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PROBLEM CONTRIBUTION

Disconjugacy and the Secant Conjecture

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Abstract We discuss the so-called secant conjecture in real algebraic geometry, and show that it follows from another interesting conjecture, about disconjugacy of vector spaces of real polynomials in one variable.

Keywords Disconjugacy · Wronskian · Schubert calculus

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Let V be a real vector space of dimension n whose elements are real functions on an interval $[a, b]$. The space V is called *disconjugate* if one of the following equivalent conditions is satisfied:

(a) Every $f \in V \backslash \{0\}$ has at most $n - 1$ zeros, or
(b) For every n distinct points $x_1, \ldots, x_n$ on $[a, b]$ and every basis $f_1, \ldots, f_n$ of V we have $\det(f_i(x_j)) \neq 0$.

One can replace "every basis" by "some basis" in (b) and obtain an equivalent condition.

If V is disconjugate then the determinant in (b) has constant sign which depends only on the ordering of x_j and on the choice of the basis.

A space of real functions on an open interval is called disconjugate if it is disconjugate on every closed subinterval.

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We are only interested here in spaces V consisting of polynomials.

Suppose that a positive integer d is given, and V consists of polynomials of degree a most d. Then every basis $f_1, \dots, f_n$ of V defines a real rational curve $\mathbf{RP}^1 \to \mathbf{RP}^{n-1}$ of degree d. Indeed, we can replace every $f_j(x)$ by a homogeneous polynomial $f_j^*(x_0, x_1)$ of two variables of degree d, such that $f_j(x) = f_j^*(1, x)$, and then $f_1^*, \dots, f_n^*$ define a map $f : \mathbf{RP}^1 \to \mathbf{RP}^{n-1}$ (if polynomials have a common root, divide it out).

Then the geometric interpretation of disconjugacy is:

(c) The curve f constructed from a basis in V is *convex*, that is intersects every hyperplane at most $n - 1$ times.

For every basis $f_1, \dots, f_n$ in V we can consider its Wronski determinant $W = W(f_1, \dots, f_n)$. Changing the basis results in multiplication of W by a non-zero constant, so the roots of W only depend on V.

Conjecture 1 *Suppose that all roots of W are real. Then V is disconjugate on every interval that does not contain these roots.*

This is known for $n = 2$ with arbitrary d (see below), and for $n = 3, d \le 5$ by direct verification with a computer.

This conjecture arises in real enumerative geometry (Schubert calculus), and we explain the connection. The problem of enumerative geometry we are interested in is the following:

Let $m \ge 2$ and $p \ge 2$ be given integers. Suppose that mp linear subspaces of dimension p in general position in $\mathbf{C}^{m+p}$ are given. How many linear subspaces of dimension m intersect all of them non-trivially?

The answer was obtained by Schubert in 1886 and it is

$$d(m, p) = \frac{1!2! \dots (p-1)!(mp)!}{m!(m+1)! \dots (m+p-1)!}.$$

Now suppose that all those given subspaces are real. Does it follow that all p-subspaces that intersect all of them non-trivially are real? The answer is negative, and we are interested in finding a geometric condition on the given p-subspaces that ensure that all $d(m, p)$ subspaces of dimension m that intersect all the given p-subspaces non-trivially are real.

One such condition was proposed by B. and M. Shapiro. Let $F(x) = (1, x, \dots, x^d)$, $d = m + p - 1$ be a rational normal curve, a. k. a. moment curve. Suppose that the given p-spaces are osculating F at some real points $F(x_j)$. This means that subspaces X_j are spanned by the (row)-vectors $F(x_j), F'(x_j), \dots, F^{(p-1)}(x_j)$ for some real x_j, $1 \le j \le mp$. Then all m-subspaces that intersect all X_j non-trivially are real.

This was conjectured by B. and M. Shapiro and proved by Mukhin, Tarasov and Varchenko (MTV) Mukhin et al. (2009). Earlier it was known for $n = 2$ Eremenko and Gabrielov (2002), and in Eremenko and Gabrielov (2011) a simplified elementary proof for the case $n = 2$ was given.

