equal to one since x is totally degenerate. Thus by applying a symplectic linear change of coordinates we can bring $d\varphi|_x$ arbitrarily close to the identity. Then φ is also C^1-close to id near x. Let us identify $(M \times M, \omega \oplus (-\omega))$ near (x, x) with a neighborhood of the zero section in T^*M near x, and hence the graph of φ with the graph of dF for a germ of a smooth function F near x. The function F is a generating function of φ. Clearly, x is an isolated critical point of F and $d^2F(x) = O(\|d\varphi|_x - I\|)$.

Furthermore, similarly to Example 3.1, we have an isomorphism

$$\mathrm{HF}_*(x) = \mathrm{HM}_{*+n}(F, x),$$

and thus $\mathrm{HM}_{2n}(F, x) \neq 0$. It is routine to show that an isolated critical point x of a function F is a local maximum if and only if $\mathrm{HM}_{2n}(F, x) \neq 0$. The generating function F is not quite a Hamiltonian generating φ, but it is not hard to turn F into such a Hamiltonian K_t and check that K_t inherits the properties of F. □

Returning to the proof of Theorem 3.9, we apply the lemma to the SDM orbit x and observe that the local loop $\varphi_H^t \circ (\varphi_K^t)^{-1}$ has zero Maslov index and hence is contractible. It is not hard to show that every local contractible loop extends to a global contractible loop; see Ginzburg (2010, Lemma 2.8). In other words, we can extend the Hamiltonian K_t from Lemma 3.10 to a global Hamiltonian such that $\varphi_K = \varphi$, not only near x but on the entire manifold M.

With this in mind, let us reset the notation. Replacing H by K but retaining the original notation, we can say that for every $\eta > 0$ there exists a Hamiltonian H such that

- $\varphi_H = \varphi$;
- x is a constant periodic orbit of H, and H_t has an isolated local maximum at x for all t;
- $\|d^2 H_t(x)\| < \eta$ for all t.

Furthermore, we can always assume that all such Hamiltonians H are related to each other and to the original Hamiltonian via global loops with zero action and zero Maslov index. Thus, in particular, $c = \mathcal{A}_H(x) = H(x)$ is independent of the choice of H above, and all Hamiltonians have the same filtered Floer homology. Therefore, it is sufficient to prove the theorem for any of these Hamiltonians H with arbitrarily small Hessian $d^2H(x)$.

To avoid technical difficulties and illuminate the idea of the proof, let us asume that $d^2H(x) = 0$ and, of course, that H_t has, as above, a strict local maximum at x for all t. This case, roughly speaking, corresponds to an SDM x with $d\varphi|_x = id$.

To prove (3.2) for a given $\epsilon > 0$, we will use the standard squeezing argument, i.e., we will bound H from above and below by two autonomous Hamiltonians $H_\pm$ as in Fig. 1 and calculate the Floer homology of $kH_\pm$.

In a Darboux neighborhood U of x, the Hamiltonians $H_\pm$ are rotationally symmetric. The Hamiltonian H_+ is constant and equal to c near x on a ball of radius r and then sharply decreases to some c' which is close to c and attained on the sphere of radius R. Then, after staying constant on a spherical shell, H_+ increases to some value c_+, to accommodate H, and becomes constant. The radii $r < R$ depend on ϵ; namely, we require that $\pi R^2 < \epsilon$. The Hamiltonian H_- is a bump function decreasing from its maximum c at x to a large negative value c_-. Thus we have

$$H_+ \geq H \geq H_-.$$

We require H_- to have a strict maximum at x with $d^2H_-(x) = 0$. Then the local Floer homology of $H_-^{\natural k}$ is equal to $\mathbb{Z}_2$ and concentrated in degree n, i.e., x is also an SDM for H_- and all its iterations.

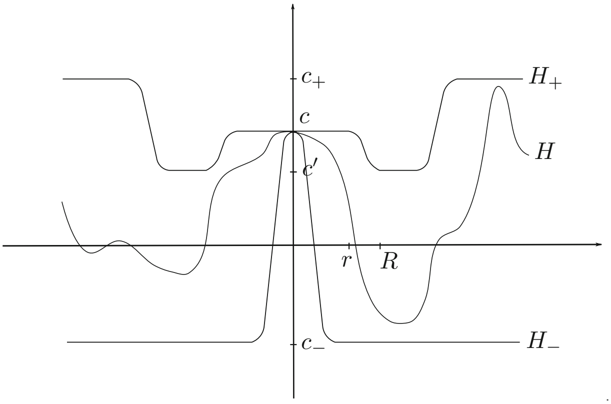

Fig. 1 The functions $H_\pm$

Now, for any $a < b$ outside $\mathcal{S}(H^{\natural k})$ and $\mathcal{S}(H_\pm^{\natural k})$, we have the maps

$$\mathrm{HF}_*^{(a,\,b)}\left(H_+^{\natural k}\right) \to \mathrm{HF}_*^{(a,\,b)}\left(H^{\natural k}\right) \to \mathrm{HF}_*^{(a,\,b)}\left(H_-^{\natural k}\right)$$

induced by monotone homotopies, where $\mathrm{HF}_*^{(a,\,b)}\left(H_\pm^{\natural k}\right) = \mathrm{HF}_*^{(a,\,b)}\left(kH_\pm\right)$ since $H_\pm$ are autonomous Hamiltonians. Therefore, it is sufficient to prove that the map

$$\mathrm{HF}_{n+1}^{(kc+\delta,\,kc+\epsilon)}\left(kH_+\right) \to \mathrm{HF}_{n+1}^{(kc+\delta,\,kc+\epsilon)}\left(kH_-\right) \tag{3.3}$$

is non-zero for some δ in the range $(0,\,\epsilon)$.

To this end, let us assume first that $H_\pm$, as above, are functions on $\mathbb{R}^{2n}$ constant outside a neighborhood of $x = 0$. The filtered Floer homology of $H_\pm$ is still defined for any interval $(a,\,b)$ not containing $c_\pm$. Moreover, a decreasing homotopy H^s from H_+ to H_- through functions constant outside a compact set induces a map in Floer homology even when the value of H^s at infinity passes through $(a,\,b)$. Then we have an isomorphism

$$\mathbb{Z}_2 \cong \mathrm{HF}_{n+1}^{(kc+\delta,\,kc+\epsilon)}\left(kH_+\right) \xrightarrow{\cong} \mathrm{HF}_{n+1}^{(kc+\delta,\,kc+\epsilon)}\left(kH_-\right) \cong \mathbb{Z}_2 \tag{3.4}$$

when k is sufficiently large and $\delta > 0$ is sufficiently small. Namely, k is so large that $k(c - c') > \pi R^2$. This is the origin of the requirement $k > k_\epsilon$. Then the homology of $kH_\pm$ is generated by the periodic orbit closest to x, and δ is chosen so that $kc + \delta$ is smaller than the action of kH_- on this orbit.

The isomorphism (3.4) is established by a straightforward analysis of periodic orbits and easily follows from the calculation carried out already in Ginzburg and Gürel (2004). It is based on two facts: that

$$\mathbb{Z}_2 \cong \mathrm{HF}_{n}^{(kc-\delta,\,kc+\delta)}\left(kH_+\right) \xrightarrow{\cong} \mathrm{HF}_{n}^{(kc-\delta,\,kc+\delta)}\left(kH_-\right) \cong \mathbb{Z}_2$$

is an isomorphism when $\delta > 0$ is small and that $\mathrm{HF}_{n+1}^{(a,\,b)}\big(kH_{\pm}\big) = 0$ for every sufficiently large interval $(a,\, b)$ containing kc and contained in $(kc_{-},\, kc_{+})$.

