The Index of a Local Boundary Value Problem for Strongly Callias-Type Operators

Maxim Braverman[1] · **Pengshuai Shi[2]**

Abstract

We consider a complete Riemannian manifold M whose boundary is a disjoint union of finitely many complete connected Riemannian manifolds. We compute the index of a local boundary value problem for a strongly Callias-type operator on M. Our result extends an index theorem of D. Freed to non-compact manifolds, thus providing a new insight on the Hořava–Witten anomaly.

Keywords Callias · Index · Boundary value problem · Chiral anomaly · Cobordism

Mathematics Subject Classification 58J28 · 58J30 · 58J32 · 19K56

1 Introduction

A mathematical description of many anomalies in quantum field theory is given by index theorems for boundary value problems, cf. (Atiyah and Singer 1984; Bär and Strohmaier 2016; Freed 1986; Witten 1985) (see Bertlmann 2001, Ch. 11) for more details). A new type of anomalies, related to index computation on an *odd dimensional* space $\mathbb{R}^{10} \times [0, 1]$, was discovered by Hořava and Witten in (1996). Freed (1998), replaced $\mathbb{R}^{10}$ with a compact manifold, and proved a new index theorem for a *local boundary value problem* on a compact manifold with boundary, which explains the Hořava-Witten anomaly, but only for this compact case. It is desirable to give a math-

✉ Maxim Braverman
maximbraverman@neu.edu
http://www.math.neu.edu/~braverman/

Pengshuai Shi
pengshuai.shi@gmail.com

1   Department of Mathematics, Northeastern University, Boston, MA 02115, USA

2   Beijing International Center for Mathematical Research (BICMR), Beijing (Peking) University, Beijing 100871, China

ematically rigorous construction of such an index on non-compact manifolds with non-compact boundary.

A systematic treatment of boundary value problems for strongly Callias-type operators on non-compact manifolds with non-compact boundary was given in Braverman and Shi (2017b), where we extended the approach of Bär and Ballmann (2012) to non-compact setting. In particular, we defined elliptic boundary conditions and proved that the corresponding boundary value problem is Fredholm. We emphasize that these results are valid for so called strongly Callias-type operators — Dirac operators coupled with an electric potential satisfying certain growth conditions at infinity.

One advantage of the approach to boundary value problems in Bär and Ballmann (2012), Braverman and Shi (2017b) is that it unifies local and non-local (eg. Atiyah–Patodi–Singer) boundary conditions. In Braverman and Shi (2017a, b) we studied the index defined by (generalized) Atiyah–Patodi–Singer boundary conditions on manifolds with non-compact boundary. In Shi (2018) the second author studied the Calderón projection in the non-compact situation and obtained an expression of the index of an elliptic boundary value problem in terms of the relative index of the Calderón projection and the projection onto the boundary conditions. In the current paper we study local boundary conditions for Callias-type operators and prove a non-compact analogue of Freed's index theorem (Freed 1998, Theorem B).

We now give a brief description of our results.

Let M be a complete Riemannian manifold with non-compact boundary ∂M. We assume that $\partial M = \bigsqcup_{j=1}^{k} N_j$ is a disjoint union of finitely many connected components. Then each N_j is a complete manifold without boundary. Let E be an (ungraded) Dirac bundle over M and let D denote the Dirac operator on E. Let $\mathcal{D} = D + \Psi$ be a formally self-adjoint Callias-type operator on M. We impose slightly stronger conditions on the growth of the potential Ψ and call the operators satisfying these conditions *strongly Callias-type*. On manifolds without boundary these conditions guarantee that $\mathcal{D}$ has a discrete spectrum.

The restriction $\mathcal{A}_j$ ($j = 0, \ldots, k$) of $\mathcal{D}$ to the boundary component N_j is a self-adjoint strongly Callias-type operator on N_j and, hence, has discrete spectrum. In particular, it is Fredholm. Moreover, the Clifford multiplication by the unit normal vector to the boundary defines a grading on $E_{N_j} := E|_{N_j}$ and $\mathcal{A}_j$ is odd with respect to this grading. We denote by ind $\mathcal{A}_j$ its index.

Set $B_j^{\pm} := L^2(N_j, E_{N_j}^{\pm})$. Let $\epsilon := (\epsilon_1, \ldots, \epsilon_k)$ where $\epsilon_j = \pm$ and set $B^{\epsilon} := \bigoplus_{j=1}^{k} B_j^{\epsilon_j}$. Then $B^{\epsilon} \subset L^2(\partial M, E_{\partial M})$, where $E_{\partial M} := E|_{\partial M}$. We use the result in Sect. 3 to show that B^{ϵ} defines an elliptic boundary condition for $\mathcal{D}$. Hence the operator

$$\mathcal{D}_{B^{\epsilon}} : \left\{ u \in L^2(M, E) : u|_{\partial M} \in B^{\epsilon} \right\} \rightarrow L^2(M, E). \tag{1.1}$$

is Fredholm.

Our main result in this paper (cf. Theorem 5.4) is the following generalization of of (Freed 1998, Theorem B):

$$\operatorname{ind} \mathcal{D}_{B^{\epsilon}} = \sum_{\{j \,:\, \epsilon_j = +\}} \operatorname{ind} \mathcal{A}_j = - \sum_{\{j \,:\, \epsilon_j = -\}} \operatorname{ind} \mathcal{A}_j. \tag{1.2}$$

If all ϵ_j have the same sign, then (1.2) becomes

$$\operatorname{ind} \mathcal{D}_{B^\epsilon} \;=\; \sum_{j=1}^{k} \operatorname{ind} \mathcal{A}_j \;=\; 0.$$

This gives a new proof of the cobordism invariance of the index of Callias-type operators, cf. (Braverman and Shi 2016).

Freed (1998), only considers the case when the dimension of M is odd. This is because the index of an elliptic differential operator on a *compact* odd-dimensional manifold without boundary vanishes and the compact analogue of (1.2) is trivial when $\dim M =$ even. This is not the case for the index of Callias-type operators on non-compact manifolds. In fact, the odd-dimensional case is very interesting and is the subject of the celebrated Callias-type index theorem (Anghel 1993; Bunke 1995). That is why we don't assume that the dimension of M is odd.

2 Boundary Value Problems for Manifolds with Non-compact Boundary

In the beginning of this section we briefly recall the notion of strongly Callias-type operator and define the scale of Sobolev spaces defined by such an operator. We then recall the definition of an elliptic boundary value problem for a strongly Callias-type operator (Braverman and Shi 2017b) and define its index.

2.1 Strongly Callias-Type Operator

Let M be a complete Riemannian manifold (possibly with boundary) and let $E \to M$ be a Dirac bundle over M, cf. (Lawson and Michelsohn 1989, Definition II.5.2). In particular, E is a Hermitian vector bundle endowed with a Clifford multiplication $c : T^*M \to \operatorname{End}(E)$ and a compatible Hermitian connection ∇^E. Let $D : C^\infty(M, E) \to C^\infty(M, E)$ be the Dirac operator defined by the connection ∇^E. Let $\Psi \in \operatorname{End}(E)$ be a self-adjoint bundle map (called a *Callias potential*). Then

$$\mathcal{D} \;:=\; D + \Psi$$

is a formally self-adjoint Dirac-type operator on E and

$$\mathcal{D}^2 = D^2 + \Psi^2 + [D, \Psi]_+, \tag{2.1}$$

where $[D, \Psi]_+ := D \circ \Psi + \Psi \circ D$ is the anticommutator of the operators D and Ψ.

