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RESEARCH CONTRIBUTION

Cyclohedron and Kantorovich–Rubinstein Polytopes

Filip D. Jevtić[1,3] · **Marija Jelić**[2] ·
Rade T. Živaljević[3]

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Abstract We show that the cyclohedron (Bott–Taubes polytope) W_n arises as the polar dual of a Kantorovich–Rubinstein polytope $KR(\rho)$, where ρ is an explicitly described quasi-metric (asymmetric distance function) satisfying strict triangle inequality. From a broader perspective, this phenomenon illustrates the relationship between a nestohedron $\Delta_{\widehat{\mathcal{F}}}$ (associated to a building set $\widehat{\mathcal{F}}$) and its non-simple deformation $\Delta_{\mathcal{F}}$, where $\mathcal{F}$ is an *irredundant* or *tight basis* of $\widehat{\mathcal{F}}$ (Definition 21). Among the consequences are a new proof of a recent result of Gordon and Petrov (Arnold Math. J. 3(2):205–218, 2017) about f-vectors of generic Kantorovich–Rubinstein polytopes and an extension of a theorem of Gelfand, Graev, and Postnikov, about triangulations of the type A, positive root polytopes.

Keywords Kantorovich-Rubinstein polytopes · Lipschitz polytope · Cyclohedron · Nestohedron · Unimodular triangulations · Metric spaces

1 Introduction

Motivated by the classic Kantorovich–Rubinstein theorem, Vershik (2015) described a canonical correspondence between finite metric spaces (X, ρ) and convex polytopes

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✉ Rade T. Živaljević
rade@mi.sanu.ac.rs

1 Department of Mathematical Sciences, University of Texas at Dallas, Richardson, USA

2 Faculty of Mathematics, University of Belgrade, Belgrade, Serbia

3 Mathematical Institute SASA, Belgrade, Serbia

in the vector space $V_0(X) \subset \mathbb{R}^X$ of all signed measures on X with total mass equal to 0. More explicitly, each finite metric space (X, ρ) is associated a *fundamental polytope* $KR(\rho)$ (Kantorovich–Rubinstein polytope) spanned by $e_{x,y} = \frac{e_x - e_y}{\rho(x,y)}$ where $\{e_x\}_{x\in X}$ is the canonical basis in $\mathbb{R}^X$.

Kantorovich–Rubinstein polytope $KR(\rho)$ can be also described as the dual of the *Lipschitz polytope* $Lip(\rho)$ where,

$$Lip(\rho) = \left\{ f \in \mathbb{R}^X \mid (\forall x, y \in X)\; f(x) - f(y) \leqslant \rho(x, y) \right\}, \tag{1}$$

and two functions $f, g \in \mathbb{R}^X$ are considered equal if they differ by a constant.

Vershik (2015) raised a general problem of studying (classifying) finite metric spaces according to the combinatorics of their fundamental polytopes.

Gordon and Petrov (2017) in a recent paper proved a very interesting result that the f-vector of the Kantorovich–Rubinstein polytope $KR(\rho)$ is one and the same for all sufficiently generic metrics on X. They obtained this result as a byproduct of a careful combinatorial analysis of face posets of Lipschitz polytopes. The invariance of the f-vector of $KR(\rho)$ can be also deduced from the fact that the *type A root polytope* $Root_n := \mathrm{Conv}(\mathcal{A}_n)$, where $\mathcal{A}_n = \{e_i - e_j \mid 1 \leqslant i \neq j \leqslant n\}$, is unimodular in the sense of De Loera et al. (2010, Definition 6.2.10) (see also the outline in Sect. 2.3).

Our point of departure was an experimentally observed fact that the *generic f-vector* of Gordon and Petrov coincides with the f-vector of (the dual of) the cyclohedron (Bott–Taubes polytope) W_n. At first sight this is an unexpected phenomenon since W_n° itself is not centrally symmetric and therefore cannot arise as a Kantorovich–Rubinstein polytope $KR(\rho)$ (unless ρ is allowed to be a quasi-metric!).

The symmetry of a metric is a standard assumption in the usual formulations of the Kantorovich–Rubinstein theorem, see for example Villani (2003, Section 1.2). However this condition is not necessary. (The proof of this fact is implicit in Villani (2009, Section 5), see *Particular Case 5.4.* on page 68). More importantly the 'radial vertex perturbation' (Sect. 2.2) of a metric may affect its symmetry, so the extension of $KR(\rho)$ to quasi-metrics may be justified both by the 'optimal transport' and the 'convex polytopes' point of view.

We prove two closely related results which both provide explanations why the cyclohedron W_n (and its dual polytope W_n°) appear in the context of generic Kantorovich–Rubinstein polytopes and triangulations of the type A root polytope.

In the first result (Theorem 14), we construct a map $\phi_n : W_n^\circ \to Root_n$ which is simplicial on the boundary $\partial(W_n^\circ)$ and maps bijectively $\partial(W_n^\circ)$ to the boundary $\partial(Root_n)$ of the root polytope. (In particular we obtain a triangulation of $\partial(Root_n)$ parameterized by faces of W_n.)

This construction is purely combinatorial and diagrammatic in nature. It relies on a combinatorial description of W_n as a graph associahedron (Devadoss 2003) and describes simplices in $\partial(W_n^\circ)$ as *admissible families* of intervals (arcs) in the cycle graph C_n.

Theorem 14 can also be seen as an extension of a result of Gelfand et al. (1996, Theorem 6.3) who described a coherent triangulation of the type A, positive root polytope $Root_n^+ = \mathrm{Conv}\{e_i - e_j \mid 1 \leqslant i \leqslant j \leqslant n\}$. For illustration, the *standard* *trees* depicted in Gelfand et al. (1996, Figure 6.1) may be interpreted as our *admissible families of arcs* (as exemplified in Fig. 3) where all arcs are oriented from left to right.

In the second result (Theorem 31 and Proposition 34) we prove the existence and than explicitly construct a canonical quasi-metric ρ such that the associated Kantorovich–Rubinstein polytope $KR(\rho)$ is a geometric realization of the polytope W_n°.

This result has a more geometric flavor since it relies on a *nestohedron representation* (Postnikov 2009; Feichtner and Sturmfels 2005) of the cyclohedron as the Minkowski sum $W_n = \Delta_{\widehat{\mathcal{F}}} = \sum_{F\in\widehat{\mathcal{F}}} \Delta_F$ of simplices. In this approach the relationship between the cyclohedron W_n and the dual $(Root_n)^\circ$ of the root polytope is seen as a special case of a more general construction linking a nestohedron $\Delta_{\widehat{\mathcal{F}}}$ and its Minkowski summand $\Delta_{\mathcal{F}}$, where $\widehat{\mathcal{F}}$ is a building set and $\mathcal{F}$ its *irredundant basis* (Definition 21).

In Sect. 6 we briefly outline a different plan (suggested by a referee) for constructing quasi-metrics of "cyclohedral type". This approach relies on the analysis of the combinatorial structure of Lipschitz polytopes for generic measures, as developed in Gordon and Petrov (2017).

In 'Concluding remarks' (Sect. 7) we discuss the significance of Theorems 14 and 31. For example we demonstrate (Sect. 7.1) how the motivating result of Gordon and Petrov (2017, Theorem 1) can be deduced from the known results about the f-vectors of cyclohedra. We also offer a glimpse into potentially interesting future developments including the study of 'tight pairs' $(\widehat{\mathcal{F}}, \mathcal{F})$ of hypergraphs (Sect. 7.2) and the 'canonical quasitoric manifolds' associated to *combinatorial quasitoric pairs* (W_n, ϕ_n) (Sect. 7.3).

2 Preliminaries

2.1 Kantorovich–Rubinstein Polytopes

Let (X, ρ), $|X| = n$, be a finite metric space and let $V(X) := \mathbb{R}^X \cong \mathbb{R}^n$ be the associated vector space of real valued functions (weight distributions, signed measures) on X. In particular, $V_0(X) := \{\mu \in V(X) \mid \mu(X) = 0\}$ is the vector subspace of measures with total mass equal to zero, while $\Delta_X := \{\mu \in V(X) \mid \mu(X) = 1 \text{ and } (\forall x \in X)\, \mu(\{x\}) \geqslant 0\}$ is the simplex of probability measures.

Let $\mathcal{T}_\rho(\mu, \nu)$ be the cost of optimal transportation of measure μ to measure ν, where the cost of transporting the unit mass from x to y is $\rho(x, y)$. Then, cf. Vershik (2013) and Villani (2003), there exists a norm $\|\cdot\|_{KR}$ on $V_0(X)$ (called the Kantorovich–Rubinstein norm), such that,

$$\mathcal{T}_\rho(\mu, \nu) = \|\mu - \nu\|_{KR},$$

for each pair of probability measures $\mu, \nu \in \Delta_X$. By definition, the Kantorovich–Rubinstein polytope $KR(\rho)$, or the *fundamental polytope* (Vershik 2015), associated to (X, ρ), is the corresponding unit ball in $V_0(X)$,

$$KR(\rho) = \{x \in V_0(X) \mid \|x\|_{KR} \leqslant 1\}. \tag{2}$$

The following explicit description for $KR(\rho)$ can be deduced from the Kantorovich–Rubinstein theorem (Theorem 1.14 in Villani 2003),

$$KR(\rho) = \text{Conv}\left\{\frac{e_x - e_y}{\rho(x, y)} \mid x \in X\right\}, \tag{3}$$

where $\{e_x\}_{x \in X}$ is the canonical basis in $\mathbb{R}^X$.

Problem 1 (Vershik 2015) *Study and classify metric spaces according to combinatorial properties of their Kantorovich–Rubinstein polytopes.*

2.2 Root Polytopes

The convex hull of the roots of a classical root system is called a root polytope. In particular the *type A root polytope*, associated to the root system of type A_{n-1}, is the following polytope (Fig. 1),

$$Root_n = \text{Conv}\{e_i - e_j \mid 1 \leqslant i \neq j \leqslant n\}. \tag{4}$$

It immediately follows from (4) that the root polytope admits the following Minkowski sum decomposition,

$$Root_n = \Delta + \nabla = \Delta + (-\Delta) = \Delta - \Delta, \tag{5}$$

where $\Delta = \Delta_n = \text{Conv}\{e_i\}_{i=1}^n$ and $\nabla = -\Delta = \text{Conv}\{-e_i\}_{i=1}^n$.

