Arnold Math J. (2017) 3:83–95
DOI 10.1007/s40598-016-0056-4

RESEARCH CONTRIBUTION

On Postsingularly Finite Exponential Maps

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Received: 5 December 2015 / Revised: 15 August 2016 / Accepted: 6 September 2016 /
Published online: 19 September 2016

Abstract We consider parameters λ for which 0 is preperiodic under the map $z \mapsto \lambda e^z$. Given k and l, let $n(r)$ be the number of λ satisfying $0 < |\lambda| \le r$ such that 0 is mapped after k iterations to a periodic point of period l. We determine the asymptotic behavior of $n(r)$ as r tends to ∞.

Keywords Entire function · Singular value · Exponential function · Periodic point · Preperiodic point · Postcritically finite · Misiurewicz map · Nevanlinna characteristic

1 Introduction and Main Result

Let $E_\lambda(z) = \lambda e^z$ where $\lambda \in \mathbb{C}\backslash\{0\}$. We are interested in parameters λ for which 0 is preperiodic. Note that 0 is the only singularity of the inverse function of E_λ. Functions for which all singularities of the inverse are preperiodic are called *postsingularly finite*. The term *Misiurewicz map* is also used for such functions. We do not discuss their role in complex dynamics here, but refer to Benini (2011), Devaney and Jarque (1997), Devaney et al. (2005), Hubbard et al. (2009), Jarque (2011), Laubner et al. (2008) and Schleicher and Zimmer (2003) as a sample of papers dealing with postsingularly finite exponential maps.

For $k, l \in \mathbb{N}$ we thus consider parameters λ such that

$$E_\lambda^k(0) = E_\lambda^{k+l}(0) \tag{1.1}$$

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while

$$E_\lambda^i(0) \neq E_\lambda^j(0) \quad \text{for } 0 < i < j < k+l. \tag{1.2}$$

We denote by $n(r)$ the number of all λ contained in $\{z\colon 0 < |z| \leq r\}$ which satisfy (1.1) and (1.2). If $k = l = 1$, then the set of all $\lambda \neq 0$ satisfying (1.1) and (1.2) is equal to $\{2\pi i m\colon m \in \mathbb{Z}\backslash\{0\}\}$. Thus $n(r) \sim r/\pi$ as $r \to \infty$.

For $m \in \mathbb{N}$ we put $f_m(z) = E_z^m(0)$. Thus $f_1(z) = z$ and $f_{m+1}(z) = ze^{f_m(z)}$.

Theorem *Let k, l and $n(r)$ be as above. If $k + l \geq 3$, then*

$$n(r) \sim \frac{1}{\sqrt{2\pi^3}} f_{k+l-1}(r)\sqrt{f_{k+l-2}(r)} \quad \textit{as } r \to \infty.$$

The theorem will be proved using Nevanlinna theory. We refer to Goldberg and Ostrovskii (2008) and Hayman (1964) for the terminology and basic results of this theory. In particular, $T(r, f)$ denotes the Nevanlinna characteristic of a meromorphic function f.

Nevanlinna theory makes it natural to consider

$$N(r) = \int_0^r \frac{n(t)}{t} dt$$

besides $n(r)$.

The theorem will be a consequence of the following two propositions.

Proposition 1 *Let k, l and $N(r)$ be as above. Then there exists a subset E of $(0, \infty)$ which has finite measure such that*

$$N(r) \sim T(r, f_{k+l}) \quad \textit{as } r \to \infty, \ r \notin E. \tag{1.3}$$

We note that this proposition suffices to show that $n(r) \to \infty$ as $r \to \infty$. This means that given $k, l \in \mathbb{N}$ there exists infinitely parameters λ such that (1.1) and (1.2) hold.

Proposition 2 *Let $m \geq 3$. Then*

$$T(r, f_m) \sim \frac{1}{\sqrt{2\pi^3}} \frac{f_{m-1}(r)}{\sqrt{f_{m-2}(r)} \prod_{j=1}^{m-3} f_j(r)}. \tag{1.4}$$

These propositions will be proved in Sects. 2 and 3, before we show in Sect. 4 how the above theorem follows from them. We will see there that (1.3) actually holds without the exceptional set E. In fact, the exceptional set in Nevanlinna's second fundamental theorem and thus in Proposition 1 does not occur when the Nevanlinna characteristic grows sufficiently regularly, and the required regularity is provided by Proposition 2.

