

# The Flapping Birds in the Pentagonram Zoo

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**Abstract:** We study the  $(k + 1, k)$  diagonal map for  $k = 2, 3, 4, \dots$ . We call this map  $\Delta_k$ . The map  $\Delta_1$  is the pentagram map and  $\Delta_k$  is a generalization.  $\Delta_k$  does not preserve convexity, but we prove that  $\Delta_k$  preserves a subset  $B_k$  of certain star-shaped polygons which we call  $k$ -birds. The action of  $\Delta_k$  on  $B_k$  seems similar to the action of  $\Delta_1$  on the space of convex polygons. We show that some classic geometric results about  $\Delta_1$  generalize to this setting.

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**Key words and phrases:** pentagram map, geometric dynamics, projective geometry, polygons

# 1 Introduction

## 1.1 Context

When you visit the pentagram zoo you should certainly make the pentagram map itself your first stop. This old and venerated animal has been around since the place opened up and it is very friendly towards children. When defined on convex pentagons, this map has a very long history. See e.g. [15]. In modern times [19], the pentagram is defined and studied much more generally. The easiest case to explain is the action on convex  $n$ -gons. One starts with a convex  $n$ -gon  $P$ , for  $n \geq 5$ , and then forms a new convex  $n$ -gon  $P'$  by intersecting the consecutive diagonals, as shown Figure 1.1 below.

The magic starts when you iterate the map. One of the first things I proved in [19] about the pentagram map is the successive iterates shrink to a point. Many years later, M. Glick [3] proved that this limit point is an algebraic function of the vertices, and indeed found a formula for it. See also [9] and [1].

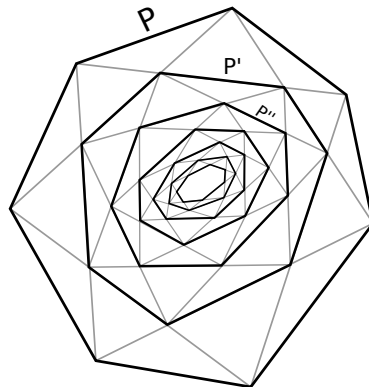


Figure 1.1: The pentagram map iterated on a convex 7-gon  $P$ .

Forgetting about convexity, the pentagram map is generically defined on polygons in the projective plane over any field except for  $\mathbb{Z}/2$ . In all cases, the pentagram map commutes with projective transformations and thereby defines a birational map on the space of  $n$ -gons modulo projective transformations. The action on this moduli space has a beautiful structure. As shown in [17] [18], and independently in [23], the pentagram

map is a discrete completely integrable system when the ground field is the reals. ([23] also treats the complex case.) Recently, M. Weinreich [24] generalized the integrability result, to a large extent, to fields of positive characteristic.

The pentagram map has many generalizations. See for example [2], [14], [16], [10], [11], [6]. The paper [2] has the first general complete integrability result. The authors prove the complete integrability of the  $(k, 1)$  diagonal maps, i.e. the maps obtained by intersecting successive  $k$ -diagonals. Figure 1.3 below shows the  $(3, 1)$  diagonal map. (Technically, [2] concentrates on what happens when these maps act on so-called *corrugated polygons* in higher dimensional Euclidean spaces.) The paper [6] proves an integrability result for a very wide class of generalizations, including the ones we study below. (Technically, for the maps we consider here, the result in [6] does not establish the algebraic independence of invariants needed for complete integrability.) The pentagram map and its many generalizations are related to a number of topics: alternating sign matrices [20], dimers [5], cluster algebras [4], the KdV hierarchy [12], [13], spin networks [2], Poisson Lie groups [8], Lax pairs [23], [10], [11], [6], [8], and so forth. The zoo has many cages and sometimes you have to get up on a tall ladder to see inside them.