By definition, $Root_n$ is the Kantorovich–Rubinstein polytope associated to the metric ρ where $\rho(x, y) = 1$ for each $x \neq y$. Conversely, in light of (3), each Kantorovich–Rubinstein polytope can be seen as a *radial, vertex perturbation* of the root polytope $Root_n$.

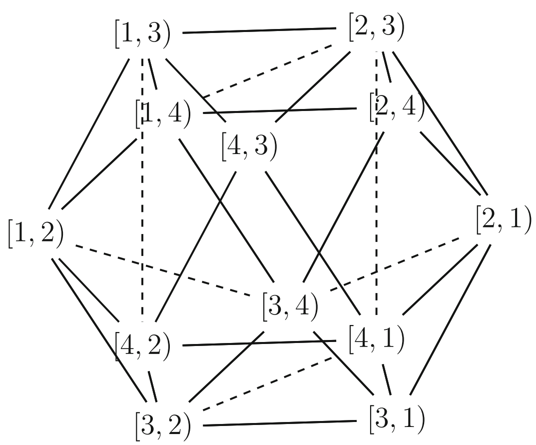


Fig. 1 The boundary triangulation of $Root_4$ described in Theorem 14

2.3 Unimodular Triangulations and Equidecomposable Polytopes

A triangulation of a convex polytope Q is tacitly assumed to be without new vertices. A triangulation of the boundary sphere $\partial(Q)$ of Q is referred to as a *boundary triangulation*. Each triangulation of Q produces the associated boundary triangulation (but not the other way around).

The f-vector of a triangulation is the f-vector of the associated simplicial complex. Different triangulations of either the polytope Q or its boundary $\partial(Q)$ may have different face numbers, so in general the f-vector of a triangulation is not uniquely determined by the polytope Q. The simplest examples illustrating this phenomenon are the bipiramid over a triangle and the 3-dimensional cube (the latter admits triangulations with both 5 and 6, three dimensional simplices).

The polytopes, all of whose triangulations have the same face numbers (f-vectors), are called *equidecomposable*, see Bayer (1993) or De Loera et al. (2010, Section 8.5.3). A notable class of equidecomposable polytopes are lattice polytopes which are *unimodular* in the sense that each full dimensional simplex spanned by its vertices has the same volume, see Definition 6.2.10 and Section 9.3 in De Loera et al. (2010). Unimodularity of a polytope immediately implies that the top dimensional face numbers are independent of a triangulation. In light of Theorem 8.5.19. from De Loera et al. (2010), this condition guarantees that the polytope is equidecomposable, i.e. that the f-vector is the same for all triangulations.

A notable example of an equidecomposable polytope is the product of two simplices, see De Loera et al. (2010, Section 6.2). As a consequence of (5), each face of the root polytope $Root_n$ is a product of two simplices. From here we immediately deduce that all boundary triangulations of $Root_n$ have the same f-vector.

Gordon and Petrov (2017) observed that each Kantorovich–Rubinstein polytope $KR(\rho)$, for a sufficiently generic metric ρ, induces a *regular boundary triangulation* of the root polytope $Root_n$. This observation allowed them to determine the f-vector of a generic K–R polytope, and to obtain some other qualitative and quantitative information about these polytopes.

Our Theorem 31 identifies this f-vector as the f-vector of the polytope W_n°, dual to the f-vector of an $(n-1)$-dimensional cyclohedron.

2.4 Kantorovich–Rubinstein Polytopes for Quasi-Metrics

Each Kantorovich–Rubinstein polytope associated to a metric ρ is centrally symmetric (as a consequence of the symmetry $\rho(x, y) = \rho(y, x)$ of the metric ρ). The cyclohedron W_n is not centrally symmetric, so it is certainly not one of the Kantorovich–Rubinstein polytopes. However, as a consequence of Theorem 31, it arises as a *generalized K–R polytope* associated to a not necessarily symmetric distance function (quasi-metric).

Definition 2 A non-negative function $\rho : X \times X \to \mathbb{R}^+$ is a *quasi-metric* (asymmetric distance function) if,

1. $(\forall x, y \in X)\, (\rho(x, y) = 0 \Leftrightarrow x = y)$;

2. $(\forall x, y, z \in X)\ \rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$.

The Kantorovich–Rubinstein polytope $KR(\rho)$, associated to a quasi-metric ρ, is defined by the same formula (3) as its symmetric counterpart.

Many basic facts remain true for generalized K–R polytopes. For illustration, here is a result which extends (with the same proof) a result of Melleray et al. (2014, Lemma 1).

Proposition 3 *Let X be a finite set. Assume that $\rho : X \times X \to \mathbb{R}_{\geqslant 0}$ is a non-negative function such that $\rho(x, y) = 0$ if and only if $x = y$. Let $KR(\rho)$ be the polytope defined by the Eq. (3). Then ρ is a quasi-metric on X if and only if none of the points $e_{x,y} = \frac{e_x - e_y}{\rho(x,y)}$ (for $x \neq y$) is in the interior of $KR(\rho)$.*

3 Preliminaries on the Cyclohedron W_n

3.1 Face Lattice of the Cyclohedron W_n

The face lattice $\mathcal{F}(W_n)$ of the $(n-1)$-dimensional *cyclohedron* W_n (Bott–Taubes polytope) admits two closely related combinatorial description.

In the first description Stasheff (1997), similar to the description of the $(n-1)$-dimensional associahedron K_n (Stasheff polytope), the lattice $\mathcal{F}(W_n)$ arises as the collection of all partial cyclic bracketing of a word $x_1 x_2 \cdots x_n$.

Carr and Devadoss (2006), in a more general approach, view both polytopes K_n and W_n as instances of the so called *graph associahedra*. In this approach, W_n is described as the graph associahedron $\mathcal{P}\Gamma$ corresponding to the graph $\Gamma = C_n$ (cycle on n vertices), where $\mathcal{F}(W_n)$ is the collection of all *valid tubings* on Γ, see Carr and Devadoss (2006), Devadoss (2003) for details.

The equivalence of the 'bracketing' and 'tubing' description is easily established, see for example Carr and Devadoss (2006, Lemma 2.3) or Markl (1999, Lemma 1.4). Recall that 'graph associahedra' are a specialization of *nestohedra*, see Feichtner and Sturmfels (2005), Postnikov (2009), or Buchstaber and Panov (2015, Section 1.5). In this more general setting, the 'valid tubings' appear under the name of 'nested sets' associated to a chosen 'building set'. A related class of polytopes was studied from a somewhat different perspective by Došen and Petrić (2011).

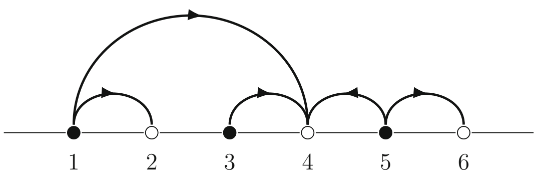


Fig. 2 Admissible family of oriented arcs

We use in this paper a slightly modified description of the poset $\mathcal{F}(W_n)$ which allows us to use pictorial description of 'valid tubings' (partial bracketings), see Fig. 2. A similar description was used by Gelfand et al. (1996), where these pictorial

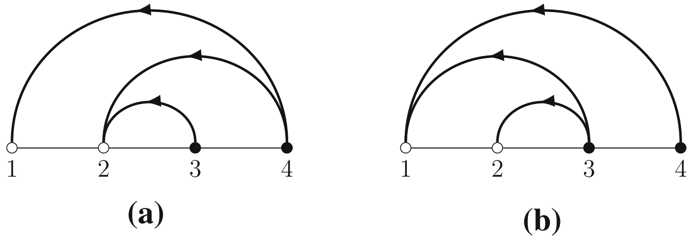


Fig. 3 Two facets of W_4° (vertices of W_4): **a** $\mathcal{T}_1 = \{[3, 2), [4, 2), [4, 1)\}$ **b** $\mathcal{T}_2 = \{[3, 2), [3, 1), [4, 1)\}$

representations appeared in the form of the so called 'standard trees', see Gelfand et al. (1996, Section 6).

The vertex set of the cycle graph C_n is the set $V(C_n) = [n] := \{1, 2, \ldots, n-1, n\}$ of vertices of a regular n-gon, inscribed in a unit circle S^1. We adopt a (counterclockwise) circular order $\prec$ (respectively $\preceq$) on the circle S^1, so in particular $[x, y] = \{z \in S^1 \mid x \preceq z \preceq y\}$ is a closed arc (interval) in S^1 (similarly $[x, y) = \{z \in S^1 \mid x \preceq z \prec y\}$, etc.). (By convention, $[x, x] = \{x\}$ and $[x, x) = \emptyset$.)

By definition $[x, y]^0 := [x, y] \cap [n]$ (similarly, $[x, y)^0 := [x, y) \cap [n]$) are intervals restricted to the set $[n]$ of vertices of the n-gon. If $i \neq j$ are two distinct vertices (elements of $V(C_n) = [n]$), then $[i, j-1]^0$ is precisely the tube (in the sense of Carr and Devadoss 2006) associated to the interval $[i, j)$. Conversely, each tube $[u, v]^0$ (a proper connected component in the graph C_n) is associated a half-open interval $[u, v+1)$ in the circle S^1. A moment's reflection reveals that each valid tubing (in the sense of Carr and Devadoss 2006) corresponds to an admissible family of half-open intervals, in the sense of the following definition.

Definition 4 A collection $\mathcal{T} = \{I_1, I_2, \ldots, I_k\}$ of half-open intervals $I_j = [a_j, b_j)$ (where $\{a_j, b_j\} \subset [n]$ and $a_j \neq b_j$ for each j) is *admissible* if for each $1 \leqslant i < j \leqslant k$, exactly one of the following two alternatives is true,

1. If $I_i \cap I_j \neq \emptyset$ then either $I_i \subsetneq I_j$ or $I_j \subsetneq I_i$;
2. $I_i \cap I_j = \emptyset$ and $I_i \cup I_j$ is not an interval (meaning that neither $b_j = a_i$ nor $b_i = a_j$).

Proposition 5 *The face lattice $\mathcal{F}(W_n)$ of the $(n-1)$-dimensional* cyclohedron W_n *is isomorphic to the poset of all admissible collections $\mathcal{T} = \{I_j\}_{j=1}^k$ of half-open intervals in S^1 with endpoints in $[n]$. Individual arcs (half-open intervals) correspond to facets of W_n while the empty set is associated to the polytope W_n itself.*

Remark 6 The dual W_n° of the cyclohedron is a simplicial polytope. It follows from Proposition 5 that the face poset of the boundary $\partial(W_n^\circ)$ of W_n° is isomorphic to the simplicial complex with vertices $V = \{[i, j) \mid 1 \leqslant i \neq j \leqslant n\}$ (all half-open intervals with endpoints in $[n]$) where $\mathcal{T} \subset V$ is a simplex if and only if $\mathcal{T}$ is an admissible family of half-open intervals (Definition 4).