It remains to transplant this calculation from $\mathbb{R}^{2n}$ to a closed manifold M. The key to this is the fact that the action interval in question is sufficiently small. This enables one to localize a calculation of filtered Floer homology by essentially turning action localization to spatial localization. A general framework for this process, developed in Ginzburg and Gürel (2009b), is as follows. Let S be a shell in M, i.e., a region between two hypersurfaces and bounding a contractible domain V in M. (To be more precise, V is bounded in M by a connected component of ∂S and $S \cap V = \emptyset$. The contractibility assumption can be significantly relaxed.) Furthermore, let F be a Hamiltonian which we require to be constant on the shell. For any interval $I = (\alpha,\, \beta)$ not containing $F|_S$, consider the subspace of the Floer complex generated by the orbits of F in V with cappings also contained in V. If necessary, we perturb F in V to make sure that the orbits with action in I are non-degenerate. Then there exists a constant $\epsilon(S) > 0$ such that, when $|I| = \beta - \alpha < \epsilon(S)$, this subspace of the Floer complex is actually a subcomplex and, moreover, a direct summand. [This is an immediate consequence of the fact that a holomorphic curve crossing S must have energy bounded away from zero by some constant $\epsilon(S)$.] Furthermore, continuation maps respect this decomposition as long as the Hamiltonians remain constant on S. (However, the value of the Hamiltonians on S can enter the interval I during the homotopy.) Let us denote the resulting Floer homology by $\mathrm{HF}_*^I(F;\, V)$.

We apply this construction to $H_{\pm}$ with S being the spherical shell where $H_+ = c'$ and V being the ball of radius R enclosed by this shell. (Hence we also need H_- to be constant outside V.) As a consequence, (3.4) turns into an isomorphism

$$\mathbb{Z}_2 \cong \mathrm{HF}_{n+1}^{(kc+\delta,\, kc+\epsilon)}\big(kH_+;\, V\big) \xrightarrow{\cong} \mathrm{HF}_{n+1}^{(kc+\delta,\, kc+\epsilon)}\big(kH_-;\, V\big) \cong \mathbb{Z}_2,$$

entering the map (3.3) as a direct summand. Hence (3.3) is also non-zero.

The general case where we only have $\|d^2H(x)\| < \eta$ is handled in a similar way, but the construction of $H_{\pm}$ is considerably more involved and the choice of the modified Hamiltonians H requires more attention; see Ginzburg (2010) and Ginzburg and Gürel (2009b). □

Interestingly, no other proof of the Conley conjecture is known for general symplectic manifolds. A more conceptual or just plain different argument may shed new light on the nature of the phenomena considered here and is likely to have other applications. [For the torus, a different proof is given in the original work Hingston (2009) and then in Mazzucchelli (2013). However, it is not clear to us how to translate that proof to symplectic topological language.]

Remark 3.12 The part of the proof that does not go through when M is irrational is the last step, the localization. The difficulty is that the action spectrum is dense in this case, and necessarily some of the recappings of degenerate trivial orbits of F in S have actions in I. Thus it is not obvious how to define the Floer homology localized in V. This problem is circumvented in Hein (2012) by considering the Hamiltonians which have a slight slope in S rather than being constant. With this modification, the localization procedure goes through, although the underlying reason for the localization is now different; see Hein (2012) and Usher (2009).

4 Reeb Flows

4.1 General Discussion

The collection of all closed symplectic manifolds breaks down into two classes: those for which the Conley conjecture holds and those for which the Conley conjecture fails. Of course, the non-trivial assertion is then that, as we have seen, the former class is non-empty and even quite large. The situation with closed contact manifolds is more involved even if we leave aside such fundamental questions as the Weinstein conjecture and furthermore focus exclusively on the contact homological properties of the manifold.

First of all, there is a class of contact manifolds for which every Reeb flow has infinitely many simple closed orbits because the rank of the contact or symplectic homology grows as a function of the index or of some other parameter related to the order of iteration. This phenomenon is studied in, e.g., Colin and Honda (2013), Hryniewicz and Macarini (2015) and McLean (2012) and the results generalize and are inspired by a theorem from Gromoll and Meyer (1969), establishing the existence of infinitely many closed geodesics for manifolds whose free loop space homology grows. (A technical but important fact closely related to Theorem 3.2 and underpinning the proof is that the iterates of a given orbit can make only bounded contributions to the homology; see Ginzburg and Gürel (2010), Gromoll and Meyer (1969), Hryniewicz and Macarini (2015) and McLean (2012) for various incarnations of this result.) By Vigué-Poirrier and Sullivan (1976), Abbondandolo and Schwarz (2006), Salamon and Weber (2006) and Viterbo (1999), among contact manifolds in this class are the unit cotangent bundles ST^*M whenever $\pi_1(M) = 0$ and the algebra $\mathrm{H}^*(M;\mathbb{Q})$ is not generated by one element, and some others; Colin and Honda (2013), Hryniewicz and Macarini (2015) and McLean (2012). As is already pointed out in Sect. 2.1, this homologically forced existence of infinitely many Reeb orbits has very different nature from the Hamiltonian Conley conjecture where there is no homological growth.

Then there are contact manifolds admitting Reeb flows with finitely many closed orbits. Among these are, of course, the standard contact spheres and, more generally, the pre-quantization circle bundles over symplectic manifolds admitting torus actions with isolated fixed points; see Gürel (2015, Example 1.13). Note that the class of such pre-quantization circle bundles includes the Katok–Ziller flows, i.e., Finsler metrics with finitely many closed geodesics on S^n and on some other manifolds; see Katok (1973) for the original construction and also Ziller (1983). Another important group of examples also containing the standard contact spheres arises from contact toric manifolds; see Abreu and Macarini (2012). These two classes (pre-quantization circle bundles and contact toric manifolds) overlap, but do not entirely coincide. Although this is not obvious, Reeb flows with finitely many periodic orbits may have non-trivial dynamics, e.g., be ergodic; see Katok (1973).

Finally, as is shown in Ginzburg et al. (2014), there is a *non-empty* class of contact manifolds for which every Reeb flow (meeting certain natural index conditions) has infinitely many simple closed orbits, although there is no obvious homological growth—the rank of the relevant contact homology remains bounded. One can expect this class to be quite large, but at this point such unconditional existence of infinitely many closed Reeb orbits has only been proved for the pre-quantization circle bundles of certain aspherical manifolds; see Theorem 4.1. The proof of this theorem is quite similar to its Hamiltonian counterpart.

This picture is, of course, oversimplified and not even close to covering the entire range of possibilities, even on the homological level. For instance, hypothetically, the Reeb flows for overtwisted contact structures have infinitely many simple closed orbits, but where should one place such contact structures in our "classification"? [See Eliashberg (1998) and Yau (2006) for a proof of the existence of one closed orbit in this case.]

One application of Theorem 4.1 is the existence of infinitely many simple periodic orbits for all low energy levels of twisted geodesic flows on surfaces of genus $g \geq 2$ with non-vanishing magnetic field; see Sect. 5.

Just as in the Hamiltonian setting, the mean indices or the actions and the mean indices of simple periodic orbits of Reeb flows must, in many instances, satisfy certain resonance relations when the number of closed orbits is finite. The mean index resonance relations for the standard contact sphere were discovered by Viterbo in Viterbo (1989), and the Morse–Bott case for geodesic flows was considered in Rademacher (1989). Viterbo's resonance relations were generalized to non-degenerate Reeb flows on a broad class of contact manifolds in Ginzburg and Kerman (2010). These resonance relations resemble the equality between two expressions for the Euler characteristic of a closed manifold: the homological one and the one using indices of zeroes of a vector field. The role of the homological expression is now taken by the mean Euler characteristic of the contact homology of the manifold, introduced in van Koert (2005), and the sum of the indices is replaced by the sum of certain local invariants of simple closed orbits. The degenerate case of the generalized Viterbo resonance relations was studied in Ginzburg and Gören (2015), Hryniewicz and Macarini (2015) and Long et al. (2014) and the Morse–Bott case in Espina (2014). There are also variants of resonance relations involving both the actions and the mean indices; see Gürel (2015) and also Ekeland (1984) and Ekeland and Hofer (1987).

