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Integrable geodesic flows with simultaneously diagonalisable quadratic integrals

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Abstract: We show that if n functionally independent commutative quadratic in momenta integrals for the geodesic flow of a Riemannian or pseudo-Riemannian metric on an n-dimensional manifold are simultaneously diagonalisable at the tangent space to every point, then they come from the Stäckel construction, so the metric admits orthogonal separation of variables.

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1 Introduction

We work locally, on a smooth n-dimensional pseudo-Riemannian manifold (M, g) of any signature. By geodesic flow we understand the Hamiltonian system on T^*M generated by the Hamiltonian

$$H(x,p) = \tfrac{1}{2} g^{ij} p_i p_j.$$

We will study the situation when the geodesic flow admits n, including the Hamiltonian, integrals

$$\overset{1}{I}(x,p) = 2H\,,\ \overset{2}{I}(x,p)\,,\ \dots\,,\ \overset{n}{I}(x,p)$$

such that the following conditions are fulfilled:

1. The integrals are quadratic in momenta, that is, $\overset{\alpha}{I}(x,p) = \overset{\alpha}{K}{}^{ij} p_i p_j$. In particular, $g^{ij} = \overset{1}{K}{}^{ij}$. We assume without loss of generality that the (2,0)-tensor fields $\overset{\alpha}{K}{}^{ij}$ are symmetric in upper indexes.
2. At almost every point $x \in M$, there exists a basis in T_xM such that, for every $\alpha = 1, \dots, n$, the matrix $\left(\overset{\alpha}{K}{}^{ij}(x)\right)$ is diagonal.
3. The differentials of the integrals are linearly independent at least at one[1] point of T^*M.

In many publications on this topic, e.g. in [BCR02, Eis34, Kiy97], it is assumed that for almost every point $x \in M$ the restrictions of the tensor fields $\overset{\alpha}{K}{}^{ij}$, $\alpha = 1, \dots, n$, to T_xM are linearly independent. Our main result, Theorem 1 below, shows that this assumptions follows from conditions (1,2,3):

Theorem 1. *Under the assumptions above, for almost every point x the restrictions of the tensor fields $\overset{\alpha}{K}{}^{ij}$, $\alpha = 1, \dots, n$, to T_xM are linearly independent. In particular, for a*

[1]Using ideas of [KM16], is is easy to show that linear independence of the differentials of polynomial in momenta integrals at one point implies their linear independence at almost every point, provided the manifold is connected

generic linear combination $I = \sum_{\alpha=2}^{n} \lambda_\alpha \overset{\alpha}{I}$ *of the integrals, the corresponding (1,1)-tensor field* $K^i_j := K^{si} g_{sj}$*, where* $K^{ij} = \sum_{\alpha=2}^{n} \lambda_\alpha \overset{\alpha}{K}{}^{ij}$*, has* n *different eigenvalues.*

In dimension $n = 3$, Theorem 1 and Corollary 1.1 below were proven in [Aga24, Theorem 2] by other methods.

Corollary 1.1. *Assume the integrals* $\overset{\alpha}{I}$ *satisfy the conditions (1,2,3) above and in addition are in involution with respect to the standard Poisson bracket. Then, near almost every point, the metric* g *and the integrals come from the Stäckel construction.*

In view of Theorem 1, Corollary 1.1 follows from [KM80, Theorem 6], [Kiy97, Proposition 1.1.3], [BCR02, Theorem 8.6] or, possibly,[2] from A. Thimm 1976. In these references, it was shown that n quadratic functionally independent integrals in involution such that the corresponding Killing tensors are simultaneously diagonalisable at every tangent space and such that at least one of the Killing tensors with one index raised by the metric has n different eigenvalues, come from the Stäckel construction which we recall below.

As mentioned above, the difference between our conditions (1,2,3) and the assumptions used in [KM80, Theorem 6], [Kiy97, Proposition 1.1.3] or [BCR02, Theorem 8.6] is as follows: in [KM80, Theorem 6], [Kiy97, Proposition 1.1.3] or [BCR02, Theorem 8.6] it was assumed that one of the Killing tensors, with one index raised by the metric, has n different eigenvalues. We do not have this condition as an assumption and prove that it follows from other assumptions.

