



EA-Matrix Integrals of Associative Algebras and Equivariant Localization

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Abstract

The EA-matrix integrals, introduced in Barannikov (Comptes Rendus Math 348:359–362, 2006), are studied in the case of graded associative algebras with odd or even scalar product. I prove that the EA-matrix integrals for associative algebras with scalar product are integrals of equivariantly closed differential forms with respect to the Lie algebra $gl_N(A)$.

Keywords Mirror symmetry · Gromov–Witten invariants · Noncommutative varieties · Batalin–Vilkovisky formalism

1 Introduction

The theory of periods of noncommutative varieties, depending on commutative parameters, was introduced in Barannikov (2000). The analogue of top-degree holomorphic form in this setting was shown in *loc.cit.* to be certain element of semi-infinite subspace of negative cyclic homology. The integrals of this element satisfy the second order equation with respect to the parameters of deformations of the varieties. It was proven in *loc.cit.* that the generating function of genus zero Gromov–Witten invariants of complete intersection in $\mathbb{C}P^d$ with trivial canonical class coincides with the coefficient of this second order equation for the family of mirror varieties. This approach had singled out the A_∞ -algebras/categories, satisfying cyclic homology analogue of degeneration of Hodge to de Rham spectral sequence, as the proper definition of (smooth and compact) noncommutative varieties.

To R. K. Gordin on the occasion of his 70th birthday.

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The EA-matrix integrals were introduced in Barannikov (2006) as a set of periods of associative, more generally A_∞ -algebras or noncommutative varieties, depending on *noncommutative* parameters:

$$\mathcal{F}(Y) = \int_\Gamma \exp\left(\text{Tr}\langle Y, X \rangle + \frac{1}{3!}m_{\tilde{A}}\left(X, \frac{\partial}{\partial X}\right)\right) \prod_{\alpha,i,j} dX_i^{\alpha,j}$$

$\tilde{A} = A \otimes q_N / \tilde{A} = A \otimes gl_N$ in even/odd scalar product case, here q_N is the odd matrix algebra, see *loc.cit.* It was shown in theorem 3 in *loc.cit.* that the matrix Airy integral from Kontsevich (1992) corresponds in this way to the simplest associative algebra of one dimension $A = \{e|e^2 = e\}$.

The usual varieties correspond here to A_∞ -algebras of endomorphisms of generators of their $D^b(\text{Coh})$ -categories.

The asymptotic expansion of EA-matrix integrals via BV formalism was shown in Barannikov (2006, 2009) to define, as a sum over generalized ribbon graphs, a generating function for series of cohomology classes of compactified moduli spaces of curves of all genus. A particular example is the formula for cohomology-valued generating function for products of ψ -classes, $\psi_i = c_1(T_{p_i}^*)$, in the cohomology $H^*(\bar{\mathcal{M}}_{g,n})$ calculated by the stable ribbon graph complex (*loc.cit.*):

$$\sum_{\sum d_i=d} \psi_1^{d_1} \dots \psi_n^{d_n} \prod_{i=1}^n \frac{(2d_i - 1)!!}{\lambda_i^{(2d_i+1)}} = \left[\sum_{G \in \Gamma_{g,n}^{dec,odd}} G \frac{2^{-\chi(G)}}{|\text{Aut}(G)|} \prod_{e \in \text{Edge}(G)} \frac{1}{\lambda_{i(e)} + \lambda_{j(e)}} \right] \tag{1.1}$$

where the sum on the right is over *stable ribbon* oriented graphs of genus g with n numbered punctures, with $2d + n$ edges, and such that at each vertex the cyclically ordered subsets of edges have arbitrary *odd* cardinality.

In this paper it is proven that the EA-matrix integrals for associative algebras with scalar product are integrals of equivariantly closed differential forms with respect to the Lie algebra $gl_N(A)$. This generalizes and clarifies the similar result with respect to the Lie algebra gl_N from Barannikov (2010). The localization formula for the $gl_N(A)$ -action then leads in Barannikov (2019) to calculation of these EA-matrix integrals via determinants and τ -functions of integrable hierarchies.

The paper is dedicated to Raphael Kalmanovich Gordin, with gratitude.