Leaving aside the exact form of the resonance relations, we only mention here some of their applications. The first one, in dynamics, is a contact analog of Theorem 2.2: the generic existence of infinitely many simple closed orbits for a large class of Reeb flows; see Ginzburg and Gürel (2009c) and also Ekeland (1984), Rademacher (1994) and Hingston (1984) for related earlier results. Another application, also in dynamics, is to the proof of the existence of at least two simple closed Reeb orbits on the standard contact S^3. This result is further discussed in the next section; see Theorem 4.3. (We refer the reader to Gürel (2015) for some other applications in dynamics.) Finally, on the topological side, the resonance relations can be used to calculate the mean Euler characteristic, Espina (2014).

4.2 Contact Conley Conjecture

Consider a closed symplectic manifold (M, ω) such that the form ω or, to be more precise, its cohomology class $[\omega]$ is *integral*, i.e., $[\omega] \in \mathrm{H}^2(M; \mathbb{Z})/\operatorname{Tor}$. Let $\pi : P \to M$ be an S^1-bundle over M with first Chern class $-[\omega]$. The bundle P admits an S^1-invariant 1-form α_0 such that $d\alpha_0 = \pi^*\omega$ and $\alpha_0(R_0) = 1$, where R_0 is the vector field generating the S^1-action on P. In other words, when we set $S^1 = \mathbb{R}/\mathbb{Z}$ and identify the Lie algebra of S^1 with $\mathbb{R}$, the form α_0 is a connection form on P with curvature ω. (Note our sign convention.)

Clearly, α_0 is a contact form with Reeb vector field R_0, and the connection distribution $\xi = \ker \alpha_0$ is a contact structure on P. Up to a gauge transformation, ξ is independent of the choice of α_0. The circle bundle P equipped with this contact structure is usually referred to as a *pre-quantization circle bundle* or a Boothby–Wang bundle. Also, recall that a degree two (real) cohomology class on P is said to be *atoroidal* if its integral over any smooth map $\mathbb{T}^2 \to P$ is zero. (Such a class is necessarily aspherical.) Finally, in what follows, we will denote by $\mathfrak{f}$ the free homotopy class of the fiber of π.

The main tool used in the proof of Theorem 4.1 stated below is the cylindrical contact homology. As is well known, to have this homology defined for a contact form α on any closed contact manifold P one has to impose certain additional requirements on the closed Reeb orbits of α. (See Bourgeois 2009; Eliashberg et al. 2000 and references therein for the definition and a detailed discussion of contact homology.) Namely, following Ginzburg et al. (2014), we say that a non-degenerate contact form α is *index–admissible* if its Reeb flow has no contractible closed orbits with Conley–Zehnder index $2-n$ or $2-n\pm 1$, where $\dim P = 2n+1$. In general, α or its Reeb flow is index–admissible when there exists a sequence of non-degenerate index–admissible forms C^1-converging to α.

This requirement is usually satisfied when (P, α) has some geometrical convexity properties. For instance, the Reeb flow on a strictly convex hypersurface in $\mathbb{R}^{2m}$ is index–admissible, Hofer et al. (1998). Likewise, as is observed in Benedetti (2014), the twisted geodesic flow on a low energy level for a symplectic magnetic field on a surface of genus $g \geq 2$ is index–admissible; see Sect. 5 for more details. Finally, let us call a closed Reeb orbit x non-degenerate (or weakly non-degenerate, SDM, etc.) if its Poincaré return map is non-degenerate (or, respectively, weakly non-degenerate, SDM, etc.), cf. Sect. 2.2.1. (The Poincaré return map is the map, or rather the germ of a map, $\Sigma \to \Sigma$ defined on a small cross section at $x(0)$ and sending a point $z \in \Sigma$ to the first intersection of the Reeb orbit through z with Σ.)

Theorem 4.1 (Contact Conley conjecture, Ginzburg et al. 2014) *Assume that*

(i) *M is aspherical, i.e., $\pi_r(M) = 0$ for all $r \geq 2$, and*
(ii) *$c_1(\xi) \in \mathrm{H}^2(P; \mathbb{R})$ is atoroidal.*

Let α be an index–admissible contact form on the pre-quantization bundle P over M, supporting ξ. Then the Reeb flow of α has infinitely many simple closed orbits with contractible projections to M. Assume furthermore that the Reeb flow has finitely many periodic orbits in the free homotopy class $\mathfrak{f}$ of the fiber and that these orbits are *weakly non-degenerate. Then for every sufficiently large prime k the Reeb flow of α has a simple closed orbit in the class $\mathfrak{f}^k$, and all classes $\mathfrak{f}^k$ are distinct.*

It follows from Theorem 4.1 and the discussion below that, when the Reeb flow of α is weakly non-degenerate, the number of simple periodic orbits of the Reeb flow of α with period (or equivalently action) less than $a > 0$ is bounded from below by $C_0 \cdot a/\ln a - C_1$, where $C_0 = \inf \alpha(R_0)$ and C_1 depends only on α. As mentioned in Sect. 2.2, this is a typical lower growth bound in the Conley conjecture type results. Note also that the weak non-degeneracy requirement here plays a technical role and probably can be eliminated.

The key to the proof of Theorem 4.1 is the observation that, as a consequence of (i), all free homotopy classes $\mathfrak{f}^k$, $k \in \mathbb{N}$, are distinct and hence give rise to an $\mathbb{N}$-grading of the cylindrical contact homology of (P, α). (In fact, it would be sufficient to assume that $[\omega]$ is aspherical and $\pi_1(M)$ is torsion free; both of these requirements follow from (i).) This grading plays essentially the same role as the order of iteration in the Hamiltonian Conley conjecture. With this observation in mind, the proof of the weakly non-degenerate case is quite similar to its Hamiltonian counterpart. [Condition (ii) is purely technical and most likely can be dropped.]

To complete the proof, one then has to deal with the case where the Reeb flow of α has a simple SDM orbit, i.e., a simple isolated orbit with an SDM Poincaré return map. This is also done similarly to the Hamiltonian case, but there are some nuances.

Consider a closed contact manifold $(P^{2n+1}, \ker\alpha)$ with a strong symplectic filling (W, ω), i.e., W is a compact symplectic manifold such that $P = \partial W$ with $\omega|_P = d\alpha$ and a natural orientation compatibility condition is satisfied. Let $\mathfrak{c}$ be a free homotopy class of loops in W.

Theorem 4.2 (Ginzburg et al. 2013, 2014) *Assume that the Reeb flow of α has a simple closed SDM orbit in the class $\mathfrak{c}$ and one of the following requirements is met*:

- *W is symplectically aspherical and $\mathfrak{c} = 1$, or*
- *ω is exact and $c_1(TW) = 0$ in $H^2(W; \mathbb{Z})$.*

Then the Reeb flow of α has infinitely many simple periodic orbits.

This result is a contact analog of Theorem 3.8. Theorem 4.1 readily follows from the first case of Theorem 4.2 where we take the pre-quantization disk bundle over M as W. [Here we only point out that $\pi_2(W) = \pi_2(M) = 0$ since M is aspherical and refer the reader to Ginzburg et al. (2014) for more details.]

