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RESEARCH EXPOSITION

Building Thermodynamics for Non-uniformly Hyperbolic Maps

Vaughn Climenhaga[1] · **Yakov Pesin**[2]

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Abstract We briefly survey the theory of thermodynamic formalism for uniformly hyperbolic systems, and then describe several recent approaches to the problem of extending this theory to non-uniform hyperbolicity. The first of these approaches involves Markov models such as Young towers, countable-state Markov shifts, and inducing schemes. The other two are less fully developed but have seen significant progress in the last few years: these involve coarse-graining techniques (expansivity and specification) and geometric arguments involving push-forward of densities on admissible manifolds.

Keywords Thermodynamic formalism · Non-uniform hyperbolicity · Equilibrium states · Phase transitions

Mathematics Subject Classification Primary 37D25 · 37D35

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✉ Yakov Pesin
pesin@math.psu.edu

Vaughn Climenhaga
climenha@math.uh.edu

[1] Department of Mathematics, University of Houston, Houston, TX 77204, USA

[2] Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA

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1 Introduction

1.1 The General Setting

Thermodynamic formalism, i.e., the formalism of equilibrium statistical physics, was adapted to the study of dynamical systems in the classical works of Ruelle (1972, 1978), Sinaĭ (1968, 1972), and Bowen (1970, 1974, 2008). It provides an ample collection of methods for constructing invariant measures with strong statistical properties. In particular, this includes constructing a certain **"physical"** measure known as the **SRB measure** (for Sinai–Ruelle–Bowen).

The general ideas can be given as follows. Let (X, d) be a compact metric space and $f: X \to X$ a continuous map of finite topological entropy. Fix a continuous function $\varphi: X \to \mathbb{R}$, which we will refer to as a **potential**. Denote by $\mathcal{M}(f)$ the space of all f-invariant Borel probability measures on X. Given $\mu \in \mathcal{M}(f)$, the **free energy** of the system with respect to μ is

$$E_\mu(\varphi) := -\left(h_\mu(f) + \int_X \varphi\, d\mu\right),$$

where $h_\mu(f)$ is the Kolmogorov–Sinai (measure-theoretic) entropy of (X, f, μ). Optimizing over all invariant measures gives the **topological pressure**

$$P(\varphi) := -\inf_{\mu\in\mathcal{M}(f)} E_\mu(\varphi) = \sup_{\mu\in\mathcal{M}(f)} \left(h_\mu(f) + \int_X \varphi\, d\mu\right),$$

and a measure achieving this extremum is called an **equilibrium measure** (or **equilibrium state**). Note that it suffices to take the infimum (supremum) over the space $\mathcal{M}^e(f) \subset \mathcal{M}(f)$ of **ergodic** measures.

The **variational principle** relates the definition of pressure as an extremum over invariant measures to an alternate definition in terms of growth rates. Given $\varepsilon > 0$ and $n \in \mathbb{N}$, a set $E \subset X$ is (n, ε)-**separated** if points in E can be distinguished at a scale ε within n iterates; more precisely, if for every $x, y \in E$ with $x \neq y$, there is $0 \le k \le n$ such that $d(f^k x, f^k y) \ge \varepsilon$. Then one has

$$P(\varphi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_{\substack{E\subset X \\ (n,\varepsilon)\text{-sep.}}} \sum_{x\in E} e^{S_n\varphi(x)}, \tag{1.1}$$

where

$$S_n\varphi(x) := \sum_{k=0}^{n-1} \varphi(f^k x). \tag{1.2}$$

The sum in (1.1) is a **partition sum** that quantifies "weighted orbit complexity at spatial scale ε and time scale n"; $P(\varphi)$ represents the growth rate of this complexity as time increases. In the particular case $\varphi = 0$, the value $P(0)$ is the topological entropy $h_{\text{top}}(f)$ of the map f.

Thermodynamic formalism is most useful when the system possesses some degree of hyperbolic behavior, so that orbit complexity increases exponentially. The most complete results are available when f is uniformly hyperbolic; we discuss these in Sect. 1.2. In this article we focus on non-uniformly hyperbolic systems, and we discuss the general picture in Sect. 1.3. Our emphasis will be on general techniques rather than on specific examples. In particular, we discuss Markov models (including Young towers) in Sects. 2–4, coarse-graining techniques (based on expansivity and specification) in Sect. 5, and push-forward (geometric) approaches (based on newly introduced standard pairs approach) in Sect. 6.

1.2 Uniformly Hyperbolic Maps (Sinai, Ruelle, Bowen)

1.2.1 General Thermodynamic Results

We refer the reader to (Katok and Hasselblatt 1995; Brin and Stuck 2002) for fundamentals of uniform hyperbolicity theory and to (Bowen 2008; Parry and Pollicott 1990) for a complete description of thermodynamic formalism for uniformly hyperbolic systems. Consider a compact smooth Riemannian manifold M and a C^1 diffeomorphism $f\colon M \to M$. A compact invariant set $\Lambda \subset M$ is called **hyperbolic** if for every $x \in \Lambda$ the tangent space $T_x M$ admits an invariant splitting $T_x M = E^s(x) \oplus E^u(x)$ into **stable** and **unstable** subspaces with uniform contraction and expansion: this means that there are numbers $c > 0$ and $0 < \lambda < 1$ such that for every $x \in \Lambda$:

(1) $\|df^n v\| \le c\lambda^n \|v\|$ for $v \in E^s(x)$ and $n \ge 0$;
(2) $\|df^{-n} v\| \le c\lambda^n \|v\|$ for $v \in E^u(x)$ and $n \ge 0$.

One can show that the subspaces E^s and E^u depend Hölder continuously on x; in particular, there is $k > 0$ such that $\angle(E^s(x), E^u(x)) \ge k$ for every $x \in \Lambda$.

Moving from the tangent bundle to the manifold itself, for every $x \in \Lambda$ one can construct local **stable** $V^s(x)$ and **unstable** $V^u(x)$ **manifolds** (also called **leaves**) through x which are tangent to $E^s(x)$ and $E^u(x)$ respectively and depend Hölder continuously on x (Katok and Hasselblatt 1995, Sect. 6.2). In particular, there is $\varepsilon > 0$ such that for any $x, y \in \Lambda$ for which $d(x, y) \le \varepsilon$ one has that the intersection $V^s(x) \cap V^u(y)$ consists of a single point (here $d(x, y)$ denotes the distance between points x and y induced by the Riemannian metric on M). We denote this point by $[x, y]$.

A hyperbolic set Λ is called **locally maximal** if there is a neighborhood U of Λ such that for any invariant set $\Lambda' \subset U$ we have that $\Lambda' \subset \Lambda$. In other words, $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$. One can show that a hyperbolic set Λ is locally maximal if and only if for any $x, y \in \Lambda$ which are sufficiently close to each other, the point $[x, y]$ lies in Λ (Katok and Hasselblatt 1995, Sect. 6.4).

Given a locally maximal hyperbolic set and a Hölder continuous potential function, thermodynamic formalism produces unique equilibrium measures with strong ergodic properties: before stating the theorem we recall some notions from ergodic theory for the reader's convenience. Let (X, μ) be a Lebesgue space with a probability measure μ and $T\colon X \to X$ an invertible measurable transformation that preserves μ.

(1) **The Bernoulli property.** Let Y be a finite set and ν a probability measure on Y (that is, a probability vector). One can associate to (Y, ν) the two-sided Bernoulli shift $\sigma : Y^{\mathbb{Z}} \to Y^{\mathbb{Z}}$ defined by $(\sigma y)_n = y_{n+1}$, $n \in \mathbb{Z}$; this preserves the measure κ given as the direct product of $\mathbb{Z}$ copies of ν. We say that (T, μ) is a Bernoulli automorphism (or "has the Bernoulli property") if (T, μ) is metrically isomorphic to the Bernoulli shift (σ, κ) associated to some Lebesgue space (Y, ν) and we also say that μ is a Bernoulli measure.[1]

(2) **Decay of correlations.** Let $\mathcal{H}$ be a class of square-integrable *test* functions $X \to \mathbb{R}$ and define

$$\mathrm{Cor}_n(h_1, h_2) := \left| \int h_1(T^n(x)) h_2(x)\, d\mu - \int h_1(x)\, d\mu \int h_2(x)\, d\mu \right|.$$

We say that (T, μ) has

- *exponential decay of correlations (EDC)* with respect to $\mathcal{H}$ if there is $0 < \theta < 1$ satisfying: for every $h_1, h_2 \in \mathcal{H}$ there is $K = K(h_1, h_2) > 0$ such that for every $n > 0$

$$\mathrm{Cor}_n(h_1, h_2) \le K\theta^n;$$

- *polynomial decay of correlations (PDC)* with respect to $\mathcal{H}$ if there is $\alpha > 0$ satisfying: for every $h_1, h_2 \in \mathcal{H}$ there is $K = K(h_1, h_2) > 0$ such that for every $n > 0$

$$\mathrm{Cor}_n(h_1, h_2) \le K n^{\alpha}.$$

(3) **The Central Limit Theorem.** Say that a measurable function h is cohomologous to a constant if there is a measurable function g and a constant c such that $h = g \circ T - g + c$ almost everywhere. We say that the transformation T satisfies the Central Limit Theorem (CLT) for functions in a class $\mathcal{H}$ if for any $h \in \mathcal{H}$ that is not cohomologous to a constant, there exists $\gamma > 0$ such that

$$\mu\left\{ x : \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left(h(T^i(x)) - \int h\, d\mu \right) < t \right\} \to \frac{1}{\gamma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\tau^2/2\gamma^2}\, d\tau.$$

Before stating the formal result, we point out that uniformly hyperbolic systems (and many non-uniformly hyperbolic ones) satisfy various other statistical properties, which we do not discuss in detail in this survey. These include large deviations principles (Orey and Pelikan 1988; Young 1990; Kifer 1990; Pfister and Sullivan 2005; Melbourne and Nicol 2008; Rey-Bellet and Young 2008; Climenhaga et al. 2013), Borel–Cantelli lemmas (Chernov and Kleinbock 2001; Dolgopyat 2004; Kim 2007;

[1] More generally, one can take (Y, ν) to be a Lebesgue space, so ν is metrically isomorphic to Lebesgue measure on an interval together with at most countably many atoms. For all the cases we discuss, it suffices to take Y finite.

Gouëzel 2007; Gupta et al. 2010; Haydn et al. 2013), the almost sure invariant principle (Denker and Philipp 1984; Melbourne and Nicol 2005, 2009), and many more besides.

Theorem 1.1 *Let Λ be a locally maximal hyperbolic set for f, and assume that $f|\Lambda$ is topologically transitive.*[2] *Then for any Hölder continuous potential φ, the following are true:*

(1) ***Existence:*** *there is an equilibrium measure μ_φ.*
(2) ***Uniqueness:*** *μ_φ is the only equilibrium measure for φ.*
(3) ***Ergodic and statistical properties:***
   (a) the Bernoulli property: there is $A \subset \Lambda$ and $n > 0$ such that the sets $f^k(A)$, $0 \le k < n$ are (essentially) disjoint and cover Λ, $f^n(A) = A$, and $(f^n|A, \mu_\varphi)$ has the Bernoulli property;
   (b) exponential decay of correlations: there are A, n as above such that $(f^n|A, \mu_\varphi)$ has EDC with respect to the class of Hölder continuous functions.
   (c) the Central Limit Theorem: μ_φ satisfies the CLT with respect to the class of Hölder continuous functions.

The proof of Theorem 1.1 uses the fact that $f|\Lambda$ can be represented by a **subshift of finite type** via a **Markov partition**. Recall that a $p \times p$ transition matrix[3] A determines a subshift of finite type (SFT) (Σ_A, σ) as the (left) shift $\sigma(\omega)_i = \omega_{i+1}$ on the space Σ_A of two-sided infinite sequences $\omega = (\omega_i) \in \{1, \dots, p\}^{\mathbb{Z}}$ which are admissible with respect to A; that is, for which $a_{\omega_i \omega_{i+1}} = 1$ for all $i \in \mathbb{Z}$.

Recall also that a finite partition $\mathcal{R} = \{R_1, \dots, R_p\}$ of Λ is a Markov partition if the following are true.

(1) The diameter $\operatorname{diam} \mathcal{R} = \max_{1 \le i \le p} \operatorname{diam} R_i$ is sufficiently small; this guarantees that $\mathcal{R}$ is generating so the coding map $\pi : \Sigma_A \to X$ introduced below is well-defined.
(2) $R_i = \overline{\operatorname{int} R_i}$[4] and for any $1 \le i, j \le p$, $i \ne j$ we have that $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset$; this guarantees that the coding map is injective away from the boundaries.
(3) Each set R_i is a **rectangle**, i.e., for any $x, y \in R_i$ we have that $z = [x, y] \in R_i$; this is the **local product structure** (or **hyperbolic product structure**) of the partition elements.
(4) The **Markov property**: for each $x \in \Lambda$, if $x \in R_i$ and $f(x) \in R_j$ for some $1 \le i, j \le p$, then

$$f(V^s(x) \cap R_i) \subset V^s(f(x)) \cap R_j,$$
$$f^{-1}(V^u(f(x)) \cap R_j) \subset V^u(x) \cap R_i.$$

The first construction of Markov partitions was obtained by Adler and Weiss (1967, 1970), and independently by Berg (1967), in the particular case of hyperbolic automor-

[2] This means that there is a point $x \in \Lambda$ whose trajectory is everywhere dense, i.e., $\Lambda = \overline{\{f^n x : n \in \mathbb{Z}\}}$. An equivalent definition is that for any two non-empty open sets U and V there is $n \in \mathbb{Z}$ such that $f^n(U) \cap V \ne \emptyset$.

[3] That is, a matrix whose entries a_{ij} are each equal to 0 or 1.

[4] Here $\operatorname{int} R_i$ means the interior of the set R_i in the relative topology.

phisms of the 2-torus. They observed that the map allowed a symbolic representation by a subshift of finite type and that this can be used to study its ergodic properties. Sinai realized that existence of Markov partitions is a rather general phenomenon and he constructed Markov partitions for general Anosov diffeomorphisms, see Sinaĭ (1968). Furthermore, in Sinaĭ (1972) he observed the analogy between the symbolic models of Anosov diffeomorphisms and lattice gas models in physics—the starting point in developing the thermodynamic formalism. Finally, in the more general setting of locally maximal hyperbolic sets Markov partitions were constructed by Bowen (1970).

Markov partitions allow one to obtain a symbolic representation of the map $f|\Lambda$ by subshifts of finite type. More precisely, let $\mathcal{R} = \{R_1, \dots, R_p\}$ be a finite Markov partition of Λ. Consider the subshift of finite type (Σ_A, σ) with the transition matrix A whose entries are given by $a_{ij} = 1$ if $f(\operatorname{int} R_i) \cap \operatorname{int} R_j \neq \emptyset$ and $a_{ij} = 0$ otherwise. One can show that for every $\omega = (\omega_i) \in \Sigma_A$ the intersection $\bigcap_{i \in \mathbb{Z}} f^{-i}(R_{\omega_i})$ is not empty and consists of a single point $\pi(\omega)$. This defines the **coding map** $\pi : \Sigma_A \to \Lambda$, which is characterized by the fact that $f^i(\pi(\omega)) \in R_{\omega_i}$ for all $i \in \mathbb{Z}$ (thus ω "codes" the orbit of $\pi(\omega)$).