Definition 2.1 We call $\mathcal{D}$ a *self-adjoint strongly Callias-type operator* if

(1) $[D, \Psi]_+$ is a zeroth order differential operator, i.e. a bundle map;

(2) for any $R > 0$, there exists a compact subset $K_R \subset M$ such that

$$\Psi^2(x) \; - \; \big|[D, \Psi]_+(x)\big| \; \geq \; R \tag{2.2}$$

for all $x \in M \backslash K_R$. In this case, the compact set K_R is called an *R-essential support* of $\mathcal{D}$.

2.2 Restriction to the Boundary

Assume that the Riemannian metric g^M is *product near the boundary*, that is, there exists a neighborhood $U \subset M$ of the boundary which is isometric to the cylinder

$$Z_r \; := \; [0, r) \times \partial M. \tag{2.3}$$

In the following we identify U with Z_r and denote by t the coordinate along the axis of $[0, r)$. Then the inward unit conormal to the boundary is given by $\tau = dt$.

Furthermore, we assume that the Dirac bundle E is *product near the boundary*. This means that the Clifford multiplication $c : T^*M \to \mathrm{End}(E)$ and the connection ∇^E have product structure on Z_r, cf. (Braverman and Shi 2017b, §3.7). Then the restriction of D to Z_r takes the form

$$D \; = \; c(\tau)(\partial_t + A), \tag{2.4}$$

where $A : C^\infty(\partial M, E_{\partial M}) \to C^\infty(\partial M, E_{\partial M})$ is a self-adjoint Dirac-type operator which anticommutes with $c(\tau)$:

$$c(\tau) \circ A \; = \; - A \circ c(\tau). \tag{2.5}$$

Let $\mathcal{D} = D + \Psi$ be a self-adjoint strongly Callias-type operator. Suppose Ψ does not depend on t on Z_r. Then the restriction of $\mathcal{D}$ to Z_r is given by

$$\mathcal{D} \; = \; c(\tau)(\partial_t + \mathcal{A}), \tag{2.6}$$

where $\mathcal{A} := A - c(\tau)\Psi : C^\infty(\partial M, E_{\partial M}) \to C^\infty(\partial M, E_{\partial M})$ is the *restriction of $\mathcal{D}$ to the boundary*.

Condition (1) of Definition 2.1 is equivalent to the condition that Ψ anticommutes with the Clifford multiplication: $\big[c(\xi), \Psi\big]_+ = 0$, for all $\xi \in T^*M$. It follows that $c(\tau)\Psi \in \mathrm{End}(E_{\partial M})$ is a self-adjoint bundle map which anticommutes with $c(\tau)$. Hence, using (2.5), we obtain

$$c(\tau) \circ \mathcal{A} \; = \; - \mathcal{A} \circ c(\tau). \tag{2.7}$$

In addition, $\mathcal{A}$ is a strongly Callias-type operator, cf. Lemma 3.12 of Braverman and Shi (2017b). In particular, it has discrete spectrum.

Definition 2.2 We say that a self-adjoint strongly Callias-type operator $\mathcal{D}$ is *product near the boundary* if the Dirac bundle E is product near the boundary and the restriction of the Callias potential Ψ to Z_r does not depend on t. The operator $\mathcal{A}$ of (2.6) is called the *restriction of $\mathcal{D}$ to the boundary*.

Throughout the paper we assume that $\mathcal{D}$ is product near the boundary.

Remark 2.3 The index theory of a boundary value problem on a compact manifold with boundary can be extended to the non-product case, (Braverman and Maschler 2017; Gilkey 1993). In this case there is a correction to the index formula given by the integral of a *transgression form* over the boundary. Though in certain situations one can slightly release the product condition even in the non-compact setting, cf. [7], in general an elliptic boundary value problem non-compact manifold with non-product structure near the boundary is not Fredholm. This is because the integral of the transgression form might diverge. It would be very interesting to find a general enough condition on the shape of the manifold near the boundary, which would guarantee the Fredholmness.

2.3 The Grading on the Boundary

Let $E_{\partial M}^{\pm} \subset E_{\partial M}$ denote the span of the eigenvectors of $c(\tau)$ associated with eigenvalue $\pm i$. Then $E_{\partial M} = E_{\partial M}^+ \oplus E_{\partial M}^-$. By (2.7), with respect to this decomposition

$$\mathcal{A} = \begin{pmatrix} 0 & \mathcal{A}^- \\ \mathcal{A}^+ & 0 \end{pmatrix}, \tag{2.8}$$

where $\mathcal{A}^{\pm} := \mathcal{A}|_{E_{\partial M}^{\pm}}$.

2.4 Sobolev Spaces

We recall the definition of Sobolev spaces $H_{\mathcal{A}}^s(\partial M, E_{\partial M})$ of sections over ∂M which depend on the boundary operator $\mathcal{A}$, cf. (Braverman and Shi 2017b, §3.13).

Definition 2.4 Set

$$C_{\mathcal{A}}^{\infty}(\partial M, E_{\partial M})$$
$$:= \left\{ \mathbf{u} \in C^{\infty}(\partial M, E_{\partial M}) : \left\| (\mathrm{id} + \mathcal{A}^2)^{s/2} \mathbf{u} \right\|_{L^2(\partial M, E_{\partial M})}^2 < +\infty \text{ for all } s \in \mathbb{R} \right\}.$$

For all $s \in \mathbb{R}$ we define the *Sobolev $H_{\mathcal{A}}^s$-norm* on $C_{\mathcal{A}}^{\infty}(\partial M, E_{\partial M})$ by

$$\|\mathbf{u}\|_{H_{\mathcal{A}}^s(\partial M, E_{\partial M})}^2 := \left\| (\mathrm{id} + \mathcal{A}^2)^{s/2} \mathbf{u} \right\|_{L^2(\partial M, E_{\partial M})}^2. \tag{2.9}$$

The Sobolev space $H_{\mathcal{A}}^s(\partial M, E_{\partial M})$ is defined to be the completion of $C_{\mathcal{A}}^{\infty}(\partial M, E_{\partial M})$ with respect to this norm.