2 Proof of Proposition 1

For a meromorphic function f and $a \in \mathbb{C}$ or—more generally—a meromorphic function a satisfying $T(r, a) = o(T(r, f))$, a so-called *small* function, we denote by

$\overline{n}(r, a, f)$ the number zeros of $f - a$ in the disk $\{z\colon |z| \leq r\}$. Here we ignore multiplicities; that is, multiple zeros are counted only once. (The notation $n(r, a, f)$ is used in Nevanlinna theory when multiplicities are counted.) One may also take $a = \infty$, in which case we count the poles of f.

As usual in Nevanlinna theory, we put

$$\overline{N}(r, a, f) = \int_0^r \frac{\overline{n}(t, a, f) - \overline{n}(0, a, f)}{t} dt + \overline{n}(0, a, f) \log r$$

and we denote by $S(r, f)$ any quantity that satisfies $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside some exceptional set of finite measure.

The following result [see Hayman (1964, Theorem 2.5)] is a simple consequence of Nevanlinna's second fundamental theorem.

Lemma 1 *Let f be a meromorphic function and let a_1, a_2, a_3 be distinct small functions (or constants in $\mathbb{C} \cup \{\infty\}$). Then*

$$T(r, f) \leq \sum_{j=1}^{3} \overline{N}(r, a_j, f) + S(r, f).$$

We remark that Yamanoi (2004) proved that if $\varepsilon > 0$, $q \geq 3$ and $a_1, \ldots, a_q$ are small functions, then

$$(q - 2 - \varepsilon)T(r, f) \leq \sum_{j=1}^{3} \overline{N}(r, a_j, f)$$

outside some exceptional set, but this result lies much deeper.

We shall need that if $j < k$, then f_j is a small function with respect to f_k; that is,

$$T(r, f_j) = o(T(r, f_k)) \quad \text{as} \ \ r \to \infty \ \text{if} \ \ j < k. \tag{2.1}$$

Of course, this follows directly from Proposition 2, but it is also an immediate consequence of the result [see Hayman (1964, Lemma 2.6)] that if f and g are transcendental entire functions, then

$$T(r, f) = o(T(r, f \circ g)) \quad \text{as} \ \ r \to \infty.$$

Alternatively, we could use that

$$T(r, g) = o(T(r, f \circ g)) \quad \text{as} \ \ r \to \infty.$$

The latter result is an exercise in Hayman's book (1964, p. 54). For a thorough discussion of these and related result we also refer to a paper by Clunie (1970).

Proof of Proposition 1 We denote by $\overline{n}_A(r)$ the number of parameters λ in $\{z\colon 0 < |z| \leq r\}$ which satisfy (1.1) and by $\overline{n}_B(r)$ the number of those λ in $\{z\colon 0 < |z| \leq r\}$ for which there exist $i, j \in \mathbb{N}$ satisfying $0 < i < j < k + l$ and $E^i_\lambda(0) = E^j_\lambda(0)$; that is, $f_i(\lambda) = f_j(\lambda)$. We also put

$$\overline{N}_A(r) = \int_0^r \frac{\overline{n}_A(t)}{t} dt \quad \text{and} \quad \overline{N}_B(r) = \int_0^r \frac{\overline{n}_B(t)}{t} dt.$$

Then $n(r) = \overline{n}_A(r) - \overline{n}_B(r)$ and

$$N(r) = \overline{N}_A(r) - \overline{N}_B(r). \tag{2.2}$$

We apply Lemma 1 with $f = f_{k+l}$, $a_1 = 0$, $a_2 = f_k$ and $a_3 = \infty$. Note that the choice $a_2 = f_k$ is admissible by (2.1). We have $\overline{N}(r, 0, f_{k+l}) = \log r$ and $\overline{N}(r, \infty, f_{k+l}) = 0$. Noting that $\overline{N}(r, f_k, f_{k+l})$ and $\overline{N}_A(r)$ count the same points, except that 0 is counted in $\overline{N}(r, f_k, f_{k+l})$ but not in $\overline{N}_A(r)$, we see that $\overline{N}(r, f_k, f_{k+l}) = \overline{N}_A(r) + \log r$. We thus deduce from Lemma 1 that

$$T(r, f_{k+l}) \leq \overline{N}_A(r) + S(r, f_{k+l}).$$

On the other hand, the first fundamental theorem of Nevanlinna theory and (2.1) imply that

$$\begin{aligned}\overline{N}_A(r) &= \overline{N}(r, f_k, f_{k+l}) - \log r \leq T(r, f_{k+l} - f_k) + O(1) \\ &\leq T(r, f_{k+l}) + T(r, f_k) + O(1) = (1 + o(1))T(r, f_{k+l}).\end{aligned}$$