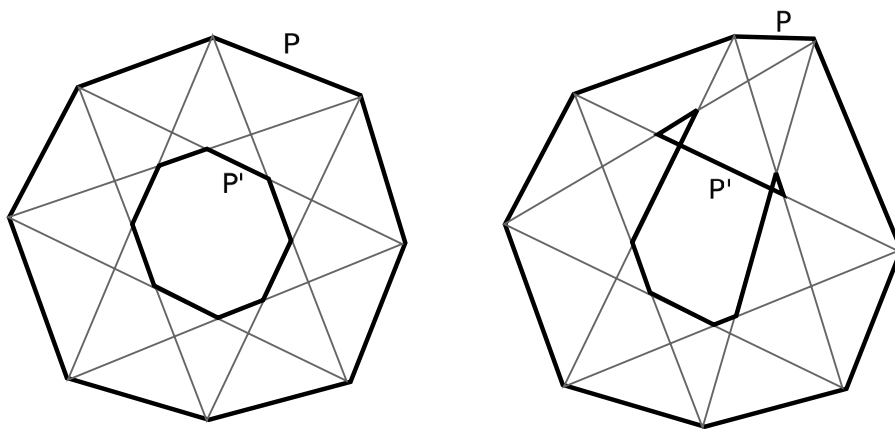


Figure 1.2: The  $(3, 1)$ -diagonal map acting on 8-gons.

The algebraic side of the pentagram zoo is extremely well developed, but the *geometric side* is hardly developed at all. In spite of all the algebraic results, we don't really know,

geometrically speaking, much about what the pentagram map and its relatives really do to polygons.

Geometrically speaking, there seems to be a dichotomy between convexity and non-convexity. The generic pentagram orbit of a projective equivalence class of a convex polygon lies on a smooth torus, and you can make very nice animations. What you will see, if you tune the power of the map and pick suitable representatives of the projective classes, is a convex polygon sloshing around as if it were moving through water waves. If you try the pentagram map on a non-convex polygon, you see a crazy erratic picture no matter how you try to normalize the images. The situation is even worse for the other maps in the pentagram zoo, because these generally do not preserve convexity. Figure 1.2 shows how the  $(3, 1)$ -diagonal map does not necessarily preserve convexity, for instance. See [21], [22] for more details.

If you want to look at pentagram map generalizations, you have to abandon convexity. However, in this paper, I will show that sometimes there are geometrically appealing replacements. The context for these replacements is the  $(k + 1, k)$ -diagonal map, which I call  $\Delta_k$ , acting on what I call  $k$ -birds.  $\Delta_k$  starts with the polygon  $P$  and intersects the  $(k + 1)$ -diagonals which differ by  $k$  clicks. (We will give a more formal definition in the next section.)  $\Delta_k$  is well (but not perfectly) understood algebraically [6]. Geometrically it is not well understood at all.

## 1.2 The Maps and the Birds

**Definition of a Polygon:** For us, a *polygon* is a choice of both vertices and the edges connecting them. Each polygon  $P$  we consider will all be *planar*, in the sense that there is some projective transformation that maps  $P$ , both vertices and edges, to the affine patch. Our classical example is a regular  $n$ -gon, with the obvious short edges chosen.

**The Maps:** Given a polygon  $P$ , let  $P_a$  denote the  $(a)$ th vertex of  $P$ . Let  $P_{ab}$  be the line

through  $P_a$  and  $P_b$ . The vertices of  $\Delta_k(P)$  are

$$P_{j,j+k+1} \cap P_{j+1,j-k}. \quad (1)$$

Here the indices are taken mod  $n$ . Figure 1.3 shows this for  $(k, n) = (2, 7)$ . The polygons in Figure 1.3 are examples of a concept we shall define shortly, that of a  $k$ -bird.

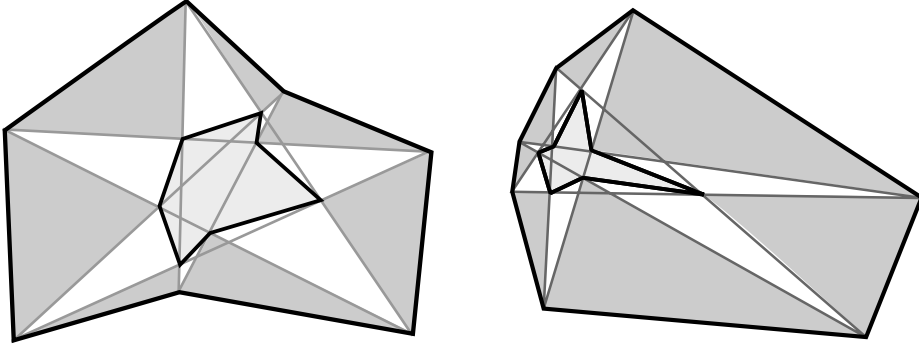


Figure 1.3:  $\Delta_2$  acting on 2-birds.

We should say a word about how the edges are defined. In the case for the regular  $n$ -gon we make the obvious choice, discussed above. In general, we define the class of polygons we consider in terms of a homotopy from the regular  $n$ -gon. So, in general, we make the edge choices so that the edges vary continuously.