2.5 The Hybrid Soblev Spaces

For $I \subset \mathbb{R}$, let $P_I^{\mathcal{A}} : L^2(\partial M, E_{\partial M}) \to L^2(\partial M, E_{\partial M})$ be the spectral projection onto the span of the eigenvectors of $\mathcal{A}$ with eigenvalues in I. It's easy to see that P_I extends to a continuous projection on $H_{\mathcal{A}}^s(\partial M, E_{\partial M})$ for all $s \in \mathbb{R}$. We set

$$H_I^s(\mathcal{A}) := P_I^{\mathcal{A}}(H_{\mathcal{A}}^s(\partial M, E_{\partial M})) \subset H_{\mathcal{A}}^s(\partial M, E_{\partial M}).$$

Definition 2.5 For $a \in \mathbb{R}$, we define the *hybrid* Sobolev space

$$
\begin{aligned}
\check{H}(\mathcal{A}) &:= H_{(-\infty, a)}^{1/2}(\mathcal{A}) \oplus H_{[a, \infty)}^{-1/2}(\mathcal{A}) \subset H_{\mathcal{A}}^{-1/2}(\partial M, E_{\partial M}), \\
\hat{H}(\mathcal{A}) &:= H_{(-\infty, a)}^{-1/2}(\mathcal{A}) \oplus H_{[a, \infty)}^{1/2}(\mathcal{A}) \subset H_{\mathcal{A}}^{-1/2}(\partial M, E_{\partial M})
\end{aligned}
\tag{2.10}
$$

with $\check{H}$-norm or $\hat{H}$-norm

$$
\begin{aligned}
\|\mathbf{u}\|_{\check{H}(\mathcal{A})}^2 &:= \left\| P_{(-\infty, a)}^{\mathcal{A}} \mathbf{u} \right\|_{H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})}^2 + \left\| P_{[a, \infty)}^{\mathcal{A}} \mathbf{u} \right\|_{H_{\mathcal{A}}^{-1/2}(\partial M, E_{\partial M})}^2, \\
\|\mathbf{u}\|_{\hat{H}(\mathcal{A})}^2 &:= \left\| P_{(-\infty, a)}^{\mathcal{A}} \mathbf{u} \right\|_{H_{\mathcal{A}}^{-1/2}(\partial M, E_{\partial M})}^2 + \left\| P_{[a, \infty)}^{\mathcal{A}} \mathbf{u} \right\|_{H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})}^2.
\end{aligned}
$$

The space $\check{H}(\mathcal{A})$ or $\hat{H}(\mathcal{A})$ is independent of the choice of a. They are dual to each other. Note from (2.7) that $c(\tau)$ induces an isomorphism between $\check{H}(\mathcal{A})$ and $\hat{H}(\mathcal{A})$. By Theorem 3.39 of Braverman and Shi (2017b), the hybrid space $\check{H}(\mathcal{A})$ coincides with the space of restrictions to the boundary of sections of E which lie in the maximal domain of $\mathcal{D}$.

2.6 Elliptic Boundary Value Problems

We are now ready to define elliptic boundary conditions for $\mathcal{D}$.

Definition 2.6 An *elliptic boundary condition* for $\mathcal{D}$ is a closed subspace $B \subset \check{H}(\mathcal{A})$ such that both B and its adjoint boundary value space

$$B^{\mathrm{ad}} := \left\{ \mathbf{v} \in \check{H}(\mathcal{A}) : \left(\mathbf{u}, c(\tau) \mathbf{v} \right) = 0 \text{ for all } \mathbf{u} \in B \right\} \tag{2.11}$$

are subspaces of $H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$.

We have shown in Braverman and Shi (2017b) that, if $\mathcal{D}$ is product near the boundary, then an elliptic boundary value problem $\mathcal{D}_B$ is Fredholm. Its *index* is defined by

$$\mathrm{ind}\, \mathcal{D}_B := \dim \ker \mathcal{D}_B - \dim \ker \mathcal{D}_{B^{\mathrm{ad}}} \in \mathbb{Z}.$$

3 Elliptic Boundary Conditions and Fredholm Pairs

In Shi (2018), the second author studied the relationship between Atiyah–Patodi–Singer index and the Cauchy data spaces using the method of Fredholm pairs. In

this section, we provide a new description of elliptic boundary conditions from the perspective of Fredholm pairs. The main result is similar in spirit to Definition 7.5 of Bär and Ballmann (2012).

3.1 Fredholm Pair of Subspaces

We recall the notion of a Fredholm pair of subspaces. Let Z be a Hilbert space. A pair (X, Y) of closed subspaces of Z is called a *Fredholm pair* if

  (i) $\dim(X \cap Y) < \infty$;
 (ii) $X + Y$ is a closed subspace of Z;
(iii) $\operatorname{codim}(X + Y) := \dim Z/(X + Y) < \infty$.

The *index* of a Fredholm pair (X, Y) is defined to be

$$\operatorname{ind}(X, Y) := \dim(X \cap Y) - \operatorname{codim}(X + Y) \in \mathbb{Z}.$$

3.2 Fredholm Pairs Associated to an Elliptic Boundary Condition

Let $\mathcal{D} : C^\infty(M, E) \to C^\infty(M, E)$ be a self-adjoint strongly Callias-type operator on a manifold with non-compact boundary. Let $\mathcal{A} : C^\infty(\partial M, E_{\partial M}) \to C^\infty(\partial M, E_{\partial M})$ be the restriction of $\mathcal{D}$ to ∂M. Recall that by Definition 2.6 a closed subspace B of $\check{H}(\mathcal{A})$ is an elliptic boundary condition if $B \subset H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$ and $B^{\mathrm{ad}} \subset H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$. Since the $H^{1/2}_{\mathcal{A}}$-norm is stronger than the $\check{H}$-norm, in this case B is also closed in $H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$. Generally, if B is a closed subspace of $H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$, then we define

$$B^* := B^0 \cap H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M}), \tag{3.1}$$

where $B^0 \subset H^{-1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$ is the annihilator of B. The main result of this section is the following equivalent definition of elliptic boundary conditions.

Theorem 3.1 *A subspace $B \subset H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$ is an elliptic boundary condition for $\mathcal{D}$ if and only if*

(1) *B is closed in $H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$,*
(2) *$(H^{1/2}_{[0,\infty)}(\mathcal{A}), B)$, $(H^{1/2}_{(-\infty,0)}(\mathcal{A}), B^*)$ are Fredholm pairs in $H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$, and*
(3) *$\operatorname{ind}(H^{1/2}_{[0,\infty)}(\mathcal{A}), B) = -\operatorname{ind}(H^{1/2}_{(-\infty,0)}(\mathcal{A}), B^*)$.*

If B satisfies (1), (2) and (3), then $B^{\mathrm{ad}} = c(\tau)B^*$.

A very typical elliptic boundary condition is the (generalized) Atiyah–Patodi–Singer boundary condition $B = H^{1/2}_{(-\infty,a)}(\mathcal{A})$ for some $a \in \mathbb{R}$. In this case $B^* = H^{1/2}_{[a,\infty)}(\mathcal{A})$. One immediately sees that both $(H^{1/2}_{[0,\infty)}(\mathcal{A}), H^{1/2}_{(-\infty,a)}(\mathcal{A}))$ and $(H^{1/2}_{(-\infty,0)}(\mathcal{A}), H^{1/2}_{[a,\infty)}(\mathcal{A}))$ are Fredholm pairs in $H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$, and

$$\text{ind}(H^{1/2}_{[0,\infty)}(\mathcal{A}), H^{1/2}_{(-\infty,a)}(\mathcal{A})) = -\dim L^2_{[a,0)}(\mathcal{A})$$

$$= -\text{ind}(H^{1/2}_{(-\infty,0)}(\mathcal{A}), H^{1/2}_{[a,\infty)}(\mathcal{A})) \quad \text{for } a < 0$$

or

$$\text{ind}(H^{1/2}_{[0,\infty)}(\mathcal{A}), H^{1/2}_{(-\infty,a)}(\mathcal{A})) = \dim L^2_{[0,a)}(\mathcal{A})$$

$$= -\text{ind}(H^{1/2}_{(-\infty,0)}(\mathcal{A}), H^{1/2}_{[a,\infty)}(\mathcal{A})) \quad \text{for } a \geq 0.$$

We break the proof of Theorem 3.1 into several steps which occupy the next two subsections.