The proof of Theorem 4.2 uses the filtered linearized contact homology. To be more specific, denote by $\mathrm{HC}_*^{(a,\,b)}(\alpha;\, W, \mathfrak{c}^k)$ the linearized contact homology of (P, α) with respect to the filling (W, ω) for the action interval $(a,\, b)$ and the free homotopy class $\mathfrak{c}^k$, graded by the Conley–Zehnder index. Set $\Delta = \Delta(x)$ and $c = \mathcal{A}(x)$ where x is the SDM orbit from the theorem. Similarly to the Hamiltonian case (cf. Theorem 3.9), one first shows that, under the hypotheses of the theorem, for any $\epsilon > 0$ there exists $k_\epsilon \in \mathbb{N}$ such that

$$\mathrm{HC}_{k\Delta+n+1}^{(kc+\delta_k,\, kc+\epsilon)}(\alpha;\, W, \mathfrak{c}^k) \neq 0 \text{ for all } k > k_\epsilon \text{ and some } \delta_k < \epsilon. \tag{4.1}$$

Theorem 4.2 is a consequence of (4.1) (see Ginzburg et al. 2014), although the argument is less obvious than its Hamiltonian counterpart for, say, symplectic CY manifolds.

The proof of (4.1) given in Ginzburg et al. (2013) follows the same path as the proof of Theorem 3.9. Namely, we squeeze the form α between two contact forms $\alpha_\pm$ constructed using the Hamiltonians $H_\pm$ near the SDM orbit, calculate the relevant contact homology for $\alpha_\pm$ (or rather a direct summand in it), and show that the map in contact homology induced by the cobordism from α_+ to α_- is non-zero. This map factors through $\mathrm{HC}^{(kc+\delta_k,\, kc+\epsilon)}_{k\Delta+n+1}(\alpha; W, \mathfrak{c}^k)$, and hence this group is also non-trivial.

Note that Theorem 4.2 as stated, without further assumptions on $\mathfrak{c}$, affords no control on the free homotopy classes of the simple orbits or their growth rate. A related point is that, at the time of this writing, there seems to be no satisfactory version of Theorem 4.2 which would not rely on the existence of the filling W. The difficulty is that without a filling one is forced to work with cylindrical contact homology to prove a variant of (4.1), but then it is not clear if the forms $\alpha_\pm$ can be made index–admissible without additional assumptions on α along the lines of index–positivity. Such a filling–free version of the theorem would, for instance, enable one to eliminate the weak non-degeneracy assumption in the growth assertion in Theorem 4.1. Another serious limitation of Theorem 4.2 is that the SDM orbit is required to be simple. This condition, which is quite restrictive but probably purely technical, is used in the proof in a crucial way to construct the forms $\alpha_\pm$.

Another application of Theorem 4.2 considered in Ginzburg et al. (2013) (and also in Ginzburg and Gören 2015; Liu and Long 2013) is the following result originally proved in Cristofaro-Gardiner and Hutchings (2012).

Theorem 4.3 *The Reeb flow of a contact form α supporting the standard contact structure on S^3 has at least two simple closed orbits.*

In fact, a much stronger result holds. Namely, every Reeb flow on a closed three-manifold has at least two simple closed Reeb orbits. This fact is proved in Cristofaro-Gardiner and Hutchings (2012) using the machinery of embedded contact homology and is outside the scope of this survey. The idea of the proof from Ginzburg et al. (2013) is that if a Reeb flow on the standard contact S^3 had only one simple closed orbit x, this orbit would be an SDM, and, by Theorem 4.2, the flow would have infinitely many periodic orbits. Showing that x is indeed an SDM requires a rather straightforward index analysis with one non-trivial ingredient used to rule out a certain index pattern. In Ginzburg et al. (2013), this ingredient is strictly three-dimensional and comes from the theory of finite energy foliations (see Hofer et al. 1995, 1996). The argument in Ginzburg and Gören (2015); Liu and Long (2013) uses a variant of the resonance relation for degenerate Reeb flows proved in Ginzburg and Gören (2015) and Long et al. (2014). Theorem 4.2 can also be applied to give a simple proof, based on the same idea, of the result from Bangert and Long (2010) that any Finsler geodesic flow on S^2 has at least two closed geodesics; see Ginzburg and Gören (2015). [Of course, this fact also immediately follows from Cristofaro-Gardiner and Hutchings (2012).]

Interestingly, no multiplicity results along the lines of Theorem 4.3 have been proved in higher dimensions without restrictive additional assumptions on the contact form. Conjecturally, every Reeb flow on the standard contact sphere S^{2n-1} has at least

n simple closed Reeb orbits. This conjecture has been proved when the contact form comes from a strictly convex hypersurface in $\mathbb{R}^{2n}$ and the flow is non-degenerate or $2n \leq 8$; see Long and Zhu (2002), Long (2002) and Wang (2013) and references therein. In the degenerate strictly convex case, the lower bound is $\lfloor n/2 \rfloor + 1$. Without any form of a convexity assumption, it is not even known if a Reeb flow on the standard contact S^5 must have at least two simple closed orbits. It is easy to see, however, that a non-degenerate Reeb flow on the standard S^{2n-1} has at least two simple closed orbits; see, e.g., Gürel (2015, Rmk. 3.3).

We conclude this section by pointing out that the machinery of contact homology which the proof of Theorem 4.1 relies on is yet to be fully put on a rigorous basis.

5 Twisted Geodesic Flows

The results from Sect. 4 have immediate applications to the dynamics of twisted geodesic flows. These flows give a Hamiltonian description of the motion of a charge in a magnetic field on a Riemannian manifold.

To be more precise, consider a closed Riemannian manifold M and let σ be a closed 2-form (a *magnetic field*) on M. Let us equip T^*M with the twisted symplectic structure $\omega = \omega_0 + \pi^*\sigma$, where ω_0 is the standard symplectic form on T^*M and $\pi\colon T^*M \to M$ is the natural projection, and let K be the standard kinetic energy Hamiltonian on T^*M arising from the Riemannian metric on M. The Hamiltonian flow of K on T^*M governs the motion of a charge on M in the magnetic field σ and is referred to as a *twisted geodesic* or *magnetic flow*. In contrast with the geodesic flow (the case $\sigma = 0$), the dynamics of a twisted geodesic flow on an energy level depends on the level. In particular, when M is a surface of genus $g \geq 2$, the example of the horocycle flow shows that a symplectic magnetic flow need not have periodic orbits on all energy levels. Furthermore, the dynamics of a twisted geodesic flow also crucially depends on whether one considers low or high energy levels, and the methods used to study this dynamics further depend on the specific properties of σ, i.e., on whether σ is assumed to be exact or symplectic or, when M is a surface, non-exact but changing sign, etc.

The existence problem for periodic orbits of a charge in a magnetic field was first addressed in the context of symplectic geometry by V.I. Arnold in the early 80s; see Arnold (1986, 1988). Namely, Arnold proved that, as a consequence of the Conley–Zehnder theorem, Conley and Zehnder (1983a), a twisted geodesic flow on $\mathbb{T}^2$ with symplectic magnetic field has periodic orbits on all energy levels when the metric is flat and on all low energy levels for an arbitrary metric, Arnold (1988). It is still unknown if the latter result can be extended to all energy levels; however it was generalized to all surfaces in Ginzburg (1987).

Example 5.1 Assume that M is a surface and let $\sigma = f\,dA$, where dA is an area form. Assume furthermore that f has a non-degenerate critical point at x. Then it is not hard to see that essentially by the inverse function theorem the twisted geodesic flow on a low energy level has a closed orbit near the fiber over x.