We are interested in the following generalization of this result.

Secant Conjecture. *Suppose that each of the mp subspaces X_j, $1 \le j \le mp$ is spanned by p distinct real vectors $F(x_{j,k})$, $0 \le k \le p - 1$, and that the sets of points*

$\{x_{j,k} : 0 \le k \le p-1\}$ *are separated, that is* $x_{j,k} \in I_j$*, where* $I_j \subset \mathbf{RP}^1$ *are disjoint intervals. Then all* m*-subspaces which intersect all* X_j *non-trivially are real.*

This is known when $p = 2$, Eremenko et al. (2006) and has been tested on a computer for $p = 3$ and small m, Hillar et al. (2010), Garcia-Puente et al. (2012). The special case when the groups $\{x_{j,k}\}_{k=0}^{p-1}$ form arithmetic progressions, $x_{j,k} = x_{j,0} + kh$ has been established Mukhin et al. (2009).

Next we show how the Secant Conjecture follows from Conjecture 1 and the results of MTV.

Let us represent an m-subspace Y that intersects all subspaces X_j as the zero set of p linear forms, and use the coefficients of these forms as coefficients of p polynomials $f_0, \ldots, f_{p-1}$. Then the condition that Y intersects some X_j is equivalent to linear dependence of the p vectors $f_i(x_{j,m})_{m=0}^{p-1}$, $i = 0, \ldots, p-1$. That is

$$\det(f_i(x_{j,m}))_{i,m=0}^{p-1} = 0.$$

These equations for $j = 1, \ldots, mp$ define the subspaces Y, and we have to prove that all solutions are real.

Let I_j be the intervals with disjoint closures which contain the $x_{j,k}$. We may assume without loss of generality that $\infty \notin I_j$. We place on each I_j a point y_j, and consider the $d(m, p)$ real rational curves $\mathbf{R} \to \mathbf{RP}^p$ with inflection points at y_j. These curves exist by the MTV theorem, and they depend continuously on the y_j.

Let $f = (f_0 \ldots, f_{p-1})$ be one of these curves. Fix $k \in \{1, \ldots, mp\}$. For all $j \ne k$, fix all $y_j \in I_j$. When y_k moves in I_k from the left end to the right end, the determinant $\det(f_i(x_{k,m}))_{i,m=0}^{p-1}$ must change sign, in view of Conjecture 1. So this determinant is 0 for some position of y_k on I_k.

Then it follows by a well-known topological argument that one can choose all $y_j \in I_j$ in such a way that $\det(f_i(x_{j,m})) = 0$ for all j.

Thus we have constructed $d(m, p)$ real solutions of the secant problem. As the total number of solutions is also $d(m, p)$, for generic data, we obtain the result.

Proof of Conjecture 1 *for* $n = 2$. We have two real polynomials f_1 and f_2, such that $f_1' f_2 - f_1 f_2'$ has only real zeros. This means that the rational function $F = f_1/f_2 : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is real and all its critical points are real. Let $I \subset \mathbf{R}$ be a closed interval without critical points. Then F is a local homeomorphism on I, so $F(I + i\epsilon)$ belongs to one of the half-planes $\mathbf{C}\backslash\mathbf{R}$, for all sufficiently small $\epsilon > 0$. Suppose without loss of generality that it belongs to the upper half-plane H. Let D be the component of $F^{-1}(H)$ that contains $I + i\epsilon$. Then D is a region in H with piecewise analytic boundary, and $I \subset \partial D$. The map $F : D \to H$ is a covering because it is proper and has no critical points. As H is simply connected, D must be simply connected and $F : D \to H$ must be a conformal homeomorphism. Then $F^{-1} : H \to D$ is a conformal homeomorphism. As ∂D is locally connected, this homeomorphism extends to $F^{-1} : \overline{H} \to \overline{D}$. This last map must be injective because this is a left inverse of a function. Thus $F^{-1} : \overline{H} \to \overline{D}$ is a homeomorphism. Then $F : \overline{D} \to \overline{H}$ must be also a homeomorphism, in particular F is injective on I.

This implies that the linear span of f_1, f_2 is disconjugate.

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