3.3 Proof of the "if" Direction

We apply the arguments of (Shi 2018, Subsection 3.3). Suppose $(H^{1/2}_{[0,\infty)}(\mathcal{A}), B)$ is a Fredholm pair in $H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$. Write B in the following direct sum of the pair of transversal subspaces

$$B = \left(H^{1/2}_{[0,\infty)}(\mathcal{A}) \cap B\right) \dotplus V,$$

where V is some closed subspace of $H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$. Let $\pi_<$ (resp. $\pi_\geq$) be the projection of V onto $H^{1/2}_{(-\infty,0)}(\mathcal{A})$ (resp. $H^{1/2}_{[0,\infty)}(\mathcal{A})$) along $H^{1/2}_{[0,\infty)}(\mathcal{A})$ (resp. $H^{1/2}_{(-\infty,0)}(\mathcal{A})$). Then $\pi_<$ is injective and

$$\text{range } \pi_< = \left(H^{1/2}_{[0,\infty)}(\mathcal{A}) + B\right) \cap H^{1/2}_{(-\infty,0)}(\mathcal{A})$$

is closed (in both $H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$ and $\check{H}(\mathcal{A})$). By closed graph theorem, $\pi_<$ has a bounded inverse $\iota_< : \text{range } \pi_< \to V$. One then has a bounded operator $\phi := \pi_\geq \circ \iota_< : \text{range } \pi_< \to \text{range } \pi_\geq$. Then $V = \text{graph}(\phi)$, and, hence,

$$B = \left(H^{1/2}_{[0,\infty)}(\mathcal{A}) \cap B\right) \dotplus \text{graph}(\phi).$$

Let $\check{\phi}$ be the composition

$$\text{range } \pi_< \overset{\phi}{\to} \text{range } \pi_\geq \hookrightarrow H^{-1/2}_{[0,\infty)}(\mathcal{A}).$$

Viewed as a map from a closed subspace of $\check{H}(\mathcal{A})$ to $\check{H}(\mathcal{A})$, $\check{\phi}$ is a bounded operator. Note that B can also be written as

$$B = \left(H^{1/2}_{[0,\infty)}(\mathcal{A}) \cap B\right) \dotplus \text{graph}(\check{\phi}) \subset \check{H}(\mathcal{A}). \tag{3.2}$$

Since the first summand is finite-dimensional, B is closed in $\check{H}(\mathcal{A})$.

It now remains to show that $B^{\text{ad}} \subset H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$. We need the following lemma.

Lemma 3.2 $(H^{-1/2}_{[0,\infty)}(\mathcal{A}), B)$ *is a Fredholm pair in* $\check{H}(\mathcal{A})$ *and*

$$\mathrm{ind}\left(H^{-1/2}_{[0,\infty)}(\mathcal{A}), B\right) = \mathrm{ind}\left(H^{1/2}_{[0,\infty)}(\mathcal{A}), B\right).$$

Proof Let $\Pi_<$ denote the projection from $\check{H}(\mathcal{A})$ onto $H^{1/2}_{(-\infty,0)}(\mathcal{A})$ along $H^{1/2}_{[0,\infty)}(\mathcal{A})$. Then $\Pi_<$ is an orthogonal projection with respect to the scalar product on $\check{H}(\mathcal{A})$. Its restriction to $H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$ is also the orthogonal projection (with respect to the scalar product on $H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$) from $H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$ onto $H^{1/2}_{(-\infty,0)}(\mathcal{A})$. Since $(H^{1/2}_{[0,\infty)}(\mathcal{A}), B)$ is a Fredholm pair in $H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$, by (Shi 2018, Proposition 3.5), $\Pi_<|_B : B \to H^{1/2}_{(-\infty,0)}(\mathcal{A})$ is a Fredholm operator and

$$\mathrm{ind}\left(H^{1/2}_{[0,\infty)}(\mathcal{A}), B\right) = \mathrm{ind}\,\Pi_<|_B.$$

Note that B is also closed in $\check{H}(\mathcal{A})$. It then follows that $(H^{-1/2}_{[0,\infty)}(\mathcal{A}), B)$ is a Fredholm pair in $\check{H}(\mathcal{A})$ and

$$\mathrm{ind}\left(H^{-1/2}_{[0,\infty)}(\mathcal{A}), B\right) = \mathrm{ind}\,\Pi_<|_B = \mathrm{ind}(H^{1/2}_{[0,\infty)}(\mathcal{A}), B).$$

$\square$

Now we use the hypothesis that $(H^{1/2}_{(-\infty,0)}(\mathcal{A}), B^*)$ is a Fredholm pair in $H^{1/2}_{\mathcal{A}}(\partial M, E_{\partial M})$. Applying the above discussions similarly, one concludes that B^* is a closed subspace of $\hat{H}(\mathcal{A})$. Moreover, $(H^{-1/2}_{(-\infty,0)}(\mathcal{A}), B^*)$ is a Fredholm pair in $\hat{H}(\mathcal{A})$ satisfying

$$\mathrm{ind}\left(H^{-1/2}_{(-\infty,0)}(\mathcal{A}), B^*\right) = \mathrm{ind}\left(H^{1/2}_{(-\infty,0)}(\mathcal{A}), B^*\right). \tag{3.3}$$

Recall that B is closed in $\check{H}(\mathcal{A})$. As a subspace of $\check{H}(\mathcal{A})$, we denote its annhilator to be

$$\hat{B}^0 := B^0 \cap \hat{H}(\mathcal{A}). \tag{3.4}$$

Clearly, $B^* \subset \hat{B}^0$. From Lemma 3.2 and (Shi 2018, Proposition 3.5), we obtain that $(H^{-1/2}_{(-\infty,0)}(\mathcal{A}), \hat{B}^0)$ is a Fredholm pair in $\hat{H}(\mathcal{A})$ and

$$\mathrm{ind}\left(H^{-1/2}_{(-\infty,0)}(\mathcal{A}), \hat{B}^0\right) = -\mathrm{ind}\left(H^{-1/2}_{[0,\infty)}(\mathcal{A}), B\right) = -\mathrm{ind}\left(H^{1/2}_{[0,\infty)}(\mathcal{A}), B\right). \tag{3.5}$$

Combining (3.3), (3.5) and Theorem 3.1 (3) yields that

$$\mathrm{ind}\left(H^{-1/2}_{(-\infty,0)}(\mathcal{A}), B^*\right) = \mathrm{ind}\left(H^{-1/2}_{(-\infty,0)}(\mathcal{A}), \hat{B}^0\right).$$

Now using (Shi 2018, Lemma 4.1), we finally obtain $B^* = \hat{B}^0$. Since $B^{\mathrm{ad}} = c(\tau)\hat{B}^0$, one deduces that $B^{\mathrm{ad}} \subset H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$. Therefore B is an elliptic boundary condition for $\mathcal{D}$.