Since Arnold's work, the problem has been studied in a variety of settings. We refer the reader to, e.g., Ginzburg (1994) for more details and references prior to 1996 and to, e.g., Asselle and Benedetti (2014), Abbondandolo et al. (2014), Contreras et al. (2004), Ginzburg et al. (2014) and Kerman (1999) for some more recent results and references.

Here we focus exclusively on the case where the magnetic field form σ is symplectic (i.e., non-vanishing when $\dim M = 2$), and we are interested in the question of the existence of periodic orbits on low energy levels. In this setting, in all dimensions, the existence of at least one closed orbit with contractible projection to M was proved in Ginzburg and Gürel (2009a) and Usher (2009).

Furthermore, when σ is symplectic, we can also think of M as a symplectic submanifold of (T^*M, ω) and K as a Hamiltonian on T^*M attaining a Morse–Bott non-degenerate local minimum $K = 0$ at M. Thus we can treat the problem of the existence of periodic orbits on a low energy level $P_\epsilon = \{K = \epsilon\}$ as a generalization of the classical Moser–Weinstein theorem (see Moser 1976; Weinstein 1973), where an isolated non-degenerate minimum is replaced by a Morse–Bott non-degenerate minimum and the critical set is symplectic. This is the point of view taken in Kerman (1999) and then in, e.g., Ginzburg and Gürel (2004, 2009a). To prove the existence of a periodic orbit on every low energy level one first shows that almost all low energy levels carry a periodic orbit with mean index in a certain range depending only on $\dim M$ and having contractible projection to M; see, e.g., Ginzburg and Gürel (2004) and Schlenk (2006). This fact does not really require M to be symplectic; it is sufficient to assume that $\sigma \neq 0$. Then a Sturm theory type argument is used in Ginzburg and Gürel (2009a) and Usher (2009) to show that long orbits must necessarily have high index, and hence, by the Arzela–Ascoli theorem, every low energy level carries a periodic orbit. At this step, the assumption that the Hessian d^2K is positive definite on the normal bundle to M becomes essential, cf. Ginzburg and Gürel (2004, Sect. 2.4).

There are also multiplicity results. Already in Arnold (1986, 1988), it was proved that when $M = \mathbb{T}^2$ and σ is symplectic, there are at least three (or four in the non-degenerate case) periodic orbits on every low energy level P_ϵ. Furthermore, Arnold also conjectured that the lower bounds on the number of periodic orbits are governed by Morse theory and Lusternik–Schnirelmann theory as in the Arnold conjecture whenever σ is symplectic and $\epsilon > 0$ is small enough. These lower bounds were then proved for surfaces in Ginzburg (1987).

For the torus the proof is particularly simple. Let us fix a flat connection on $P_\epsilon = \mathbb{T}^2 \times S^1$. When $\epsilon > 0$ is small, the horizontal sections are transverse to X_K, and one can show that the resulting Poincaré return map is a Hamiltonian diffeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$; see, e.g., Ginzburg (1987) and Levi (2003). Now it remains to apply the Conley–Zehnder theorem. Note that this argument captures only the short orbits, i.e., the orbits in the homotopy class of the fiber. Likewise, the proof for other surfaces in Ginzburg (1987) captures only the orbits that stay close to a fiber and wind around it exactly once. In higher dimensions, however, it is not entirely clear how to define such short orbits. The difficulty arises from the fact that d^2K has several "modes" in every fiber, and the modes can vary significantly and bifurcate from one fiber to another. Furthermore, the Weinstein–Moser theorem provides a hypothetical lower bound which is different from the one coming from the Arnold conjecture perspective; see Kerman (1999). Without a distinguished class of short orbits to work with, one is forced to consider all periodic orbits and, already for $M = \mathbb{T}^2$, use the Conley conjecture type results in place of the Arnold conjecture. Hypothetically, as is observed in Ginzburg and Gürel (2009a), every low energy level should carry infinitely many simple periodic orbits, at least when (M, σ) is a symplectic CY manifold. This is still a conjecture when dim $M > 2$, but in dimension two the question has been recently settled in Ginzburg et al. (2014). Namely, we have

Theorem 5.2 (Ginzburg et al. 2014) *Assume that M is a surface of genus $g \geq 1$ and σ is symplectic. Then for every small $\epsilon > 0$, the flow of K has infinitely many simple periodic orbits on P_ϵ with contractible projections to M. Moreover, assume that the flow has finitely many periodic orbits in the free homotopy class $\mathfrak{f}$ of the fiber. Then for every sufficiently large prime k there is a simple periodic orbit in the class $\mathfrak{f}^k$, and all such classes are distinct.*

When $M = \mathbb{T}^2$, the theorem immediately follows from Arnold's cross section argument once one uses the Conley conjecture for $\mathbb{T}^2$ (proved in Franks and Handel 2003) instead of the Conley–Zehnder theorem; see Ginzburg and Gürel (2009a). When $g \geq 2$, Theorem 5.2 (almost) follows from Theorem 4.1 since P_ϵ has contact type and the flow is index–admissible as observed in Benedetti (2014). The part that is not a consequence of Theorem 4.1 is the existence of a simple periodic orbit in the class $\mathfrak{f}^k$ for a large prime k without any non-degeneracy assumptions. This is proved by applying the second case of Theorem 4.2 to the disjoint union $P_\epsilon \sqcup P_E$, where E is large, with the filling W formed by the part of T^*M between these two energy levels, and $\mathfrak{c} = \mathfrak{f}$. The proof of Theorem 5.2 heavily relies on the machinery of cylindrical contact homology via its dependence on Theorem 4.1. Note, however, that in the present setting one might be able to circumvent foundational difficulties by using automatic transversality results from Hutchings and Nelson (2014). Alternatively, one could work with the linearized contact homology or the equivariant symplectic homology for the filling W, entirely avoiding foundational problems in the latter case.

Two difficulties arise in extending Theorem 5.2 to higher dimensions. One is that the energy levels do not have contact type, and hence the standard contact or symplectic homology techniques are not applicable. This difficulty seems to be more technical than conceptual: using Sturm theory as in Ginzburg and Gürel (2009a) one can still associate to a level a variant of symplectic homology generated by periodic orbits on the level. A more serious obstacle is the lack of filtration by the free homotopy classes $\mathfrak{f}^k$, which plays a central role in the proof.

There seems to be no reason to expect Theorem 5.2 to hold for S^2. However, no counterexamples are known. For instance, let us consider the round metric on S^2 and a non-vanishing magnetic field σ symmetric with respect to rotations about the z axis. The twisted geodesic flow on every energy level is completely integrable. It would be useful and illuminating to analyze this flow and check if it has infinitely many periodic orbits on every (low or high) energy level.

It is conceivable that for any magnetic field, every sufficiently low energy level carries infinitely many periodic orbits. For exact magnetic fields on closed surfaces this is proved for almost all low energy levels in Abbondandolo et al. (2014) by methods from the "classical calculus of variations"; see, e.g., Asselle and Benedetti (2015) for related results and further references. It would be extremely interesting to understand this phenomenon of "almost existence of infinitely many periodic orbits" from a symplectic topology perspective and generalize it to higher dimensions. Furthermore, even in dimension two, no examples of magnetic flows with finitely many periodic orbits on arbitrarily low energy levels are known. For instance, it is not known if the completely integrable twisted geodesic flow on S^2 with an exact S^1-invariant magnetic field σ has infinitely many periodic orbits on only almost all low energy levels or in fact on all such levels. (Note that the Katok–Ziller flows from Katok 1973; Ziller 1983 correspond to high energy levels.)