Proposition 1.2 *The map π has the following properties:*

(1) π is Hölder continuous;
(2) π is a conjugacy between the shift σ and the map $f|\Lambda$, i.e., $(f|\Lambda) \circ \pi = \pi \circ \sigma$;
(3) π is one-to-one on the set $\Sigma' \subset \Sigma$ which consists of points ω for which the trajectory of the point $\pi(\omega)$ never hits the boundary of the Markov partition.

Consider a Hölder continuous potential φ on Λ. By Proposition 1.2, the function $\tilde{\varphi}$ on Σ_A given by $\tilde{\varphi}(\omega) = \varphi(\pi(\omega))$ is Hölder continuous. Thus in order to prove Theorem 1.1 it suffices to study thermodynamic formalism for Hölder continuous potentials for SFTs. The starting point for this theory is the following result of Parry (1964), which uses Perron–Frobenius theory to deal with the case $\varphi = 0$. The corresponding equilibrium measure is the measure of maximal entropy (MME) for which $h_\mu(f) = h_{\text{top}}(f)$.

Theorem 1.3 *Let A be a transition matrix such that $A^n > 0$ for some $n \in \mathbb{N}$, and let Σ_A be the corresponding SFT.*

(1) The topological entropy of Σ_A is $\log \lambda$, where $\lambda > 1$ is the maximal eigenvalue of A guaranteed by the Perron–Frobenius theorem.
*(2) Let v be a positive right eigenvector for (A, λ) (so $Av = \lambda v$); then the matrix P given by $P_{ij} = A_{ij} \frac{v_j}{\lambda v_i}$ is **stochastic** (its rows are probability vectors), so it defines **transition probabilities** for a **Markov chain**.*
*(3) Let h be a positive left eigenvector for (A, λ), normalized so that $\pi_i = h_i v_i$ defines a probability vector π. Then π is the unique probability vector with $\pi P = \pi$, and the unique MME for Σ_A is the **Markov measure** defined by*

$$\mu[\omega_1 \cdots \omega_n] = \pi_{\omega_1} P_{\omega_1 \omega_2} \cdots P_{\omega_{n-1} \omega_n}.$$

Theorem 1.3 was adapted to non-zero potentials by Ruelle (1968, 1976), replacing the transition matrix with a **transfer operator**. Ruelle's version of the Perron–Frobenius

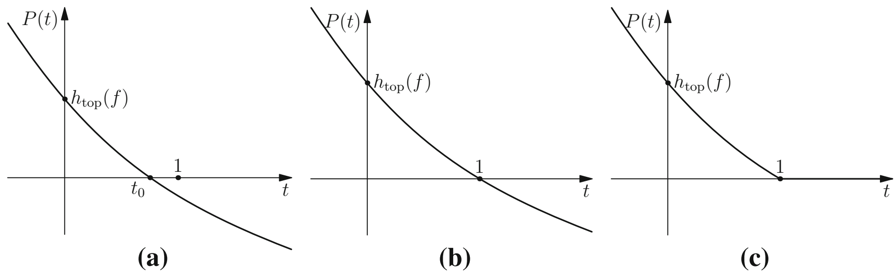

Fig. 1 The pressure function for **a** typical hyperbolic sets; **b** a hyperbolic attractor; **c** a non-uniformly hyperbolic map with a phase transition

theorem for this transfer operator is at the heart of the classical results in thermodynamic formalism for SFTs, and hence, for uniformly hyperbolic systems. Roughly speaking the idea is the following.

(1) Replace the two-sided SFT Σ_A with its one-sided version Σ_A^+, and define the transfer operator associated to φ on $C(\Sigma_A^+)$ by[5]

$$(\mathcal{L}_\varphi f)(x) = \sum_{\sigma y = x} e^{\varphi(y)} f(y).$$

(2) Show that $\mathcal{L}_\varphi$ has a largest eigenvalue λ and that the rest of the spectrum lies inside a disc with radius $<\lambda$ (the **spectral gap** property).
(3) Instead of the left and right eigenvalues h and v, find a positive eigenfunction $h \in C(\Sigma_A^+)$ for $\mathcal{L}_\varphi$, and an eigenmeasure $\nu \in \mathcal{M}(\Sigma_A^+)$ for the dual $\mathcal{L}_\varphi^*$.
(4) Obtain the unique equilibrium state as $d\mu = h\, d\nu$.

We stress that this result (and hence Theorem 1.1) may not hold if the the potential function fails to be Hölder continuous, see Hofbauer (1977), Sarig (2001a), Pesin and Zhang (2006).

1.2.2 Thermodynamic Formalism for the Geometric t-Potential

Returning from SFTs to the setting of uniformly hyperbolic smooth systems, the most significant potential function is the **geometric t-potential**: a family of potential functions $\varphi_t(x) := -t \log |\mathrm{Jac}(df|E^u(x))|$ for $t \in \mathbb{R}$. Since the subspaces $E^u(x)$ depend Hölder continuously on x, the potential φ_t is Hölder continuous for each t whenever f is $C^{1+\alpha}$; in particular, it admits a unique equilibrium measure μ_t. Furthermore, the **pressure function** $P(t) := P(\varphi_t)$ is well defined for all t, is convex, decreasing, and real analytic in t, as in Fig. 1a.

There are certain values of t that are particularly important.

- When $t = 0$, we obtain the topological entropy $h_{top}(f)$ as $P(0)$, and the unique MME as μ_0.

[5] It is instructive to consider the case $\varphi = 0$ and write down the action of $\mathcal{L}_0$ on the (finite-dimensional) space of functions constant on 1-cylinders, where the action is given by the (transpose of the) transition matrix A.

- Since P is strictly decreasing and has $P(0) > 0$ and $P(t) \to -\infty$ as $t \to \infty$, there is a unique number $t_0 > 0$ for which $P(t_0) = 0$. The equation $P(t) = 0$ is called **Bowen's equation**. In the two-dimensional case its root is the Hausdorff dimension of $\Lambda \cap V^u(x)$[6] and the equilibrium measure μ_{t_0} achieves this Hausdorff dimension (i.e., is the measure of maximal dimension) (Bowen 1979; Ruelle 1982; McCluskey and Manning 1983).

To further study the properties of the pressure function (and t_0 in particular) we recall the notion of the Lyapunov exponent. Given $x \in \Lambda$ and $v \in T_x M$, define the **Lyapunov exponent**

$$\chi(x, v) = \limsup_{n\to\infty} \frac{1}{n} \log \|df^n v\|.$$

For every $x \in \Lambda$ the function $\chi(x, \cdot)$ takes on finitely many values $\chi_1(x) \leq \cdots \leq \chi_d(x)$ where $d = \dim M$. The functions $\chi_i(x)$ are Borel and are invariant under f; in particular, if μ is an ergodic measure, then $\chi_i(x) = \chi_i(\mu)$ is constant almost everywhere for each $i = 1, \ldots, d$, and the numbers $\chi_i(\mu)$ are called the **Lyapunov exponent of the measure** μ. If none of these numbers is equal to zero, μ is called a **hyperbolic measure**;[7] note that when Λ is a hyperbolic set for f, every invariant measure supported on Λ is hyperbolic. The Margulis–Ruelle inequality (see Ruelle 1979; Barreira and Pesin 2013) says that

$$h_\mu(f) \leq \sum_{i:\chi_i(\mu)\geq 0} \chi_i(\mu) \tag{1.3}$$

and in particular implies that $t_0 \leq 1$, since the sum in (1.3) is equal to $-\int \varphi_1 \, d\mu$, and hence $h_\mu(f) + \int \varphi_1 \, d\mu \leq 0$ for every ergodic μ.

1.2.3 Hyperbolic Attractors

We consider the particular case when Λ is a **topological attractor** for f. This means that there is a neighborhood $U \supset \Lambda$ such that $\overline{f(U)} \subset U$ and $\Lambda = \bigcap_{n\geq 0} f^n(U)$. It is not difficult to see that for every $x \in \Lambda$, the local unstable manifold $V^u(x)$ is contained in Λ;[8] the same is true for the global unstable manifold through x. Therefore, the attractor contains all the global unstable manifolds of its points. On the other hand the intersection of Λ with stable manifolds of its points is usually a Cantor set.

In the case when Λ is a hyperbolic attractor we have that $t_0 = 1$ (see Bowen 2008), so $P(t)$ is as in Fig. 1b. The equilibrium state μ_1 is a hyperbolic ergodic measure for which the Margulis–Ruelle inequality (1.3) becomes equality. By Ledrappier and Young (1985), this implies that μ_1 has absolutely continuous conditional measures along unstable manifolds; that is, there is a collection $\mathcal{R}$ of local unstable manifolds V^u and a measure η on $\mathcal{R}$ such that μ_1 can be written as

6 The value of the Hausdorff dimension does not depend on x.

7 It is assumed that some of these numbers are positive while others are negative.

8 Indeed, for any $y \in V^u(x)$ the trajectory of y, $\{f^n(y)\}_{n\in\mathbb{Z}}$ lies in U and hence, must belong to Λ since it is locally maximal.

$$\mu_1(E) = \int_{\mathcal{R}} \mu_{V^u}(E)\, d\eta(V^u) \tag{1.4}$$

where the measures μ_{V^u} are absolutely continuous with respect to the leaf volumes m_{V^u}. A hyperbolic measure μ satisfying (1.4) is said to be a **Sinai–Ruelle–Bowen** (SRB) measure, and it can be shown that such measures are **physical**: the set of **generic** points

$$G_\mu := \left\{ x \in M \mid \frac{1}{n}\sum_{k=0}^{n-1} \varphi(f^k(x)) \to \int \varphi\, d\mu \text{ for all continuous } \varphi\colon M \to \mathbb{R} \right\}$$

has positive volume, and so μ is the appropriate invariant measure for studying "physically relevant" trajectories. The discussion above shows that when Λ is a hyperbolic attractor, SRB measures are precisely the equilibrium states for the geometric potential φ_1.

1.3 Non-uniformly Hyperbolic Maps

1.3.1 Definition of Non-uniform Hyperbolicity

A $C^{1+\alpha}$ diffeomorphism f of a compact smooth Riemannian manifold M is **non-uniformly hyperbolic** on an invariant Borel subset $S \subset M$ if there are a measurable df-invariant decomposition of the tangent space $T_xM = E^s(x) \oplus E^u(x)$ for every $x \in S$ and measurable f-invariant functions $\varepsilon(x) > 0$ and $0 < \lambda(x) < 1$ such that for every $0 < \varepsilon \le \varepsilon(x)$ one can find measurable functions $c(x) > 0$ and $k(x) > 0$ satisfying for every $x \in S$:

(1) $\|df^n v\| \le c(x)\lambda(x)^n \|v\|$ for $v \in E^s(x)$, $n \ge 0$;
(2) $\|df^{-n} v\| \le c(x)\lambda(x)^n \|v\|$ for $v \in E^u(x)$, $n \ge 0$;
(3) $\angle(E^s(x), E^u(x)) \ge k(x)$;
(4) $c(f^m(x)) \le e^{\varepsilon|m|}c(x)$, $k(f^m(x)) \ge e^{-\varepsilon|m|}k(x)$, $m \in \mathbb{Z}$.

The last property means that the estimates in (1) and (2) can deteriorate but with sub-exponential rate.

If μ is an invariant measure for f with $\mu(S) = 1$, then by the Multiplicative Ergodic theorem, if for almost every $x \in S$ the Lyapunov exponents at x are all nonzero, i.e., μ is a hyperbolic measure, then f is non-uniformly hyperbolic on S.

1.3.2 Possibility of Phase Transitions and Non-hyperbolic Behavior

A general theory of thermodynamic formalism for non-uniformly hyperbolic maps is far from being complete, although certain examples here are well-understood. They include one-dimensional maps, where the pressure function $P(t) = P(\varphi_t)$ associated with the family of geometric potentials may behave as in the uniformly hyperbolic case, or may exhibit new phenomena such as **phase transitions** (points of non-differentiability where there is more than one equilibrium measure). The latter is illustrated in Fig. 1c and is most thoroughly studied for the Manneville–Pomeau map $x \mapsto x + x^{1+\alpha} \pmod 1$, where $\alpha \in (0, 1)$ controls the degree of **intermittency** at the neutral fixed point. In this example one has the following behavior (Pianigiani 1980; Thaler 1980, 1983; Lopes 1993; Pollicott and Weiss 1999; Liverani et al. 1999; Young 1999; Sarig 2002; Hu 2004).

- **Hyperbolic behavior for** $t < 1$: the pressure function $P(t)$ is real analytic and decreasing on $(-\infty, 1)$, and for every t in this range, the geometric t-potential φ_t has a unique equilibrium measure μ_t, which is Bernoulli, has EDC, and satisfies the CLT with respect to the class of Hölder continuous potentials. In a nutshell, for $t \in (-\infty, 1)$, the thermodynamics of this system is just as in the case of uniform hyperbolicity.
- **Phase transition at** $t = 1$: the pressure function $P(t)$ is non-differentiable at $t = 1$, and φ_1 has two ergodic equilibrium measures. One of these is the **absolutely continuous invariant probability measure** μ_1 (which plays the role of SRB measure), and the other is the point mass δ_0 on the neutral fixed point.[9] The measure μ_1 is Bernoulli and decay of correlations is **polynomial** (in particular, subexponential).
- **Non-hyperbolic behavior for** $t > 1$: for every $t \in (1, \infty)$, the unique equilibrium state for φ_t is the point mass δ_0, which has zero entropy and zero Lyapunov exponent.

Similar results for the geometric t-potential are available for other classes of one-dimensional maps (e.g., unimodal and multimodal maps) and rather specific higher-dimensional examples (e.g., polynomial and rational maps and (piecewise) non-uniformly expanding maps); in some of these examples phase transitions occur while others are without phase transitions. As a small sample of the recent literature on the topic, we mention only (Bruin and Keller 1998; Makarov and Smirnov 2000; Oliveira 2003; Alves et al. 2005; Przytycki and Rivera-Letelier 2007; Pesin and Senti 2008; Bruin and Todd 2008, 2009; Dobbs 2009; Iommi and Todd 2010; Przytycki and Rivera-Letelier 2011; Li and Rivera-Letelier 2014a, b), as well as the comprehensive and far-reaching discussion of thermodynamics for interval maps with critical points in Dobbs and Todd (2015).

Our goal in the rest of this paper is not to discuss these results, which rely on the specific structure of the examples being studied (or on the absence of a contracting direction); rather, we want to discuss the recently developed techniques for studying multi-dimensional non-uniformly hyperbolic systems, with particular emphasis on recent results that have the potential to be applied very generally, although they do not yet give as complete a picture as the one outlined above. These general results have been obtained in the last few years and represent an actively evolving area of research.

[9] For $\alpha \in (0, 1)$ the measure μ_1 is finite but for $\alpha \geq 1$, a new phenomenon occurs: the intermittent behavior becomes strong enough that while there is still an absolutely continuous invariant measure, it is infinite. At the same time, the pressure function for $\alpha \geq 1$ becomes differentiable at $t = 1$, and the measure δ_0 becomes the unique equilibrium measure.