**The Birds:** Given an  $n$ -gon  $P$ , we let  $P_{a,b}$  denote the line containing the vertices  $P_a$  and  $P_b$ . We call  $P$   $k$ -nice if  $n > 3k$ , and  $P$  is planar, and the 4 lines

$$P_{i,i-k-1}, \quad P_{i,i-k}, \quad P_{i,i+k}, \quad P_{i,i+k+1} \quad (2)$$

are distinct for all  $i$ . It is not true that the generic  $n$ -gon is  $k$ -nice, because there are open sets of non-planar polygons. (Consider a neighborhood of  $P$ , where  $P$  the regular 100-gon with the opposite choice of edges.) However, the generic perturbation of a planar  $n$ -gon is also  $k$ -nice.

We call  $P$  a  $k$ -bird if  $P$  is the endpoint of a path of  $k$ -nice  $n$ -gons that starts with the regular  $n$ -gon. We let  $B_{k,n}$  be the subspace of  $n$ -gons which are  $k$ -birds. Note that  $B_{k,n}$

contains the set of convex  $n$ -gons, and the containment is strict when  $k > 1$ . As Figure 1.3 illustrates, a  $k$ -bird need not be convex for  $k \geq 2$ . We will show that  $k$ -birds are always star-shaped, and in particular embedded. As we mentioned above, we use the homotopic definition of a  $k$ -bird, to define the edges of  $\Delta_k(P)$  when  $P$  is a  $k$ -bird.

**Example:** The homotopy part of our definition looks a bit strange, but it is necessary. To illustrate this, we consider the picture further for the case  $k = 1$ . In this case, a 1-bird must be convex, though the 1-niceness condition just means planar and locally convex. Figure 1.4 shows how we might attempt a homotopy from the regular octagon to a locally convex octagon which essentially wraps twice around a quadrilateral. The little grey arrows give hints about how the points are moved. At some times, the homotopy must break the 1-niceness condition. The two grey polygons indicate failures and the highlighted vertices indicate the sites of the failures. There might be other failures as well; we are taking some jumps in our depiction.

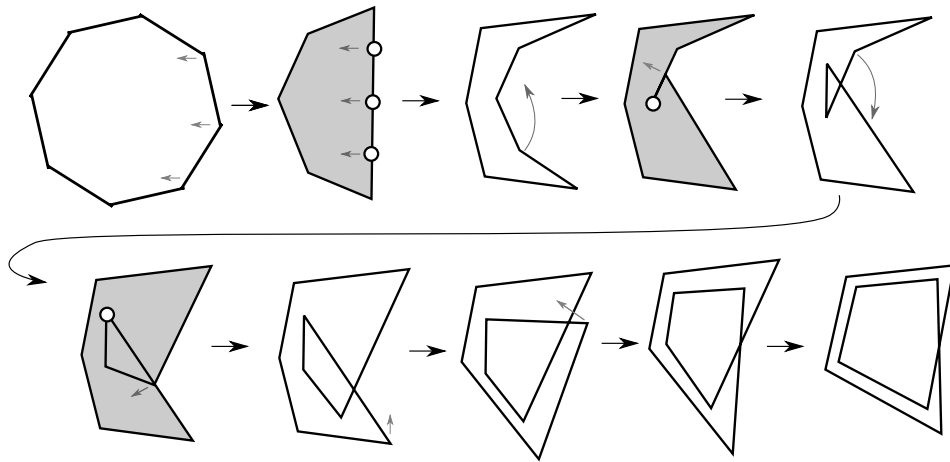


Figure 1.4: A homotopy that cannot stay 1-nice.

One could make similar pictures when  $k \geq 1$ , but the pictures might be harder to understand.

### 1.3 The Main Result

Given an embedded planar polygon  $P$ , let  $P^I$  denote the interior of region bounded by  $P$ . We say that  $P$  is *strictly star shaped* with respect to  $x \in P^I$  if each ray emanating from  $x$  intersects  $P$  exactly once. More simply, we say that  $P$  is *strictly star shaped* if it is strictly star shaped with respect to some point  $x \in P^I$ . Here is the main result.