Let us recall the Stäckel[3] construction following [Eis34, St1]. Take a non-degenerate $n \times n$ matrix $S = (S_{ij})$ with S_{ij} being a function of the i-th variable x^i only . Next, consider the functions $\overset{\alpha}{I}$, $\alpha = 1, \dots, n$, given by the following system of linear equations

$$S\mathbb{I} = \mathbb{P}, \tag{1}$$

[2]By [Kli78, Note on page 185] the diploma thesis of A. Thimm 1976, which we were not able to find, contains this result

[3]The construction appeared already in [Lio49, §§13-14], see also discussion in [Lö90, pp. 703–705]

where $\mathbb{I} = \left(\overset{1}{I}, \overset{2}{I}, \ldots, \overset{n}{I}\right)^\top$ and $\mathbb{P} = \left(p_1^2, p_2^2, \ldots, p_n^2\right)^\top$. It is known that the functions $\overset{\alpha}{I}$ are in involution. Taking one of them (say, the first one, provided the inverse matrix to S has no zeros in the first raw) as twice the Hamiltonian of the metric, one obtains an integrable geodesic flow whose integrals satisfy the conditions (1,2,3). Corollary 1.1 says that locally, near almost every points, there exist no other examples of geodesic flows admitting n independend quadratic in momenta integrals in involution, such that the corresponding Killing tensors are simultaneously diagonalisable at almost every tangent space.

It is known that metrics coming from the Stäckel construction admit orthogonal separation of variables in the Hamilton-Jacobi equations, so the equation for their geodesics can be locally solved in quadratures [BKM25, KKM18]. Namely, J. Liouville [Lio49] and, independently, P. Stäckel [St1] has shown that the metrics are precisely those admitting othogonal separation of variables. L. P. Eisenhart, in his widely cited and very influential paper [Eis34], has shown that locally the metrics coming from the Stäckel construction are precisely those whose geodesic flows admit n functionally independent integrals in involution satisfying the following conditions: the integrals are quadratic in momenta, the corresponding matrices are simultaneously diagonalisable in a coordinate system, and at every point the corresponding matrices are linearly independent. In [BCR02, KM80, Kiy97] it was shown that the assumption that the integrals are simultaneously diagonalisable in a coordinate system may be replaced by a weaker assumption that the matrices of the integrals are diagonalisable in a frame. Our result further improves the result of Eisenhart and shows that the condition that the matrices of the integrals are linearly independent at each point is not necessary as this assumption follows from other conditions.

2 Proof of Theorem 1

Under the assumptions (1,2,3) from Section 1, near almost every point, there exist smooth vector fields $v_1(x), \ldots, v_n(x) \in T_xM$ such that they are linearly independent at every tangent space and such that the metric g and the matrices $\overset{\alpha}{K}{}^{ij}$ are diagonal in the basis $(v_1, ..., v_n)$. After re-arranging and re-scaling the vectors v_i, there exists $m \in \mathbb{N}$, $m \leq n$, $k_1, ..., k_m \in \mathbb{N}$ with $k_1 + \cdots + k_m = n$ and smooth local functions $g_1, .., g_m, \overset{\alpha}{\rho}_1, \ldots, \overset{\alpha}{\rho}_m$, $\alpha \in \{2, \ldots, n\}$, on M such that the Hamiltonian and the integrals $\overset{\alpha}{I}$ with $\alpha = 2, \ldots, n$ are given by the formulas

$$\begin{aligned} 2H &= V_1 + V_2 + \cdots + V_m \\ \overset{\alpha}{I} &= \overset{\alpha}{\rho}_1 V_1 + \overset{\alpha}{\rho}_2 V_2 + \cdots + \overset{\alpha}{\rho}_m V_m \end{aligned} \tag{2}$$