1.3.3 Different Types of Equilibrium Measures

Before describing the general methods, we recall some basic notions from non-uniform hyperbolicity; see Barreira and Pesin (2007) for more complete definitions and properties. Let M be a compact smooth manifold and $f: M \to M$ a $C^{1+\alpha}$ diffeomorphism. Recall that a point $x \in M$ is called **Lyapunov–Perron regular** if for any basis $\{v_1, \ldots, v_p\}$ of $T_x M$,

$$\liminf_{n\to\pm\infty} \frac{1}{n} \log V(n) = \limsup_{n\to\pm\infty} \frac{1}{n} \log V(n) = \sum_{i=1}^{p} \chi_i(x, v_i),$$

where $V(n)$ is the volume of the parallelepiped built on the vectors $\{df^n v_1, \ldots, df^n v_p\}$.

Let $\mathcal{R}$ be the set of all Lyapunov–Perron regular points. The Multiplicative Ergodic theorem claims that this set has full measure with respect to any invariant measure. Consider now the set $\Gamma \subset \mathcal{R}$ of points for which all Lyapunov exponents are nonzero, and let $\mathcal{M}^e(f, \Gamma) \subset \mathcal{M}^e(f)$ be the set of all ergodic measures that give full weight to the set Γ; these are hyperbolic measures and they form the class of measures where it is reasonable to attempt to recover some of the theory of uniformly hyperbolic systems.

Let φ be a measurable potential function; note that we cannot a priori assume more than measurability if we wish to include the family of geometric potentials, since in general the unstable subspace varies discontinuously and so φ_t is not a continuous function.[10] Consider the **hyperbolic pressure** defined by using only hyperbolic measures:

$$P_\Gamma(\varphi) := -\inf_{\mu\in\mathcal{M}^e(f,\Gamma)} E_\mu(\varphi). \tag{1.5}$$

Say that μ_φ is a **hyperbolic equilibrium measure** if $-E_{\mu_\varphi}(\varphi) = P_\Gamma(\varphi)$. For the Manneville–Pomeau example above, we have $P_\Gamma(\varphi_t) = P(\varphi_t)$ for every $t \in \mathbb{R}$, and the equilibrium measure μ_t is the unique hyperbolic equilibrium measure for every $t \leq 1$,[11] while for $t > 1$ there is no hyperbolic equilibrium measure, since δ_0 has zero Lyapunov exponent.

One could also fix a threshold $h > 0$ and consider the set $\mathcal{M}^e(f, \Gamma, h)$ of all measures in $\mathcal{M}^e(f, \Gamma)$ whose entropies are greater than h; restricting our attention to measures from this class gives the restricted pressure[12]

[10] On the other hand, for surface diffeomorphisms Sarig (2013) constructed Markov partitions with countably many partition elements (see Sect. 3 below), and showed Sarig (2011) that the function φ_t can be lifted to a function on the symbolic space that is globally well-defined and is Hölder continuous. This can be used to study equilibrium measures for this function.

[11] Note that for $t = 1$ it is no longer the unique equilibrium measure, but it is the only hyperbolic one.

[12] Because $\mathcal{M}^e(f, \Gamma, h)$ is not compact, the existence of an optimizing measure in (1.6) becomes a more subtle issue. Although it may happen that the value of $P_\Gamma^h(\varphi)$ is achieved by a measure μ whose entropy may not be greater than h, the restriction to measures in the class $\mathcal{M}^e(f, \Gamma, h)$ is often made to ensure a certain "liftability" condition, which may still be satisfied by μ; see Theorem 2.3 and the discussion in that section.

$$P_\Gamma^h(\varphi) := -\inf_{\mu\in\mathcal{M}^e(f,\Gamma,h)} E_\mu(\varphi). \tag{1.6}$$

For the Manneville–Pomeau example, we have for every $t \in \mathbb{R}$,[13]

$$\lim_{h\to 0} P_\Gamma^h(\varphi_t) = P_\Gamma(\varphi_t) = P(\varphi_t).$$

In addition to the use of μ_t^h to approximate non-hyperbolic measures by hyperbolic ones, the above approach is also useful when one can identify a (not necessarily invariant) subset $\mathcal{A} \subset X$ of **"bad"** points away from which the dynamics exhibits good hyperbolic behavior; then putting $h > h_{\text{top}}(f, \mathcal{A})$ guarantees that we consider only measures to which $\mathcal{A}$ is invisible.[14] This concept originated in the work of Buzzi on piecewise invertible continuous maps of compact metric spaces Buzzi (1999),[15] but it is reasonable to consider it in other situations.[16]

One could also impose a threshold in other ways. For example, one could fix a reference potential ψ and a threshold $p < P(\psi)$, then restrict attention to the set $\mathcal{M}^e(f, \Gamma, \psi, p)$ of all measures in $\mathcal{M}^e(f, \Gamma)$ for which $-E_\mu(\psi) > p$. Optimizing $E_\mu(\varphi)$ over this restricted set of measures gives another notion of thresholded equilibrium states that may be useful; again, it is often natural to take $p = P_S(\varphi)$ as the topological pressure of f on a (not necessarily invariant) subset $S \subset M$ of bad points. Another approach would be to consider only measures whose Lyapunov exponents are sufficiently large; it may be that this is a more natural approach in certain settings. We stress that while restricting the class of invariant measures using thresholds for the topological pressure or Lyapunov exponents seem to be natural it is yet to be shown to be a working tool in effecting thermodynamic formalism.

1.3.4 Outline of the Paper

A direct application of the uniformly hyperbolic approach in the non-uniformly hyperbolic setting is hopeless in general; we cannot expect to have finite Markov partitions.[17]

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[13] This is reminiscent of the use of Katok horseshoes to approximate (with respect to entropy) an arbitrary system with a uniformly hyperbolic one Katok (1980), which was recently generalized to pressure by Sánchez-Salas (2015).

[14] Note that since $\mathcal{A}$ is not assumed to be invariant, one should use the definition of the topological entropy based on the Carathéodory construction of dimension-like characteristics for dynamical systems (Bowen 1973; Pesin 1997).

[15] An important goal there was to study the notion of h-**isomorphism**, which asks for two systems to have (measure-theoretically) conjugate subsystems that carry all ergodic measures with large enough entropy, even if the whole systems are not conjugate.

[16] For example, if the set $\mathcal{A}$ is an elliptic island and the potential function is sufficiently large on $\mathcal{A}$, then the equilibrium measure may be a zero entropy measure sitting outside the set with non-zero Lyapunov exponents. Putting any positive threshold removes this measure from consideration.

[17] Indeed, if a map possesses a Markov partition, then its topological entropy is the logarithm of an algebraic number, which should certainly not be expected in general. On the other hand, in the presence of a hyperbolic invariant measure μ of positive entropy, there are *horseshoes* with finite Markov partitions whose entropy approximates the entropy of μ Katok (1980), but these have zero μ-measure.

However, in many cases it is possible to use the symbolic approach by finding a **countable Markov partition**, or the related tools of a **Young tower** or a more general **inducing scheme**; these are discussed in Sects. 2–4. This approach is challenging to apply completely, but can help establish existence and uniqueness of equilibrium measures and study their statistical properties including decay of correlations and the CLT.

A second approach is to avoid the issue of building a Markov partition by adapting Bowen's **specification** property to the non-uniformly hyperbolic setting; this is discussed in Sect. 5. This is similar to the symbolic approach in that one uses a "coarse-graining" of the system to make counting arguments borrowed from statistical physics, but sidesteps the issue of producing a Markov structure. The price paid for this added flexibility is that while existence and uniqueness can be obtained with specification-based techniques, there does not seem to be a direct way to obtain strong statistical properties without first establishing some sort of Markov structure.

A third approach, which we discuss in Sect. 6, is geometric and is based on pushing forward the leaf volume on unstable manifolds by the dynamics. More generally, one can work with approximations to unstable manifolds by **admissible** manifolds and use measures which have positive densities with respect to the leaf volume as reference measures. Such pairs of admissible manifolds and densities are called **standard** and working with them has proven to be quite a useful technique in various problems in dynamics.[18] So far the geometric approach can be used to establish existence of SRB measures for uniformly hyperbolic and some non-uniformly hyperbolic attractors and one can also use a version of this method to construct equilibrium measures for uniformly hyperbolic sets, see Sect. 6; the questions of uniqueness and statistical properties using this approach as well as construction of equilibrium measures for non-uniformly hyperbolic systems are still open.

In the remainder of this paper we describe the three approaches just listed in more detail, and discuss their application to open problems in the thermodynamics of non-uniformly hyperbolic systems.

2 Markov Models for Non-uniformly Hyperbolic Maps I: Young Diffeomorphisms

2.1 Earlier Results: One-dimensional and Rational Maps

In one form or another, the use of Markov models with countably many states to study non-uniformly hyperbolic systems dates back to the late 1970s and early 1980s, when Hofbauer (1979, 1981a, b) used a countable-state Markov model to study equilibrium states for piecewise monotonic interval maps. Indeed, such models for β-transformations were studied already in 1973 by Takahashi (1973).

In Jakobson (1981) Jakobson initiated the study of thermodynamics of unimodal interval maps by constructing absolutely continuous invariant measures (acim) for the family of quadratic maps $f_a(x) = 1 - ax^2$ whenever $a \in \Delta$, where Δ is a set of

[18] This notion was introduced by Chernov and Dolgopyat (2009).

parameters with positive Lebesgue measure. First we discuss in Sect. 2.2 the extensions of Jakobson's result to study SRB measures by what have become known as **Young towers**. Then in Sect. 3 we discuss the study of general equilibrium states in the setting of topological Markov chains with countably many states, which generalizes the SFT theory from Sect. 1.2. Finally, in Sect. 4 we discuss the use of **inducing schemes** to apply this theory to the thermodynamics of smooth examples.

2.2 Young Towers and Gibbs–Markov–Young Structures

2.2.1 Tower Constructions in Dynamical Systems

Roughly speaking, a tower construction begins with a **base** set Λ, a map $G\colon \Lambda \to \Lambda$, and a **height** function $R\colon \Lambda \to \mathbb{N}$. Then the tower is constructed as $\tilde{\Lambda} := \{(z, n) \in \Lambda \times \{0, 1, 2, \dots\} : n < R(z)\}$, and a map $g\colon \tilde{\Lambda} \to \tilde{\Lambda}$ is defined by $g(z, n) = (z, n+1)$ whenever $n + 1 < R(z)$, and $g(z, R(z) - 1) = (F(z), 0)$. Typically one requires that the dynamics of the return map G can be coded by a full shift, or a Markov shift on a countable set of states. To study a dynamical system $f\colon X \to X$ using a tower, one defines a coding map $\pi : \tilde{\Lambda} \to X$ such that $f \circ \pi = \pi \circ g$; this coding map is usually not surjective (the tower does not cover the entire space), and so we will ultimately need to give some "largeness" condition on the tower. It is important to distinguish between the case when $\pi(\Lambda)$ is disjoint from $\pi(\tilde{\Lambda} \setminus \Lambda)$, so that the height R is the first return time to the base $\pi(\Lambda)$, and the case when R is not the first return time.

Tower constructions for which the height of the tower is the first return time to the base of the tower are classical objects in ergodic theory and were considered in works of Kakutani, Rokhlin, and others. Towers for which the height of the tower is not the first return time appeared in the paper by Neveu (1969) under the name of **temps d'arret** and in the context of dynamical systems in the paper by Schweiger (1975, 1979) under the name **jump transformation** (which are associated with some **fibered systems**; see also the paper by Aaronson et al. 1993 for some general results on ergodic properties of Markov fibered systems and jump transformations).

A tower construction is implicitly present in Jakobson's proof of existence of physical measures for quadratic maps. The first significant use of the tower approach beyond the one-dimensional setting came in the study of the Hénon map

$$f_{a,b}(x, y) = (1 - ax^2 + y, bx), \tag{2.1}$$

which for $b \approx 0$ can be viewed as a two-dimensional extension of a unimodal map with parameter a. Building on their alternate proof of Jakobson's theorem in Benedicks and Carleson (1985), Benedicks and Carleson showed in (1991) that when b is sufficiently close to 0, there is a set $\Delta_b \subset \mathbb{R}$ of positive Lebesgue measure such that $f_{a,b}$ has a topologically transitive attractor for every $a \in \Delta_b$. Soon afterwards, Benedicks and Young established existence of an SRB measure for these examples (Benedicks and Young 1993); their approach also gives exponential decay of correlations and the CLT (Benedicks and Young 1995).

The general structure behind these results was developed in Young (1998) and has come to be known as a **Young tower**,[19] or a **Gibbs–Markov–Young structure**. The principal feature of a Young tower is that the induced map on the base of the tower is conjugate to the full shift on the space of two-sided sequences over countable alphabet. This allows one to use some recent results on thermodynamics of this symbolic map to establish existence and uniqueness of equilibrium measures for the original map and study their ergodic properties.

2.2.2 Young Diffeomorphisms

A $C^{1+\alpha}$ diffeomorphism f of a compact smooth manifold M is called **Young diffeomorphism** if it admits a **Young tower**. This tower has a particular structure which is characterized as follows:

- The base Λ of the tower has **hyperbolic product structure** which is generated by continuous families $\mathbf{V}^u = \{V^u\}$ and $\mathbf{V}^s = \{V^s\}$ of local unstable and stable manifolds.
- The induced map has the Markov property, is uniformly hyperbolic and has uniform bounded distortion.
- The intersection of at least one unstable manifold with the base of the tower has positive leaf volume[20] and the integral of the height of the tower against leaf volume is finite.

In particular, the tower codes a positive volume part of the system (but not necessarily all trajectories) by a countable state Markov shift.

A formal description of the Young tower is as follows. There are two continuous families $\mathbf{V}^u = \{V^u\}$ and $\mathbf{V}^s = \{V^s\}$ of local unstable and stable manifolds, respectively, with the property that each V^s meets each V^u transversely in a single point and $\Lambda = (\bigcup V^u) \cap (\bigcup V^s)$; a union of some of the manifolds V^u is called a **u-set**, a union of some of the manifolds V^s is called an **s-set**. One asks for Λ to have the following properties; here $C, \eta > 0$ and $\beta \in (0, 1)$ are constants.

(P1) Positive measure: each $V^u \cap \Lambda$ has positive leaf volume m_{V^u}.

(P2) Markov structure: there are (countably many) pairwise disjoint s-sets $\Lambda_i^s \subset \Lambda$ and numbers $R_i \in \mathbb{N}$ such that

- $\Lambda \setminus \bigcup_i \Lambda_i^s$ is m_{V^u}-null for all V^u;
- $\Lambda_i^u = f^{R_i}(\Lambda_i^s)$ is a u-set in Λ;
- for every $x \in \Lambda_i^s$,

$$\begin{aligned} f^{R_i}(V^s(x)) &\subset V^s(f^{R_i}(x)),\\ f^{R_i}(V^u(x)) &\supset V^u(f^{R_i}(x)),\\ f^{-R_i}(V^s(f^{R_i}(x)) \cap \Lambda_i^u) &= V^s(x) \cap \Lambda, \end{aligned}$$

[19] It is worth mentioning that a major achievement of Young (1998) was to establish exponential decay of correlations for billiards with convex scatterers, which is an example of a uniformly hyperbolic system with discontinuities; we will not discuss such examples further in this paper.