Combining the last two equations we find that

$$\overline{N}_A(r) = T(r, f_{k+l}) + S(r, f_{k+l}). \tag{2.3}$$

The first fundamental theorem also yields that

$$\begin{aligned}\overline{N}_B(r) &\leq \sum_{0<i<j<k+l} N(r, f_i, f_j) \leq \sum_{0<i<j<k+l} T(r, f_j - f_i) + O(1) \\ &\leq \sum_{0<i<j<k+l} T(r, f_j) + T(r, f_i) + O(1) = O\left(\sum_{0<j<k+l} T(r, f_j)\right)\end{aligned}$$

so that

$$\overline{N}_B(r) = o(T(r, f_{k+l})) \tag{2.4}$$

by (2.1). The conclusion now follows from (2.2)–(2.4). □

Remark The ideas used in the above proof are similar to those employed by Baker [see Baker (1960) or Hayman (1964, Section 2.8)] in his proof that a transcendental entire function has periodic points of period p for all $p \in \mathbb{N}$, with at most one exception. His conjecture that $p = 1$ is the only possible exception was proved in Bergweiler (1991).

3 Proof of Proposition 2

An exercise in Hayman's book (1964, p. 7) is to show that

$$T\left(r, e^{e^z}\right) \sim \frac{e^r}{\sqrt{2\pi^3 r}}.$$

The computations here are similar, but somewhat more involved.

The proof of Proposition 2 we give below is self-contained, but we note that using results of Hayman (1956) the proof can be shorted. More specifically, Lemmas 3 and 4 below can be replaced by a reference to results of this paper; see the remark at the end of this section.

We define

$$a_k(r) = \frac{d \log f_k(r)}{d \log r} = \frac{r f_k'(r)}{f_k(r)} \quad \text{and} \quad b_k(r) = \frac{d\ a_k(r)}{d \log r} = r a_k'(r).$$

We also put

$$F_k(z) = \prod_{j=1}^{k} f_j(z), \tag{3.1}$$

with $F_0(z) = 1$.

Lemma 2 *Let $k \geq 2$. Then*

$$a_k(r) \sim F_{k-1}(r) \quad \text{and} \quad b_k(r) \sim F_{k-1}(r) F_{k-2}(r) = f_{k-1}(r) F_{k-2}(r)^2.$$

Proof Since $z f_k'(z) = f_k(z) + f_k(z) z f_{k-1}'(z)$ we see by induction that

$$z f_k'(z) = \sum_{m=0}^{k-1} \prod_{l=0}^{m} f_{k-l}(z) = F_k(z) \sum_{j=0}^{k-1} \frac{1}{F_j(z)}.$$

Hence

$$a_k(r) = F_{k-1}(r) \sum_{j=0}^{k-1} \frac{1}{F_j(r)} \sim F_{k-1}(r)$$

as claimed. The asymptotics for $b_k(r)$ follow from this by a straightforward calculation. □

By $\log f_k$ we denote the branch of the logarithm which is real on the positive real axis.