The following proposition shows that admissible families (in the sense of Definition 4) can be naturally interpreted as directed trees (directed forests)

Proposition 7 *Each admissible family $\mathcal{T} = \{[i_\nu, j_\nu)\}_{\nu=1}^k$ of arcs can be interpreted as a digraph, where $[i_\nu, j_\nu) \in \mathcal{T}$ defines an oriented edge from i_ν to j_ν (as in Figs. 2,* 3). *If the orientation of arcs is neglected, than we obtain an ordinary graph* $\Gamma_{\mathcal{T}}$. *We claim that for each admissible family* $\mathcal{T}$, *the associated graph* $\Gamma_{\mathcal{T}}$ *has no cycles.*

Proof Indeed, suppose that $[u_1, v_1), (v_1, u_2], [u_2, v_2), \ldots, [u_s, v_s), (v_s, u_1]$ is a minimal cycle in $\Gamma_{\mathcal{T}}$. We may assume without loss of generality that $u_1 \prec v_1 \prec u_2$ (in the counterclockwise circular order on S^1). From here we deduce that remaining indices also follow the circular order,

$$u_1 \prec v_1 \prec u_2 \prec v_2 \prec \cdots \prec u_s \prec v_s \prec u_1, \tag{6}$$

otherwise two different arcs would cross (which would violate the assumption that $\mathcal{T}$ is admissible). Moreover, for the same reason, the sequence (6) winds around the circle S^1 only once. This however leads to a contradiction since the intervals $I = (v_1, u_2]$ and $J = (v_2, u_3]$ would have a non-empty intersection, while neither $I \subset J$ nor $J \subset I$ (a contradiction with Definition 4). □

Definition 8 Let $a = [i, j)$ be a half-open circle interval. By definition $s(a) = i$ is the source of a and $t(a) = j$ is the sink or the terminal point of a. For an admissible family $\mathcal{T} = \{[i_\nu, j_\nu) \mid \nu \in [k]\}$ of intervals (arcs), the associated source and sink sets are,

$$s(\mathcal{T}) = \{i_\nu \mid \nu \in [k]\} \quad \text{and} \quad t(\mathcal{T}) = \{j_\nu \mid \nu \in [k]\}\,.$$

Note that, as a consequence of Definition 4, $s(\mathcal{T}) \cap t(\mathcal{T}) = \emptyset$ for each admissible family $\mathcal{T}$ of intervals.

3.2 Automorphism Group of the Cyclohedron

Each automorphism of a graph Γ induces an automorphism of the associated graph associahedron $\mathcal{P}_\Gamma$. The group of all automorphisms of the cycle graph C_n is the dihedral group D_{2n} of order $2n$. It immediately follows that both the $(n-1)$-dimensional cyclohedron W_n and its polar polytope W_n° are invariant under the action of the dihedral group D_{2n}.

Let $D_{2n} \cong \langle \alpha, \beta \,|\, \alpha^n = \beta^2 = e, \beta\alpha\beta = \alpha^{n-1} \rangle$ be a standard presentation of the group D_{2n} where α is the cyclic permutation of C_n (corresponding to the rotation of the regular polygon through the angle $2\pi/n$) and β is the involution (reflection) which keeps the vertex n fixed.

Then the action of D_{2n} on W_n and W_n° can be described as follows.

Proposition 9 *Suppose that* C_n *is the cycle graph, realized as a regular polygon inscribed in the unit circle. Let* $[i, j)$ *be a half-open interval representing a vertex (face) of the polytope* W_n° *(respectively polytope* W_n*). Then* $\alpha([i, j)) := [i+1, j+1)$ *and* $\beta([i, j)) := [n-j+1, n-i+1)$.

3.3 Canonical Map ϕ_n

The associahedron K_n may be described as the *secondary polytope* Gelfand et al. (1994), associated to all subdivisions of a convex $(n+2)$-gon by configurations of non-crossing diagonals. It was shown by Simion (2003) that a similar description exists for W_n, provided we deal only with centrally symmetric configurations. The polytopes K_n and W_n are sometimes referred to as the *type A* and *type B* associahedra. This classification emphasizes a connection with type A or B root systems, the corresponding hyperplane arrangements etc. In this section we relate W_n to the root system of type A_{n-1}, in other words W_n may also be interpreted as a 'type A associahedron'.

Let $\{e_i\}_{i=1}^n$ be the standard basis in $\mathbb{R}^n$ and let $\mathcal{A}_n = \{e_i - e_j\}_{1\leqslant i\neq j\leqslant n}$ be the associated root system of type A_{n-1}. The type A root polytope is introduced in Sect. 2.2 as the convex hull $Root_n = \mathrm{conv}\{e_i - e_j \mid 1 \leq i \neq j \leq n\}$ of the set $\mathcal{A}_n$ of all roots. (We warn the reader that this terminology may not be uniform, for example the root polytopes introduced in Gelfand et al. (1996), Postnikov (2009) deal only with the set $\mathcal{A}_n^+ = \{e_i - e_j\}_{1\leqslant i<j\leqslant n}$ of positive roots.)

The following definition introduces a canonical map which links the (dual of the) cyclohedron W_n to the root system of type A_{n-1}, via the root polytope $Root_n$. Recall (Proposition 5) that the boundary $\partial(W_n^\circ)$ of the polytope dual to the cyclohedron is the simplicial complex of all admissible half-open intervals in S^1 with endpoints in $[n]$.

Definition 10 The map,

$$\phi_n : \partial(W_n^\circ) \longrightarrow \partial(Root_n) \tag{7}$$

is defined as the simplicial (affine) extension of the map which sends the interval $[i, j)$ (vertex of W_n°) to $\phi_n([i, j)) := e_i - e_j \in \mathbb{R}^n$.

Proposition 11 *The map $\phi_n : \partial(W_n^\circ) \to \partial(Root_n)$, introduced in Definition 10, is one-to-one on faces, i.e. it sends simplices of $\partial(W_n^\circ)$ to non-degenerate simplices in the boundary $\partial(Root_n)$ of the root polytope.*

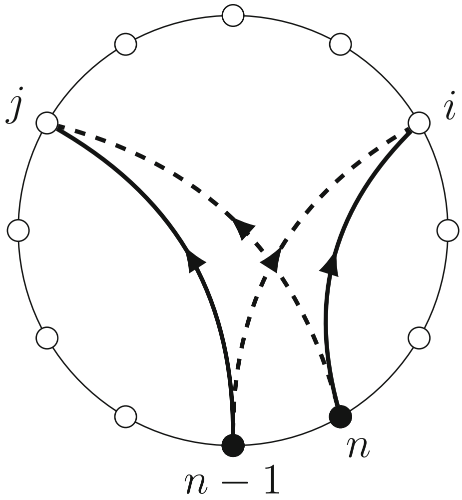
Proof A face of $\partial(W_n^\circ)$ corresponds to an admissible family $\mathcal{T} = \{[i_\nu, j_\nu)\}_{\nu=1}^k$. The associated digraph (also denoted by $\mathcal{T}$) is a directed forest (by Proposition 7). The associated unoriented graph $\Gamma_{\mathcal{T}}$ is a bipartite graph with the shores $P = \{i_1, \ldots, i_k\}$ and $Q = \{j_1, \ldots, j_k\}$ which has no cycles, i.e. $\Gamma_{\mathcal{T}}$ is a forest. The elements of the corresponding set $\phi_n(\mathcal{T}) = \{e_{i_\nu} - e_{j_\nu}\}_{j=1}^k$ of vectors may be interpreted as some of the vertices of the product of simplices $\Delta_P \times \Delta_Q$. By Lemma 6.2.8 from De Loera et al. (2010, Section 6.2) these vertices are affinely independent. This implies that ϕ_n must be one-to-one on $\mathcal{T}$. □

Proposition 11 is a very special case of Proposition 15, which claims global injectivity of the canonical map ϕ_n. The following example illustrates one of the main reasons why the ϕ_n-images of different simplices have disjoint interiors.

Example 12 By inspection of Fig. 1 (which illustrates the case $n = 4$ of Proposition 15), we observe that the images of different triangles (admissible triples) $\mathcal{T}_1$ and $\mathcal{T}_2$, have disjoint interiors. For example let, as in Fig. 3,

$$\mathcal{T}_1 = \{[3, 2), [4, 2), [4, 1)\} \quad \text{and} \quad \mathcal{T}_2 = \{[3, 2), [3, 1), [4, 1)\}.$$

Fig. 4 Admissible triangles with a common edge



Suppose that the interiors of their images have a non-empty intersection, i.e. assume that there is a solution of the equation,

$$\alpha_{3,2}(e_3 - e_2) + \alpha_{4,2}(e_4 - e_2) + \alpha_{4,1}(e_4 - e_1)$$
$$= \beta_{3,2}(e_3 - e_2) + \beta_{3,1}(e_3 - e_1) + \beta_{4,1}(e_4 - e_1)$$

By rearranging the terms we obtain,

$$\alpha'(e_4 - e_3) + \alpha''(e_3 - e_2) + \alpha'''(e_2 - e_1)$$
$$= \beta'(e_4 - e_3) + \beta''(e_3 - e_2) + \beta'''(e_2 - e_1)$$

However, this is impossible since α's and β's are positive and $\alpha'' > \alpha' > \alpha''' > 0$ while $\beta'' > \beta''' > \beta' > 0$.

Remark 13 The argument used in the previous example is sufficiently general to cover the case of triangles (admissible triples) $\mathcal{T}_1$ and $\mathcal{T}_2$, which share a common edge (as in Fig. 4). Indeed, by setting $i = 1$, $j = 2$ and $n = 4$ this case is reduced to Example 12.

4 Cyclohedron and the Root Polytope I

The following theorem is together with Theorem 31 one of the central results of the paper. Informally speaking, it says that there exists a triangulation of the boundary of the $(n-1)$-dimensional type A root polytope $Root_n$ parameterized by proper faces of the $(n-1)$-dimensional cyclohedron.

Theorem 14 *The map $\phi_n : \partial(W_n^\circ) \to \partial(Root_n)$, introduced in Definition 10, is a piecewise linear homeomorphism of boundary spheres of polytopes W_n° and $Root_n$.*

The map ϕ_n sends bijectively vertices of $\partial(W_n^\circ)$ to vertices of the polytope $Root_n$, while higher dimensional faces of $Root_n$ are triangulated by images of simplices from $\partial(W_n^\circ)$.