[20] It follows that every local unstable manifold intersects the base in a set of positive leaf volume.

$$f^{R_i}(V^u(x) \cap \Lambda_i^s) = V^u(f^{R_i}(x)) \cap \Lambda;$$

(**P3**) Defining the **recurrence (induced) time** $R\colon \bigcup_i \Lambda_i^s \to \Lambda$ by $R|\Lambda_i^s = R_i$ and the **induced map** $F(x) = f^{R(x)}(x)$, we have that for all $n \geq 1$

- **Forward contraction on** V^s: if x, y are in the same leaf V^s, then $d(F^n x, F^n y) \leq C\beta^n d(x, y)$.
- **Backward contraction on** V^u: if x, y are in the same leaf V^u and the same s-set Λ_i^s, then $d(F^{-n}x, F^{-n}y) \leq C\beta^n d(Fx, Fy)$.
- **Bounded distortion:** if x, y are in the same leaf V^u and the same s-set Λ_i^s then

$$\log \frac{|\det dF^u(x)|}{|\det dF^u(y)|} \leq Cd(Fx, Fy)^\eta.$$

Our description of Young tower follows Pesin et al. (2016b) and differs from the original description in Young (1998). Most importantly, we do not require that the map f contracts distances along local stable manifolds uniformly with an exponential rate and neither does the inverse map f^{-1} along local unstable manifolds but that this requirement holds with respect to the induced map F (see (**P3**)). We stress that in constructing SRB and equilibrium measures on Young towers and studying their ergodic properties these extra requirements on the maps f and f^{-1} are not needed and that there are examples in which the map f contracts distances along local stable manifolds uniformly with a polynomial rate, see Sect. 2.3.2.

2.2.3 SRB Measures for Young Diffeomorphisms

Once a tower structure has been found, the strength of the conclusions one can draw depends on the rate of decay of the **tail of the tower**; that is, the speed with which $m_{V^u}\{x \in V^u \mid R(x) > T\} \to 0$ as $T \to \infty$ for $V^u \in \mathbf{V}^u$. We say that with respect to the measure m_{V^u} the tower has

- **integrable tails** if

$$\int R\, dm_{V^u} < \infty;$$

- **exponential tails** if for some $C, a > 0$ and $T \geq 1$,

$$m_{V^u}\{x \mid R(x) > T\} < Ce^{-aT}; \tag{2.2}$$

- **polynomial tails** if for some $C, a > 0$ and $T \geq 1$,

$$m_{V^u}\{x \mid R(x) > T\} < CT^{-an}.$$

Theorem 2.1 (Young 1998) *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact manifold M admitting a Young tower. Assume that*

(1) there is local unstable manifold V^u such that

$$m_{V^u}\left(\bigcup_{i\geq 1}\bar{\Lambda}_i\backslash\Lambda_i\right)=0; \tag{2.3}$$

(2) the tower has integrable tails.

Then f has an SRB measure μ.

To describe ergodic properties of the SRB measure one needs an extra condition. We say that the tower satisfies the **arithmetic condition** if the greatest common denominator of of the set of integers $\{R_i\}$ is one.[21]

Theorem 2.2 (Young 1998) *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact manifold M admitting a Young tower. Assume that the tower satisfies* (2.3)*, the arithmetic condition and has exponential (respectively, polynomial) tails. Then (f, μ) is Bernoulli, has exponential (respectively, polynomial) decay of correlations and satisfies the CLT with respect to the class of functions which are Hölder continuous on Λ.*

Note that even without the arithmetic condition one still obtains the "exponential decay up to a period" result stated earlier in Theorem 1.1 (1.1).

In Young (1999), Young gave an extension of the results from Young (1998) that applies in a more abstract setting, giving existence of an invariant measure that is absolutely continuous with respect to some reference measure (not necessarily Lebesgue). She also provided a condition on the height of the tower that guarantees a polynomial upper bound for the decay of correlations. The corresponding polynomial lower bound (showing that Young's bound is optimal) was obtained by Sarig (2002) and Gouëzel (2004).

The flexibility in the reference measure makes Young's result suitable for studying existence, uniqueness and ergodic properties of equilibrium measures other than SRB measures (although this was not done in Young 1999). In particular, this is used in the proof of Statement 2 of Theorem 2.3 below; we discuss such questions more in Sects. 3, 4.

Just as the Hénon maps can be studied as a "small" two-dimensional extension of the unimodal maps, Theorems 2.1 and 2.2 can be applied to more general 'strongly dissipative' maps that are obtained as 'small' two-dimensional extensions of one-dimensional maps; this is carried out in Wang and Young (2001, 2008).

Aside from such strongly dissipative maps, Young towers have been constructed for some partially hyperbolic maps where the center direction is non-uniformly contracting (Castro 2004) or expanding (Alves and Pinheiro 2010; Alves and Li 2015); the latter papers are built on earlier results for non-uniformly expanding maps where one does not need to worry about the stable direction (Alves et al. 2005; Gouëzel 2006). In both cases existence (and uniqueness) of an SRB measure was proved first (Bonatti and Viana 2000; Alves et al. 2000) via other methods closer to the push-forward geometric

[21] The tower $\tilde{\Lambda}$ admits a natural countable Markov partition (see Young 1998) and the arithmetic condition is equivalent to the requirement that the corresponding Markov shift is topologically mixing.

approach that we discuss in Sect. 6, so the achievement of the tower construction was to establish exponential decay of correlations and the CLT. These results only cover the SRB measure and do not consider more general equilibrium states.

2.2.4 Thermodynamics of Young Diffeomorphisms for the Geometric t-Potential

Let f be a $C^{1+\alpha}$ Young diffeomorphism of a compact smooth manifold M. Consider the set Λ with hyperbolic product structure. Let Λ_i^s be the collections of s-sets and R_i the corresponding inducing times. Set

$$Y = \bigcup_{k\geq 0} f^k(\Lambda).$$

This is a forward invariant set for f. For every $y \in Y$ the tangent space at y admits an invariant splitting $T_yM = E^s(y) \oplus E^u(y)$ into stable and unstable subspaces. Thus we can consider the geometric t-potential $\varphi_t(y)$ which is well defined for $y \in Y$ and is a Borel (but not necessarily continuous) function for every $t \in \mathbb{R}$. We consider the class $\mathcal{M}(f, Y)$ of all invariant measures μ supported on Y, i.e., for which $\mu(Y) = 1$. It follows that $\mu(\Lambda) > 0$, so that μ 'charges' the base of the Young tower. Further, given a number $h > 0$, we denote by $\mathcal{M}(f, Y, h)$ the class of invariant measures $\mu \in \mathcal{M}(f, Y)$ for which $h_\mu(f) > h$.

The following result describes existence, uniqueness, and ergodic properties of equilibrium measures. Given $n > 0$, denote by

$$S_n := \text{Card}\{\Lambda_i^s : R_i = n\}.$$

Theorem 2.3 (see Pesin et al. 2016b; Melbourne and Terhesiu 2014) *Assume that the Young tower satisfies:*

(1) for all large n

$$S_n \leq e^{hn}, \tag{2.4}$$

where $0 < h < h_{\mu_1}(f)$ is a constant and μ_1 is the SRB measure for f;

(2) the set $\bigcup_{i\geq 1}(\bar{\Lambda}_i \setminus \Lambda_i)$ supports no invariant measure that gives positive weight to any open set.[22]

*Then there is $t_0 < 0$ such that for $t_0 \leq t < 1$ there exists a measure μ_t which is a unique equilibrium measure for φ_t among all **liftable** measures (see the remark below). If in addition, the tower satisfies the arithmetic condition,*[23] *then (f, μ_t) is Bernoulli, has exponential decay of correlations and satisfies the CLT with respect to a class of potential functions which contains all Hölder continuous functions on Y.*

[22] This condition is stronger than the corresponding condition (2.3).

[23] This requirement should be added to Theorem 4.5, Statement 2 of Theorem 4.7 and Statement 3 of Theorem 7.1 in Pesin et al. (2016b).

Remark 1. The requirement (2.4) means that the number of s-sets in the base of the tower can grow exponentially but with rate slower than the metric entropy of the SRB measure. This is a strong requirement on the Young tower, but it is known to hold in some examples, see Sect. 2.3 below.

2. For $t = 1$, the SRB measure μ_1 may not have exponential decay of correlations; this is the case for the Manneville–Pomeau map where the decay is polynomial. See Sect. 1.3.2 and also Sect. 2.3 for more details.
3. We stress that the measures μ_t are equilibrium measures within the class of measures that can be **lifted** to the tower: recall that an invariant measure μ supported on Y is called **liftable** if there is a measure ν supported on Λ and invariant under the induced map F such that the number

$$Q_\nu = \int_\Lambda R \, d\nu \tag{2.5}$$

is finite, and for any measurable set $E \subset Y$,

$$\mu(E) = \mathcal{L}(\nu)(E) := \frac{1}{Q_\nu} \sum_{i \geq 0} \sum_{k=0}^{R_i - 1} \nu(f^{-k}(E) \cap \Lambda_i^s). \tag{2.6}$$

In particular, $\mu_t = \mathcal{L}(\nu_t)$ for some measure ν_t which is an equilibrium (and indeed, Gibbs) measure for the induced map F.
Under the condition 2.4 every measure with entropy $>h$ is liftable. In general, it is shown in Zweimüller (2005) that if $R \in L^1(Y, \mu)$ then μ is liftable. In particular, if the return time R is the first return time to the base of the tower, then every measure that charges the base of the tower is liftable.
4. The proof of exponential decay of correlations and the CLT is based on showing the exponential tails property of the measure ν_t[24] (see Pesin et al. 2016b, Theorem 4.5) and then applying results from Melbourne and Terhesiu (2014).[25]
5. For a $C^{1+\alpha}$ diffeomorphism f there may exist several Young towers with bases Λ_k, $k = 1, \ldots, m$, such that the corresponding sets Y_k are disjoint. For each k, Theorem 2.3 gives a number $t_{0k} < 0$ and for every $t_{0k} < t < 1$ the equilibrium measure μ_{tk} for the geometric potential φ_t. This measure is unique within the class of measures μ for which $\mu(Y_k) = 1$ and $h_\mu(f) > h$ where $0 < h < h_{\mu_1}(f)$.[26] Setting $t_0 = \max_{1 \leq k \leq m} t_{0k}$, for every $t_0 < t < 1$ we obtain the measure μ_t such that $\mu_t | Y_k = \mu_{tk}$. If for every measure μ with $h_\mu(f) > h$, we have that $\mu(Y_k) > 0$ for some $1 \leq k \leq m$, then the measure μ_t is the unique equilibrium measure for φ_t within the class of invariant measures with large entropy. This is the case in the two examples described in Sect. 2.3.

[24] See (2.2) where one should replace the leaf volume with the measure ν_t.

[25] In Melbourne and Terhesiu (2014) the authors considered only expanding maps and Young towers with polynomial tails, however, their results can easily be extended to invertible maps and Young towers with exponential tails.

[26] Note that both h and $h_{\mu_1}(f)$ do not depend on k.

6. It is known that $t = 1$ can be a phase transition, that is the pressure function $P(t)$ is not differentiable and there are more than one equilibrium measures for φ_1. However, it is not known whether phase transitions can occur for $t < t_0$.
7. Theorem 2.3 is a corollary of a more general result establishing thermodynamics for maps admitting inducing schemes of hyperbolic type, see Theorem 4.1.

2.3 Examples of Young Diffeomorphisms

We describe two examples of Young diffeomorphisms for which Theorem 2.3 applies.

2.3.1 A Hénon-like Diffeomorphism at the First Bifurcation

The first example is Hénon-like diffeomorphisms of the plane at the first bifurcation parameter. For parameters a, b consider the Hénon map $f_{a,b}$ given by (2.1). It is shown in Bedford and Smillie (2004), Bedford et al. (2006), Cao et al. (2008) that for each $0 < b \ll 1$ there exists a uniquely defined parameter $a^* = a^*(b)$ such that the non-wandering set for $f_{a,b}$ is a uniformly hyperbolic horseshoe for $a > a^*$ and the parameter a^* is the first parameter value for which a homoclinic tangency between certain stable and unstable manifolds appears.

Theorem 2.4 (Senti and Takahasi 2013, 2016, Theorem A) *For any bounded open interval $I \subset (-1, +\infty)$ there exists $0 < b_0 \ll 1$ such that if $0 \le b < b_0$ then*

(1) the map $f_{a^*(b),b}$ is a Young diffeomorphism;
(2) there exists a unique equilibrium measure for the geometric t-potential and for all $t \in I$.

2.3.2 The Katok Map

We describe the Katok map (1979) (see also Barreira and Pesin 2013), which can be thought of as an invertible and two-dimensional analogue of the Manneville–Pomeau map. Consider the automorphism of the 2-torus given by the matrix $T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and then choose $0 < \alpha < 1$ and a function $\psi : [0, 1] \mapsto [0, 1]$ satisfying:

- ψ is of class C^∞ except at zero;
- $\psi(u) = 1$ for $u \ge r_0$ and some $0 < r_0 < 1$;
- $\psi'(u) > 0$ for every $0 < u < r_0$;
- $\psi(u) = (ur_0)^\alpha$ for $0 \le u \le \frac{r_0}{2}$.

Let $D_r = \{(s_1, s_2) : {s_1}^2 + {s_2}^2 \le r^2\}$ where (s_1, s_2) is the coordinate system obtained from the eigendirections of T. Consider the system of differential equations in D_{r_0}

$$\dot{s}_1 = s_1 \log \lambda, \quad \dot{s}_2 = -s_2 \log \lambda, \tag{2.7}$$

where $\lambda > 1$ is the eigenvalue of T. Observe that T is the time-1 map of the flow generated by the system of equations (2.7).

We slow down trajectories of (2.7) by perturbing it in D_{r_0} as follows:

$$\dot{s}_1 = s_1\psi({s_1}^2 + {s_2}^2)\log\lambda, \quad \dot{s}_2 = -s_2\psi({s_1}^2 + {s_2}^2)\log\lambda.$$

This generates a local flow, whose time-1 map we denote by g. The choices of ψ and r_0 guarantee that the domain of g contains D_{r_0}. Furthermore, g is of class C^∞ in D_{r_0} except at the origin and it coincides with T in some neighborhood of the boundary ∂D_{r_0}. Therefore, the map

$$G(x) = \begin{cases} T(x) & \text{if} \quad x \in \mathbb{T}^2 \setminus D_{r_0}, \\ g(x) & \text{if} \quad x \in D_{r_0} \end{cases}$$

defines a homeomorphism of the torus, which is a C^∞ diffeomorphism everywhere except at the origin.