6 Beyond the Conley Conjecture

6.1 Franks' Theorem

Even when the Conley conjecture fails, the existence of infinitely many simple periodic orbits is, as we have already seen, a generic feature of Hamiltonian diffeomorphisms (and Reeb flows) for a broad class of manifolds. There is, however, a different and more interesting, from our perspective, phenomenon responsible for the existence of infinitely many periodic orbits. The starting point here is a celebrated theorem of Franks.

Theorem 6.1 (Franks 1992, 1996) *Any Hamiltonian diffeomorphism φ of S^2 with at least three fixed points has infinitely many simple periodic orbits.*

In fact, the theorem, already in its original form, was proved for area preserving homeomorphisms. This aspect of the problem is outside the scope of the paper, and here we focus entirely on smooth maps. Furthermore, in the setting of the theorem, there are also strong growth results; see Franks and Handel (2003), Le Calvez (2006) and Kerman (2012) for this and other refinements of Theorem 6.1. The proof of the theorem given in Franks (1992, 1996) utilized methods from low–dimensional dynamics. Recently, a purely symplectic topological proof of the theorem was obtained in Collier et al. (2012); see also Bramham and Hofer (2012) for a different approach.
Outline of the proof from Collier et al. (2012) Arguing by contradiction and passing if necessary to an iteration of φ, we can assume that φ has finitely many periodic points, that all these points are fixed points and that there are at least three fixed points. Applying a variant of the resonance relations from Ginzburg and Kerman (2010) combined with Theorem 3.8 and a simple topological argument, it is not hard to see that there must be (at least) two fixed points x and y with irrational mean indices and at least one point z with zero mean index. Note that, since $\dim S^2 = 2$, the points x and y are elliptic and strongly non-degenerate, and z is either degenerate or hyperbolic.

In the former case, we glue together two copies of S^2 punctured at y and z by inserting narrow cylinders at the seams as in Arnold (1989, App. 9). As a result, we obtain a torus $\mathbb{T}^2$, and the Hamiltonian diffeomorphism φ gives rise to an area preserving map $\psi\colon \mathbb{T}^2 \to \mathbb{T}^2$. This map is not necessarily a Hamiltonian diffeomorphism, but it is symplectically isotopic to id and its Floer homology $\mathrm{HF}_*(\psi)$ is defined. Hence, either $\mathrm{HF}_*(\psi) = 0$ or $\mathrm{HF}_*(\psi) \cong \mathrm{H}_{*+1}(\mathbb{T}^2)$ when ψ is Hamiltonian. Now one shows that, roughly speaking, any of the points $x^{\pm} \in \mathbb{T}^2$ corresponding to x represents a non-trivial homology class of degree different from 0 and ± 1, which is impossible.

When z is a hyperbolic point, we use the points x and y to produce the torus, and again a simple Floer homological argument leads to a contradiction. Indeed, for a sufficiently large iteration of ψ, each elliptic point has large Conley–Zehnder index, since Theorem 3.8 rules out SDM points, and each hyperbolic point has even index. Moreover, hyperbolic points give rise to non-trivial homology classes (cf. Ginzburg and Gürel 2009c, Thm. 1.7). Thus $\mathrm{HF}_*(\psi) \neq 0$ but $\mathrm{HF}_1(\psi) = 0$, which is again impossible. (Alternatively, one can just apply Theorem 6.2 below to deal with this case.) □

6.2 Generalizing Franks' Theorem

Even though all proofs of Franks' theorem are purely low-dimensional, it is tempting to think of the result as a particular case of a more general phenomenon. For instance, one hypothetical generalization of Franks' theorem would be that a Hamiltonian diffeomorphism with more than necessary non-degenerate (or just homologically non-trivial in the sense of Sect. 3.1.2) fixed points has infinitely many periodic orbits. Here more than necessary is usually interpreted as a lower bound arising from some version of the Arnold conjecture. For $\mathbb{CP}^n$, the expected threshold is $n + 1$ and, in particular, it is 2 for S^2 as in Franks' theorem, cf. Hofer and Zehnder (1994, p. 263).

However, this conjectural generalization of Franks' theorem seems to be too restrictive, and from the authors' perspective it is fruitful to put the conjecture in a broader context. Namely, it appears that the presence of a fixed point that is unnecessary from a homological or geometrical perspective is already sufficient to force the existence of infinitely many simple periodic orbits. Let us now state some recent results in this direction.

Theorem 6.2 (Ginzburg and Gürel 2014) *A Hamiltonian diffeomorphism of $\mathbb{CP}^n$ with a hyperbolic periodic orbit has infinitely many simple periodic orbits.*

Here, clearly, the hyperbolic periodic orbit is unnecessary from every perspective. In contrast with Franks' theorem and the Conley conjecture type results, at the time of writing, there are no growth results in this setting. The theorem actually holds for a broader class of manifolds M, and the requirements on M can be stated purely in terms of the quantum homology of M; see Ginzburg and Gürel (2014, Thm. 1.1). Among the manifolds meeting these requirements are, in addition to $\mathbb{CP}^n$, the complex Grassmannians $\mathrm{Gr}(2; N)$, $\mathrm{Gr}(3; 6)$ and $\mathrm{Gr}(3; 7)$; the products $\mathbb{CP}^m \times P^{2d}$ and $\mathrm{Gr}(2; N) \times P^{2d}$, where P is symplectically aspherical and $d \leq m$ in the former case and $d \leq 2$ in the latter; and the monotone products $\mathbb{CP}^m \times \mathbb{CP}^m$. There is also a variant of the theorem for non-contractible hyperbolic orbits, which is applicable to, for example, the product $\mathbb{CP}^m \times P^{2d}$. Note also that the generalization of Franks' theorem to $\mathbb{CP}^n$, at least for non-degenerate Hamiltonian diffeomorphisms, would follow if one could replace in Theorem 6.2 a hyperbolic fixed point by a non-elliptic one.

Another result fitting into this context is the following.

Theorem 6.3 (Gürel 2014) *Let* $\varphi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ *be a Hamiltonian diffeomorphism generated by a Hamiltonian equal to a hyperbolic quadratic form* Q *at infinity* (*i.e., outside a compact set*) *such that* Q *has only real eigenvalues. Assume that* φ *has finitely many fixed points, and one of these points,* x*, is non-degenerate* (*or just isolated and homologically non-trivial*) *and has non-zero mean index. Then* φ *has simple periodic orbits of arbitrarily large period.*

As a consequence, regardless of whether the fixed-point set is finite or not, φ has infinitely many simple periodic orbits. In this theorem the condition that the eigenvalues of Q are real can be slightly relaxed. Conjecturally, it should be enough to require Q to be non-degenerate and x to have mean index different from the mean index of the origin for Q. However, hyperbolicity of Q is used in an essential way in the proof of the theorem. Also, interestingly, in contrast with Franks' theorem, the requirement that x is homologically non-trivial is essential and cannot be omitted, even in dimension two. As an easy consequence of Theorem 6.3, we obtain

Theorem 6.4 (Gürel 2014) *Let* $\varphi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$*, where* $2n = 2$ *or* $2n = 4$*, be a Hamiltonian diffeomorphism generated by a Hamiltonian equal to a hyperbolic quadratic form* Q *at infinity as in Theorem* 6.3. *Assume that* φ *is strongly non-degenerate and has at least two* (*but finitely many*) *fixed points. Then* φ *has simple periodic orbits of arbitrarily large period.*

In the two-dimensional case, a stronger result is proved in Abbondandolo (2001, Thm. 5.1.9). In the setting of Theorems 6.3 and 6.4, one can be more precise about which simple periods occur. Namely, for a certain integer $m > 0$, starting with a sufficiently large prime number, among every m consecutive primes, there exists at least one prime which is the period of a simple periodic orbit. Thus, as in many other results of this type, we have the lower growth bound $const \cdot k/\ln k$.