Lemma 3 *Let $k \geq 2$ and $r \geq 1$. Then*

$$\log f_k(r e^{\tau}) = \log f_k(r) + a_k(r)\tau + \frac{1}{2} b_k(r)\tau^2 + R(\tau) \tag{3.2}$$

where

$$|R(\tau)| \leq 6 \cdot 3^{3(k-1)} F_{k-1}(r) F_{k-2}(r)^2 |\tau|^3 \quad \textit{for} \quad |\tau| \leq \frac{1}{2 \cdot 3^{k-1} F_{k-2}(r)}. \tag{3.3}$$

Proof We first show by induction that if $j \in \mathbb{N}$ and $r \geq 1$, then

$$f_j(re^t) \leq (1 + 3^j F_{j-1}(r) t) f_j(r) \leq 2 f_j(r) \quad \text{for} \quad t \leq \frac{1}{3^j F_{j-1}(r)}. \tag{3.4}$$

This is clear for $j = 1$ in which case this just says that

$$re^t \leq (1 + 3t) r \leq 2r \quad \text{for} \quad t \leq \frac{1}{3}.$$

Assuming that (3.4) holds, we find that if $t \leq 1/(3^{j+1} F_j(r))$ and $r \geq 1$, then also $t \leq 1/(3^j F_{j-1}(r))$ and thus

$$\begin{aligned} f_{j+1}(re^t) &= re^t \exp f_j(re^t) \leq re^t \exp\Big((1 + 3^j F_{j-1}(r) t) f_j(r)\Big) \\ &= re^t \exp\Big(f_j(r) + 3^j F_j(r) t\Big) = f_{j+1}(r) \exp\Big((1 + 3^j F_j(r)) t\Big) \\ &\leq f_{j+1}(r) \exp\Big(2 \cdot 3^j F_j(r) t\Big) \leq f_{j+1}(r) \Big(1 + 3^{j+1} F_j(r) t\Big). \end{aligned}$$

This proves (3.4).

We put

$$h(\tau) = \log f_k(re^\tau) = \log r + \tau + f_{k-1}(re^\tau).$$

Noting that (3.2) is nothing else than the Taylor expansion of h with remainder $R(\tau)$ we deduce that (see, e.g., Ahlfors 1966, p. 126)

$$R(\tau) = \frac{\tau^3}{2\pi i} \int_{|w|=s} \frac{h(w)}{w^3 (w - \tau)} dw$$

if $s > |\tau|$. With $s = 1/(3^{k-1} F_{k-2}(r))$ we find that if $|\tau| \leq s/2$, then

$$\begin{aligned} |R(\tau)| &\leq \frac{2|\tau|^3}{s^3} \max_{|w|=s} |h(w)| \leq \frac{2|\tau|^3}{s^3} (\log r + s + f_{k-1}(re^s)) \\ &\leq \frac{2|\tau|^3}{s^3} (\log r + s + 2 f_{k-1}(r)) \leq \frac{6|\tau|^3}{s^3} f_{k-1}(r) \\ &= 6 \cdot 3^{3(k-1)} F_{k-1}(r) F_{k-2}(r)^2 |\tau|^3. \end{aligned}$$

This is (3.3). □

We have restricted to $k \geq 2$ in Lemma 3, but we note that (3.2) trivially holds for $k = 1$ with $a_1(r) = 1$, $b_1(r) = 0$ and $R(\tau) = 0$.

We will actually use Lemma 3 not for the computation of $T(r, f_k)$, but for that of

$$T(r, f_{k+1}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |f_{k+1}(re^{i\theta})| d\theta. \tag{3.5}$$

Here $\log^+ x = \max\{\log x, 0\}$. The notation $h^+(x) = \max\{h(x), 0\}$ will also be used for other functions h in the sequel.

We will split the integral in (3.5) into two parts by considering the ranges $|\theta| \le \delta(r)$ and $\delta(r) \le |\theta| \le \pi$ separately, for a suitably chosen function $\delta(r)$. It will be convenient to choose

$$\delta(r) = \frac{1}{F_{k-1}(r)^{2/5}}.$$

Then Lemma 3 can be applied for $|\theta| \le \delta(r)$, with an error term $R(i\theta)$ satisfying $R(i\theta) = o(1)$.

To deal with the range $\delta(r) \le |\theta| \le \pi$ we will use the following lemma.

Lemma 4 *If $k \ge 2$, $\delta(r) \le |\theta| \le \pi$ and r is sufficiently large, then*

$$\log |f_{k+1}(re^{i\theta})| \le \frac{f_k(r)}{f_{k-1}(r)}.$$

Proof Put $g_1(\theta) = r\cos\theta$ and $g_j(\theta) = r\exp g_{j-1}(\theta)$ for $j \ge 2$. Noting that $g_2(\theta) = re^{r\cos\theta} = |f_2(re^{i\theta})|$ and

$$|f_j(re^{i\theta})| = r\exp \mathrm{Re}(f_{j-1}(re^{i\theta})) \le r\exp |f_{j-1}(re^{i\theta})|$$

for $j \ge 3$ we see by induction that

$$|f_j(re^{i\theta})| \le g_j(\theta) \tag{3.6}$$

for all $j \ge 2$.