The map G preserves the probability measure $d\nu = \kappa_0^{-1}\kappa\, dm$ where m is the area and the density κ is defined by

$$\kappa(s_1, s_2) := \begin{cases} (\psi({s_1}^2 + {s_2}^2))^{-1} & \text{if } (s_1, s_2) \in D_{r_0}, \\ 1 & \text{otherwise} \end{cases}$$

and

$$\kappa_0 := \int_{\mathbb{T}^2} \kappa\, dm.$$

We further perturb the map G by a coordinate change ϕ in $\mathbb{T}^2$ to obtain an area-preserving C^∞ diffeomorphism. To achieve this, define a map ϕ in D_{r_0} by the formula

$$\phi(s_1, s_2) := \frac{1}{\sqrt{\kappa_0({s_1}^2 + {s_2}^2)}} \left(\int_0^{{s_1}^2 + {s_2}^2} \frac{du}{\psi(u)} \right)^{1/2} (s_1, s_2) \tag{2.8}$$

and set $\phi = \mathrm{Id}$ in $\mathbb{T}^2 \setminus D_{r_0}$. Clearly, ϕ is a homeomorphism and is a C^∞ diffeomorphism outside the origin. One can show that ϕ transfers the measure ν into the area and that the map $f = \phi \circ G \circ \phi^{-1}$ is a C^∞ diffeomorphism. This is the Katok map (Katok 1979; Barreira and Pesin 2013). One can show that the map f has nonzero Lyapunov exponents almost everywhere.[27]

Theorem 2.5 (see Pesin et al. 2016a) *The following statements hold:*

(1) the Katok map f is a Young diffeomorphism; moreover,
- *there are finitely many disjoint sets Λ_k that are bases of Young towers for which the corresponding sets Y_k cover the whole torus except for the origin;*

[27] However, there are trajectories with zero Lyapunov exponents, for example the origin is a neutral fixed point.

- *every invariant measure μ except for the Dirac measure at the origin δ_0 can be lifted to one of the towers.*

(2) For any $t_0 < 0$ one can find a small $r_0 = r_0(t_0)$ such that if the construction is carried out with this value of r_0, then for every $t_0 < t < 1$
- *there exists a unique equilibrium ergodic measure μ_t associated to the geometric potential φ_t;*
- *(f, μ_t) has exponential decay of correlations and satisfies the CLT with respect to a class of functions which includes all Hölder continuous functions on the torus;*
- *the pressure function P_t is real analytic on $(t_0, 1)$.*

(3) For $t = 1$ there exist two equilibrium measures associated to φ_1, namely the Dirac measure at the origin δ_0 and the Lebesgue measure.
(4) For $t > 1$, δ_0 is the unique equilibrium measure associated to φ_t.

3 Markov Models for Non-uniformly Hyperbolic Maps II: Countable State Markov Shifts

The thermodynamic formalism for SFTs rested on the Ruelle's version of the Perron–Frobenius theorem for finite-state topological Markov chains. For the class of two-step potential functions $\varphi(x) = \varphi(x_0, x_1)$, which includes the zero potential $\varphi = 0$, the extension of this theory to countable-state Markov shifts dates back to work of Vere-Jones (1962, 1967), Gurevič (1969, 1970, 1984), and Gurevich and Savchenko (1998); we discuss this in Sect. 3.1. For more general potential functions a sufficiently complete picture is primarily due to Sarig, and we discuss these in Sect. 3.2.

3.1 Recurrence Properties for Random Walks

Recall the form of Theorem 1.3 on existence of a unique MME for SFTs:

(1) the largest eigenvalue λ of the transition matrix A determines the topological entropy;
(2) the right eigenvector $v = (v_i)$ for λ determines a Markov chain whose transition probabilities are given by a stochastic matrix $P_{ij} = A_{ij} \frac{v_j}{\lambda v_i}$;
(3) P has a unique **stationary vector** π (which can be written in terms of left and right eigenvectors for (A, λ)), which determines a Markov measure that is the unique MME.

In the countable-state setting, existence of eigenvectors and stationary vectors is a more subtle question (although once these are found, the proof of uniqueness goes through just as in the finite-state case). The general story is well-illustrated by just considering the last step above: suppose we are given a stochastic matrix P_{ij} with countably many entries. This corresponds to a directed graph G with countably many vertices, whose edges are given weights as follows: the weight of the edge from i to j is P_{ij}. Then one can consider the Markov chain described by P as a **random walk** on G.

Existence of a stationary vector $\pi = (\pi_i)$ with $\pi P = \pi$ is determined by the **recurrence** properties of the shift Vere-Jones (1962, 1967). Suppose we start our random walk at a vertex a; one can show that the probability that we return to a infinitely many times is either 0 or 1. If the probability of returning infinitely many times is 1, then the walk is **recurrent**. Recurrence is necessary in order to have a stationary probability vector π, but it is not sufficient; one must distinguish between the case when our expected return time is finite (**positive recurrence**) and when it is infinite (**null recurrence**). If the walk is positive recurrent then there is a stationary probability vector π; if it is null recurrent then one can still find a vector π such that $\pi P = \pi$, but one has $\sum_i \pi_i = \infty$, so π cannot be normalized to a probability vector.

In fact, the trichotomy between transience, null recurrence, and positive recurrence is the key to generalizing all of Theorem 1.3 to the countable-state case Pesin (2014). The recurrence conditions can be formulated in terms of the number of loops in the graph G. Fixing a vertex a, let Z_n^* be the number of *simple* loops of length n based at a (first returns to a) and Z_n be the number of *all* loops of length n based at a (including loops which return more than once).[28]

(1) The supremum of the metric entropies is equal to the **Gurevich entropy** $h_G := \lim \frac{1}{n} \log Z_n$ (the limit exists if the graph is aperiodic; otherwise one should take the upper limit).
(2) The shift Σ_A is **recurrent** if $\sum_n e^{-nh_G} Z_n = \infty$, and **transient** if the sum is finite.[29] The eigenvectors h and v for (A, λ) exist if and only if Σ_A is recurrent.
(3) Among recurrent shifts, one must distinguish between **positive recurrence**, when $\sum_n ne^{-nh_G} Z_n^* < \infty$, and **null recurrence**, when the sum diverges. Writing $\pi_i = h_i v_i$, one has $\sum \pi_i < \infty$ if Σ_A is positive recurrent (hence, π can be normalized), and $\sum \pi_i = \infty$ if it is null recurrent. One can also characterize positive recurrent shifts as those for which $e^{nh_G} Z_n$ is bounded away from 0 and ∞, which immediately implies divergence of the sum $\sum_n e^{-nh_G} Z_n$, while null recurrent shifts are those for which $\underline{\lim}_n e^{-nh_G} Z_n = 0$ but the sum still diverges.

It is instructive to note that once a distinguished vertex a is fixed as the starting point of the loops, one can view the first return map to $[a]$ as a Young tower, and then the summability condition in positive recurrence is equivalent to the condition that the tails of the tower are integrable, which was the existence criterion in Theorem 2.1.

3.2 Non-zero Potentials

In discussing the extension to non-zero potentials on countable-state topological Markov chains, we will follow the notation, terminology, and results of Sarig (1999, 2001b, a), although the contributions of Gurevič (1984), Gurevich and Savchenko

[28] In the next section when we consider non-zero potentials, we will have to count the loops with weights coming from the potential.

[29] For some intuition behind this definition, it may be helpful to consider again a countable-state random walk: writing $\mathbb{P}(n)$ for the probability of returning to the original vertex at time n, we recall that by the Borel–Cantelli lemma, the walk is recurrent (infinitely many returns a.s.) if $\sum_n \mathbb{P}(n) = \infty$, and transient (finitely many returns a.s.) if the sum is finite.

(1998), Mauldin and Urbański (1996, 2001), Aaronson and Denker (2001), and of Fiebig et al. (2002) should also be mentioned. Sarig adapted transience, null recurrent, and positive recurrence for non-zero potential functions. The summability criterion for positive recurrence is exactly as above, except that now Z_n represents the total weight of all loops of length n and Z_n^* represents the total weight of simple loops of length n where weight is computed with respect to the potential function; more precisely

$$Z_n = Z_n(\varphi, a) = \sum_{\sigma^n(x)=x} \exp(\Phi_n(x))1_{[a]}(x)$$

and

$$Z_n^* = Z_n^*(\varphi, a) = \sum_{\sigma^n(x)=x} \exp(\Phi_n(x))1_{[\varphi_a=n]}(x),$$

where $\Phi_n(x) = \sum_{k=0}^{n-1} \varphi(f^k x)$. Furthermore, the Gurevich entropy $h_G(\sigma)$ is replaced with the **Gurevich-Sarig pressure** $P_{GS}(\sigma, \varphi)$, which is the exponential growth rate of Z_n, i.e.,

$$P_{GS}(\sigma, \varphi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n.$$

For Markov shifts with finite topological entropy, Buzzi and Sarig (2003) proved that an equilibrium measure exists if and only if the shift is positive recurrent. A good summary of the theory can be found in Sarig (2015). For our purposes the main result is the following.

Theorem 3.1 *Let Σ be a topologically mixing countable-state Markov shift with finite topological entropy, and let $\varphi\colon X \to \mathbb{R}$ be a Hölder continuous*[30] *function such that $P_{GS}(\varphi) < \infty$. Then φ is positive recurrent if and only if there are $\lambda > 0$, a positive continuous function h, and a* ***conservative measure*** *ν (i.e., a measure that allows no nontrivial wandering sets) which is finite and positive on cylinders, such that $\mathcal{L}_\varphi h = \lambda h$, $\mathcal{L}_\varphi^* \nu = \lambda \nu$, and $\int h\, d\nu = 1$. In this case the following are true.*

(1) $P_{GS}(\varphi) = \log \lambda$, and $d\mu = h\, d\nu$ defines a σ-invariant measure.
(2) If $h(\mu) < \infty$, then μ is the unique equilibrium state for φ.
(3) For every cylinder $[w] \subset \Sigma$, we have $\lambda^{-n}\nu[w]^{-1}\mathcal{L}_\varphi^n \mathbf{1}_{[w]} \to h$ uniformly on compact subsets.

The statistical properties of μ depend on the rate of convergence in the last item of Theorem 3.1, which in turn depends on how quickly $Z_n^* e^{-nP_{GS}}$ goes to 0. If it goes to zero with polynomial rate then the corresponding tower (obtained by inducing on a single state) has polynomial tails, and the equilibrium state has polynomial decay of correlations. If it goes to zero with exponential speed—that is, if Z_n^* has smaller exponential growth rate than Z_n—then the tower has exponential tails and correlations

[30] In fact Theorem 3.1 holds for the more general class of potentials with **summable variations**, but Hölder continuity is needed for the statistical properties mentioned below.

decay exponentially. In this case the shift is called **strong positive recurrent**; see Cyr and Sarig (2009) for a summary of the results in this case.

3.3 Countable-State Markov Partitions for Smooth Systems

Using Pesin theory, Sarig recently carried out a version of the construction of Markov partitions for non-uniformly hyperbolic diffeomorphisms in two dimensions. Recall that for a uniformly hyperbolic diffeomorphism $f : M \to M$, one obtains an SFT Σ and a coding map $\pi : \Sigma \to M$ such that

- π is Hölder continuous and has $f \circ \pi = \pi \circ \sigma$;
- π is onto and is 1–1 on a residual set $\Sigma' \subset \Sigma$ that has full measure for every equilibrium state of a Hölder potential on Σ.

In non-uniform hyperbolicity one must replace the SFT with a countable-state Markov shift, and also weaken some of the conclusions.

Theorem 3.2 Sarig (2013) *Let M be a compact smooth surface and $f : M \to M$ a $C^{1+\alpha}$ diffeomorphism of positive topological entropy. Fix a threshold $0 < \chi < h_{\text{top}}(f)$. Then there is a countable-state topological Markov shift Σ_χ and a coding map $\pi_\chi : \Sigma_\chi \to M$ such that*

- *π_χ is Hölder continuous and has $f \circ \pi_\chi = \pi_\chi \circ \sigma$;*
- *if μ is an ergodic f-invariant measure on M with $h_\mu(f) > \chi$, then $\mu(\pi(\Sigma_\chi)) = 1$, and moreover there is an ergodic σ-invariant measure $\hat{\mu}$ on Σ_χ such that $(\pi_\chi)_*\hat{\mu} = \mu$ and $h_{\hat{\mu}}(\sigma) = h_\mu(f)$.*

Observe that Theorem 3.2 echoes our recurring theme that in non-uniform hyperbolicity, to obtain 'good' hyperbolic-type results one often needs to ignore a 'small-entropy' part of the system. In fact the key property of the threshold χ is that by the Margulis–Ruelle inequality, any ergodic measure with $h_\mu(f) > \chi$ must have positive Lyapunov exponent at least χ. Thus for a higher-dimensional generalization of Theorem 3.2, one should expect that the natural condition would be on the Lyapunov exponents, rather than the entropy.

The analogous result to Theorem 3.2 for three-dimensional flows was proved by Lima and Sarig (2014). In both cases this can be used to deduce Bernoullicity up to finite rotations of ergodic positive entropy equilibrium states (Sarig 2011; Ledrappier et al. 2016). However, these general results do not give any information on the recurrence properties of the countable state shift, or the tail of the resulting tower, and in particular they do not provide a mechanism for verifying decay of correlations and the CLT. This is of no surprise, since at this level of generality, one should not expect to get exponential decay (or any other particular rate).

4 Markov Models for Non-uniformly Hyperbolic Maps III: Inducing Schemes of Hyperbolic Type

The study of SRB measures via Young towers generalizes to the study of equilibrium states via **inducing schemes**, which use the tower approach to model (a large part of) the system by a countable-state Markov shift, and then apply the thermodynamic results from Sect. 3. The concept of an inducing scheme in dynamics is quite broad and applies to systems which may be invertible or not, smooth or not differentiable. Every inducing scheme generates a symbolic representation by a tower which is well adapted to constructing equilibrium measures for an appropriate class of potential functions using the formalism of countable state Markov shifts. The projection of these measures from the tower are natural candidates for the equilibrium measures for the original system.

In order to use this symbolic approach to establish existence and to study equilibrium states, some care must be taken to deal with the **liftability problem** as only measures that can be lifted to the tower can be 'seen' by the tower.

One may consider inducing schemes of expanding type, or of hyperbolic type. The former were introduced in Pesin and Senti (2008) and apply to study thermodynamics of non-invertible maps (e.g., non-uniformly expanding maps) while the latter were introduced in Pesin et al. (2016b) and are used to model invertible maps (e.g,, non-uniformly hyperbolic maps). In this paper we only consider inducing schemes of hyperbolic types and we follow Pesin et al. (2016b).

Let $f: X \to X$ be a homeomorphism of a compact metric space (X, d). We assume that f has finite topological entropy $h_{\text{top}}(f) < \infty$. An **inducing scheme of hyperbolic type** for f consists of a countable collection of disjoint Borel sets $S = \{J\}$ and a positive integer-valued function $\tau : S \to \mathbb{N}$; the **inducing domain** of the inducing scheme $\{S, \tau\}$ is $W = \bigcup_{J \in S} J$, and the **inducing time** $\tau : X \to \mathbb{N}$ is defined by $\tau(x) = \tau(J)$ for $x \in J$ and $\tau(x) = 0$ otherwise. We require several conditions.