3.4 Proof of the "Only if" Direction

Let $B \subset \check{H}(\mathcal{A})$ be an elliptic boundary condition. Then condition (1) of Theorem 3.1 is automatically true and the $\check{H}$-norm is equivalent to the $H_{\mathcal{A}}^{1/2}$-norm on B. Let $\Pi_<$ (resp. $\Pi_\geq$) denote the orthogonal projection from $\check{H}(\mathcal{A})$ onto $H_{(-\infty,0)}^{1/2}(\mathcal{A})$ (resp. $H_{[0,\infty)}^{-1/2}(\mathcal{A})$). For any $\mathbf{u} \in B$, there exists a constant $C > 0$ such that

$$\|\mathbf{u}\|_{H_{\mathcal{A}}^{1/2}} \leq C\|\mathbf{u}\|_{\check{H}(\mathcal{A})} = C\big(\|\Pi_<\mathbf{u}\|_{H_{\mathcal{A}}^{1/2}} + \|\Pi_\geq\mathbf{u}\|_{H_{\mathcal{A}}^{-1/2}}\big).$$

By (Braverman and Shi 2017b, Theorem 3.19), the map

$$\Pi_\geq \; : \; B \; \subset \; H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M}) \; \to \; H_{[0,\infty)}^{1/2}(\mathcal{A}) \; \hookrightarrow \; H_{[0,\infty)}^{-1/2}(\mathcal{A})$$

is compact. Using (Bär and Ballmann 2012, Proposition A.3), one concludes that $\Pi_< : B \to H_{(-\infty,0)}^{1/2}(\mathcal{A})$ has finite-dimensional kernel and closed image. In other words, $H_{[0,\infty)}^{1/2}(\mathcal{A}) \cap B$ is finite-dimensional and $H_{[0,\infty)}^{1/2}(\mathcal{A}) + B$ (resp. $H_{[0,\infty)}^{-1/2}(\mathcal{A}) + B$) is a closed subspace of $H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$ (resp. $\check{H}(\mathcal{A})$). By the fact that B is an elliptic boundary condition,

$$B^{\mathrm{ad}} = c(\tau)\hat{B}^0 \; \subset \; H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M}),$$

where $\hat{B}^0$ is defined in (3.4). Viewed as a subspace of $\check{H}(\mathcal{A})$, the annihilator of $H_{[0,\infty)}^{-1/2}(\mathcal{A}) + B$ is

$$H_{(-\infty,0)}^{-1/2}(\mathcal{A}) \cap \hat{B}^0 \; \cong \; H_{(0,\infty)}^{-1/2}(\mathcal{A}) \cap B^{\mathrm{ad}} = H_{(0,\infty)}^{1/2}(\mathcal{A}) \cap B^{\mathrm{ad}}.$$

By the same reason as above,

$$+\infty \; > \; \dim\left(H_{(0,\infty)}^{1/2}(\mathcal{A}) \cap B^{\mathrm{ad}}\right) = \mathrm{codim}\left(H_{[0,\infty)}^{-1/2}(\mathcal{A}) + B\right).$$

Therefore $(H_{[0,\infty)}^{-1/2}(\mathcal{A}), B)$ is a Fredholm pair in $\check{H}(\mathcal{A})$.

Using the same argument as in the proof of Lemma 3.2, we can prove the following

Lemma 3.3 $(H_{[0,\infty)}^{1/2}(\mathcal{A}), B)$ *is a Fredholm pair in* $H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$ *and*

$$\mathrm{ind}\left(H_{[0,\infty)}^{1/2}(\mathcal{A}), B\right) = \mathrm{ind}\left(H_{[0,\infty)}^{-1/2}(\mathcal{A}), B\right). \tag{3.6}$$

Since B is an elliptic boundary condition, from the discussion above, $\hat{B}^0 \subset H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$. Thus

$$B^* = \hat{B}^0 \cap H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M}) = \hat{B}^0.$$

By the fact that $(H_{[0,\infty)}^{-1/2}(\mathcal{A}), B)$ is a Fredholm pair in $\check{H}(\mathcal{A})$, one concludes that $(H_{(-\infty,0)}^{-1/2}(\mathcal{A}), B^*)$ is a Fredholm pair in $\hat{H}(\mathcal{A})$. It then follows from Lemma 3.3 that $(H_{(-\infty,0)}^{1/2}(\mathcal{A}), B^*)$ is a Fredholm pair in $H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$ and

$$
\begin{aligned}
&\operatorname{ind}\left(H_{(-\infty,0)}^{1/2}(\mathcal{A}), B^*\right) \\
&\quad = \operatorname{ind}\left(H_{(-\infty,0)}^{-1/2}(\mathcal{A}), B^*\right) = -\operatorname{ind}\left(H_{[0,\infty)}^{-1/2}(\mathcal{A}), B\right) = -\operatorname{ind}\left(H_{[0,\infty)}^{1/2}(\mathcal{A}), B\right),
\end{aligned}
$$

so conditions (2) and (3) of Theorem 3.1 are verified. We thus complete the proof of Theorem 3.1. $\qquad\square$

3.5 An Index Formula Regarding Fredholm Pairs

Let B_1 and B_2 be two elliptic boundary conditions for $\mathcal{D}$. Then $\mathcal{D}_{B_1}$ and $\mathcal{D}_{B_2}$ are Fredholm operators. By Theorem 3.1, $(H_{[0,\infty)}^{1/2}(\mathcal{A}), B_1)$ and $(H_{[0,\infty)}^{1/2}(\mathcal{A}), B_2)$ are Fredholm pairs in $H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$. The following theorem, which generalizes (Braverman and Shi 2017b, Proposition 5.8) and can be compared with (Bär and Ballmann 2012, Theorem 8.15) (where the boundary is compact), computes the difference of the two elliptic boundary value problems in terms of the indexes of Fredholm pairs.

Theorem 3.4 *Suppose $\mathcal{D}$ is a strongly Callias-type operator on M which is product near the boundary. Let $B_1, B_2 \subset H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$ be elliptic boundary conditions for $\mathcal{D}$. Then*

$$\operatorname{ind} \mathcal{D}_{B_1} - \operatorname{ind} \mathcal{D}_{B_2} = \operatorname{ind}\left(H_{[0,\infty)}^{1/2}(\mathcal{A}), B_1\right) - \operatorname{ind}\left(H_{[0,\infty)}^{1/2}(\mathcal{A}), B_2\right). \quad (3.7)$$

Let $B_2^\perp$ be the orthogonal complement of B_2 in $H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$. If $(B_2^\perp, B_1)$ is a Fredholm pair in $H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$, then

$$\operatorname{ind} \mathcal{D}_{B_1} - \operatorname{ind} \mathcal{D}_{B_2} = \operatorname{ind}(B_2^\perp, B_1). \quad (3.8)$$

Remark 3.5 Note that the hypothesis that $(B_2^\perp, B_1)$ is a Fredholm pair is essential. In Sect. 4 (cf. Remark 4.5), we will provide an example that the hypothesis does not hold.