Theorem 6.2 and, with some extra work, Theorem 6.3 imply the case of Franks' theorem where φ is assumed to have a hyperbolic periodic orbit, e.g., when φ is non-degenerate. Furthermore, it is conceivable that one could prove Franks' theorem as a consequence of Theorem 6.2. Such a proof would certainly be of interest, but it would most likely be much more involved than the argument in Collier et al. (2012).

Let us now turn to non-contractible orbits. Recall that for a (time-dependent) Hamiltonian flow φ_H^t generated by a Hamiltonian $H\colon S^1 \times M \to \mathbb{R}$ there is a one-to-one correspondence between the one-periodic orbits of φ_H^t and the fixed points of $\varphi = \varphi_H$. Furthermore, as is easy to see from the proof of the Arnold conjecture, the free homotopy class of an orbit x is independent of the Hamiltonian generating the time-one map φ. Thus the notion of a contractible one-periodic orbit (or even a "contractible fixed point") of φ is well-defined. Of course, the same applies to k-periodic orbits.

On a closed symplectic manifold M a Hamiltonian diffeomorphism need not have non-contractible one-periodic orbits. Indeed, the Hamiltonian Floer homology vanishes for any non-trivial free homotopy class when M is compact, since all one-periodic orbits of a C^2-small autonomous Hamiltonian are its critical points (hence contractible). Thus, from a homological perspective, non-contractible periodic orbits are totally unnecessary.

To state our next result, recall that a symplectic form ω on M is said to be atoroidal if for every map $v\colon \mathbb{T}^2 \to M$, the integral of ω over v vanishes. We have

Theorem 6.5 (Gürel 2013) *Let M be a closed symplectic manifold equipped with an atoroidal symplectic form ω. Assume that a Hamiltonian diffeomorphism φ of M has a non-degenerate one-periodic orbit x with homology class $[x] \neq 0$ in $\mathrm{H}_1(M;\mathbb{Z})/\operatorname{Tor}$ and that the set of one-periodic orbits in the class $[x]$ is finite. Then, for every sufficiently large prime p, the Hamiltonian diffeomorphism φ has a simple periodic orbit in the homology class $p[x]$ and with period either p or p', where p' is the first prime greater than p.*

Thus the number of simple non-contractible periodic orbits with period less than or equal to k, or the number of distinct homology classes represented by such orbits, is bounded from below by $const \cdot k/\ln k$. An immediate consequence of the theorem is that φ has infinitely many simple periodic orbits with homology classes in $\mathbb{N}[x]$ whether or not the set of one-periodic orbits (in the class $[x]$) is finite. Moreover, in this theorem, as in Theorem 6.3, the non-degeneracy condition can be relaxed and replaced by the much weaker requirement that x is isolated and homologically non-trivial. Finally, in both theorems, the orbit x need not be one-periodic; the theorems (with obvious modifications) still hold when x is just a periodic orbit.

Among the manifolds meeting the requirements of Theorem 6.5 are, for instance, closed Kähler manifolds with negative sectional curvature and, more generally, any closed symplectic manifold with $[\omega]|_{\pi_2(M)} = 0$ and hyperbolic $\pi_1(M)$. Furthermore, Hamiltonian diffeomorphisms having a periodic orbit in a non-trivial homology class exist in abundance. It is plausible that a C^1-generic, or even C^∞-generic, Hamiltonian diffeomorphism has an orbit in a non-trivial homology class when the fundamental group (or the first homology group) of M is large enough; see Tal (2013) for some possibly relevant results for surfaces. However, as is easy to see, already for $M = \mathbb{T}^2$, a fixed Hamiltonian diffeomorphism need not have non-contractible periodic orbits (e.g., φ_H for a small bump function H), and even C^∞-generically one cannot prescribe the homology class of an orbit in advance; Ginzburg and Gürel (2015) and Gürel (2013).

Hypothetically, one can expect an analog of the theorem to hold when the condition that ω is atoroidal is omitted or relaxed, e.g., replaced by the requirement that (M,ω) is toroidally monotone. We refer the reader to Ginzburg and Gürel (2015) for some further results in this direction.

The proofs of all these theorems are based on the same idea that an unnecessary periodic orbit is a seed creating infinitely many periodic orbits. In Theorems 6.3 and 6.5 the argument is that, roughly speaking, the change in filtered Floer homology, for a carefully chosen action range (and/or degree), between different iterations of φ requires new simple periodic orbits to be created. The proof of Theorem 6.2 relies on a result, perhaps of independent interest, asserting that the energy needed for a Floer connecting trajectory of an iterated Hamiltonian to approach a hyperbolic orbit and cross its fixed neighborhood cannot become arbitrarily small: it is bounded away from zero by a constant independent of the order of iteration. Then the product structure in quantum homology is used to show that there must be Floer connecting trajectories with energy converging to zero for some sequence of iterations unless new periodic orbits are created.

6.3 Reeb Flows, Symplectomorphisms and All That

The conjectures discussed in Sect. 6.2 have obvious analogs for Reeb flows and symplectomorphisms.

6.3.1 Reeb Flows Revisited

Just like Hamiltonian diffeomorphisms, Reeb flows with "unnecessary" periodic orbits can be expected to have infinitely many simple periodic orbits. However, at the time of this writing, there is little evidence supporting this conjecture, and all the relevant results are three-dimensional. The most notable one is a theorem, proved in Hofer et al. (1998), asserting that the Reeb flow on a strictly convex hypersurface in $\mathbb{R}^4$ has either two or infinitely many periodic orbits. In fact, more generally, this is true for the so-called dynamically convex contact forms on S^3. Conjecturally, this "two-or-infinitely-many" alternative should hold for all contact forms supporting the standard contact structure on S^3, which could be thought of as a three-dimensional analog of Franks' theorem; see Hofer et al. (2003) for some other related results.

The existence of infinitely many closed geodesics on S^2 also fits perfectly into the framework of this conjecture; see Bangert (1993) and Franks (1992) and also Hingston (1993) and Hingston (1997) and the references therein for the original argument. Indeed, the classical Lusternik–Schnirelmann theorem asserts the existence of at least three closed geodesics on S^2, i.e., at least one more than is necessary from the Floer–theoretic perspective, cf. Katok (1973) and Ziller (1983). The geodesic flow on S^2, interpreted as a Reeb flow on the standard contact $\mathbb{RP}^3$, should then have infinitely many simple (i.e., non-iterated) closed orbits or, in other words, infinitely many geometrically distinct closed geodesics on S^2. In fact, one can reprove the existence of infinitely many closed geodesics in exactly this way using the variant of the Lusternik–Schnirelmann theorem from Grayson (1989) as the starting point and then the results from Hryniewicz et al. (2014) and Ginzburg et al. (2013) on the symplectic side of the problem; see the latter reference for more details.

Finally, another aspect of this question is related to the so-called perfect Reeb flows. Let us call a non-degenerate Reeb flow on a contact manifold *perfect* if the differential in the contact homology complex vanishes for some choice of the auxiliary data, cf. Bourgeois et al. (2007). (Thus this definition depends on the type of the contact homology used.) For instance, a Reeb flow is perfect (for every auxiliary data) when all closed orbits have Conley–Zehnder index of the same parity; we refer the reader to Gürel (2015) for numerous examples of perfect Reeb flows. One can think of non-perfect Reeb flows as those with unnecessary periodic orbits. In Gürel (2015) an upper bound on the number of simple periodic orbits of perfect Reeb flows is established for many contact manifolds under some (minor) additional assumptions. For S^{2n-1}, as expected, the upper bound is n. However, in general, it is not even known if a perfect Reeb flow on the standard contact S^{2n-1}, $2n-1 \geq 5$, must have finitely many simple periodic orbits or, if it does, whether this number is independent of the flow. [For S^3, this is proved in Bourgeois et al. (2007) and reproved in Gürel (2015).]