(**I1**) For any $J \in S$ we have $f^{\tau(J)}(J) \subset W$ and $\bigcup_{J \in S} f^{\tau(J)}(J) = W$. Moreover, $f^{\tau(J)}|J$ can be extended to a homeomorphism of a neighborhood of J.

This condition allows one to define the **induced map** $F : W \to W$ by setting $F|J := f^{\tau(J)}|J$ for each $J \in S$. If τ is the first return time to W, then all images $f^{\tau(J)}(J)$ are disjoint. However, in general the sets $f^{\tau(J)}(J)$ corresponding to different $J \in S$ may overlap. In this case the map F may not be invertible.

(**I2**) For every bi-infinite sequence $\underline{a} = (a_n)_{n \in \mathbb{Z}} \in S^{\mathbb{Z}}$ there exists a **unique** sequence $\underline{x} = \underline{x}(\underline{a}) = (x_n = x_n(\underline{a}))_{n \in \mathbb{Z}}$ such that
  (a) $x_n \in \overline{J_{a_n}}$ and $f^{\tau(J_{a_n})}(x_n) = x_{n+1}$;
  (b) if $x_n(\underline{a}) = x_n(\underline{b})$ for all $n \leq 0$ then $\underline{a} = \underline{b}$.

This condition allows one to define the **coding map** $\pi : S^{\mathbb{Z}} \to \bigcup \overline{J}$ by $\pi(\underline{a}) := x_0(\underline{a})$. Within the full shift $\sigma : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ we consider the set

$$\check{S} := \{\underline{a} \in S^{\mathbb{Z}} \mid x_n(\underline{a}) \in J_{a_n} \text{ for all } n \in \mathbb{Z}\}.$$

For any $\underline{a} \in S^{\mathbb{Z}} \setminus \check{S}$ there exists $n \in \mathbb{Z}$ such that $\pi \circ \sigma^n(\underline{a}) \in \overline{J_{a_n}} \setminus J_{a_n}$. In particular, if all $J \in S$ are closed then $S^{\mathbb{Z}} \setminus \check{S} = \emptyset$; however, this need not always be the case. It follows from (**I1**) and (**I2**) that the map π has the following properties:

(1) π is well defined, continuous and for all $\underline{a} \in S^{\mathbb{Z}}$ one has $\pi \circ \sigma(\underline{a}) = f^{\tau(J)} \circ \pi(\underline{a})$ where $J \in S$ is such that $\pi(\underline{a}) \in \bar{J}$;

(2) π is one-to-one on $\check{S}$ and $\pi(\check{S}) = W$;
(3) if $\pi(\underline{a}) = \pi(\underline{b})$ for some $\underline{a}, \underline{b} \in \check{S}$ then $a_n = b_n$ for all $n \geq 0$.

Proving the existence and uniqueness of equilibrium measures requires some additional condition on the inducing scheme $\{S, \tau\}$:

(**I3**) The set $S^{\mathbb{Z}} \setminus \check{S}$ supports no (ergodic) σ-invariant measure which gives positive weight to any open subset.

This condition is designed to ensure that every equilibrium measure for the shift is supported on $\check{S}$ and its projection by π is thus supported on W and is F-invariant. This projection is a natural candidate for the equilibrium measure for F.

Set $Y = \{f^k(x) \mid x \in W, 0 \leq k \leq \tau(x) - 1\}$. Note that Y is forward invariant under f. This can be thought of as the region of X that is 'swept out' as W is carried forward under the dynamics of f; in particular, it contains all trajectories that intersect the base W.

Let φ be a potential function. Existence of an equilibrium measure for φ is obtained by first studying the problem for the induced system (F, W) and the **induced potential** $\overline{\varphi}\colon W \to \mathbb{R}$ defined by (1.2). The study of existence and uniqueness of equilibrium measures for the induced system (F, W) is carried out by conjugating the induced system to the two-sided full shift over the countable alphabet S. This requires that the potential function $\Phi := \bar{\varphi} \circ \pi$ be well defined on $S^{\mathbb{Z}}$. To this end we require that

(**P1**) the induced potential $\overline{\varphi}$ can be extended by continuity to a function on $\bar{J}$ for every $J \in S$.

Denote the potential induced by the **normalized** potential $\varphi - P_L(\varphi)$ by

$$\varphi^+ := \overline{\varphi - P_L(\varphi)} = \overline{\varphi} - P_L(\varphi)\tau$$

and let $\Phi^+ := \varphi^+ \circ \pi$.

Theorem 4.1 (see Pesin et al. 2016b) *Let $\{S, \tau\}$ be an inducing scheme of hyperbolic type satisfying Conditions (**I1**)–(**I3**) and φ a potential satisfying Condition (**P1**). Assume that*

- *Φ has strongly summable variations;*
- *$P_{GS}(\Phi) < \infty$ and $P_{GS}(\Phi^+) < \infty$;*
- $\sup_{a \in S^{\mathbb{Z}}} \Phi^+(a) < \infty$.

Then

(1) There exists a σ-invariant ergodic measure ν_{Φ^+} for Φ^+;
(2) If $h_{\nu_{\Phi^+}}(\sigma) < \infty$, then ν_{Φ^+} is the unique equilibrium measure for Φ^+;
(3) If $h_{\nu_{\Phi^+}}(\sigma) < \infty$, then the measure $\nu_{\varphi^+} := \pi_* \nu_{\Phi^+}$ is a unique F-invariant ergodic equilibrium measure for φ^+;
(4) If $P_{GS}(\Phi^+) = 0$ and $Q_{\nu_{\varphi^+}} < \infty$, then $\mu_\varphi = \mathcal{L}(\nu_{\varphi^+})$ is the unique equilibrium ergodic measure in the class $\mathcal{M}_L(f, Y)$ of liftable measures (see (2.5) and (2.6)).

The following result describes ergodic properties of equilibrium measures. assume that ν_{φ^+} has **exponential tails** (see (2.2)): there exist $C > 0$ and $0 < \theta < 1$ such that for all $n > 0$,

$$\nu_{\varphi^+}(\{x \in W : \tau(x) \geq n\}) \leq C\theta^n.$$

Theorem 4.2 (see Pesin et al. 2016b) *Under the conditions of Theorem 4.1 assume that*

- *the induced function $\overline{\varphi}$ on W is locally Hölder continuous;*
- *the tower has exponential tails with respect to the measure ν_{φ^+} that is there exist $C > 0$ and $0 < \theta < 1$ such that for all $n > 0$,*

$$\nu_{\varphi^+}(\{x \in W : \tau(x) \geq n\}) \leq C\theta^n;$$

(compare to (2.2));
- *the tower satisfies the arithmetic condition.*[31]

Then (f, μ_φ) has exponential decay of correlations and satisfies the CLT with respect to the class of functions whose induced functions on W are bounded locally Hölder continuous.

We describe some **verifiable** conditions on the potential function φ under which the assumptions of Theorem 4.1 hold:

(**P2**) there exist $C > 0$ and $0 < r < 1$ such that for any $n \geq 1$

$$V_n(\phi) := V_n(\Phi) \leq Cr^n,$$

where

$$V_n(\Phi) := \sup_{[b_{-n+1}, \cdots, b_{n-1}]} \sup_{\underline{a}, \underline{a}' \in [b_{-n+1}, \cdots, b_{n-1}]} \{|\Phi(\underline{a}) - \Phi(\underline{a}')|\}$$

is the n variation of Φ;

(**P3**) $\sum_{J \in S} \sup_{x \in J} \exp \bar{\varphi}(x) < \infty$;

(**P4**) there exists $\epsilon > 0$ such that

$$\sum_{J \in S} \tau(J) \sup_{x \in J} \exp(\varphi^+(x) + \epsilon\tau(x)) < \infty.$$

The following result is a corollary of Theorems 4.1 and 4.2.

Theorem 4.3 (see Pesin et al. (2016b)) *Let $\{S, \tau\}$ be an inducing scheme of hyperbolic type satisfying Conditions (**I1**)–(**I3**). Assume that the potential function φ satisfies Conditions (**P1**)–(**P4**). Then*

(1) there exists a unique equilibrium measure μ_φ for φ among all measures in $\mathcal{M}_L(f, Y)$; the measure μ_φ is ergodic;

(2) if $\nu_{\varphi^+} = \mathcal{L}^{-1}(\mu_\varphi)$ has exponential tail and the tower satisfies the arithmetic condition, then (f, μ_φ) has exponential decay of correlations and satisfies the CLT with respect to a class of functions whose corresponding induced functions on W (see (1.2)) are bounded locally Hölder continuous functions.

[31] This requirement should be added to Theorem 4.6 in Pesin et al. (2016b).

5 Coarse-Graining, Expansivity, and Specification

5.1 Uniform Expansivity and Specification

Let X be a compact metric space and $f: X \to X$ a homeomorphism; given $\varepsilon > 0$ and $x \in X$, the set

$$\Gamma_\varepsilon(x) := \{y \in X \mid d(f^n x, f^n y) < \varepsilon \text{ for all } n \in \mathbb{Z}\} \tag{5.1}$$

contains all points whose trajectory stays within ε of the trajectory of x for all time. The map f is **expansive** if there is $\varepsilon > 0$ such that $\Gamma_\varepsilon(x) = \{x\}$ for every $x \in X$; that is, if any two distinct trajectories eventually separate at scale ε. Uniformly hyperbolic systems can easily be shown to be expansive, and expansivity is a sufficient condition for **existence** of an equilibrium measure for any continuous potential φ; indeed, the standard proof of the variational principle (Walters 1982, Theorem 8.6) gives a construction of such a measure. The idea is that one "coarse-grains" the system at scale ε and builds a measure that is appropriately distributed over all trajectories that separate by ε within n iterates; sending $n \to \infty$ and using expansivity one guarantees that this measure is an equilibrium state.

To show that this equilibrium state is **unique**, Bowen used the following **specification property** of uniformly hyperbolic systems: for every $\varepsilon > 0$ there is $\tau \in \mathbb{N}$ such that any collection of finite-length orbit segments can be ε-shadowed by a single orbit that takes τ iterates to transition from one segment to the next. More precisely, if we associate $(x, n) \in X \times N$ to the orbit segment $x, f(x), \ldots, f^{n-1}(x)$ and write

$$B_n(x, \varepsilon) = \{y \in X \mid d(f^k x, f^k y) \le \varepsilon \text{ for all } 0 \le k < n\}$$

for the **Bowen ball** of points that shadow (x, n) to within ε for those n iterates, then specification requires that for every $(x_1, n_1), \ldots, (x_k, n_k)$ there is $y \in X$ such that $y \in B_{n_1}(x_1, \varepsilon)$, then $f^{n_1+\tau}(y) \in B_{n_2}(x_2, \varepsilon)$, and in general

$$f^{\sum_{i=0}^{j-1}(n_i+\tau)}(y) \in B_{n_j}(x, \varepsilon) \text{ for all } 1 \le j \le k. \tag{5.2}$$

Mixing Axiom A systems satisfy specification; this is a consequence of the mixing property together with the shadowing lemma.

A continuous potential $\varphi: X \to \mathbb{R}$ satisfies the **Bowen property** if there is $K \in \mathbb{R}$ such that $|S_n\varphi(x) - S_n\varphi(y)| < K$ whenever $y \in B_n(x, \varepsilon)$, where $S_n\varphi(x) = \sum_{j=0}^{n-1} \varphi(f^j x)$. The following theorem summarizes the classical results due to Bowen on systems with specification Bowen (1974).[32]

Theorem 5.1 *If (X, f) is an expansive system with specification and φ is a potential with the Bowen property, then there is a unique equilibrium measure μ. This includes*

[32] In fact, Bowen required the slightly stronger property that the shadowing point y in (5.2) be periodic, but this is only necessary for the part of his results dealing with periodic orbits, which we omit here.

the case when $f|\Lambda$ is topologically mixing and uniformly hyperbolic, and φ is Hölder continuous.

5.2 Non-uniform Expansivity and Specification

Various weaker versions of the specification property have been introduced in the literature. The one which is most relevant for our purposes first appeared in Climenhaga and Thompson (2012) for MMEs in the symbolic setting, and was developed in Climenhaga and Thompson (2013, 2014, 2016) to a version that applies to smooth maps and flows.

Given $\varepsilon > 0$, consider the **'non-expansive set'** $\mathrm{NE}(\varepsilon) = \{x \in X \mid \Gamma_\varepsilon(x) \neq \{x\}\}$, where $\Gamma_\varepsilon(x)$ is as in (5.1). Note that (X, f) is expansive if and only if $\mathrm{NE}(\varepsilon) = \emptyset$. The **pressure of obstructions to expansivity** is[33]

$$P^\perp_{\exp}(\varphi) = \lim_{\varepsilon\to 0} \sup_{\mu\in\mathcal{M}^e(f)} \left\{ h_\mu(f) + \int \varphi\, d\mu \mid \mu(\mathrm{NE}(\varepsilon)) = 1 \right\}. \tag{5.3}$$

In particular, expansive systems have $P^\perp_{\exp}(\varphi) = -\infty$. It follows from the results in Climenhaga and Thompson (2016) that the condition $P^\perp_{\exp}(\varphi) < P(\varphi)$ is enough for existence of an equilibrium measure. For uniqueness, we need to weaken the notion of specification. The idea behind this is to only require the specification property (5.2) to hold for a certain 'good' **collection of orbit segments** $\mathcal{G} \subset X \times \mathbb{N}$ (and similarly for the Bowen property). One must also require $\mathcal{G}$ to be large enough, which in this case means that there are collections of orbit segments $\mathcal{P}, \mathcal{S} \subset X \times \mathbb{N}$ that have small pressure compared to the whole system, but are sufficient to generate $X \times \mathbb{N}$ from $\mathcal{G}$ by adding prefixes from $\mathcal{P}$ and suffixes from $\mathcal{S}$.[34]

Let us make this more precise. A **decomposition of the space of orbit segments** consists of $\mathcal{P}, \mathcal{G}, \mathcal{S} \subset X \times \mathbb{N}$ and functions $p, g, s \colon X \times \mathbb{N} \to \mathbb{N} \cup \{0\}$ such that $(p + g + s)(x, n) = n$ and

$$\begin{aligned} (x, p(x,n)) &\in \mathcal{P}, \\ (f^{p(x,n)}(x), g(x,n)) &\in \mathcal{G}, \\ (f^{(p+g)(x,n)}(x), s(x,n)) &\in \mathcal{S}. \end{aligned}$$

The following is (Climenhaga and Thompson 2016, Theorem 5.5).

Theorem 5.2 *Let X be a compact metric space, $f\colon X \to X$ a homeomorphism, and $\varphi\colon X \to \mathbb{R}$ a continuous function. Suppose that $P^\perp_{\exp}(\varphi) < P(\varphi)$ and $X \times \mathbb{N}$ admits a decomposition $(\mathcal{P}, \mathcal{G}, \mathcal{S})$ with the following properties:*

[33] The idea of ignoring measures sitting on $\mathrm{NE}(\varepsilon)$ was introduced earlier by Buzzi and Fisher (2013).