**Theorem 1.1.** *Let  $k \geq 2$  and  $n > 3k$  and  $P \in B_{k,n}$ . Then*

1.  *$P$  is strictly star-shaped, and in particular embedded.*
2.  *$\Delta_k(P) \subset P^I$ .*
3.  *$\Delta_k(B_{k,n}) = B_{k,n}$ .*

**Remark:** The statement that  $n > 3k$  is present just for emphasis.  $B_{n,k}$  is by definition empty when  $n \leq 3k$ . The restriction  $n > 3k$  is necessary. Figure 1.5 illustrates what would be a counter-example to Theorem 1.1 for the pair  $(k, n) = (3, 9)$ . The issue is that a certain triple of 4-diagonals has a common intersection point. This does not happen for  $n > 3k$ . See Lemma 3.6.

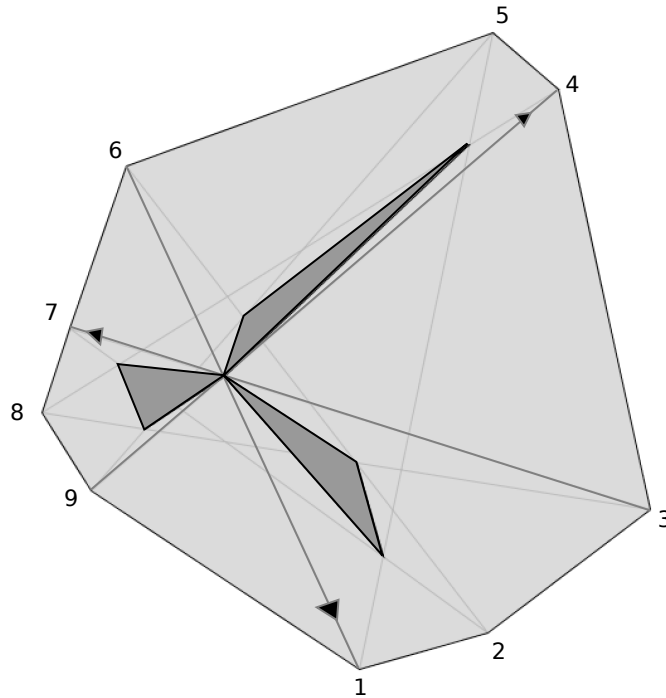


Figure 1.5:  $\Delta_3$  acting on a certain convex 9-gon.

## 1.4 The Energy

We will deduce Statements 1 and 2 of Theorem 1.1 in a geometric way. The key to proving Statement 3 is a natural quantity associated to a  $k$ -bird. We let  $\sigma_{a,b}$  be the slope of the line  $P_{a,b}$  and we define the *cross ratio*

$$\chi(a, b, c, d) = \frac{(a - b)(c - d)}{(a - c)(b - d)}. \quad (3)$$

We define

$$\chi_k(P) = \prod_{i=1}^n \chi(i, k, P), \quad \chi(i, k, P) = \chi(\sigma_{i,i-k}, \sigma_{i,i-k-1}, \sigma_{i,i+k+1}, \sigma_{i,i+k}) \quad (4)$$

Here we are taking the cross ratio the slopes the lines involved in our definition of  $k$ -nice. When  $k = 1$  this is the familiar invariant  $\chi_1 = OE$  for the pentagram map  $\Delta_1$ . See [19], [20], [17], [18]. When  $n = 3k + 1$ , a suitable star-relabeling of our polygons converts  $\Delta_k$  to  $\Delta_1$  and  $\chi_k$  to  $1/\chi_1$ . So, in this case  $\chi_k \circ \Delta_k = \chi_k$ . Figure 1.5 illustrates this for  $(k, n) = (3, 10)$ . Note that the polygons suggested by the dots in Figure 1.5 are not convex. Were we to add in the edges we would get a highly non-convex pattern.

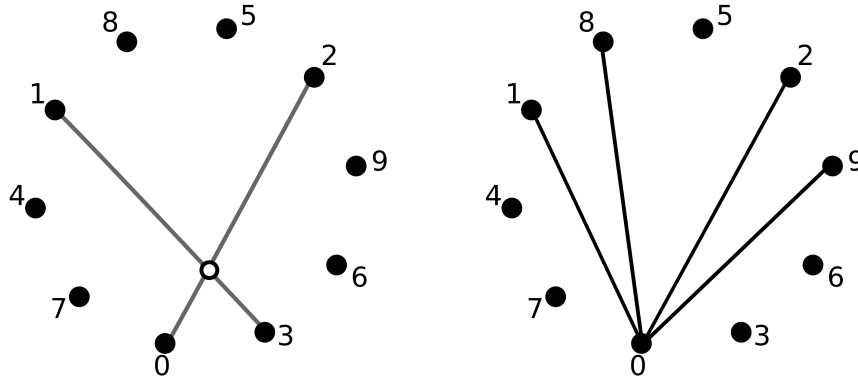


Figure 1.6: A star-relabeling converts  $\Delta_1$  to  $\Delta_3$  and  $1/\chi_1$  to  $\chi_3$ .