Proof We first show that (3.7) can be implied by (3.8). Since both $(H_{[0,\infty)}^{1/2}(\mathcal{A}), B_1)$ and $(H_{[0,\infty)}^{1/2}(\mathcal{A}), B_2)$ are Fredholm pairs in $H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$, by (3.8),

$$\text{ind}\,\mathcal{D}_{B_1} - \text{ind}\,\mathcal{D}_{\text{APS}} = \text{ind}(H_{[0,\infty)}^{1/2}(\mathcal{A}), B_1),$$

$$\text{ind}\,\mathcal{D}_{B_2} - \text{ind}\,\mathcal{D}_{\text{APS}} = \text{ind}(H_{[0,\infty)}^{1/2}(\mathcal{A}), B_2),$$

where $\mathcal{D}_{\text{APS}}$ denotes the APS boundary value problem for $\mathcal{D}$. Then (3.7) is verified by taking the difference of the two equations.

We now prove (3.8). Since $(B_2^{\perp}, B_1)$ is a Fredholm pair in $H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$, we can adapt the idea of Sect. 3.3 to write B_1 in the following form

$$B_1 = (B_2^{\perp} \cap B_1) \dot{+} \text{graph}(\phi),$$

where ϕ is a bounded operator from $(B_2^{\perp} + B_1) \cap B_2$ to $B_2^{\perp}$. For $0 \le s \le 1$, let

$$B_{1,s} = (B_2^{\perp} \cap B_1) \dot{+} \text{graph}(s\phi).$$

Then $B_{1,1} = B_1$ and $B_{1,s}$ is an elliptic boundary condition for each s. Moreover, $(B_2^{\perp}, B_{1,s})$ is a Fredholm pair in $H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$ and

$$\text{ind}(B_2^{\perp}, B_{1,s}) = \text{ind}(B_2^{\perp}, B_1), \qquad \text{for any } s \in [0, 1].$$

Consider the family of Fredholm operators $\mathcal{D}_{B_{1,s}}$. Applying the arguments of (Bär and Ballmann 2012, Theorem 8.12) indicates that

$$\text{ind}\,\mathcal{D}_{B_{1,s}} = \text{ind}\,\mathcal{D}_{B_1}, \qquad \text{for any } s \in [0, 1].$$

Thus without loss of generality, one can assume that

$$B_1 = B_{1,0} = (B_2^{\perp} \cap B_1) \oplus ((B_2^{\perp} + B_1) \cap B_2).$$

Let X be the orthogonal complement of $B_2^{\perp} + B_1$ in $H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$. Note that X is a finite-dimensional space, so $B_1 \oplus X$ is still an elliptic boundary condition for $\mathcal{D}$. Since $B_1, B_2 \subset B_1 \oplus X$, using the idea of (Bär and Ballmann 2012, Corollary 8.8), we have

$$\text{ind}\,\mathcal{D}_{B_1} - \text{ind}\,\mathcal{D}_{B_1 \oplus X} = -\dim X = -\text{codim}(B_2^{\perp} + B_1),$$

$$\text{ind}\,\mathcal{D}_{B_2} - \text{ind}\,\mathcal{D}_{B_1 \oplus X} = -\dim(B_2^{\perp} \cap B_1).$$

Taking the difference of the two equations yields (3.8). $\qquad\qquad\square$

4 A Local Boundary Value Problem for Strongly Callias-Type Operators

In this section we introduce a local boundary condition for $\mathcal{D}$ and show that it is elliptic.

4.1 Splitting of the Vector Bundle on the Boundary

From now on we assume that there is given an orthogonal decomposition

$$E_{\partial M} = \tilde{E}_{\partial M}^+ \oplus \tilde{E}_{\partial M}^- \tag{4.1}$$

such that

$$\mathcal{A} = \begin{pmatrix} 0 & \mathcal{A}^- \\ \mathcal{A}^+ & 0 \end{pmatrix} \tag{4.2}$$

with respect to the grading (4.1). Here $\mathcal{A}^- = (\mathcal{A}^+)^*$.

The grading (4.1) might or might not be induced by the Clifford multiplication $c(\tau)$. One of the main and most natural examples of such grading is the grading defined in Sect. 2.3. This is the grading considered by Freed in (1998).

4.2 Boundary Value Space Induced by a Graded Bundle

By the decomposition (4.1), each $\mathbf{u} \in H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$ has the form $\mathbf{u} = (\mathbf{u}^+, \mathbf{u}^-)$. Consider closed subspaces

$$B^+ = \left\{ (\mathbf{u}^+, 0) \in H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M}) \right\}, \qquad B^- = \left\{ (0, \mathbf{u}^-) \in H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M}) \right\} \tag{4.3}$$

of $H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$. Then $H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M}) = B^+ \oplus B^-$.

Proposition 4.1 $(H_{[0,\infty)}^{1/2}(\mathcal{A}), B^+)$ *is a Fredholm pair in* $H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$ *and*

$$\mathrm{ind}\left(H_{[0,\infty)}^{1/2}(\mathcal{A}), B^+ \right) = \dim \ker \mathcal{A}^+. \tag{4.4}$$

Similarly, $(H_{[0,\infty)}^{1/2}(\mathcal{A}), B^-)$ *is a Fredholm pair and*

$$\mathrm{ind}\left(H_{[0,\infty)}^{1/2}(\mathcal{A}), B^- \right) = \dim \ker \mathcal{A}^-. \tag{4.5}$$

The proof of the proposition is based on the following

Lemma 4.2 *We have*

$$H_{[0,\infty)}^{1/2}(\mathcal{A}) \cap B^+ = \left\{ (\mathbf{u}^+, 0) : \mathbf{u}^+ \in \ker \mathcal{A}^+ \right\}. \tag{4.6}$$

In particular, $H_{[0,\infty)}^{1/2}(\mathcal{A}) \cap B^+ \cong \ker \mathcal{A}^+$.

Proof Since $\mathcal{A}$ is anti-diagonal with respect to the grading $E_{\partial M} = \tilde{E}_{\partial M}^+ \oplus \tilde{E}_{\partial M}^-$, one readily sees that if $\mathbf{v} = (\mathbf{v}^+, \mathbf{v}^-)$ is an eigenvector of $\mathcal{A}$ associated with eigenvalue λ, then $\tilde{\mathbf{v}} := (\mathbf{v}^+, -\mathbf{v}^-)$ is an eigenvector of $\mathcal{A}$ associated with eigenvalue $-\lambda$.

Each $\mathbf{u} = (\mathbf{u}^+, 0) \in H_{[0,\infty)}^{1/2}(\mathcal{A}) \cap B^+$ has an expansion into a sum of eigenvectors $\mathbf{u} = \sum a_j \mathbf{v}_j$, where $a_j \in \mathbb{C}$ and $\mathbf{v}_j = (\mathbf{v}_j^+, \mathbf{v}_j^-)$ is an eigenvector of $\mathcal{A}$ associated with eigenvalue $\lambda_j \geq 0$. Then $\mathbf{u}^+ = \sum a_j \mathbf{v}_j^+$ and $0 = \sum a_j \mathbf{v}_j^-$. It follows that $\mathbf{u} = \sum a_j \tilde{\mathbf{v}}_j$, where as above $\tilde{\mathbf{v}}_j = (\mathbf{v}_j^+, -\mathbf{v}_j^-)$. Since $\tilde{\mathbf{v}}_j$'s are eigenvectors associated with non-positive eigenvalues we conclude that $\mathbf{u} \in H_{(-\infty,0]}^{1/2}(\mathcal{A})$. Since

$$H_{(-\infty,0]}^{1/2}(\mathcal{A}) \cap H_{[0,\infty)}^{1/2}(\mathcal{A}) = \ker \mathcal{A},$$

we obtain $\mathbf{u} = (\mathbf{u}^+, 0) \in \ker \mathcal{A}$. Thus $\mathbf{u}^+ \in \ker \mathcal{A}^+$. It follows that $H_{[0,\infty)}^{1/2}(\mathcal{A}) \cap B^+ \subset \{ (\mathbf{u}^+, 0) : \mathbf{u}^+ \in \ker \mathcal{A}^+ \}$. The opposite inclusion is obvious. $\qquad\square$

Proof of Proposition 4.1 We only prove (4.4). The proof of the other equality is analogous.