[34] One can sum up the situation by saying that "the pressure of obstructions to specification is small". A related idea of studying shift spaces for which "the entropy of constraints is small" appeared in Buzzi (2005), where Buzzi studied **shifts of quasi-finite type**. For more details on the relationship between the two notions, see Climenhaga (2015), especially Theorem 1.4.

*(**I**) $\mathcal{G}$ has specification at every scale;*
*(**II**) φ has the Bowen property on $\mathcal{G}$;*
*(**III**) $P(\mathcal{P} \cup \mathcal{S}, \varphi) < P(\varphi)$.*

Then f has a unique equilibrium measure μ_φ.

We describe two examples for which Theorem 5.2 applies. One of them is the Mañé example Mañé (1978), which was introduced as an example of a robustly transitive diffeomorphism that is not Anosov. This "derived from Anosov" example is obtained by taking a 3-dimensional hyperbolic toral automorphism with one unstable direction and performing a pitchfork bifurcation in E^{cs} near the fixed point so that E^c becomes weakly expanding in that neighborhood. One obtains a partially hyperbolic diffeomorphism with a splitting $E^s \oplus E^c \oplus E^u$ such that E^c "contracts on average" with respect to the Lebesgue measure; this falls under the results in Castro (2004) mentioned above, and its inverse map (for which E^c "expands on average") is covered by Alves and Pinheiro (2010), Alves and Li (2015).

Now given any Hölder continuous potential $\varphi\colon \mathbb{T}^3 \to \mathbb{R}$, it is shown in Climenhaga et al. (2015) that there is a C^1-open class of Mañé examples for which this potential has a unique equilibrium state. In particular, when f is C^2, there is an interval $(t_0, t_1) \supset [0, 1]$ such that the geometric t-potential $-t \log|\det(df|E^{cu})|$ has a unique equilibrium state for every $t \in (t_0, t_1)$, and φ_1 is the unique SRB measure.

A related second example is the Bonatti–Viana example introduced in Bonatti and Viana (2000). Here one takes a 4-dimensional hyperbolic toral automorphism with $\dim E^s = \dim E^u = 2$, and makes two perturbations, one in the E^s-direction and another one in the E^u-direction. The first perturbation creates a pitchfork bifurcation as above in E^s and then "mixes up" the two directions in E^s so that there is no invariant subbundle of E^s; the second perturbation does a similar thing to E^u. One obtains a map with a dominated splitting $E^{cs} \oplus E^{cu}$ that has no uniformly hyperbolic subbundles.

The same approach as above works for the Bonatti–Viana examples, which have a dominated splitting but are not partially hyperbolic; see Climenhaga et al. (2015). In this case the presence of non-uniformity in both the stable and unstable directions makes tower constructions more difficult, and no Gibbs-Markov-Young structure has been built for these examples. Earlier results on thermodynamics of these examples (and the Mañé examples) were given in Buzzi et al. (2012), Buzzi and Fisher (2013), which proved existence of a unique MME. These results make strong use of the semi-conjugacy between the examples and the original toral automorphisms, and in particular do not generalize to equilibrium states corresponding to non-zero potentials.

Finally, the flow version of Theorem 5.2 can be applied to geodesic flow in nonpositive curvature. Geodesic flow in negative curvature is one of the classical examples of an Anosov flow Anosov (1969), and in particular it has unique equilibrium states with strong statistical properties.[35] If M is a smooth rank 1 manifold with nonpositive curvature, then its geodesic flow is non-uniformly hyperbolic. Bernoullicity of the regular component of the Liouville measure was shown by Pesin (1977). It was shown by Knieper (1998) that there is a unique measure of maximal entropy; his proof

[35] Although the issue of decay of correlations is more subtle because it is a flow, not a map; see Dolgopyat (1998), among others.

uses powerful geometric tools and does not seem to generalize to non-zero potentials. Using non-uniform specification, Knieper's result can be extended to the geometric t-potential for $t \approx 0$, and when $\dim M = 2$, it works for any $t \in (-\infty, 1)$, showing that the pressure function is differentiable on this interval and we recover the same picture as for Manneville–Pomeau Burns et al. (2016).

In each of the above examples, the basic idea is as follows: one identifies a "bad set" $B \subset X$ with the properties that

(1) the system has uniformly hyperbolic properties outside of B;
(2) trajectories spending all (or almost all) of their time in B carry small pressure relative to the whole system.

For the Mañé and Bonatti–Viana examples, B is the neighborhood where the perturbation is carried out; for the geodesic flow, B is a small neighborhood of the singular set.

Given an orbit segment (x, n), let $G(x, n) = \frac{1}{n}\#\{0 \le k < n \mid f^k x \notin B\}$ be the proportion of time that the orbit segment spends in the "good" part of the system.[36] A decomposition of the space of orbit segments $X \times \mathbb{N}$ is obtained by fixing a threshold $\gamma > 0$ and taking

$$\mathcal{P} = \mathcal{S} = \{(x, n) \mid G(x, n) < \gamma\},$$
$$\mathcal{G} = \{(x, n) \mid G(x, k) \ge \gamma, G(f^k x, n - k) \ge \gamma \text{ for all } 0 \le k \le n\}.$$

Indeed, given any $(x, n) \in X \times \mathbb{N}$, one can take p and s to be maximal such that $(x, p) \in \mathcal{P}$ and $(f^{n-s}x, s) \in \mathcal{S}$, and use additivity of G along orbit segments to argue that $(f^p x, n - p - s) \in \mathcal{G}$, which yields a decomposition $X \times \mathbb{N} = \mathcal{P}\mathcal{G}\mathcal{S}$. Then one makes the following arguments to apply Theorem 5.2.

- Assumption (1) above leads to hyperbolic estimates along trajectories in $\mathcal{G}$, which can be used to prove specification for $\mathcal{G}$ (condition (**I**) in Theorem 5.2) and the Bowen property on $\mathcal{G}$ for Hölder continuous potentials (condition (**II**)).
- Assumption (2) gives the pressure estimate $P(\mathcal{P} \cup \mathcal{S}, \varphi) < P(\varphi)$ from (**III**).
- The expansion estimates along $\mathcal{G}$ and the pressure estimates on $\mathcal{P}$ and $\mathcal{S}$ also yield $P^{\perp}_{\exp}(\varphi) < P(\varphi)$.

This approach establishes existence and uniqueness, and yields some statistical properties such as large deviations estimates. However, it does not yet give stronger statistical results such as a rate of decay of correlations, or the CLT. In the setting when X is a shift space with non-uniform specification, results along these lines have recently been established Climenhaga (2015) by using conditions (**I**)–(**III**) (or closely related ones) to build a tower with exponential tails, but it is not yet clear how this result extends to the smooth setting.

[36] For flows one should make the obvious modifications, replacing $\mathbb{N}$ by $[0, \infty)$ and cardinality with Lebesgue measure.

6 The Geometric Approach

6.1 Geometric Construction of SRB Measures

6.1.1 Idea of Construction

Having discussed constructions of SRB and equilibrium measures via Markov dynamics (SFTs and Young towers) and via coarse-graining (expansivity and specification), we turn our attention now to a third approach, which is in some sense more natural and more simple-minded. The first two approaches addressed not just existence but also questions of uniqueness and statistical properties; the price to be paid for these stronger results is that the construction of a tower (or even the verification of non-uniform specification) may be difficult in many examples. The approach that we now describe is best suited to prove existence, rather than uniqueness or statistical properties, but has the advantage that it seems easier to verify.

We start by discussing SRB measure, which for dissipative systems plays the role of Lebesgue measure in conservative systems and is the most natural measure from the physical point of view. So in trying to find an SRB measure, it is natural to start with Lebesgue measure itself; while it may not be invariant, we will follow the standard Bogolubov–Krylov procedure of taking a non-invariant measure m, average it under the dynamics to produce the sequence

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k m, \tag{6.1}$$

and pass to a weak*-convergent subsequence $\mu_{n_j} \to \mu$; then μ is f-invariant. If we do this starting with Lebesgue measure as our reference measure m, then it is reasonable to expect that the limiting measure will have something to do with Lebesgue, and may even be an SRB measure.[37]

At an intuitive level, this approach is consistent with Viana's conjecture (1998) that nonzero Lyapunov exponents imply existence of an SRB, since this should be exactly the setting in which the iterates of Lebesgue spread out along the unstable manifolds and converge in average to a measure that is absolutely continuous in the unstable direction. Now we describe how it can be made precise.

6.1.2 Uniform Geometry: Uniform and Partial Hyperbolicity

In the uniformly hyperbolic setting, this approach can be carried out as follows. Let $\mathcal{R}$ be the set of all **standard pairs** (W, ρ), where W is a small piece of unstable manifold and $\rho\colon W \to (0, \infty)$ is integrable with respect to m_W, the leaf volume on W. Let $\mathcal{M}^{\text{ac}}$ be the set of all (not necessarily invariant) probability measures μ on the manifold M that can be expressed as

[37] In general, though, the measure μ may be quite trivial—just consider the point mass at an attracting fixed point.

$$\mu(E) = \int_{\mathcal{R}} \int_{W \cap E} \rho(x)\, dm_W(x)\, d\zeta(W, \rho) \tag{6.2}$$

for some measure ζ on $\mathcal{R}$; in other words, μ admits a decomposition (in the sense of Fubini's theorem) along unstable local manifolds, in which conditional measures are leaf-volumes. Then one can show that $\mathcal{M}^{\rm ac} \cap \mathcal{M}(f)$ is precisely the set of SRB measures for f. Moreover, Lebesgue measure m is in $\mathcal{M}^{\rm ac}$ and thus since images of unstable manifolds can be decomposed into small pieces of unstable manifolds, we have $f_*^k m \in \mathcal{M}^{\rm ac}$ for all $k \in \mathbb{N}$, so the averaged measures given by (6.1) are in $\mathcal{M}^{\rm ac}$ as well.

In order to pass to the limit and obtain $\mu \in \mathcal{M}^{\rm ac}$ one needs a little more control. Fixing $K > 0$, let $\mathcal{R}_K$ be the set of all standard pairs (W, ρ) such that W has size at least $1/K$, and $\rho\colon W \to [1/K, K]$ is Hölder continuous with constant K. Then defining $\mathcal{M}_K^{\rm ac}$ using (6.2) with $\mathcal{R}_K$ in place of $\mathcal{R}$, one can show that $\mathcal{M}_K^{\rm ac}$ is weak* compact and is f_*-invariant for large enough K. This is basically a consequence of the Arzelà–Ascoli theorem and the fact that f uniformly expands unstable manifolds; in particular it relies strongly on the uniform hyperbolicity assumption. Then $\mu_n \in \mathcal{M}_K^{\rm ac}$ for all n by invariance, and by compactness, $\mu = \lim \mu_{n_j} \in \mathcal{M}_K^{\rm ac} \cap \mathcal{M}(f)$ is an SRB measure. Thus we have the following statement.

Theorem 6.1 *Let Λ be a hyperbolic attractor for f and assume that $f|\Lambda$ is topologically transitive. If the reference measure m is the restriction of the Lebesgue measure to a neighborhood of Λ, then the sequence of measures* (6.1) *converges and the limit measure is the unique SRB measure for f.*

Now consider the setting where f is partially hyperbolic, i.e., for every point x the tangent space splits $T_x M = E^s(x) \oplus E^c(x) \oplus E^u(x)$ into stable, central, and unstable subspaces respectively with uniform contraction along $E^s(x)$, uniform expansion along $E^u(x)$, and possible contractions and/or expansions along $E^c(x)$ with rates which are weaker than the corresponding rates along $E^s(x)$ and $E^u(x)$.

In the situation where the centre-unstable direction E^{cu} is only non-uniformly expanding more care must be taken with the above approach because $\mathcal{M}_K^{\rm ac}$ may no longer be f_*-invariant: even if W is a "sufficiently large" local unstable manifold, its image $f(W)$ may be smaller than $1/K$, and similarly the Hölder constant of the density ρ can get worse under the action of f_* if W is contracted by f.

The solution is to use **hyperbolic times**, which were introduced by Alves (2000). Roughly speaking, a time n is hyperbolic for a point x if $df^k|E^u(f^{n-k}x)$ is uniformly expanding for every $0 \le k \le n$. If W is a local unstable manifold around x and n is a hyperbolic time for x, then $f^n(W)$ contains a large neighborhood of $f^n(x)$, and the density ρ behaves well under f_*^n. Thus from the point of view of the construction above, the key property of hyperbolic times is that if H_n is the set of all points x for which n is a hyperbolic time, then the measures

$$\nu_n := \frac{1}{n} \sum_{k=0}^{n-1} f_*^k(m|H_k) \tag{6.3}$$

are all contained in $\mathcal{M}_K^{\mathrm{ac}}$ (after rescaling to obtain a probability measure). As long as $\nu_n \not\to 0$, one concludes that $\mu = \lim \nu_{n_k} \in \mathcal{M}(f)$ has some ergodic component in $\mathcal{M}_K^{\mathrm{ac}}$, which must be an SRB measure. To get the lower bound on the total weight of ν_n, one needs a lower bound on $\frac{1}{n}\sum_{k=0}^{n-1} m(H_k)$, which can be obtained using Pliss' lemma as long as a positive Lebesgue measure set of points have positive Lyapunov exponents along E^{cu}.

One can also construct the SRB measure by beginning "within the attractor": instead of using Lebesgue measure on M as the starting point for the sequence (6.1), one can let m^u be leaf volume along a local unstable manifold and then consider the sequence

$$\nu_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k m^u(x). \tag{6.4}$$

If the attractor Λ is uniformly hyperbolic, the sequence of measures 6.4 converges and the limit measure is a unique SRB measure for f. In the partially hyperbolic setting, it was shown by Pesin and Sinaĭ (1983) that every limit measure ν of the sequence of measures 6.4 is a ***u*-measure** on Λ: that is, the conditional measures it generates on local unstable manifolds are absolutely continuous with respect to the leaf-volume on these manifolds. What prevents ν from being an SRB measure in general is that the Lyapunov exponents in the central direction can be positive or zero.

Several results are available that establish existence (and in some cases uniqueness) of SRB measures under some additional requirements on the action of the system along the central direction E^c or central-unstable direction E^{cu}. For example the case of systems with mostly contracting central directions was carried out in Bonatti and Viana (2000), Burns et al. (2008) and with mostly expanding central directions in Alves et al. (2000). A more general case of systems whose central direction is weakly expanding was studied in Alves et al. (2014).

In these settings one at least has a dominated splitting, which gives the system various uniform geometric properties, even if the dynamics is non-uniform. To extend this approach to settings where the geometry is non-uniform (no dominated splitting, stable and unstable directions vary discontinuously) some new tools are needed. An important observation (which holds in the uniform case as well) is that for many purposes we can replace $V^u(x)$ itself with a local manifold passing through x that is C^1-close to $V^u(x)$. Such a manifold is called **admissible**, and in the next section will develop the machinery of standard pairs, the class of measures $\mathcal{M}^{\mathrm{ac}}$, and the sequences of measures (6.4) using admissible manifolds in place of unstable manifolds.