Since $\cos\theta \le 1 - \theta^2/4$ for $|\theta| \le 1$ we have

$$\begin{aligned} g_2(\theta) = re^{r\cos\theta} &\le re^r \exp\left(-r\frac{\theta^2}{4}\right) \\ &= f_2(r)\exp\left(-\frac{F_1(r)}{4}\theta^2\right) \quad \text{for } |\theta| \le 1. \end{aligned} \tag{3.7}$$

We shall show by induction that if $j \ge 2$ and $r \ge 1$, then

$$g_j(\theta) \le f_j(r)\exp\left(-\frac{F_{j-1}(r)}{2^j}\theta^2\right) \quad \text{for } |\theta| \le \frac{1}{\sqrt{F_{j-2}(r)}}. \tag{3.8}$$

Note that (3.7) says that this holds for $j = 2$. Suppose now that $j \ge 2$ and that (3.8) holds. Let $|\theta| \le 1/\sqrt{F_{j-1}(r)}$. Then $|\theta| \le 1/\sqrt{F_{j-2}(r)}$ since $r \ge 1$. Noting that $e^{-x} \le 1 - x/2$ for $0 \le x \le 1$ we obtain

$$g_{j+1}(\theta) = r \exp g_j(\theta) \le r \exp\left(f_j(r) \exp\left(-\frac{F_{j-1}(r)}{2^j}\theta^2 \right)\right)$$
$$\le r \exp\left(f_j(r) \left(1 - \frac{F_{j-1}(r)}{2^{j+1}}\theta^2 \right)\right) = f_{j+1}(r) \exp\left(-\frac{F_j(r)}{2^{j+1}}\theta^2 \right).$$

Hence (3.8) holds for all $j \ge 2$.

Suppose now that $\delta(r) \le |\theta| \le \pi$. Then

$$\log |f_{k+1}(re^{i\theta})| \le \log g_{k+1}(\theta) \le \log g_{k+1}(\delta(r)) = g_k(\delta(r)) + \log r$$

by (3.6). Since $\delta(r) = 1/F_{k-1}(r)^{2/5} \le 1/\sqrt{F_{k-2}(r)}$ for large r we deduce from the last inequality and (3.8) that

$$\log |f_{k+1}(re^{i\theta})| \le f_k(r) \exp\left(-\frac{F_{k-1}(r)}{2^k}\delta(r)^2 \right) + \log r$$
$$= f_k(r) \exp\left(-\frac{F_{k-1}(r)^{1/5}}{2^k} \right) + \log r \le \frac{f_k(r)}{f_{k-1}(r)},$$

if r is sufficiently large. □

Lemma 5

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} e^{-x^2} \cos^+(tx)dx = \frac{1}{\sqrt{\pi}}.$$

Proof Integration by parts yields

$$\int_{-\infty}^{\infty} e^{-x^2} \cos^+(tx)dx = \int_{-\infty}^{\infty} e^{-x^2} 2x \int_0^x \cos^+(ty)dy\, dx.$$

Since

$$\int_0^x \cos^+(ty)dy \sim \frac{x}{\pi} \quad \text{as } t \to \infty,$$

locally uniformly in $\mathbb{R}\backslash\{0\}$, we obtain

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} e^{-x^2} \cos^+(tx)dx = \frac{2}{\pi} \int_{-\infty}^{\infty} e^{-x^2} x^2 dx = \frac{1}{\sqrt{\pi}}$$

as claimed. □

Proof of Proposition 2 It follows from Lemma 3 that

$$f_k(re^{i\theta}) = f_k(r) \exp\left(i a_k(r)\theta - \tfrac{1}{2} b_k(r)\theta^2 \right) (1 + S(\theta)) \quad \text{for } |\theta| \le \delta(r), \tag{3.9}$$