The proof of Theorem 14 is given in the following two sections. Its main part is the proof of the injectivity of the canonical map ϕ_n.

4.1 Injectivity of the Map ϕ_n

Proposition 11 can be interpreted as a result claiming local injectivity of the map $\phi_n : \partial(W_n^\circ) \to \partial(Root_n)$. Our central result in this section is Proposition 15, which establishes global injectivity of this map and provides a key step in the proof of Theorem 14.

Recall that $\phi_n : \partial(W_n^\circ) \to \partial(Root_n)$ is defined as the simplicial map such that $\phi_n([i, j)) = e_i - e_j$ for each pair $i \neq j$. More explicitly, if $x \in \partial W_n^\circ$ is a convex combination of arcs (intervals),

$$x = \sum_{[i,j)\in\mathcal{T}} \lambda_{i,j}\, [i, j) \tag{8}$$

(where $\mathcal{T}$ is the associated admissible family) then,

$$\phi_n(x) = \sum_{[i,j)\in\mathcal{T}} \lambda_{i,j}(e_i - e_j). \tag{9}$$

We will usually assume that the representation (8) is minimal ($x \in int(\mathcal{T})$) which means that the weights $\{\lambda_{i,j}\}$ satisfy the conditions $\sum_{[i,j)\in T} \lambda_{i,j} = 1$ and $(\forall i, j)\, \lambda_{i,j} > 0$.

Proposition 15 *The map $\phi_n : \partial(W_n^\circ) \to \partial(Root_n)$ is injective.*

Proof Suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are two admissible families of intervals (representing two faces of ∂W_n°). We want to show that $\mathcal{T}_1$ and $\mathcal{T}_2$ must be equal if,

$$\phi_n(int(\mathcal{T}_1)) \cap \phi_n(int(\mathcal{T}_2)) \neq \emptyset. \tag{10}$$

(Note that this observation immediately reduces Proposition 15 to Proposition 11.) Condition (10) says that there exist $\{\alpha_{ij} \mid [i, j) \in \mathcal{T}_1\}$,$\{\beta_{ij} \mid [i, j) \in \mathcal{T}_2\}$ such that,

$$(\forall\, [i, j) \in \mathcal{T}_1)\, \alpha_{ij} > 0 \tag{11}$$

$$(\forall\, [i, j) \in \mathcal{T}_2)\, \beta_{ij} > 0 \tag{12}$$

$$\sum_{[i,j)\in\mathcal{T}_1} \alpha_{ij} = 1 = \sum_{[i,j)\in\mathcal{T}_2} \beta_{ij} \tag{13}$$

$$\sum_{[i,j)\in\mathcal{T}_1} \alpha_{ij} \left(e_i - e_j\right) = \sum_{[i,j)\in\mathcal{T}_2} \beta_{ij} \left(e_i - e_j\right) \tag{14}$$

Our objective is to show that conditions (11)–(14) imply $\mathcal{T}_1 = \mathcal{T}_2$ and $\alpha_{i,j} = \beta_{i,j}$ for each interval $[i, j) \in \mathcal{T}_1 = \mathcal{T}_2$.

We begin with the observation that Proposition 15 is trivially true for $n = 3$. (In this case both $\partial((W_3)^\circ)$ and $\partial(Root_3)$ are boundaries of a hexagon.) This is sufficient to start an inductive proof. However note that we already know (Fig. 1, Example 12, and Remark 13) that Proposition 15 is also true in the case $n = 4$.

The proof is continued by induction on the parameter $\nu := |\mathcal{T}_1| + |\mathcal{T}_2| + n$. More precisely, we show that if there is a counterexample $(\mathcal{T}_1, \mathcal{T}_2)$ on $[n]$ then there is a counterexample $(\mathcal{T}_1', \mathcal{T}_2')$ on $[n']$ such that $\nu' = |\mathcal{T}_1'|+|\mathcal{T}_2'|+n' < |\mathcal{T}_1|+|\mathcal{T}_2|+n = \nu$.
Step 1: Without loss of generality we are allowed to assume that,

$$s(\mathcal{T}_1) = s(\mathcal{T}_2) = I \quad \text{and} \quad t(\mathcal{T}_1) = t(\mathcal{T}_2) = J. \tag{15}$$

Indeed, it follows from Eq. (14) that I (respectively J) collects the indices i (respectively the indices j) where e_i appears with a positive coefficient (e_j appears with a negative coefficient).

Moreover, we assume that,

$$I \cup J = [n]. \tag{16}$$

Otherwise, there exists an element $i \in [n]$ which is neither source nor terminal point of an interval in $\mathcal{T}_1 \cup \mathcal{T}_2$. In this case the vertex i can be deleted and n can be replaced by a smaller number n'.
Step 2: Let us assume that either I or J contains two consecutive elements, for example let $\{i, i + 1\} \subset I$ for some $i \in [n]$. The proof in the case $\{i, i + 1\} \subset J$ is similar (alternatively we can apply the automorphism β from Proposition 9 which reverses the orientation of arcs).

By cyclic relabelling, in other words by applying repeatedly the automorphism α from Proposition 9, we may assume that $i = n - 1$ and $i + 1 = n$.

Let $L_n : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the linear map such that $L_n(e_j) = e_j$ for $j = 1, \dots, n-1$ and $L_n(e_n) = e_{n-1}$. On applying the map L_n to both sides of the equality (14) we obtain a new relation,

$$\sum_{[i,j)\in\mathcal{T}_1'} \alpha'_{ij} \left(e_i - e_j\right) = \sum_{[i,j)\in\mathcal{T}_2'} \beta'_{ij} \left(e_i - e_j\right). \tag{17}$$

For better combinatorial understanding of the relation (17), we note that a combinatorial-geometric counterpart of the map L_n is the operation of collapsing the interval $[n - 1, n]$ (in the circle S^1) to the point $n - 1$.

It is not difficult to describe the effect of the collapsing operation (CO) on the admissible families $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfying the condition $\{n - 1, n\} \subset s(\mathcal{T}_1) = s(\mathcal{T}_2)$.

Lemma 16 *The collapsing operation CO sends each admissible family $\mathcal{T}$ with the property $\{n - 1, n\} \subset s(\mathcal{T})$ to an admissible family $\mathcal{T}'$ on the vertex set $[n - 1] = \{1, \dots, n - 1\}$. Moreover, under this condition, $CO([i, j)) = [i, j)$ if $i \neq n$, while $CO([n, j)) = [n - 1, j)$.*

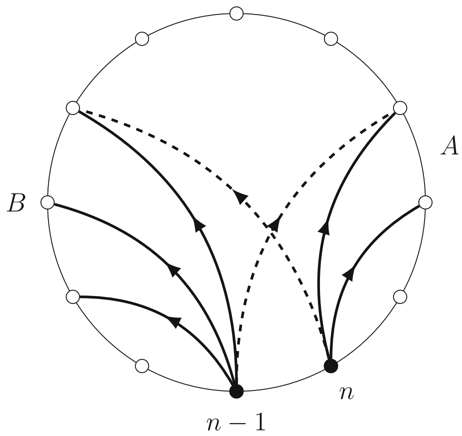


Fig. 5 Admissible family of arcs before the collapse of the interval $[n-1, n]$

Each admissible family $\mathcal{T}$, satisfying the condition $\{n-1, n\} \subset s(\mathcal{T})$, has a decomposition $\mathcal{T} = \mathcal{T}^a \uplus \mathcal{T}^b \uplus \mathcal{T}^c$ where,

$$\mathcal{T}^a = \{[i, j) \in \mathcal{T} \mid i = n\}, \quad \mathcal{T}^b = \{[i, j) \in \mathcal{T} \mid i = n-1\}, \quad \mathcal{T}^c = \mathcal{T} \backslash (\mathcal{T}^a \cup \mathcal{T}^b). \tag{18}$$

Let $A = A(\mathcal{T}) := \{j \in [n] \mid [n, j) \in \mathcal{T}\}$ and $B = B(\mathcal{T}) := \{j \in [n] \mid [n-1, j) \in \mathcal{T}\}$. Note that the sets $A(\mathcal{T})$ and $B(\mathcal{T})$ are either disjoint or have exactly one point in common. (Figure 5 shows how the common point arises as the end-point of one of the dotted arcs.)

It follows from Lemma 16 that the admissible family $\mathcal{T}' := CO(\mathcal{T})$ admits the decomposition,

$$\mathcal{T}' = CO(\mathcal{T}) = \mathcal{T}^c \uplus \mathcal{T}^{ab} \tag{19}$$

where $\mathcal{T}^{ab} := \{[n-1, j) \mid j \in A(\mathcal{T}) \cup B(\mathcal{T})\}$. This analysis and a comparison of equalities (14) and (17) lead to the following observations.

1. By the induction hypothesis, the equality (17) leads to the conclusion that $\mathcal{T}_1' = \mathcal{T}_2'$ and $\alpha'_{i,j} = \beta'_{i,j}$ for each pair (i, j) such that $[i, j) \in \mathcal{T}_1' = \mathcal{T}_2'$;
2. $\mathcal{T}_1' = \mathcal{T}_2'$ together with (19) implies $T_1^c = T_2^c$ and $\alpha'_{i,j} = \alpha_{i,j} = \beta_{i,j} = \beta'_{i,j}$ for each pair (i, j) such that $[i, j) \in \mathcal{T}_1^c = \mathcal{T}_2^c$. By canceling these terms in (14) we obtain the following equality,

$$\sum_{[i,j)\in \mathcal{T}_1^a \cup T_1^b} \alpha_{ij} \left(e_i - e_j\right) = \sum_{[i,j)\in \mathcal{T}_2^a \cup T_2^b} \beta_{ij} \left(e_i - e_j\right) \tag{20}$$

3. This cancelation in (20) can be continued. The only indices, end-points of oriented arcs, that remain unaffected by the cancelation belong to the set $W := (A(\mathcal{T}_1) \cap B(\mathcal{T}_1)) \cup (A(\mathcal{T}_2) \cap B(\mathcal{T}_2))$.
4. The only case that requires further analysis is the case when $W = \{i, j\}$ has precisely two elements (Fig. 5). In this case we obtain a contradiction by the argument used in Example 12 (Remark 13 and Fig. 4).