6.1.3 Non-uniform Geometry: Effective Hyperbolicity

The difficulties encountered in the geometrically non-uniform setting can be overcome by the machinery of 'effective hyperbolicity' from Climenhaga and Pesin (2016), Climenhaga et al. (2016). This approach has the advantage that the requirements on the system appear weaker, and much closer to the Viana conjecture. The drawback of this approach is that it is currently out of reach to use it to establish exponential (or even polynomial) decay of correlations and the CLT.

Let U be a neighborhood of the attractor Λ for a $C^{1+\epsilon}$ diffeomorphism, and consider a forward invariant set $S \subset U$ on which there are two measurable cone families $K^s(x) = K^s(x, E^s(x), \theta)$ and $K^u(x) = K^u(x, E^s(x), \theta)$ that are

- **invariant**: $\overline{Df(K^u(x))} \subset K^u(fx)$ and $\overline{Df^{-1}(K^s(fx))} \subset K^s(x)$;
- **transverse**: $T_xM = E^s(x) \oplus E^u(x)$.

Given $x \in S$, consider the **expansion and contraction coefficients**

$$\lambda^u(x) := \inf\{\log \|Df(v)\| \mid v \in K^u(x), \|v\| = 1\},$$
$$\lambda^s(x) := \sup\{\log \|Df(v)\| \mid v \in K^s(x), \|v\| = 1\},$$

and the **defect from domination**

$$\Delta(x) := \max(0, \tfrac{1}{\epsilon}(\lambda^s(x) - \lambda^u(x));$$

note that $\Delta(x) = 0$ whenever $\lambda^u(x) > \lambda^s(x)$, so the defect only comes into play when the stable cone expands more than the unstable cone. The **coefficient of effective hyperbolicity** is

$$\lambda(x) := \min(\lambda^u(x) - \Delta(x), -\lambda^s(x)); \tag{6.5}$$

thus $\lambda(x) > 0$ whenever the system "behaves hyperbolically" at x, while $\lambda(x) \leq 0$ when one of the following happens:

- some stable vectors expand (so $-\lambda^s(x) < 0$); or
- some unstable vectors contract (so $\lambda^u(x) < 0$); or
- the defect from domination is greater than the expansion in the unstable cone (so $\lambda^u(x) - \Delta(x) < 0$).

Let $\alpha(x)$ be the angle between the cones $K^s(x)$ and $K^u(x)$, and given a threshold $\bar{\alpha} > 0$, let

$$\rho_{\bar{\alpha}}(x) := \overline{\lim_{n\to\infty}} \frac{1}{n} \#\{0 \leq k < n \mid \alpha(f^k x) < \bar{\alpha}\}$$

be the asymptotic upper bound on how often the angle drops below that threshold. Notice that in the case of a dominated splitting, $\alpha(x)$ is uniformly bounded away from 0, so there is $\bar{\alpha} > 0$ with $\rho_{\bar{\alpha}}(x) = 0$ for every x; however, for a system with non-uniform geometry it may be the case that every $\bar{\alpha} > 0$ has points with $\rho_{\bar{\alpha}}(x) > 0$.

With the above notions in mind, we consider the following set of points:

$$S' = \left\{ x \in S \mid \underline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=0}^{n-1} \lambda(f^k x) > 0 \text{ and } \lim_{\bar{\alpha}\to 0} \rho_{\bar{\alpha}}(x) = 0 \right\}.$$

Thus S' contains points for which the average asymptotic rate of effective hyperbolicity is positive, and for which the asymptotic frequency with which the angle between the cones degenerates can be made arbitrarily small. Then we have the following result, which is a step towards Viana's conjecture.

Theorem 6.2 (Climenhaga et al. 2016) *If S' has positive volume then f has an SRB measure.*

6.1.4 Idea of Proof

The construction of an SRB measure in the setting of Theorem 6.2 follows the same averaging idea as in Sects. 6.1.1, 6.1.2: if μ_n is the sequence of measures given by (6.1), then one wish to show that a uniformly large part of μ_n lies in the set of "uniformly absolutely continuous" measures $\mathcal{M}_K^{\mathrm{ac}}$.

In this more general setting the definition of $\mathcal{M}_K^{\mathrm{ac}}$ is significantly more involved. Broadly speaking, in the definition of $\mathcal{R}$ we must replace unstable manifolds W with **admissible manifolds**; an admissible manifold W through a point $x \in S$ is a smooth submanifold such that $T_x W \subset K^u(x)$ and W is the graph of a function $\psi : B^u(r) \subset E^u(x) \to E^s(x)$, such that $D\psi$ is uniformly bounded and is uniformly Hölder continuous. The Hölder constant for $D\psi$ can be thought of as the "curvature" of W.

When the geometry is uniform as in the previous setting, the image of an admissible manifold W is itself admissible; this is essentially the classical Hadamard–Perron theorem. In the more general case this is no longer true; although $f^n(W)$ contains an admissible manifold, its size and curvature may vary with time n, with the size becoming arbitrarily small and the curvature arbitrarily large. In this setting a version of the Hadamard–Perron theorem was proved in Climenhaga and Pesin (2016) that gives good bounds on $f^n(W)$ when n is an **effective hyperbolic time** for $x \in W$; that is, when

$$\sum_{j=k}^{n-1} \lambda(f^j x) \geq \chi(n-k)$$

for every $0 \leq k < n$, where $\chi > 0$ is a fixed **rate of effective hyperbolicity**.

The set of effective hyperbolic times is a subset of the set of hyperbolic times; the extra conditions in the definition of effective hyperbolic time guarantee that we can control the dynamics of f on the manifold itself, not just the dynamics of df on the tangent bundle. In the uniform geometry setting from earlier, this extension came for free for hyperbolic times.

With the notion of effective hyperbolic times, the approach outlined in Sects. 6.1.1, 6.1.2 can be carried out. One must add some more conditions to the collection $\mathcal{R}$; most notably, one must fix $n \in \mathbb{N}$ and then consider only admissible manifolds W for which

$$d(f^{-k}(x), f^{-k}(y)) \leq Ce^{-\chi k} d(x, y) \quad \text{for all} \quad 0 \leq k \leq n \quad \text{and} \quad x, y \in W,$$

and then define $\mathcal{M}_{K,n}^{\mathrm{ac}}$ using only this class of admissible manifolds. In addition to size of W and regularity of ρ, the constant K must also be chosen to govern the curvature of W, but we omit details here. The point is that the set $\mathcal{M}_{K,n}^{\mathrm{ac}}$ is compact, but not f_*-invariant, and so the proof of Theorem 6.2 can be completed via the following steps.

(1) Writing H_n for the set of points with n as an effective hyperbolic time, use Pliss' lemma and the assumption that S' has positive volume to show that H_n has positive Lebesgue measure on average.
(2) Use the effective Hadamard–Perron theorem from Climenhaga and Pesin (2016) to show that $\nu_n := \frac{1}{n}\sum_{k=0}^{n-1} f_*^k \in \mathcal{M}_{K,n}^{\mathrm{ac}}$, and use the bound from the previous step to get a lower bound on the total weight of ν_n.
(3) Write $\mu_n = \nu_n + \zeta_n$ and argue from general principles that if $\mu_{n_k} \to \mu$, then μ has an ergodic component in $\mathcal{M}^{\mathrm{ac}}$; moreover, this ergodic component is hyperbolic and f-invariant, so it is an SRB measure.

6.2 Constructing Equilibrium Measures

A natural next step is to extend the above procedure to study general equilibrium states, and not just SRB measures. The direct analogue of the previous section has not yet been fully developed, and we describe instead a related approach that is also based on studying how densities transform under the dynamics.

First consider the case of a piecewise expanding interval map, and the question of finding an SRB measure. In this case there is no stable direction, and so we do not have to keep track of the "shape" of unstable manifolds, or admissible manifolds; indeed, a local unstable manifold is just a small piece of the interval, and an SRB measure is just an invariant measure that is absolutely continuous with respect to Lebesgue. Thus the entire problem is reduced to the following question: given a (not necessarily invariant) absolutely continuous measure $\mu \ll m$, how is the density function of its image $f_*\mu$ related to the density function of μ? One ends up defining a **transfer operator** $\mathcal{L}$ with the property that if $d\mu = h\,dx$, then $d(f_*\mu) = (\mathcal{L}h)\,dx$. Questions about the existence of an absolutely continuous invariant measure, and its statistical properties, can be reduced to questions about the transfer operator $\mathcal{L}$.

The central issue in studying $\mathcal{L}$ is the problem of finding a Banach space $\mathcal{B}$ (of functions) on which $\mathcal{L}$ acts "with good spectral properties". Generally speaking this means that 1 is a simple eigenvalue of $\mathcal{L}$ (so there is a unique fixed point $h = \mathcal{L}h$, which corresponds to the unique absolutely continuous invariant measure) and the rest of the spectrum of $\mathcal{L}$ lies inside a disc of radius $r < 1$, which guarantees exponential decay of correlations and other statistical properties.

For piecewise expanding interval maps, this was accomplished by Lasota and Yorke (1974), and the approach can be adapted to equilibrium states for other potential functions by considering a transfer operator that depends on the potential in an appropriate way. A thorough account of this approach is given in Baladi (2000).

The mechanism that drives this approach is that the expansion of the dynamics acts to "smooth out" the density function; irregularities in the function h are made milder by passing to $\mathcal{L}h$. (The precise meaning of this statement depends on the particular choice of Banach space $\mathcal{B}$, and is encoded by the Lasota–Yorke inequality, which we do not pursue further here.) But this means that one runs into problems when going from expanding interval maps to hyperbolic diffeomorphisms, where there is a non-trivial stable direction; the contracting dynamics in the stable direction make irregularities in the function worse!

In the classical approach to uniformly hyperbolic systems, this was dealt with by passing to a symbolic coding by an SFT (as described after Theorem 1.1) and then replacing the two-sided SFT $\Sigma \subset A^{\mathbb{Z}}$ by its one-sided version $\Sigma^+ \subset A^{\mathbb{N}}$. As described after Theorem 1.3, the transfer operator $\mathcal{L}$ has an eigenfunction $h \in C(\Sigma^+)$, and its dual $\mathcal{L}^*$ has an eigenmeasure $\nu \in C(\Sigma^+)^*$; combining them gives the equilibrium state $d\mu = h\, d\nu$. Note that positive indices of an element of Σ code the future of a trajectory, while negative indices code the past, and so dynamically, passing from Σ to Σ^+ can be interpreted as "forgetting the past". Geometrically, this means that we conflate points lying on the same local stable manifold; taking a quotient in the stable direction eliminates the problem described in the previous paragraph, where contraction in the stable direction exacerbates irregularities in the density function.

More recent work has shown that this problem can be addressed without the use of symbolic dynamics. The key is to consider a Banach space $\mathcal{B}$ whose elements are not functions, but are rather objects that behave like functions in the unstable direction, and like **distributions** in the stable direction. For SRB measures, this was carried out in Blank et al. (2002), Gouëzel and Liverani (2006), Baladi and Tsujii (2007). A further generalization to equilibrium states for other potential functions was given in Gui and Li (2008); as with expanding interval maps, this requires working with a transfer operator $\mathcal{L}$ that depends on the potential. Moreover, instead of distributions along the stable direction, one must consider a certain class of "generalized differential forms". We refer the reader to (Gui and Li 2008, Sect. 7) for a comparison of this approach to equilibrium states and other related approaches, including the technique of "standard pairs".

It remains an open problem to extend this approach to the non-uniformly hyperbolic setting.

6.3 Ergodic Properties

An important open question is to study uniqueness and statistical properties of the SRB measure produced in Theorem 6.2, or of any equilibrium states that may be produced by an analogous result for other potentials. One potential approach is to study the standard pairs (W, ρ) and derive statistical properties via **coupling** techniques, as was done by Chernov and Dolgopyat in another setting (Chernov and Dolgopyat 2009).[38] One might also hope to adapt the functional analytic approach from Sect. 6.2 into the non-uniformly hyperbolic setting and obtain statistical properties this way. For now, though, we only mention results on Bernoullicity and hyperbolic product structure.

6.3.1 SRB Measures

By a result of Ledrappier (1984), a hyperbolic SRB measure has at most countably many ergodic components and every hyperbolic SRB measure is Bernoulli up to a finite period. It follows that there may exist at most countably many ergodic SRB

[38] Coupling techniques are also at the heart of Young's tower results for subexponential mixing rates (Young 1999).

measures on Λ. One way to ensure uniqueness of SRB measures is to show that its every ergodic component is open (mod 0) in the topology of Λ and that $f|\Lambda$ is topologically transitive.

6.3.2 Equilibrium Measures

Let μ be a hyperbolic ergodic measure for a $C^{1+\alpha}$ diffeomorphism f. Given $\ell > 0$, consider the regular set Γ_ℓ, which consists of points $x \in \Gamma$ whose local stable $V^s(x)$ and unstable $V^u(x)$ manifolds have size at least $1/\ell$. For $x \in \Gamma_\ell$ and some sufficiently small $r > 0$ let $R_\ell(x, r) = \bigcup_{y \in A^u(x)} V^s(y)$ be a **rectangle** at x, where $A^u(x)$ is the set of points of intersection of $V^u(x)$ with local stable manifolds $V^s(z)$ for $z \in \Gamma_\ell \cap B(x, r)$. We denote by

- $\pi : V^u(z_1) \to V^u(z_2)$ with $z_1, z_2 \in R_\ell(x, r) \cap \Gamma_\ell$ the **holonomy map** generated by local stable manifolds;
- $\mu^u(z)$ the conditional measure generated by μ on local unstable manifolds $V^u(z)$.

Say that μ has a **direct product structure** if the holonomy map is absolutely continuous with the Jacobian uniformly bounded away from 0 and ∞ on $R_\ell(x, r)$.

Conjecture 6.3 If μ is a hyperbolic ergodic equilibrium measure for the geometric t-potential for a $C^{1+\alpha}$ diffeomorphism f, then μ has a direct product structure.

If true, this would imply that μ has some "nice" ergodic properties; for example, it has at most countably many ergodic components. Similar results have recently been established (using the symbolic approach) for two-dimensional diffeomorphisms and three-dimensional flows (Sarig 2011; Ledrappier et al. 2016).

We conclude with a conjecture on the relationship between effective hyperbolicity (from Sect. 6.1) and decay of correlations. Suppose that Λ is an attractor with trapping region U, and that we have invariant measurable transverse cone families defined Lebesgue-a.e. in U, with the property that there is $\chi > 0$ for which

$$S' = \left\{ x \in U \mid \varliminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda(f^k x) > \chi \text{ and } \varlimsup_{\bar{\alpha}\to 0} \rho_{\bar{\alpha}}(x) = 0 \right\}$$

has full Lebesgue measure in U. Consider for each $N \in \mathbb{N}$ the set

$$X_N = \left\{ x \in U \mid \sum_{k=0}^{n-1} \lambda(f^k x) > \chi n \text{ for all } n > N \right\},$$

and note that the assumption on S' guarantees that $m(U \setminus X_N) \to 0$ as $N \to \infty$.

Conjecture 6.4 If $m(U \setminus X_N)$ decays exponentially in N, then the SRB measure μ produced by Theorem 6.2 has exponential decay of correlations.

Some support for this conjecture is provided by the fact that the analogous result for partially hyperbolic attractors with mostly expanding central direction was proved in Alves and Li (2015).

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