where

$$|S(\theta)| = \left| e^{R(i\theta)} - 1 \right| \le 2|R(i\theta)|$$
$$\le 12 \cdot 3^{3(k-1)} F_{k-1}(r) F_{k-2}(r)^2 \delta(r)^3 = 12 \cdot 3^{3(k-1)} \frac{F_{k-2}(r)^2}{F_{k-1}(r)^{1/5}}$$

for large r and hence $S(\theta) = o(1)$ as $r \to \infty$. This implies that

$$\mathrm{Re}\big(f_k(re^{i\theta})\big) = f_k(r)\exp\big(-\tfrac{1}{2}b_k(r)\theta^2\big)\cos(a_k(r)\theta) + o\big(f_k(r)\exp\big(-\tfrac{1}{2}b_k(r)\theta^2\big)\big)$$

and thus

$$\mathrm{Re}^+\Big(f_k(re^{i\theta})\Big) = f_k(r)\exp\Big(-\tfrac{1}{2}b_k(r)\theta^2\Big)\,\big(\cos^+(a_k(r)\theta) + o(1)\big) \quad \text{for } |\theta| \le \delta(r),$$

where the term $o(1)$ is uniform in θ.

We conclude that

$$\begin{aligned}
&\int_{-\delta(r)}^{\delta(r)} \log^+ |f_{k+1}(re^{i\theta})|d\theta \\
&= f_k(r)\int_{-\delta(r)}^{\delta(r)} \exp\Big(-\tfrac{1}{2}b_k(r)\theta^2\Big)\,\big(\cos^+(a_k(r)\theta) + o(1)\big)\,d\theta \\
&= \frac{\sqrt{2}f_k(r)}{\sqrt{b_k(r)}}\int_{-c(r)}^{c(r)} \exp\Big(-u^2\Big)\left(\cos^+\left(\frac{\sqrt{2}a_k(r)}{\sqrt{b_k(r)}}u\right) + o(1)\right)du
\end{aligned}$$

with

$$c(r) = \frac{\sqrt{b_k(r)}\delta(r)}{\sqrt{2}} = (1+o(1))\frac{F_{k-1}(r)^{1/10}\sqrt{F_{k-2}(r)}}{\sqrt{2}} \to \infty \tag{3.10}$$

by Lemma 2. The same lemma yields that

$$\frac{a_k(r)}{\sqrt{b_k(r)}} = (1+o(1))\frac{\sqrt{F_{k-1}(r)}}{\sqrt{F_{k-2}(r)}} = (1+o(1))\sqrt{f_{k-1}(r)} \to \infty.$$

Lemma 5 now implies that

$$\int_{-\delta(r)}^{\delta(r)} \log^+ |f_{k+1}(re^{i\theta})|d\theta \sim \frac{\sqrt{2}f_k(r)}{\sqrt{\pi b_k(r)}}. \tag{3.11}$$

Since

$$\log^+ |f_{k+1}(re^{i\theta})| \le \frac{f_k(r)}{f_{k-1}(r)} = o\left(\frac{f_k(r)}{\sqrt{b_k(r)}}\right) \quad \text{for } \delta(r) \le |\theta| \le \pi$$

by Lemmas 4 and 2 we conclude that

$$\int_{-\pi}^{\pi} \log^+ |f_{k+1}(re^{i\theta})|d\theta \sim \frac{\sqrt{2}f_k(r)}{\sqrt{\pi b_k(r)}}.$$

Thus

$$
\begin{aligned}
T(r, f_{k+1}) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f_{k+1}(re^{i\theta})| d\theta \\
&\sim \frac{f_k(r)}{\sqrt{2\pi^3 b_k(r)}} \sim \frac{f_k(r)}{\sqrt{2\pi^3}\sqrt{f_{k-1}(r) F_{k-2}(r)}}
\end{aligned}
$$

by Lemma 2. The conclusion follows with $k = m - 1$. □

Remark An entire function f is called *admissible* in the sense of Hayman (1956) if $f(r) = M(r, f)$ for large r and if with

$$
a(r) = \frac{d \log M(r, f)}{d \log r} = \frac{r f'(r)}{f(r)} \quad \text{and} \quad b(r) = \frac{d\ a(r)}{d \log r} = r a'(r) \tag{3.12}
$$

there exists $\delta(r) \in (0, \pi]$ such that, as $r \to \infty$,

$$
f(re^{i\theta}) \sim f(r) \exp\Big(i a(r)\theta - \tfrac{1}{2} b(r)\theta^2 \Big) \quad \text{for } |\theta| \leq \delta(r) \tag{3.13}
$$

and

$$
f(re^{i\theta}) = \frac{o(f(r))}{\sqrt{b(r)}} \quad \text{for } \delta(r) \leq |\theta| \leq \pi. \tag{3.14}
$$

Moreover, it is assumed that $b(r) \to \infty$ as $r \to \infty$.