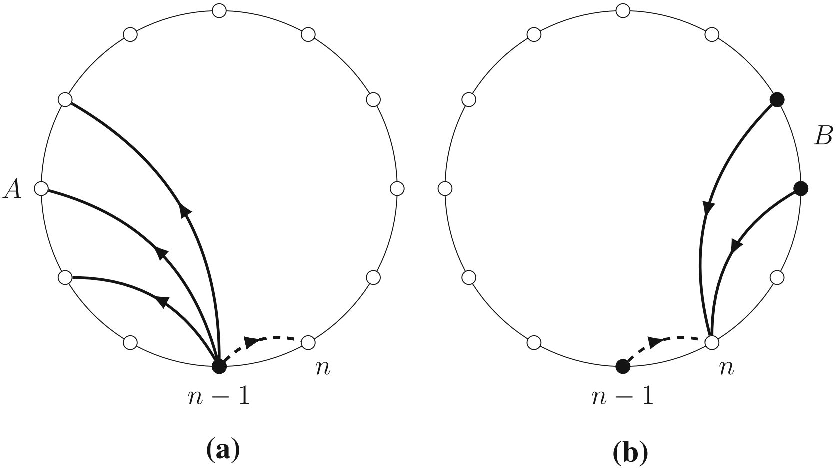


Fig. 6 Two cases of the Step 3 collapsing operation

Step 3 In this step we handle the only remaining case where neither $s(\mathcal{T}_1) = s(\mathcal{T}_2) = I$ nor $t(\mathcal{T}_1) = t(\mathcal{T}_2) = J$ have consecutive elements. In this case there must exist two consecutive indices $\{i-1, i\}$ such that $i-1 \in I$ and $i \in J$. Again, by the cyclic re-enumeration, we can assume that $i = n$ (Fig. 6, cases a, b).

For an admissible family $\mathcal{T}$, satisfying these conditions, there is a decomposition, $\mathcal{T} = \mathcal{T}^a \cup \mathcal{T}^b \cup \mathcal{T}^c$, similar to (18), where $\mathcal{T}^a = \{[i, j) \in \mathcal{T} \mid j = n\}$, $\mathcal{T}^b = \{[i, j) \in \mathcal{T} \mid i = n-1\}$, and $\mathcal{T}^c = \mathcal{T} \backslash (\mathcal{T}^a \cup \mathcal{T}^b)$. Note that $\mathcal{T}^a \cap \mathcal{T}^b$ is either empty or $\mathcal{T}^a \cap \mathcal{T}^b = \{[n-1, n)\}$. It follows that $\mathcal{T} \backslash \{[n-1, n)\} = \hat{\mathcal{T}}^a \uplus \hat{\mathcal{T}}^b \uplus \mathcal{T}^c$, where $\hat{\mathcal{T}}^a := \mathcal{T}^a \backslash \{[n-1, n)\}$ and $\hat{\mathcal{T}}^b := \mathcal{T}^b \backslash \{[n-1, n)\}$.

The key observation is that if $J_1 = [n-1, j_1) \in \hat{\mathcal{T}}^a$ and $I_1 = [i_1, n) \in \hat{\mathcal{T}}^b$, then intervals I_1 and J_1 intersect but cannot be compatible in the sense of Definition 4. It immediately follows that either $\hat{\mathcal{T}}^a = \emptyset$ or $\hat{\mathcal{T}}^b = \emptyset$.

Lemma 17 *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are (Step 3) admissible families satisfying conditions* (11)–(14) *then either,*

(A) $\quad \hat{\mathcal{T}}_1^a = \hat{\mathcal{T}}_2^a = \emptyset$ *and* $\mathcal{T}_i^b = \hat{\mathcal{T}}_i^b \cup \{[n-1, n)\}$ *for* $i = 1, 2$*, or*

(B) $\quad \hat{\mathcal{T}}_1^b = \hat{\mathcal{T}}_2^b = \emptyset$ *and* $\mathcal{T}_i^a = \hat{\mathcal{T}}_i^a \cup \{[n-1, n)\}$ *for* $i = 1, 2.$

For illustration, the case $\hat{\mathcal{T}}_1^a = \hat{\mathcal{T}}_2^b = \emptyset$ is ruled out by the following reasoning. The only way to satisfy the condition (14) is to have $[n-1, n) \in \mathcal{T}_1^b \cap \mathcal{T}_2^a$ (and $\alpha_{n-1,n} \neq 0 \neq \beta_{n-1,n}$). This is not possible, however, since by comparing the coefficients near e_{n-1} and e_n we obtain the following contradictory equalities,

$$\alpha_{n-1,n} + \sum_{[n-1,j) \in \mathcal{T}_1^b} \alpha_{n-1,j} = \beta_{n-1,n} \qquad \alpha_{n-1,n} = \beta_{n-1,n} + \sum_{[i,n) \in \mathcal{T}_2^a} \beta_{i,n}. \tag{21}$$

The proof (Step 3) is continued by observing that in both cases of Lemma 17 the coefficients $\alpha_{n-1,n}$ and $\beta_{n-1,n}$ must be equal, and the corresponding terms in (14) can be cancelled out. The proof (Step 3) is finished by applying the collapsing operator CO, which collapses the interval $[n-1, n]$ either to the left end-point $(n-1)$ (corresponding to the case A of Lemma 17) or to the right end-point n (corresponding to the case B of Lemma 17). The analysis is similar to the collapsing procedure described in Step 2 so we omit the details.

This completes the proof of Proposition 15. □

4.2 Surjectivity of the Map ϕ_n

We already know (Proposition 15) that ϕ_n is injective. Let us show that it is surjective as well.

By Proposition 15 the map ϕ_n induces an isomorphism in homology, i.e. the degree $\deg(\phi_n)$ is either $+1$ or -1. Therefore it must be an epimorphism since otherwise it would be homotopic to a constant map. □

5 Cyclohedron and the Root Polytope II

In this section we give a geometric interpretation of the map $\phi_n : \partial(W_n^\circ) \to \partial(Root_n)$, introduced in Definition 10. The key observation is that the dual $(Root_n)^\circ$ of the root polytope belongs to the *irredundant part* of the *face deformation cone* (Postnikov et al. 2008, Section 15) of the cyclohedron W_n. A more precise statement says that the pair $(W_n, (Root_n)^\circ)$ may be interpreted as a couple of polytopes $(\Delta_{\widehat{\mathcal{F}}}, \Delta_{\mathcal{F}})$, where $\widehat{\mathcal{F}}$ is a building set and $\mathcal{F}$ its irredundant basis in the sense of Definition 21. For an introduction into the theory of *nestohedra, building sets, etc.*, the reader is referred to Buchstaber and Panov (2015), Postnikov (2009) and Feichtner and Sturmfels (2005).

5.1 Building Set of the Cyclohedron W_n

It is well-known, see Feichtner and Sturmfels (2005, Section 3), Postnikov (2009), Devadoss (2003), or Buchstaber and Panov (2015, Section 1.5), that the cyclohedron W_n is a nestohedron (graph associahedron), so it has a Minkowski sum decomposition,

$$W_n = \Delta_{\widehat{\mathcal{F}}} := \sum_{F\in\widehat{\mathcal{F}}} \Delta_F, \tag{22}$$

where $\widehat{\mathcal{F}} \subset 2^{[n]}\backslash\{\emptyset\}$ is the associated *building set* (Feichtner and Sturmfels 2005; Postnikov 2009) and,

$$\Delta_F \subset \Delta_{[n]} := \mathrm{Conv}\{e_i\}_{i=1}^n \subset \mathbb{R}^n,$$

is the simplex spanned by $F \subset [n]$. The family $\widehat{\mathcal{F}}$ is in the case of W_n identified as the collection $\widehat{\mathcal{F}} = Con(C_n)$ of all connected subsets in the cycle graph C_n with n-vertices. Note that a set $Z \subset [n]$ is connected if Z is either a cyclic interval or $Z = [n]$.

The Minkowski sum $\Delta_{\mathcal{F}} := \sum_{F\in\mathcal{F}} \Delta_F$ is defined for any family (hypergraph) $\mathcal{F} \subset 2^{[n]}\backslash\{\emptyset\}$, however it is not necessarily a simple polytope, unless $\mathcal{F}$ is a building set. For this reason it is interesting to compare $\Delta_{\mathcal{F}}$ and $\Delta_{\widehat{\mathcal{F}}}$ where $\widehat{\mathcal{F}}$ is the *building closure* of $\mathcal{F}$.

Definition 18 A family $\widehat{\mathcal{F}} \supset \mathcal{F}$ is the *building closure* of a hypergraph $\mathcal{F}$ if $\widehat{\mathcal{F}}$ is the unique minimal building set which contains $\mathcal{F}$. In this case we also say that $\mathcal{F}$ is a *building basis* of the building set $\widehat{\mathcal{F}}$.

Definition 19 If $\mathcal{F} \subset 2^{[n]}\backslash\{\emptyset\}$ is a hypergraph and $X \subset [n]$ then,

$$\mathcal{F}_X := \{F \in \mathcal{F} \mid F \subset X\} \quad \text{and} \quad \mathcal{F}^X := \{F\backslash X \mid F \in \mathcal{F},\, F\backslash X \neq \emptyset\}.$$

The family $\mathcal{F}_X$ is referred to as the *restriction* of $\mathcal{F}$ to X, while $\mathcal{F}^X$ is obtained from $\mathcal{F}$ by *deletion* of the set X.

For each $X \subset [n]$ let $\phi_X : \mathbb{R}^n \to \mathbb{R}$ be the linear form $\phi_X(x) = \sum_{i\in X} x_i$ (for example $\phi_{[n]}(x) = x_1 + x_2 \cdots + x_n$). The cardinality of a family $\mathcal{F}$ is denoted by $|\mathcal{F}|$.

The following proposition, see Postnikov (2009, Proposition 7.5) or Feichtner and Sturmfels (2005, Proposition 3.12), provides a description of $\Delta_{\mathcal{F}}$ in terms of linear (in)equalities.

Proposition 20 *Suppose that $\widehat{\mathcal{F}}$ is the building closure of a family $\mathcal{F} \subset 2^{[n]}\backslash\{\emptyset\}$. Then,*

$$\Delta_{\mathcal{F}} = \left\{x \in \mathbb{R}^n \mid \phi_{[n]}(x) = |\mathcal{F}| \text{ and for each } X \in \widehat{\mathcal{F}},\ \phi_X(x) \geqslant |\mathcal{F}_X|\right\}. \tag{23}$$

Moreover, the face of $\Delta_{\mathcal{F}}$ where ϕ_X attains its minimum is isomorphic to the Minkowski sum,

$$\Delta_{\mathcal{F}_X} + \Delta_{\mathcal{F}^X}. \tag{24}$$

Definition 21 We say that a hypergraph $\mathcal{F} \subset 2^{[n]}\backslash\{\emptyset\}$ is *tight* if all inequalities in (23) are essential (irredundant), where $\widehat{\mathcal{F}}$ is the building closure of $\mathcal{F}$. We also say that $\mathcal{F}$ is a tight or *irredundant basis* of the building set $\widehat{\mathcal{F}}$.