In view of Lemma 4.2, we only need to show that

$$H_{[0,\infty)}^{1/2}(\mathcal{A}) + B^+ = H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M}). \tag{4.7}$$

Choose an arbitrary $\mathbf{u} = (\mathbf{u}^+, \mathbf{u}^-) \in H_{(-\infty,0)}^{1/2}(\mathcal{A})$. Then $\mathbf{v}_1 := (\mathbf{u}^+, -\mathbf{u}^-) \in H_{[0,\infty)}^{1/2}(\mathcal{A})$. Now let $\mathbf{v}_2 = (2\mathbf{u}^+, 0) \in B^+$. Then $-\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{u}$. So

$$H_{(-\infty,0)}^{1/2}(\mathcal{A}) \subset H_{[0,\infty)}^{1/2}(\mathcal{A}) + B^+.$$

Therefore (4.7) is true. This completes the proof of the proposition. $\qquad\square$

Since $H_{[0,\infty)}^{1/2}(\mathcal{A})$ and $H_{(-\infty,0)}^{1/2}(\mathcal{A})$, $B^\pm$ and $B^\mp$ are orthogonal to each other as subspaces of $H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$, it follows from Proposition 4.1 that

Corollary 4.3 $(H_{(-\infty,0)}^{1/2}(\mathcal{A}), B^\pm)$ *are Fredholm pairs in* $H_{\mathcal{A}}^{1/2}(\partial M, E_{\partial M})$ *and*

$$\mathrm{ind}\left(H_{(-\infty,0)}^{1/2}(\mathcal{A}), B^\pm\right) = -\dim \ker \mathcal{A}^\mp. \tag{4.8}$$

From Proposition 4.1 and Corollary 4.3, one readily sees that $B^\pm$ satisfies Theorem 3.1. Therefore we get

Theorem 4.4 $B^\pm$ *is an elliptic boundary condition for* $\mathcal{D}$, *whose adjoint boundary condition is* $c(\tau)B^\mp$.

We call $B^\pm$ the *boundary condition subject to the grading* (4.1). It is a local boundary condition for $\mathcal{D}$.

Remark 4.5 If in this situation we let B^+ and B^- be the B_1 and B_2 as in Theorem 3.4, then $(B_2^\perp, B_1) = (B^+, B^+)$ is not a Fredholm pair (cf. Remark 3.5).

Corollary 4.6 $\mathcal{D}_{B^+}$ *(resp.* $\mathcal{D}_{B^-}$*) is a Fredholm operator, whose adjoint operator is* $\mathcal{D}_{c(\tau)B^-}$ *(resp.* $\mathcal{D}_{c(\tau)B^+}$*).*

4.3 An Index Theorem

Substituting $B_1 = B^+$ and $B_2 = B^-$ in Theorem 3.4 and using Proposition 4.1, we get the following index formula for the local boundary value problem:

Theorem 4.7 *Let* $B^\pm$ *be as in Sect.* 4.2. *Then*

$$\operatorname{ind} \mathcal{D}_{B^+} \ - \ \operatorname{ind} \mathcal{D}_{B^-} = \operatorname{ind} \mathcal{A}. \tag{4.9}$$

5 Index of the Local Boundary Problem Subject to Natural Gradings

In this section we formulate our main result—the index theorem for a local boundary value problem similar to the one considered in Freed (1998). First we obtain a vanishing result for the index subject to the grading $E_{\partial M} = E_{\partial M}^+ \oplus E_{\partial M}^-$ given by the action of $c(\tau)$, cf. Sect. 2.3. As an application we obtain a new proof of the cobordism invariance of the index of a Callias-type operator. Then we assume that the boundary has multiple components $\partial M = \bigsqcup N_j$ with the local boundary condition $u|_{N_j} \in E_{\partial M}^\pm|_{N_j}$ and obtain an extension of the index theorem of Freed (1998), to our non-compact situation.

5.1 A Vanishing Result

Choose the grading (4.1) so that $c(\tau)|_{E_{\partial M}^\pm} = \pm i$ (this is, for example, the case for the spinor bundle on the boundary of an odd-dimensional manifold). Then near the boundary $\mathcal{D}$ has the form

$$\mathcal{D} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \left(\partial_t + \begin{pmatrix} 0 & \mathcal{A}^- \\ \mathcal{A}^+ & 0 \end{pmatrix} \right). \tag{5.1}$$

In this case $c(\tau)B^\pm = B^\pm$, thus $\mathcal{D}_{B^+}$ and $\mathcal{D}_{B^-}$ are adjoint operator to each other by Corollary 4.6.

Proposition 5.1 *Under the above assumption,* $\operatorname{ind} \mathcal{D}_{B^\pm} = 0$.

Proof A verbatim repetition of the arguments in the proof of (Bavnbek and Wojciechowski 1993, Theorem 21.5) shows that $\ker \mathcal{D}_{B^\pm} = \{0\}$. Since $(\mathcal{D}_{B^+})^{\mathrm{ad}} = \mathcal{D}_{B^-}$, the index of $\mathcal{D}_{B^\pm}$ vanishes.   □

Combining this proposition with Theorem 4.7, we obtain the following cobordism invariance of the index of strongly Callias-type operators (cf. Braverman and Shi 2016 where this result is proven by a different method. Yet another proof is given in (Braverman and Cecchini 2018, §2.7)):

Corollary 5.2 *Let* $\mathcal{A} : C^\infty(N, E^\pm) \to C^\infty(N, E^\mp)$ *be a graded strongly Callias-type operator on a non-compact manifold N. Suppose there exist a non-compact manifold M, a Dirac bundle $\hat{E} \to M$ and a self-adjoint strongly Callias-type operator $\mathcal{D} : C^\infty(M, \hat{E}) \to C^\infty(M, \hat{E})$ such that $\partial M = N$, $\hat{E}|_{\partial M} = E^+ \oplus E^-$ and $\mathcal{D}$ takes the form* (5.1) *near ∂M. Then* ind $\mathcal{A} = 0$.

5.2 The Case of Multiple Boundary Components

Assume that $\partial M = \bigsqcup_{j=1}^{k} N_j$ is a disjoint union of finitely many connected components. The restriction E_{N_j} of E to each connected component N_j inherits the grading (4.1):

$$E_{N_j} = E_{N_j}^+ \oplus E_{N_j}^-.$$

We denote the restriction of $\mathcal{D}$ to $E_{N_j}^\pm$ by $\mathcal{A}_j^\pm$. Then $E_{\partial M}^\pm = \bigoplus_{j=1}^{k} E_{N_j}^\pm$ and $\mathcal{A}^\pm = \bigoplus_{j=1}^{k} \mathcal{A}_j^\pm$. Let $\epsilon = (\epsilon_1, \ldots, \epsilon_k)$ with $\epsilon_j = +$ or $-$.