1.1 Notations

For a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $A = A_0 \oplus A_1$ denote via ΠA the parity inverted vector space, $(\Pi A)_0 = A_1, (\Pi A)_1 = A_0$. For an element a from $\mathbb{Z}/2\mathbb{Z}$ -graded vector space A denote by $\pi a \in \Pi A$ the same element considered with parity reversed.

2 Equivariantly Closed De Rham Differential Form

Let $A = A_0 \oplus A_1$ denotes a $\mathbb{Z}/2\mathbb{Z}$ -graded associative algebra, $\dim_k A_0 = r < \infty, \text{char}(k) = 0$, with multiplication denoted by $m_2 : A^{\otimes 2} \rightarrow A$. Let A be endowed with

odd invariant scalar product $\langle \cdot, \cdot \rangle : A_0 \otimes A_1 \rightarrow k$. The multiplication tensor can be viewed then as the $\mathbb{Z}/3\mathbb{Z}$ -cyclically invariant linear function on $(\Pi A)^{\otimes 3}$

$$m_A : (\pi a_1, \pi a_2, \pi a_3) \mapsto (-1)^{\bar{a}_2+1} \langle m_2(a_1, a_2), a_3 \rangle, m_A \in (\text{Hom}((\Pi A)^{\otimes 3}, k))^{\mathbb{Z}/3\mathbb{Z}}.$$

The odd symmetric scalar product on A corresponds to the odd anti-symmetric product $\langle \cdot, \cdot \rangle^\pi$ on ΠA :

$$\langle \pi a_1, \pi a_2 \rangle^\pi = (-1)^{\bar{a}_1+1} \langle a_1, a_2 \rangle$$

The tensor product with the matrix algebra gl_N is again naturally a $\mathbb{Z}/2\mathbb{Z}$ -graded associative algebra with the odd scalar product $A \otimes gl_N$. The cyclic tensor

$$m_{A \otimes gl_N} \in (\text{Hom}((\Pi A \otimes gl_N)^{\otimes 3}, k))^{\mathbb{Z}/3\mathbb{Z}} \tag{2.1}$$

restricted to the diagonal $\Pi A \otimes gl_N \subset (\Pi A \otimes gl_N)^{\otimes 3}$ is $GL(N)$ -invariant cubic polynomial, denoted by $m_{A \otimes gl_N}(Z)$, $Z \in \Pi A \otimes gl_N$. The associativity of the algebra A translates into the equation

$$\{m_{A \otimes gl_N}(Z), m_{A \otimes gl_N}(Z)\} = 0, \tag{2.2}$$

where $\{ \cdot, \cdot \}$ is the odd Poisson bracket corresponding to the odd anti-symmetric product $Tr|_{gl_N^{\otimes 2}} \otimes \langle \cdot, \cdot \rangle^\pi$ on $\Pi A \otimes gl_N$.

Proposition 1 *The algebra of functions on $\Pi A \otimes gl_N$ is identified naturally, preserving the odd Poisson bracket, with the algebra of polyvectors on the even affine space $\Pi A_1 \otimes gl_N$. □*

This is analogous to algebra of functions on symplectic space being identified naturally, preserving Poisson bracket, with algebra of functions on cotangent bundle of given lagrangian subspace.

Denote by $X^\alpha \in gl_N$, $P_\alpha \in \Pi gl_N$ the matrices of coordinates on $\Pi A \otimes gl_N$ corresponding to a choice of a dual pair of bases $\{e^\alpha\}$, $\{\xi_\alpha\}$ on A_0 and A_1 so that

$$Z = \sum_\alpha \pi \xi_\alpha \otimes X^\alpha + \pi e^\alpha \otimes P_\alpha \tag{2.3}$$

Then $(P_\alpha)^i_j$ corresponds to the vector field $\frac{\partial}{\partial (X^\alpha)^j_i}$ on $\Pi A_1 \otimes gl_N$. The cubic polynomial $\frac{1}{3!} m_{A \otimes gl_N}(Z)$ corresponds to the sum of the function and the bivector,

$$\frac{1}{3!} \sum_{\alpha, \beta, \gamma} (m_A)_{\alpha\beta\gamma} Tr(X^\alpha X^\beta X^\gamma) + \frac{1}{2} \sum_{\alpha, \beta, \gamma} (m_A)^{\beta\gamma}_\alpha Tr(X^\alpha P_\beta P_\gamma). \tag{2.4}$$

The odd Poisson bracket is generated by the odd second order Batalin–Vilkovisky differential Δ acting on the algebra of functions on $\Pi A \otimes gl_N$

$$\begin{aligned} \{f_1, f_2\} &= (-1)^{\bar{f}_1} (\Delta(f_1 f_2) - \Delta(f_1) f_2 + (-1)^{\bar{f}_1} f_1 \Delta(f_2)) \\ \Delta &= \sum_{\alpha, i, j} \frac{\partial^2}{\partial X_i^{\alpha, j} \partial P_{\alpha, j}^i} \end{aligned} \quad (2.5)$$

2.1 Divergence-Free Condition

Let us assume from now on that the Lie algebra A_0 is unimodular.