6.3.2 Symplectomorphisms

For symplectomorphisms, the problem of the existence of infinitely many periodic orbits breaks down into several phenomena in the same way as for Reeb flows, although even less is known. Namely, as in Sect. 4.1, we can, roughly speaking, single out three types of behavior of symplectomorphisms. First of all, some manifolds (such as $\mathbb{CP}^n$ or tori or their products) admit symplectomorphisms with finitely many periodic orbits or even, in some instances (e.g., $\mathbb{T}^{2n}$), without periodic orbits. Here a non-obvious fact is that a surface Σ_g of genus $g \geq 1$ admits a symplectic (autonomous) flow with exactly $|2-2g|$ fixed points and no other periodic orbits; see, e.g., Katok and Hasselblatt (1995, Chap. 14) and, in particular, Exercise 14.6.1 and the hint therein.

Then there are symplectomorphisms φ such that the rank of the Floer homology $\mathrm{HF}_*(\varphi^k)$ over a suitable Novikov ring Λ grows with the order of iteration k. The Floer homology groups of symplectomorphisms have been studied for close to two decades starting with Dostoglou and Salamon (1994) and Lê and Ono (1995), and the literature on the subject is quite extensive (particularly so for symplectomorphisms of surfaces); we refer the reader to, e.g., Cotton-Clay (2010) and Fel'shtyn (2012) and references therein for recent results focusing specifically on the growth of the Floer homology. Let us assume here, for the sake of simplicity, that M is symplectically aspherical or monotone and that the Floer homology $\mathrm{HF}_*(\varphi^k)$ is defined. Similarly to the results in Gromoll and Meyer (1969), Hryniewicz and Macarini (2015) and McLean (2012), we have

Proposition 6.6 *Let $\varphi\colon M \to M$ be a symplectomorphism of a closed symplectic manifold M such that the sequence $\mathrm{rk}_\Lambda\, \mathrm{HF}_*(\varphi^k)$ is unbounded. Then φ has infinitely many simple periodic orbits. Moreover, every sufficiently large prime occurs as a simple period when the number of fixed points of φ is finite and $\mathrm{rk}_\Lambda\, \mathrm{HF}_*(\varphi^k) \to \infty$.*

Proof The proposition is obvious and well known when φ is strongly non-degenerate. (See Cotton-Clay 2010; Fel'shtyn 2012 for more specific and stronger results.) The degenerate case follows from the fact that the dimension of the local Floer homology of an isolated periodic orbit remains bounded as a function of the order of iteration, as a consequence of Theorem 3.2. □

Example 6.7 Let Σ be a closed surface and $\psi\colon \Sigma \to \Sigma$ be a symplectomorphism such that $\mathrm{rk}_\Lambda\, \mathrm{HF}_*(\psi^k) \to \infty$. This is the case, for instance, when the Lefschetz number $L(\psi^k)$ grows; such symplectomorphisms exist in abundance. [Proposition 6.6 applies to ψ, but in this case a simpler argument is available: when $L(\psi^k) \to \infty$ the assertion immediately follows from the Shub–Sullivan theorem, (1974).] Let P be a symplectically aspherical manifold with $\chi(P) = 0$, such as $P = \mathbb{T}^{2n}$, and $\varphi\colon P \times \Sigma \to P \times \Sigma$ be Hamiltonian isotopic to (id, ψ). Then $\mathrm{rk}_\Lambda\, \mathrm{HF}_*(\varphi^k) \to \infty$ and, by the proposition, φ has infinitely many simple periodic orbits. However, $L(\varphi^k) = 0$, and, moreover, a symplectomorphism in the smooth or symplectic isotopy class of (id, ψ) need not have periodic orbits at all when, e.g., $P = \mathbb{T}^{2n}$.

Thirdly, there are symplectomorphisms with infinitely many simple periodic orbits, but no homological growth. Here, of course, we have the Hamiltonian Conley conjecture as a source of examples. The authors tend to think that there should be other classes of symplectomorphisms of this type, but no results to this account have so far been proved. One class of symplectomorphisms which might be worthwhile to examine is that of symplectomorphisms of $\Sigma_g \times P$ symplectically isotopic to id and with flux vanishing on $\mathrm{H}_1(P)$, where Σ_g is a surface of genus $g \geq 2$ and P is symplectically aspherical and not a point.

Finally, one can expect the presence of an unnecessary fixed or periodic point to force a symplectomorphism to have infinitely many simple periodic orbits. However, now the situation is more subtle, less is known, and there is a counterexample to this general principle. A prototypical (and simple) result of this type is that a non-degenerate symplectomorphism of $\mathbb{T}^2$ symplectically isotopic to id has infinitely many simple periodic orbits, provided that it has one fixed or periodic point; see Ginzburg and Gürel (2009c, Thm. 1.7). In other words, we have the following "zero or infinitely many" alternative: a non-degenerate symplectomorphism of $\mathbb{T}^2$ isotopic to id has either no periodic orbits or infinitely many periodic orbits. It is interesting, however, that the non-degeneracy condition cannot entirely be omitted, although it can probably be relaxed. Namely, it is easy to construct a symplectic vector field on $\mathbb{T}^2$ with exactly one (homologically trivial) zero and no periodic points; see Ginzburg and Gürel (2009c, Example 1.10). (No similar results or counterexamples for tori $\mathbb{T}^{2n}$, $2n \geq 4$, are known.) There are also analogs of Theorem 6.2 for symplectomorphisms, Batoréo (2015a, b), applicable to manifolds such as $\mathbb{CP}^n \times P^{2m}$, where P is symplectically aspherical and $m \leq n$.

Note in conclusion that when discussing symplectomorphisms in the homological framework, it would make sense to ask the question of the existence of infinitely many periodic orbits while fixing the class of symplectomorphisms Hamiltonian isotopic to each other. The reason is that Floer homology is very sensitive to symplectic isotopy but is invariant under Hamiltonian isotopy. Above, however, we have not strictly adhered to this point of view and mainly focused on the properties of the ambient manifolds. As just one implication of that viewpoint and to emphasize the difference with the Hamiltonian setting, let us point that one may expect the C^∞-generic (or even C^k-generic for a large k) existence of infinitely many periodic orbits to break down for symplectomorphisms with a fixed flux; cf. Herman (1991a, b).

Remark 6.8 In this survey, we have just briefly touched upon the question of the existence of infinitely many periodic orbits for Hamiltonian diffeomorphisms and symplectomorphisms of open manifolds and manifolds with boundary. (In this case, one, of course, has to impose some restrictions on the behavior of the map near infinity or on the boundary.) Such symplectomorphisms naturally arise in applications and in physics. For instance, the billiard maps and the time-one maps describing the motion of a particle in a time-dependent conservative force field and/or exact magnetic field are in this class. We are not aware of any new phenomena happening in this setting, and our general discussion readily translates to such maps. For instance, Hamiltonian diffeomorphisms of open manifolds can exhibit the same three types of behavior as symplectomorphisms of closed manifolds or Reeb flows. (After all, a geodesic flow is a Hamiltonian flow on the cotangent bundle.) To the best of our knowledge, there are relatively few results of symplectic topological nature concerning this class of maps; see Sect. 2.1 for some relevant references.

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