Definition 5.3 We call $B^\epsilon := \bigoplus_{j=1}^{k} B_j^{\epsilon_j}$ the *mixed boundary condition* subject to the grading (4.1), where $B_j^{\epsilon_j}$ is the local boundary condition (4.3) on each component (N_j, E_{N_j}) of the boundary.

Using the same arguments as in Sect. 4, one can show that B^ϵ is an elliptic boundary condition for $\mathcal{D}$, whose adjoint boundary condition is $B^{\bar{\epsilon}}$, where $\bar{\epsilon}_j = -\epsilon_j$. Applying Theorem 4.7 to this situation we obtain the following generalization of (Freed 1998, Theorem B):

Theorem 5.4 *Let M be a complete manifold with boundary and let $E \to M$ be a Dirac bundle over M. Let $\mathcal{D} = D + \Psi$ be a formally self-adjoint strongly Callias-type operator on E. Assume that the boundary ∂M of M is a disjoint union of finitely many connected components $\partial M = \bigsqcup_{j=1}^{k} N_j$. Fix $\epsilon := (\epsilon_1, \ldots, \epsilon_k)$ with $\epsilon_j = \pm$ and set $B^\epsilon = \bigoplus_{j=1}^{k} B_j^{\epsilon_j}$. Then*

$$\text{ind } \mathcal{D}_{B^\epsilon} = \sum_{\{j : \epsilon_j = +\}} \text{ind } \mathcal{A}_j = - \sum_{\{j : \epsilon_j = -\}} \text{ind } \mathcal{A}_j. \tag{5.2}$$

Proof The second equlity of (5.2) follows from Corollary 5.2.

We can apply Proposition 4.1 to each boundary component to conclude that $(H_{[0,\infty)}^{1/2}(\mathcal{A}_j), B_j^{\epsilon_j})$ is a Fredholm pair in $H_{\mathcal{A}_j}^{1/2}(N_j, E_{N_j})$ and

$$\text{ind } \left(H_{[0,\infty)}^{1/2}(\mathcal{A}_j), B_j^{\epsilon_j} \right) = \dim \ker \mathcal{A}_j^{\epsilon_j}.$$

As in the proof of Theorem 4.7, using (3.7) we obtain

$$\operatorname{ind} \mathcal{D}_{B^\epsilon} - \operatorname{ind} \mathcal{D}_{\mathrm{APS}} = \sum_{j=1}^{k} \operatorname{ind}(H^{1/2}_{[0,\infty)}(\mathcal{A}_j), B_j^{\epsilon_j}) = \sum_{j=1}^{k} \dim \ker \mathcal{A}_j^{\epsilon_j}$$

and

$$\operatorname{ind} \mathcal{D}_{B^-} - \operatorname{ind} \mathcal{D}_{\mathrm{APS}} = \sum_{j=1}^{k} \operatorname{ind}\left(H^{1/2}_{[0,\infty)}(\mathcal{A}_j), B_j^-\right) = \sum_{j=1}^{k} \dim \ker \mathcal{A}_j^-.$$

By Proposition 5.1, $\operatorname{ind} \mathcal{D}_{B^-} = 0$. Hence,

$$\begin{aligned}
\operatorname{ind} \mathcal{D}_{B^\epsilon} &= (\operatorname{ind} \mathcal{D}_{B^\epsilon} - \operatorname{ind} \mathcal{D}_{\mathrm{APS}}) - (\operatorname{ind} \mathcal{D}_{B^-} - \operatorname{ind} \mathcal{D}_{\mathrm{APS}}) \\
&= \sum_{j=1}^{k} (\dim \ker \mathcal{A}_j^{\epsilon_j} - \dim \ker \mathcal{A}_j^-) = \sum_{\{j\,:\,\epsilon_j=+\}} \operatorname{ind} \mathcal{A}_j.
\end{aligned}$$

This completes the proof. □

References

Anghel, N.: On the index of Callias-type operators. Geom. Funct. Anal. **3**(5), 431–438 (1993)

Atiyah, M.F., Singer, I.M.: Dirac operators coupled to vector potentials. Proc. Natl. Acad. Sci. USA Phys. Sci. **81**(8), 2597–2600 (1984)

Bär, C., Ballmann, W.: Boundary value problems for elliptic differential operators of first order. Surv. Differ. Geometr. XVII **1–78**, (2012)

Bär, C., Strohmaier, A.: A rigorous geometric derivation of the chiral anomaly in curved backgrounds. Commun. Math. Phys. **347**(3), 703–721 (2016)

Bertlmann, R.A.: Anomalies in Quantum Field Theory (International Series of Monographs on Physics). Clarendon Press, Oxford (2001)

Bavnbek, B.B., Wojciechowski, K.P.: Elliptic Boundary Problems for Dirac Operators, Mathematics: Theory and Applications. Birkhäuser Boston Inc, Boston (1993)

Braverman, M.: An index of strongly Callias operators on Lorentzian manifolds with non-compact boundary. Mathematische Zeitschrift. arXiv:1807.03851. https://doi.org/10.1007/s00209-019-02270-4

Braverman, M., Cecchini, S.: Callias-type operators in von Neumann algebras. J. Geometr. Anal. **28**(1), 546–586 (2018)

Braverman, M., Maschler, G.: Equivariant APS index for Dirac operators of non-product type near the boundary. Indiana Univ. Math. J. **68**, 435–501 (2019). arXiv:1702.08105

Braverman, M., Shi, P.: Cobordism invariance of the index of Callias-type operators. Commun. Partial Differ. Equ. **41**(8), 1183–1203 (2016)

Braverman, M., Shi, P.: APS index theorem for even-dimensional manifolds with non-compact boundary (2017a). arXiv:1708.08336 to appear. Comm. Anal. Geom

Braverman, M., Shi, P.: The Atiyah–Patodi–Singer index on manifolds with non-compact boundary (2017b). arXiv:1706.06737 [math.DG]

Bunke, U.: A K-theoretic relative index theorem and Callias-type Dirac operators. Math. Ann. **303**(2), 241–279 (1995)

Freed, D.: Determinants, torsion, and strings. Commun. Math. Phys. **107**(3), 483–513 (1986)

Freed, D.S.: Two index theorems in odd dimensions. Commun. Anal. Geom. **6**(2), 317–329 (1998)

Gilkey, P.: On the index of geometrical operators for Riemannian manifolds with boundary. Adv. Math. **102**(2), 129–183 (1993)

Hořava, P., Witten, E.: Heterotic and type I string dynamics from eleven dimensions. Nucl. Phys. B **460**(3), 506–524 (1996)

Lawson, H.B., Michelsohn, M.-L.: Spin Geometry. Princeton University Press, Princeton (1989)

Shi, P.: Cauchy data spaces and Atiyah-Patodi-Singer index on non-compact manifolds. J. Geom. Phys. **133**, 81–90 (2018)

Witten, E.: Global gravitational anomalies. Commun. Math. Phys. **100**(2), 197–229 (1985)