The following criterion for tightness of $\mathcal{F}$ is easily deduced from Proposition 20.

Proposition 22 *Let $\mathcal{F} \subset 2^{[n]}\backslash\{\emptyset\}$ be a hypergraph and let $\widehat{\mathcal{F}}$ be its building closure. Then $\mathcal{F}$ is tight if and only for each $X \in \widehat{\mathcal{F}}$,*

1. *$\mathcal{F}_X$ is connected as a hypergraph on X and,*
2. *$\mathcal{F}^X$ is connected as a hypergraph on $[n]\backslash X$.*

Actually, the first condition is automatically satisfied, as a consequence of the fact that $\widehat{\mathcal{F}}$ is the building closure of $\mathcal{F}$.

Proof By Proposition 20 [relation (24)] $\mathcal{F}$ is tight if and only if for each $X \in \widehat{\mathcal{F}}$,

$$\dim(\Delta_{\mathcal{F}_X} + \Delta_{\mathcal{F}^X}) = \dim(\Delta_{\mathcal{F}}) - 1 = n - 2.$$

It is well known, see Buchstaber and Panov (2015, Proposition 1.5.2) or Feichtner and Sturmfels (2005, Remark 3.11), that for each hypergraph $\mathcal{H} \subset 2^S \setminus \{\emptyset\}$ the dimension of the polytope $\Delta_{\mathcal{H}} = \sum_{H \in \mathcal{H}} \Delta_H$ is $|S| - c$ where c is the number of components of the hypergraph $\mathcal{H}$. (Recall that $x, y \in S$ are in the same connected component in the hypergraph $\mathcal{H}$ if there is a sequence of elements $x = z_1, z_2, \ldots, z_k = y$ in S such that each $\{z_i, z_{i+1}\}$ is contained in some 'edge' of the hypergraph $\mathcal{H}$.)

Since $\widehat{\mathcal{F}}$ is the building closure of $\mathcal{F}$, $\dim(\Delta_{\mathcal{F}_X}) = |X| - 1$. Indeed, by the proof of Feichtner and Sturmfels (2005, Lemma 3.10) $X \in \widehat{\mathcal{F}}$ if and only if X is a singleton or $\mathcal{F}_X$ is a connected hypergraph on X.

It follows that,

$$\begin{aligned}\dim(\Delta_{\mathcal{F}_X} + \Delta_{\mathcal{F}^X}) &= \dim(\Delta_{\mathcal{F}_X}) + \dim(\Delta_{\mathcal{F}^X}) \\ &= n - 2 = (|X| - 1) + (n - |X| - 1)\end{aligned}$$

if and only if $\dim(\Delta_{\mathcal{F}^X}) = n - |X| - 1$. This equality is equivalent to the condition (2) in Proposition 22. ◻

As a consequence of Proposition 22 we obtain the following result.

Proposition 23 *Let C_n be the cycle graph on n vertices and let (22) be the associated graph associahedron representation of the cyclohedron W_n, where $\widehat{\mathcal{F}} = Con(C_n)$ is the building set of all C_n-connected subsets in $[n]$. Then,*

$$\mathcal{F} := \{\{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}, \{n, 1\}\} \tag{25}$$

is a tight hypergraph on $[n]$, which is a tight (irredundant) basis of $\widehat{\mathcal{F}}$ in the sense of Definition 21. As a consequence all inequalities (23), in the corresponding description of the polytope,

$$\Delta_{\mathcal{F}} = \sum_{i=1}^{n} [e_i, e_{i+1}] \tag{26}$$

are essential (irredundant).

Proof If $X = [i, j]^0 = \{i, i + 1, \ldots, j\} \subset [n] = \mathrm{Vert}(C_n)$ is an element in $\widehat{\mathcal{F}} = Con(C_n)$ then,

$$\mathcal{F}^X = \{\{i - 1\}, \{j + 1\}\} \cup \left\{[j + 1, j + 2]^0, [j + 2, j + 3]^0, \ldots, [i - 2, i - 1]^0\right\}.$$

This hypergraph is clearly connected on its set of vertices which verifies the condition (2) in Proposition 22 and completes the proof of Proposition 23. ◻

5.2 Polar Dual of the Root Polytope

Definition 24 Let $\Delta = \Delta_A = \mathrm{Conv}(A) = \mathrm{Conv}\{a_0, a_1, \ldots, a_m\}$ be a non-degenerate simplex with vertices in $A \subset \mathbb{R}^n$. The associated Δ-zonotope $Zono(\Delta) = Zono(\Delta_A)$ is the Minkowski sum,

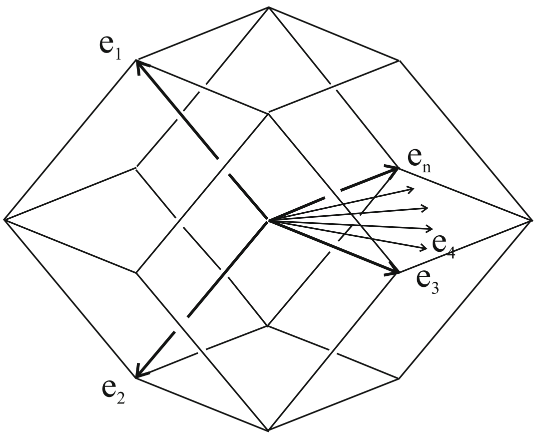


Fig. 7 Δ-zonotope $Zono(\Delta_n)$ as a generalized rhombic dodecahedron

$$Zono(\Delta_A) = [a, a_0] + [a, a_1] + \cdots + [a, a_m], \tag{27}$$

where $a := \frac{1}{m+1}(a_0 + \cdots + a_m)$. If $\Delta_n = \Delta_{[n]} = \mathrm{Conv}(\{e_1, \ldots, e_n\})$ is the simplex spanned by the orthonormal basis in $\mathbb{R}^n$, then the associated Δ-zonotope $Zono(\Delta_n) = Zono(\Delta_{[n]})$ is referred to as the standard, $(n-1)$-dimensional Δ-zonotope.

Example 25 Figure 7 depicts the standard $(n-1)$-dimensional Δ-zonotope. In the special case $n = 4$ one obtains the rhombic dodecahedron.

Definition 26 The generalized root polytope associated to a simplex $\Delta_A = \mathrm{Conv}(A) = \mathrm{Conv}\{a_1, \ldots, a_n\}$ is the polytope,

$$Root(\Delta_A) = \mathrm{Conv}\{a_i - a_j \mid 1 \leqslant i \neq j \leqslant n\}. \tag{28}$$

If W is an affine map such that $a_i = W(e_i)$ for each $i \in [n]$ then,

$$Root(W(\Delta_n)) = W(Root(\Delta_n)) = W(Root_n). \tag{29}$$

The root polytope $Root_n$ is a subset of the hyperplane $H_0 = \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$ while $Zono(\Delta_n) \subset H_n = \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = n\}$. In the following proposition we claim that the Δ-zonotope $Zono(\Delta_n^0)$, obtained by translating $Zono(\Delta_n)$ to H_0, is precisely the polar dual of the polytope $Root_n$.

Proposition 27 *The root polytope* $Root_n = \mathrm{Conv}\{e_i - e_j \mid 1 \leqslant i \neq j \leqslant n\} \subset H_0$ *is the dual (inside* H_0*) of the* Δ*-zonotope* $Zono(\Delta_n^0)$ *where* $\Delta_n^0 = \Delta_n - \frac{1}{n}(e_1 + \cdots + e_n)$,

$$(Root_n)^\circ = Zono(\Delta_n^0). \tag{30}$$

Proof The proof is an elementary exercise in the concept of duality (see Živaljević 2015, Proposition 7). Let $\hat{e}_i = e_i - \frac{1}{n}(e_1 + \cdots + e_n) \in H_0$. It is sufficient to observe that the dual of the root polytope is,

$$(Root_n)^\circ = \left\{x \in H_0 \mid |x_i - x_j| \leqslant 1,\ \text{for } 1 \leqslant i \leqslant n)\right\}, \tag{31}$$

while the two supporting hyperplanes of $Zono(\Delta_n^0)$, parallel to $\mathcal{L}_{i,j} = \text{span}\{\hat{e}_k \mid i \neq k \neq j\}$ (Fig. 7) have equations,

$$x_i - x_j = 1 \quad \text{and} \quad x_j - x_i = 1.$$

□

Lemma 28 *Suppose that K° is the polar dual of a convex body K. If $A : \mathbb{R}^d \to \mathbb{R}^d$ is a non-singular linear map then,*

$$(A(K))^\circ = B(K^\circ) \tag{32}$$

where $B = (A^*)^{-1}$. In particular if $A = (A^*)^{-1}$ is an orthogonal transformation and $\mu \neq 0$ then, $(A(K))^\circ = A(K^\circ)$ and $(\mu K)^\circ = (1/\mu)K^\circ$.

Proof If $y \in (A(K))^\circ$ then by definition,

$$\langle y, Ax\rangle = \langle A^* y, x\rangle \leqslant 1 \quad \text{for each } x \in K. \tag{33}$$

□

The following extension of Proposition 27 is recorded for the future reference.

Proposition 29 *Let $H_0 \subset \mathbb{R}^n$ be the subspace spanned by $\hat{e}_i = e_i - \frac{1}{n}(e_1 + \cdots + e_n)$ and let $\Delta_n^0 = \text{Conv}\{\hat{e}_1, \ldots, \hat{e}_n\}$. Let $A : H_0 \to H_0$ be a non-singular linear map and let $B := (A^*)^{-1}$. Then,*

$$(Root(A(\Delta_n^0)))^\circ = Zono(B(\Delta_n^0)) \tag{34}$$

Proof By definition $Root(A(\Delta_n^0)) = A(Root(\Delta_n^0))$ and $Zono(B(\Delta_n^0)) = B(Zono(\Delta_n^0))$. It follows from Proposition 27 and Lemma 28 that,

$$\begin{aligned}(Root(A(\Delta_n^0)))^\circ &= (A(Root(\Delta_n^0)))^\circ \\ &= B((Root(\Delta_n^0))^\circ) = B(Zono(\Delta_n^0)) = Zono(B(\Delta_n^0)).\end{aligned}$$

□

5.3 W_n° as a Kantorovich–Rubinstein Polytope

There is a canonical isomorphism of vector spaces $H_\nu = \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = \nu\}$ and the quotient space $\mathbb{R}^n/\mathbb{R}e$, where $e = \frac{1}{n}(e_1 + \cdots + e_n)$, which induces a canonical isomorphism between H_0 and $H_\nu = \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = \nu\}$ for each $\nu \in \mathbb{R}$.