In the formulas above, V_i are the functions on the cotangent bundle given by

$$\begin{aligned} V_1 &= (v_1)^2\varepsilon_1 + (v_2)^2\varepsilon_2 + \cdots + (v_{k_1})^2\varepsilon_{k_1}, \\ V_2 &= (v_{k_1+1})^2\varepsilon_{k_1+1} + (v_{k_1+2})^2\varepsilon_{k_1+2} + \cdots + (v_{k_1+k_2})^2\varepsilon_{k_1+k_2}, \\ &\vdots \\ V_m &= (v_{k_1+\cdots+k_{m-1}+1})^2\varepsilon_{k_1+\cdots+k_{m-1}+1} + (v_{k_1+\cdots+k_{m-1}+2})^2\varepsilon_{k_1+\cdots+k_{m-1}+2} + \cdots + (v_n)^2\varepsilon_n, \end{aligned}$$

where v_i is the linear function on the T^*M generated by the vector field v_i via the canonical identification[4] $TM \equiv T^{**}M$, and $\varepsilon_i \in \{-1, 1\}$.

The Poisson bracket of H and $I = \overset{\alpha}{I}$ reads (we omit the index α since the equations hold for any $\overset{\alpha}{I}$):

$$0 = \{2H, I\} = \sum_{i,j=1}^{m} \left(\{V_i, \rho_j\}V_j + \rho_j\{V_i, V_j\}\right). \tag{3}$$

The right hand side of (3) is a cubic polynomial in momenta so all its coefficients are zero. For every point $x \in M$, this gives us a system of linear equations on the directional derivatives $v_s(\rho_j)$ with $s \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. The coefficients and free terms of this system depend on $\rho_j(x)$, on the entries of the vector fields v_s at x, and on the derivatives of the entries of the vector fields v_s at x. Let us show that all directional derivatives $v_s(\rho_i)$ can be reconstructed from this system. We will show this for the directional derivatives $v_1(\rho_2)$ and $v_1(\rho_1)$, since this will cover two principle cases $i = j$ and $i \neq j$; the proof for all other $v_i(\rho_j)$ is completely analogous.

[4]In naive terms, we consider the vector field $v = \sum_i v^i \partial_i$ as the linear function $p \longmapsto \sum_i v^i p_i$ on T^*M. This identification of vector fields on M and linear in momenta functions on the cotangent bundle is independent of a coordinate system

In order to extract $v_1(\rho_2)$, note that the cubic in momenta component $(v_{k_1+1})^2 v_1$ shows up only in the addends $\{V_1, \rho_2\}V_2$, $\rho_1\{V_2, V_1\}$ and $\rho_1\{V_1, V_2\}$. In these addends, the coefficient containing a derivative of one of the functions ρ_s is $v_1(\rho_2)$. Thus, equating the coefficient of $(v_{k_1+1})^2 v_1$ to zero gives us $v_1(\rho_2)$ as a function of ρ_1, ρ_2 and the entries of $\{V_1, V_2\}$.

Similarly, in order to extract $v_1(\rho_1)$, we note that the cubic in momenta component $(v_1)^3$ shows up only in the addends $\{V_1, \rho_1\}V_1$ and $\rho_1\{V_1, V_2\}$. Its coefficient containing the derivatives of ρ's is $v_1(\rho_1)$. Thus, equating the coefficient of $(v_1)^3$ to zero gives us $v_1(\rho_1)$ as a function of ρ_1.

Thus, all directional derivatives $v_s(\rho_j)$ can be obtained from the system (3). Let us now view the system (3) as a linear PDE-system on unknown functions ρ_i. The coefficients of this system come from the vector fields v_s and are given by certain nonlinear expressions in the components of v_s and their derivatives. Since the directional derivatives of all functions ρ_i are expressed in the terms of the functions ρ_i, the system can be solved with respect to all derivatives of the functions ρ_i. Therefore, the initial values of the functions ρ_i at one point x_0 determine the local solution of the system. This implies that the space of solutions is at most m-dimensional. Finally, the linear vector space of the integrals $\overset{\alpha}{I}$ is at most m-dimensional. Since n of them are functionally independent by our assumptions, $n = m$ and Theorem 1 is proved.

Remark 1. *The proof of Theorem 1 is motivated by [Ben92, proof of Lemma 1.2], [Kiy97, §1.1] and [KKM24, §2].*

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