Condition 2 (Unimodularity of A_0) For any $a \in A_0$

$$\text{tr}([a, \cdot]_{A_0}) = 0 \quad (2.6)$$

Proposition 3 The unimodularity of A_0 (2.6) implies

$$\Delta m_{A \otimes gl_N}(Z) = 0. \quad (2.7)$$

□

Next proposition is the standard corollary of the Eqs. (2.2), (2.7) and the relation (2.5).

Proposition 4 The exponent of the sum (2.4) is closed under the Batalin–Vilkovisky differential

$$\Delta \exp \left(\frac{1}{3!} m_{A \otimes gl_N}(Z) \right) = 0.$$

□

2.2 Closed De Rham Differential Form

The affine space $\Pi A_1 \otimes gl_N$ has a holomorphic volume element, defined canonically up to a multiplication by a constant

$$\varpi = \lambda \prod_{\alpha, i, j} dX_i^{\alpha, j}.$$

It identifies the polyvectorfields on $\Pi A_1 \otimes gl_N$ with the de Rham differential forms $\Omega_{\Pi A_1 \otimes gl_N}$ on the same affine space via

$$\gamma \mapsto \gamma \lrcorner \varpi$$

The Batalin–Vilkovisky differential Δ corresponds then to the De Rham differential d_{DR} acting on the differential forms. By the Proposition 4 the polyvector $\exp \frac{1}{3!}(m_{A \otimes gl_N}(Z))$ defines the closed differential form

$$\begin{aligned} \Psi(X) &= \exp \left(\frac{1}{3!} \sum_{\alpha, \beta, \gamma} (m_A)_{\alpha\beta\gamma} Tr(X^\alpha X^\beta X^\gamma) \right. \\ &\quad \left. + \frac{1}{2} \sum_{\alpha, \beta, \gamma} (m_A)^{\beta\gamma} Tr(X^\alpha \frac{\partial}{\partial X^\beta} \wedge \frac{\partial}{\partial X^\gamma}) \right) \vdash \lambda \prod_{\alpha, i, j} dX_i^{\alpha, j} \\ d_{DR}\Psi(X) &= 0 \end{aligned} \tag{2.8}$$

It is a sum of the closed differential forms of degrees $rN^2, rN^2 - 2, \dots$

2.3 Equivariantly Closed Differential Form

The unimodularity (2.6) implies the invariance of ϖ under the co-adjoint action of the Lie algebra $A_0 \otimes gl_N$

$$X \mapsto [Y, X],$$

$Y \in A_0 \otimes gl_N$. Consider the $A_0 \otimes gl_N$ -equivariant differential forms on $\Pi A_1 \otimes gl_N$:

$$\Omega_{\Pi A_1 \otimes gl_N}^{A_0 \otimes gl_N} = (\Omega_{\Pi A_1 \otimes gl_N} \otimes \mathcal{O}_{A_0 \otimes gl_N})^{A_0 \otimes gl_N}.$$

The $A_0 \otimes gl_N$ -equivariant differential is given by

$$d_{A_0 \otimes gl_N} \Phi(Y) = d_{DR} \Phi - \sum_{\alpha, l, j} Y_{\alpha, j}^l (i_{[E_l^j \otimes e^\alpha, \cdot]} \Phi)$$