Hayman (1956, Theorems VI and VIII) showed that if f is admissible, then so are e^f and fP for any real polynomial P with positive leading coefficient. This implies that f_k is admissible for $k \geq 2$.

The admissibility of f_k immediately yields slightly weaker versions of Lemmas 3 and 4, but these versions are strong enough to prove Proposition 2. In fact, the arguments used in the above proof yield the following Proposition 3. Since its proof is largely analogous to that of Proposition 2, replacing Lemmas 3 and 4 by a reference to (3.13) and (3.14), we will only sketch the proof.

Proposition 3 *Let f be an admissible entire function and let $b(r)$ be defined by* (3.12). *Then*

$$
T(r, e^f) \sim \frac{1}{\sqrt{2\pi^3}} \frac{f(r)}{\sqrt{b(r)}}
$$

Sketch of proof First we note that (3.13) means that (3.9) holds with f_k replaced by f and $S(\theta) = o(1)$ for $|\theta| \leq \delta(r)$. We proceed as in the proof of Proposition 2. To see that $c(r) = \delta(r)\sqrt{b(r)/2} \to \infty$ as in (3.10) we note that we may choose $\theta = \delta(r)$ in both (3.13) and (3.14). This yields

$$
f(r) \exp\Big(-\tfrac{1}{2} b(r) \delta(r)^2 \Big) = o\left(\frac{f(r)}{\sqrt{b(r)}} \right)
$$

and hence $\exp\left(-\frac{1}{2}b(r)\delta(r)^2\right) = o(1)$, from which we deduce that $c(r) \to \infty$. We conclude that (3.11) holds with f_k replaced by f and f_{k+1} replaced by e^f; that is,

$$\int_{-\delta(r)}^{\delta(r)} \log^+ |e^{f(re^{i\theta})}| d\theta \sim \frac{\sqrt{2} f(r)}{\sqrt{\pi b(r)}}.$$

Moreover,

$$\log^+ |e^{f(re^{i\theta})}| \le |f(re^{i\theta})| = o\left(\frac{f(r)}{\sqrt{b(r)}}\right) \quad \text{for } \delta(r) \le |\theta| \le \pi$$

by (3.14). The conclusion follows directly from the last two equations. □

We note that Proposition 2 is an immediate consequence of Proposition 3.

4 Proof of the Theorem

A classical growth lemma of Borel [see Goldberg (2008, p. 90) or Hayman (1964, Lemma 2.4)] says that if $\phi\colon [r_0, \infty) \to (0, \infty)$ is a continuous, increasing function, then there exists a subset E of $[r_0, \infty)$ of finite measure such that

$$\phi\left(1 + \frac{1}{\phi(r)}\right) \le 2\phi(r) \quad \text{for } r \notin E.$$

The exceptional set in Nevanlinna's second fundamental theorem and thus the exceptional set E in Proposition 1 arise from the application of this lemma to the Nevanlinna characteristic.

If the function ϕ is sufficiently "regular", then the inequality in Borel's lemma holds for all large r. In fact, boundedness of the exceptional set E in Borel's lemma is sometimes taken as a regularity condition; see, e.g., Edrei and Fuchs (1964, p. 245). The following lemma gives a simple condition implying that the exceptional set in this lemma is bounded. While I believe that this or similar results are well-known to the experts, I have not found this lemma in the literature.