The canonical isomorphism between H_1 and H_0 sends e_i to $\hat{e}_i = e_i - \frac{1}{n}(e_1 + \cdots + e_n)$ and $\Delta_n = \mathrm{Conv}\{e_1, \ldots, e_n\}$ to $\Delta_n^0 = \mathrm{Conv}\{\hat{e}_1, \ldots, \hat{e}_n\}$.

The canonical isomorphism between H_n and H_0 identifies the polytope $\Delta_{\mathcal{F}}$, introduced in Sect. 5.1 [Eq. (26)], with the polytope,

$$\Delta_{\mathcal{F}} = \sum_{i=1}^{n} [\hat{e}_i, \hat{e}_{i+1}] = \sum_{i=1}^{n} (\hat{e}_i + [0, \hat{e}_{i+1} - \hat{e}_i]) = \sum_{i=1}^{n} [0, \hat{e}_{i+1} - \hat{e}_i] = Zono(B(\Delta_n^0)) \tag{35}$$

where $e_{n+1} := e_1$ and $B : H_0 \to H_0$ is the linear map defined by $B(\hat{e}_i) = b_i = \hat{e}_{i+1} - \hat{e}_i$. In other words the polytope $\Delta_{\mathcal{F}}$ (associated to the irredundant basis (25) of the building set $\widehat{\mathcal{F}} = Con(C_n))$ is a Δ-zonotope (generalized rhombic dodecahedron) $Zono(B(\Delta_n^0))$.

The dual of $Zono(B(\Delta_n^0))$ is by Proposition 29 a root polytope,

$$Root_n^a := \mathrm{Conv}\{a_i - a_j \mid 1 \leqslant i \neq j \leqslant n\} = Root(A(\Delta_n^0)) \tag{36}$$

where the vectors $\{a_i\}_{i=1}^n \subset H_0$ are defined by $a_i = A(\hat{e}_i)$ and $A : H_0 \to H_0$ is the linear map such that $B = (A^*)^{-1}$ $(A = (B^*)^{-1}$).

Summarizing, we record for the future reference the following proposition,

Proposition 30 *There exist vectors* $\{a_i\}_{i=1}^n \subset H_0$*, such that* $a_1 + \cdots + a_n = 0$ *and* $\mathrm{Span}(\{a_i\}_{i=1}^n) = H_0$*, which have the property that the dual of the root polytope* $Root_n^a$ [*defined by* (36)] *is the polytope* $\Delta_{\mathcal{F}}$ [*defined by* (35)].

The following theorem is the main result of Sect. 5.

Theorem 31 *There exists a quasi-metric (asymmetric distance function)* ρ *on the set* $[n]$ *such that the associated Kantorovich–Rubinstein polytope,*

$$KR(\rho) = \mathrm{Conv}\left\{\frac{e_i - e_j}{\rho(i,j)} \mid 1 \leqslant i \neq j \leqslant n\right\} \tag{37}$$

is affinely isomorphic *to the dual* W_n° *of the cyclohedron* W_n*. Moreover the distance function* ρ *satisfies a strict triangle inequality in the sense that,*

$$\rho(i,k) < \rho(i,j) + \rho(j,k) \text{ if } i \neq j \neq k \neq i.$$

Proof Let $\{a_i\}_{i=1}^n \subset H_0$ be the collection of vectors described in Proposition 30 and let $a_{i,j} = a_i - a_j$ (for $1 \leqslant i \neq j \leqslant n$) be the corresponding roots. In light of Proposition 30 the polytope $\Delta_{\mathcal{F}}$ has the following description,

$$\Delta_{\mathcal{F}} = \left\{x \in H_0 \mid \langle a_{i,j}, x\rangle \leqslant 1 \text{ for each pair } i \neq j\right\}. \tag{38}$$

All inequalities in (38) are irredundant. Moreover, the analysis from Sect. 5.1 guarantees that there exist positive real numbers $\{\alpha_{i,j} \mid 1 \leqslant i \neq j \leqslant n\}$ such that,

$$W_n = \Delta_{\widehat{\mathcal{F}}} = \left\{x \in H_0 \mid \langle a_{i,j}, x\rangle \leqslant \alpha_{i,j} \text{ for each pair } i \neq j\right\}. \tag{39}$$

From here it immediately follows that,

$$W_n^\circ = \mathrm{Conv}\left\{\frac{a_{i,j}}{\alpha_{i,j}} \mid 1 \leqslant i \neq j \leqslant n\right\}.$$

Let us show that $\rho(i,j) := \alpha_{i,j}$ is a strict quasi-metric on $[n]$. Assume that there exist three, pairwise distinct, indices $i.j.k \in [n]$ such that $\rho(i,k) \geqslant \rho(i,j) + \rho(j,k)$. Then,

$$\frac{a_{i,k}}{\rho(i,k)} \in \big[0, a_{i,k}/(\rho(i,j)+\rho(j,k))\big] \quad \text{and} \quad \frac{a_{i,k}}{\alpha_{i,k}} \in \big[0, a_{i,k}/(\alpha_{i,j}+\alpha_{j,k})\big].$$

In light of the obvious equality,

$$\frac{a_{i,k}}{\alpha_{i,j}+\alpha_{j,k}} = \frac{\alpha_{i,j}}{\alpha_{i,j}+\alpha_{j,k}}\left(\frac{a_{i,j}}{\alpha_{i,j}}\right) + \frac{\alpha_{j,k}}{\alpha_{i,j}+\alpha_{j,k}}\left(\frac{a_{j,k}}{\alpha_{j,k}}\right)$$

we observe that if both inequalities,

$$\langle a_{i,j}, x\rangle \leqslant \alpha_{i,j} \quad \text{and} \quad \langle a_{j,k}, x\rangle \leqslant \alpha_{j,k}$$

are satisfied then $\langle a_{i,k}, x\rangle \leqslant \alpha_{i,k}$. This is however in contradiction with non-redundancy of the last inequality in the representation (39). □

5.4 An Explicit Quasi-Metric Associated to a Cyclohedron

Definition 32 Let $\widehat{\mathcal{F}}$ be the building closure of a hypergraph $\mathcal{F} \subset 2^{[n]}\setminus\{\emptyset\}$. The associated *'height function'* $h_{\mathcal{F}} : \widehat{\mathcal{F}} \to \mathbb{R}$ is defined by,

$$h_{\mathcal{F}}(X) = \frac{|\mathcal{F}|}{n} - \frac{|\mathcal{F}_X|}{|X|},$$

so in particular $h_{\mathcal{F}}([n]) = 0$ for each hypergraph $\mathcal{F}$.

The inequalities (23), describing Δ_F as a subset of $H_{|\mathcal{F}|} = \{x \in \mathbb{R}^n \mid \phi_{[n]}(x) = |\mathcal{F}|\}$ can be, with the help of the height function, rewritten as follows,

$$\Delta_{\mathcal{F}} = \left\{x \in H_{|\mathcal{F}|} \mid \text{For each } X \in \widehat{\mathcal{F}}\setminus\{[n]\},\ \frac{\phi_{[n]}(x)}{n} - \frac{\phi_X(x)}{|X|} \leqslant h_{\mathcal{F}}(X)\right\}. \quad (40)$$

In particular, if $\mathcal{F} = \widehat{\mathcal{F}}$ we obtain the representation,

$$\Delta_{\widehat{\mathcal{F}}} = \left\{x \in H_{|\widehat{\mathcal{F}}|} \mid \text{For each } X \in \widehat{\mathcal{F}}\setminus\{[n]\},\ \frac{\phi_{[n]}(x)}{n} - \frac{\phi_X(x)}{|X|} \leqslant h_{\widehat{\mathcal{F}}}(X)\right\}. \quad (41)$$

Assuming that $h_{\mathcal{F}}(X) \neq 0$ for each $X \in \widehat{\mathcal{F}} \backslash \{[n]\}$, let $A_X \in H_0$ be the vector defined by,

$$A_X := \frac{1}{h_{\mathcal{F}}(X)} \left(\frac{e}{n} - \frac{e_X}{|X|} \right), \tag{42}$$

where $e_X := \sum_{i \in X} e_i$ and $e = e_{[n]} = e_1 + \cdots + e_n$. It follows that (40) and (41) can be rewritten as,

$$\Delta_{\mathcal{F}} = \left\{ x \in H_0 \mid \text{For each } X \in \widehat{\mathcal{F}} \backslash \{[n]\},\ \langle A_X, x \rangle \leqslant 1 \right\}, \tag{43}$$

$$\Delta_{\widehat{\mathcal{F}}} = \left\{ x \in H_0 \mid \text{ For each } X \in \widehat{\mathcal{F}} \backslash \{[n]\},\ \langle A_X, x \rangle \leqslant \frac{h_{\widehat{\mathcal{F}}}(X)}{h_{\mathcal{F}}(X)} \right\}. \tag{44}$$

Now we specialize to the case $\mathcal{F} := \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}\}$, so the associated building closure $\widehat{\mathcal{F}} = Con(C_n)$ is, as in Proposition 23, the building set of all C_n-connected subsets (circular intervals) in the circle graph C_n . The corresponding height functions are shown in the following lemma.

Lemma 33 *If $X = [n]$ then $h_{\mathcal{F}}(X) = h_{\widehat{\mathcal{F}}}(X) = 0$. If $X \neq [n]$ then,*

$$h_{\mathcal{F}}(X) = \frac{1}{|X|} \quad \text{and} \quad h_{\widehat{\mathcal{F}}}(X) = \frac{n^2 - n + 1}{n} - \frac{|X| + 1}{2}.$$

Recall (Sect. 3.1) that for $i, j \in \text{Vert}(C_n) = [n]$, the associated (discrete) circular interval is $[i, j]^0 := [i, j] \cap [n]$. Similarly (for $i \neq j$) $[i, j)^0 := [i, j) \cap [n]$, so $[i, j)^0 = [i, j-1]^0$ if $i \neq j$ and $[i, i)^0 = \emptyset$. Define the *"clock quasi-metric"* on $\text{Vert}(C_n)$ by,

$$d(i, j) := |[i, j)^0| = |[i, j]^0| - 1. \tag{45}$$

Proposition 34 *Let ρ be the quasi-metric on $[n] = \text{Vert}(C_n)$ defined by,*

$$\rho(i, j) = d(i, j) \frac{n^2 - n + 1}{n} - \frac{d(i, j)(d(i, j) + 1)}{2} \tag{46}$$

where d is the clock quasi-metric on $[n]$. Than the associated Kantorovich–Rubinstein polytope $KR(\rho)$ is affinely isomorphic to a polytope dual to the standard cyclohedron.