$\Phi \in \Omega_{\Pi A_1 \otimes gl_N}^{A_0 \otimes gl_N}$, where i_γ denotes the contraction operator with respect to the vector field γ , see e.g. Duflo et al. (1993). This differential corresponds, when passing to functions on $\Pi A \otimes gl_N$, to the sum

$$\begin{aligned} \Delta_{A_0 \otimes gl_N} : f(Z, Y) &\mapsto \Delta f - \frac{1}{2} Tr \langle [Y, Z], Z \rangle^\pi f, \\ f(Z, Y) &\in (\mathcal{O}_{\Pi A \otimes gl_N} \otimes \mathcal{O}_{A_0 \otimes gl_N})^{A_0 \otimes gl_N} \end{aligned}$$

of the Batalin–Vilkovisky differential and the operator of multiplication by the odd quadratic function

$$\frac{1}{2} Tr \langle [Y, Z], Z \rangle^\pi = m_{A \otimes gl_N}(Y \otimes Z \otimes Z). \tag{2.9}$$

The function depends on the equivariant parameters $Y \in A_0 \otimes gl_N$.

Theorem 1 *The product of the closed de Rham differential form $\Psi(X)$ (2.8) with the function $\exp \text{Tr}\langle Y, X \rangle$, $Y \in A_0 \otimes gl_N$, $X \in \Pi A_1 \otimes gl_N$, is $A_0 \otimes gl_N$ -equivariantly closed differential form:*

$$d_{A_0 \otimes gl_N} \left(\exp \left(\text{Tr}\langle Y, X \rangle + \frac{1}{3!} m_{A \otimes gl_N} \left(X, \frac{\partial}{\partial X} \right) \right) \vdash \lambda \prod_{\alpha, i, j} dX_i^{\alpha, j} \right) = 0$$

Proof Denote by $i_{m(X \frac{\partial}{\partial X} \frac{\partial}{\partial X})}$ the operator of contraction with the bivector field $\frac{1}{2} \sum_{\alpha, \beta, \gamma} (m_A)^{\beta\gamma} \text{Tr}(X^\alpha \frac{\partial}{\partial X^\beta} \wedge \frac{\partial}{\partial X^\gamma})$ and by $R_{\text{Tr}\langle Y, dX \rangle}$ the operator of exterior multiplication by the 1-form $\text{Tr}\langle Y, dX \rangle$ acting on differential forms,

$$R_{\text{Tr}\langle Y, dX \rangle} = [d_{DR}, i_{\text{Tr}\langle Y, X \rangle}]$$

where $i_{\text{Tr}\langle Y, X \rangle}$ is the multiplication by the linear function $\text{Tr}\langle Y, X \rangle$. Then

$$\left[i_{m(X \frac{\partial}{\partial X} \frac{\partial}{\partial X})}, R_{\text{Tr}\langle Y, dX \rangle} \right] = i_{[\cdot, Y]}$$

This is simply a particular case of the standard relation

$$[i_{\gamma_1}, Lie_{\gamma_2}] = i_{[\gamma_1, \gamma_2]}$$

for the action of polyvector fields. Notice that

$$d_{DR} e^{\text{Tr}\langle Y, X \rangle} = e^{\text{Tr}\langle Y, X \rangle} (d_{DR} + R_{\text{Tr}\langle Y, dX \rangle})$$

and that

$$R_{\text{Tr}\langle Y, dX \rangle} \exp \left(i_{m(X \frac{\partial}{\partial X} \frac{\partial}{\partial X})} \right) = \exp \left(i_{m(X \frac{\partial}{\partial X} \frac{\partial}{\partial X})} \right) (R_{\text{Tr}\langle Y, dX \rangle} + i_{[\cdot, Y]}) \tag{2.10}$$

Since $d_{DR}\Psi(X) = 0$, and $R_{\text{Tr}\langle Y, dX \rangle} \prod_{\alpha, i, j} dX_i^{\alpha, j} = 0$, therefore

$$d_{DR}(e^{\text{Tr}\langle Y, X \rangle} \Psi(X)) = i_{[\cdot, Y]} e^{\text{Tr}\langle Y, X \rangle} \Psi(X).$$

□

3 The Integral

The closed differential form $\Psi(X)$ is integrated over the cycles, which are standard in the theory of exponential integrals $\int_\Gamma \exp f$, see (Arnold et al. 1982 and references therein):

$$\Gamma \in H_*(M, \text{Re}(f) \rightarrow -\infty), \quad M = \Pi A_1 \otimes gl_N(\mathbb{C}). \tag{3.1}$$

Here f is the first term in (2.4), which is the restriction of the cubic polynomial $\frac{1}{3!}m_{A \otimes gl_N}(Z)$ on M .