Lemma 6 *Let $\phi\colon [r_0, \infty) \to (0, \infty)$ be a non-decreasing, differentiable function satisfying $\phi'(r) \le \phi(r)^{3/2}$ for all r. Then*

$$\phi\left(1 + \frac{1}{\phi(r)}\right) \sim \phi(r) \quad \text{as } r \to \infty.$$

Proof The result is trivial if $\lim_{r\to\infty} \phi(r) < \infty$. We may thus assume that $\lim_{r\to\infty} \phi(r) = \infty$. For $r \ge r_0$ we have

$$\frac{1}{\sqrt{\phi(r)}} - \frac{1}{\sqrt{\phi(r + 1/\phi(r))}} = \frac{1}{2}\int_r^{r+1/\phi(r)} \frac{\phi'(t)}{\phi(t)^{3/2}} dt \le \frac{1}{2\phi(r)}$$

and thus

$$\sqrt{\frac{\phi(r)}{\phi(r+1/\phi(r))}} \geq 1 - \frac{1}{2\sqrt{\phi(r)}},$$

from which the conclusion follows. □

A straightforward calculation shows that the right hand side of (1.4) satisfies the hypothesis—and thus the conclusion—of Lemma 6. From this it is not difficult to deduce that the exceptional set in Nevanlinna's second fundamental theorem and in Lemma 1 is bounded for $f = f_m$. This implies that no exceptional set E is required in Proposition 1. Combining this with Proposition 2 we find that under the hypotheses of Proposition 1 we have

$$N(r) \sim T(r, f_{k+l}) \sim \frac{1}{\sqrt{2\pi^3}} \frac{f_{k+l-1}(r)}{\sqrt{f_{k+l-2}(r)F_{k+l-3}(r)}} \quad \text{as } r \to \infty, \tag{4.1}$$

with $F_{k+l-3}(r)$ defined by (3.1).

To obtain a result for $n(r)$ we use the following result of London (1975/1976, p. 502).

Lemma 7 *Let $\phi, \psi : [x_0, \infty) \to (0, \infty)$ be functions satisfying*

$$\phi(x) \sim \psi(x) \quad as \ \ x \to \infty. \tag{4.2}$$

Suppose that ψ is convex and that ϕ is twice continuously differentiable, with ϕ' and ϕ'' positive and ϕ' unbounded. Suppose also that there exists a constant β such that

$$\frac{\phi''(x)\phi(x)}{\phi'(x)^2} \leq \beta \tag{4.3}$$

for all $x \geq x_0$. Then

$$\phi'(x) \sim \psi'(x) \quad as \ \ x \to \infty. \tag{4.4}$$

Here ψ' denotes either the left or the right derivative of ψ on the countable set for which these may be different.

Note that l'Hospital's rule says that (4.4) implies (4.2). Lemma 7 may be considered as a reversal of l'Hospital's rule. For this an additional hypothesis such as (4.3) is essential.

Proof of the theorem We denote the right hand side of (4.1) by $g(r)$. Since $N(r)$ is convex in $\log r$ we see that $\psi(x) = N(e^x)$ is convex in x. It is easy to see that $\phi(x) = g(e^x)$ satisfies the hypothesis of Lemma 7. In fact, it is not difficult to see that $\phi''(x)\phi(x)/\phi'(x)^2 \to 1$ as $x \to \infty$. We thus deduce from Lemma 7 that $\phi'(x) \sim \psi'(x)$ and hence that $n(r) \sim rg'(r)$. From this the conclusion follows easily using Lemma 2. □

Acknowledgements The results of this paper (except for Proposition 3) were presented in two talks in John H. Hubbard's seminar at Cornell University in the fall of 1988. They were inspired by a talk by Ben Bielefeld in this seminar about the computation of Misiurewicz parameters using the spider algorithm (Bielefeld et al. 1992; Hubbard and Schleicher 1994). The proof of Proposition 2 given below is a simplified version of the one presented in the seminar. John Hubbard's seminar was my first encounter with complex dynamics. (The purpose of my stay at Cornell University was to visit Wolfgang H. J. Fuchs, a leading figure in Nevanlinna theory; see Anderson et al. (1998) for Fuchs' life and work.) I would like to take this opportunity—albeit very belatedly—to thank John Hubbard and the participants of his seminar for igniting my interest in complex dynamics and for helpful discussions. I thank Dierk Schleicher for encouraging me to make the results of my talks in this seminar available—and I also thank him, Saikat Batabyal and the referee for useful comments on this manuscript. Finally, I remain grateful to the Alexander von Humboldt Foundation for making my stay at Cornell University possible by granting me a Feodor Lynen research fellowship.

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