Proof By definition $\rho(i, j) = h_{\widehat{\mathcal{F}}}([i, j)^0)/h_{\mathcal{F}}([i, j)^0)$. Recall that Eqs. (43) and (44) are nothing but a more explicit form of Eqs. (38) and (39). It immediately follows that $\rho(i, j) = \alpha_{i,j}$ which by Theorem 31 implies that ρ is indeed a quasi-metric on $[n]$ such that the associated Lipschitz polytope $Lip(\rho)$ is a cyclohedron. □

Remark 35 By a similar argument we already know that vectors $\{A_X \mid X \in \widehat{\mathcal{F}} \backslash \{[n]\}\}$, described by Eq. (42), form a type A root system if $\mathcal{F} := \{\{1, 2\}, \{2, 3\}, \ldots, \{n, 1\}\}$. This can be seen directly as follows. For $2 \leq i \leq n$, let $X_i := [1, i)^0$ and $A_i := A_{X_i}$. Then there is a disjoint union,

$$\{A_X \mid X \in \widehat{\mathcal{F}} \backslash \{[n]\}\} = \{A_i\}_{i=2}^n \sqcup \{-A_i\}_{i=2}^n \sqcup \{A_i - A_j \mid 2 \leq i \neq j \leq n\}.$$

This observation and a comparison of (35) and (43) provide an alternative proof of Proposition 27.

6 Alternative Approaches and Proofs

An elegant and versatile analysis of the combinatorial structure of Lipschitz polytopes, conducted by Gordon and Petrov (2017), can be with little care (but without introducing any new ideas) extended to the case of quasi-metrics.

This fact, as kindly pointed by an anonymous referee, provides a method for describing a large class of quasi-metrics which are combinatorially of "cyclohedral type".

Here we give an outline of this method. [This whole section can be seen as a short addendum to the paper (Gordon and Petrov 2017).]

A *combinatorial structure* on the (dual) pair of polytopes $Lip(\rho)$ and $KR(\rho)$ is, following Gordon and Petrov (2017, Definition 2), the collection of directed graphs $\mathcal{D}(\rho) = \{D(\alpha) \mid \alpha \text{ is a face of } KR(\rho)\}$, where for each face α of $KR(\rho)$,

$$(i, j) \in D(\alpha) \quad \Leftrightarrow \quad \frac{e_j - e_i}{\rho(i, j)} \in \alpha.$$

Following Gordon and Petrov (2017, Definition 1), a quasi-metric is *generic* if the triangle inequality is strict ($x \neq y \neq z \Rightarrow \rho(x, z) < \rho(x, y) + \rho(y, z)$) and the polytope $KR(\rho)$ is simplicial ($Lip(\rho)$ is simple).

In the case of a generic quasi-metric, the combinatorial structure $\mathcal{D}(\rho)$ is a simplicial complex whose face poset is isomorphic to the face poset of the polytope $KR(\rho)$ (see Corollary 1 and Theorem 4 in Gordon and Petrov 2017). Moreover, in this case $D(\alpha)$ is a directed forest (such that either the in-degree or the out-degree of each vertex is zero), and in particular if α is a facet then $D(\alpha)$ is a directed tree.

Following Gordon and Petrov (2017, Theorem 3) (see also Gordon and Petrov 2017, Theorem 4) a directed tree (forest) T is in $\mathcal{D}(\rho)$ if and only if it satisfies a *"cyclic monotonicity"* condition [inequality (1) in Gordon and Petrov 2017, Theorem 3], indicating that T can be thought of as an 'optimal transference plan' for the transport of the corresponding measures.

It was shown in Sect. 3 (Proposition 7) that the face poset of a cyclohedron can be also described as a poset of directed trees (corresponding to the diagrams of oriented arcs, as exemplified by Figs. 2 and 3).

From these observations arises a general plan for finding a generic quasi-metric ρ such that the associated combinatorial structure $\mathcal{D}(\rho)$ is precisely the collection of trees associated to a cyclohedron. Moreover, this approach allows us (at least in principle) to characterize all generic quasi-metrics of "combinatorial cyclohedral type".

Indeed, if $\rho = (\rho(i, j))_{1 \leq i, j \leq n}$ is an unknown quasi-metric (ranging over the space of all quasi-metric matrices), then one can characterize quasi-metrics of cyclohedral type by writing all inequalities of the type (1) in Gordon and Petrov (2017, Theorem 3) (see also the simplification provided by Gordon and Petrov (2017, Theorem 4).

Remark 36 Guided by the form of the metric ρ, described by the formula (46) (Proposition 34), the referee observed that the quasi-metric $\rho_\epsilon := d - \epsilon \cdot d^2$ (where d is the clock quasi-metric and $\epsilon > 0$ a sufficiently small number), is a good candidate for a cyclohedral quasi-metric. (The details of related calculations will appear elsewhere.)

Remark 37 The quasi-metric ρ introduced in Proposition 34 is somewhat exceptional since in this case we guarantee that $KR(\rho)$ is geometrically (affinely) and not only combinatorially (via face posets) isomorphic to a dual of a standard cyclohedron. This has some interesting consequences, for example this metric has the property that the associated Lipschitz polytope $Lip(\rho)$ is a *Delzant polytope*.

7 Concluding Remarks

7.1 The Result of Gordon and Petrov

The motivating result of Gordon and Petrov (2017, Theorem 1) says that for a generic metric ρ on a set of size $n+1$, the number of $(n-i)$-dimensional faces of the associated Lipschitz polytope [the dual of $KR(\rho)$, see the Eq. (1)] is equal to,

$$f_{n-i}(Lip(\rho)) = \binom{n+i}{i, i, n-i} = \frac{(n+i)!}{i!i!(n-i)!}.$$

The link with the combinatorics of cyclohedra, established by Theorems 14 and 31, allows us to deduce this result from the known calculations of f-vectors of these polytopes. For example Simion (2003, Proposition 1) proved that,

$$f_{i-1}(W_n) = \binom{n}{i}\binom{n+i}{i} = \frac{(n+i)!}{i!i!(n-i)!}.$$

Moreover, in light of Theorems 14 and 31, the generating series for these numbers have a new interpretation as a solution of a concrete partial differential equation, see for example Buchstaber and Panov (2015, Sections 1.7. and 1.8).

7.2 Tight Hypergraphs

The relationship between the cyclohedron and the (dual of the) root polytope is explained in Sect. 5 as a special case of the relationship between tight hypergraphs $\mathcal{F}$ and their building closures $\widehat{\mathcal{F}}$. For this reason it may be interesting to search for other examples of 'tight pairs' $(\widehat{\mathcal{F}}, \mathcal{F})$ of hypergraphs.

Example 38 Let C_n be the cycle graph on n vertices (identified with their labels [n]) and let $\prec$ be the associated (counterclockwise) cyclic order on $[n]$. For each ordered pair (i, j) of indices let $[i, j]^0 := \{i, i+1, \dots, j\}$ be the associated 'cyclic interval'. For each $2 \leq k \leq n-1$ let $BC_n^k \subset 2^{[n]} \setminus \{\emptyset\}$ be the hypergraph defined by,

$$BC_n^k := \left\{[1, k]^0, [2, k+1]^0, \dots, [n, k-1]^0\right\}. \tag{47}$$

Then BC_n^k is a tight hypergraph on $[n]$. Moreover, if $\widehat{\mathcal{F}}_{n,k} := \widehat{BC}_n^k$ is its building closure and $Q_{n,k} := \Delta_{\widehat{\mathcal{F}}_{n,k}}$ the associated simple polytope than for $k \neq k'$ the associated polytopes $Q_{n,k}$ and $Q_{n,k'}$ are combinatorially non-isomorphic.

Note that for $k = 2$ we recover the tight hypergraph described in Proposition 23 [Eq. (25)] and in this case $Q_{n,2} = W_n$. Moreover observe that,

$$\mathcal{N}_{n,2} \supsetneq \mathcal{N}_{n,3} \supsetneq \cdots \supsetneq \mathcal{N}_{n,n-1},$$

where $\mathcal{N}_{n,k}$ is the poset of nested sets in $\widehat{\mathcal{F}}_{n,k}$, which immediately implies that $Q_{n,k} \ncong Q_{n,k'}$ for $k \neq k'$.

7.3 Canonical Quasitoric Manifold Over a Cyclohedron

The cyclohedron W_n, together with the associated canonical map ϕ_n, restricted to the set of vertices of W_n° ($\leftrightarrow$ the set of facets of W_n), defines a *combinatorial quasitoric pair* (W_n, ϕ_n) in the sense of Buchstaber and Panov (2015, Definition 7.3.10). Indeed, if $F_1, \ldots, F_{n-1}$ are distinct facets of W_n such that $\cap_{i=1}^{n-1} F_i \neq \emptyset$, then the corresponding dual vertices $v_1, \ldots, v_{n-1}$ of W_n° span a simplex and the vectors $\phi(v_1), \ldots, \phi(v_{n-1})$ form a basis of the associated type A root lattice $\Lambda_n \cong \mathbb{Z}^{n-1}$ (spanned by the vertices of the root polytope $Root_n$).

We refer to the associated quasitoric manifold $M = M_{(W_n, \phi_n)}$ as the *canonical quasitoric manifold* over a cyclohedron W_n.

7.4 The Cyclohedron and the Self-Linking Knot Invariants

It may be expected that the combinatorics of the map $\phi : W_n^\circ \to Root_n$, as illustrated by Theorems 14 and 31 (and their proofs), may be of some relevance for other applications where the cyclohedron W_n played an important role. Perhaps the most interesting is the role of the cyclohedron in the combinatorics of the self-linking knot invariants (Bott and Taubes 1994; Volić 2013). Other potentially interesting applications include some problems of discrete geometry, as exemplified by the 'polygonal pegs problem' (Vrećica and Živaljević 2011) and its relatives.

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