The relative homology are the same for such f , $f \neq 0$, and for $f + Tr\langle Y, X \rangle$ since linear term is dominated by the cubic term when $|X| \rightarrow +\infty$. Choosing a real form of $A_0 \otimes gl_N(\mathbb{C})$ and taking the cycles in $H_*(M, \text{Re}(f) \rightarrow -\infty)$ invariant with respect to this Lie algebra gives natural cycles for integration of the equivariantly closed differential form $e^{Tr\langle Y, X \rangle} \psi(X)$

$$\mathcal{F}(Y) = \int_{\Gamma} \exp\left(Tr\langle Y, X \rangle + \frac{1}{3!}m_{A \otimes gl_N}\left(X, \frac{\partial}{\partial X}\right)\right) \lrcorner \prod_{\alpha,i,j} dX_i^{\alpha,j}$$

In general the integration cycles are the elements of the equivariant homology

$$H_{*, A_0 \otimes gl_N}\left(M, \text{Re}\left(\frac{1}{3!} \sum_{\alpha,\beta,\gamma} (m_A)_{\alpha\beta\gamma} Tr(X^\alpha X^\beta X^\gamma)\right) \rightarrow -\infty\right)$$

One can consider also the normalized integral

$$\widehat{\mathcal{F}}(Y) = \int_{\Gamma} \exp\left(Tr\langle Y, X \rangle + \frac{1}{3!}m_{A \otimes gl_N}\left(X, \frac{\partial}{\partial X}\right)\right) \lrcorner \prod_{\alpha,i,j} dX_i^{\alpha,j} / \mathcal{F}_{[2]}(Y) \tag{3.2}$$

where $\mathcal{F}_{[2]}(Y)$ is the corresponding Gaussian integral of the quadratic part of $f + Tr\langle Y, X \rangle$ at a critical point $(-Y)^{\frac{1}{2}}$.

Let the associative algebra A_0 has an *anti-involution* $a \rightarrow a^\dagger$

$$(ab)^\dagger = b^\dagger a^\dagger, \quad (ca)^\dagger = \bar{c} a^\dagger, \quad \text{tr}(a^\dagger) = \overline{\text{tr}(a)}, \quad (a^\dagger)^\dagger = a$$

The anti-involution defines the natural cycle for the equivariant integration. This anti-involution extends naturally to $A_0 \otimes gl_N(\mathbb{C})$. Then the Lie subalgebra of anti-hermitian elements in $A_0 \otimes gl_N(\mathbb{C})$

$$u_N(A_0) = \{Y^\dagger = -Y \mid Y \in A_0 \otimes gl_N(\mathbb{C})\}$$

is a real form of $A_0 \otimes gl_N(\mathbb{C})$. And the space of hermitian elements in the dual space

$$\Gamma = \{X^\dagger = X \mid X \in A_0^\vee \otimes gl_N(\mathbb{C})\} \tag{3.3}$$

is invariant under the action of $u_N(A_0)$. Then the ‘‘real-slice’’ Γ is the natural choice of the cycle for the equivariant integration.

The localization formula for equivariant cohomology reduces the integral of the equivariantly closed form Ω over Γ to the integral over the fixed locus F ,

$$\int_{\Gamma} \Omega = \int_F \frac{\Omega}{eu(N_F)} \tag{3.4}$$

where $eu(N_F)$ is the euler class of the normal bundle of F in Γ , see Atiyah and Bott (1984) and Duflo et al. (1993). Calculating the integral using the equivariant localization leads to generalized Vandermonde determinants and τ -functions.

Let for simplicity the algebra A with odd scalar product is the tensor product $A = A_0 \otimes q_1$ of the even associative algebra A_0 with scalar product, denoted $\eta(y_1, y_2)$, and the algebra $q_1 = \{1, \xi \mid \xi^2 = 1\}$ with the odd scalar product $\langle 1, \xi \rangle = 1$.

Assume that the natural scalar product on the Lie algebra of anti-hermitian elements in A_0 is positive definite

$$-\eta(y, y) = \eta(y, y^\dagger) > 0.$$

Otherwise one can apply to Γ a partial Wick rotation.

Then the localization formula (3.4) after some calculations leads to the following result (Barannikov 2019):

Proposition 5 *The integral (3.2), written in variables $t \in HC^*(A)$, is a τ -function of KP-type hierarchy and, in particular, satisfies the Hirota quadratic equations.*

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