In general,  $\chi_k$  is not as clearly related to  $\chi_1$ . Nonetheless, we will prove

**Theorem 1.2.**  $\chi_k \circ \Delta_k = \chi_k$ .



Theorem 1.2 is meant to hold for all  $n$ -gons, as long as all quantities are defined. There is no need to restrict to birds.

## 1.5 The Collapse Point

When it is understood that  $P \in B_{k,n}$  it is convenient to write

$$P^\ell = \Delta_k^\ell(P) \quad (5)$$

We also let  $\hat{P}$  denote the closed planar region bounded by  $P$ . Figure 1.7 below shows  $\hat{P} = \hat{P}^0, \hat{P}^1, \hat{P}^2, \hat{P}^3, \hat{P}^4$  for some  $P \in B_{4,13}$ .

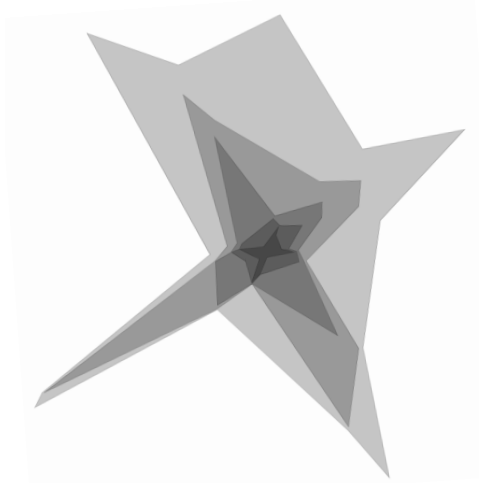


Figure 1.7:  $\Delta_4$  and its iterates acting on a member of  $B_{4,13}$ .

Define

$$\hat{P}_\infty = \bigcap_{\ell \in \mathbf{Z}} \hat{P}^\ell, \quad \hat{P}_{-\infty} = \bigcup_{\ell \in \mathbf{Z}} \hat{P}^\ell. \quad (6)$$

**Theorem 1.3.** *If  $P \in B_{k,n}$  then  $\hat{P}_\infty$  is a point and  $\hat{P}_{-\infty}$  is an affine plane.*

Our argument will show that  $P \in B_{k,n}$  is strictly star-shaped with respect to all points in  $\hat{P}^n$ . In particular, all polygons in the orbit are strictly star-shaped with respect to the collapse point  $\hat{P}_\infty$ . See Corollary 7.3.

One might wonder if some version of Glick's formula works for the  $\hat{P}_\infty$  in general. I discovered experimentally that this is indeed the case for  $n = 3k + 1$  and  $n = 3k + 2$ . See §9.2 for a discussion of this and related matters.

Here is a corollary of our results that is just about convex polygons.

**Corollary 1.4.** *Suppose that  $n > 3k$  and  $P$  is a convex  $n$ -gon. Then the sequence  $\{\Delta_k^\ell(P)\}$  shrinks to a point as  $\ell \rightarrow \infty$ , and each member of this sequence is strictly star-shaped with respect to the collapse point.*

## 1.6 The Triangulations

In §7.1 we associate to each  $k$ -bird  $P$  a triangulation  $\tau_P \subset P$ , the projective plane. Here  $\tau_P$  is an embedded degree 6 triangulation of  $P_{-\infty} - P_\infty$ . The edges are made from the segments in the  $\delta$ -diagonals of  $P$  and its iterates for  $\delta = 1, k, k + 1$ .

Figure 1.8 shows this tiling associated to a member of  $B_{5,16}$ . In this figure, the interface between the big black triangles and the big white triangles is some  $\Delta_5^\ell(P)$  for some smallish value of  $\ell$ . (I zoomed into the picture a bit to remove the boundary of the initial  $P$ .) The picture is normalized so that the line  $P_{-\infty}$  is the line at infinity. When I make these kinds of pictures (and animations), I normalize so that the ellipse of inertia of  $P$  is the unit disk.

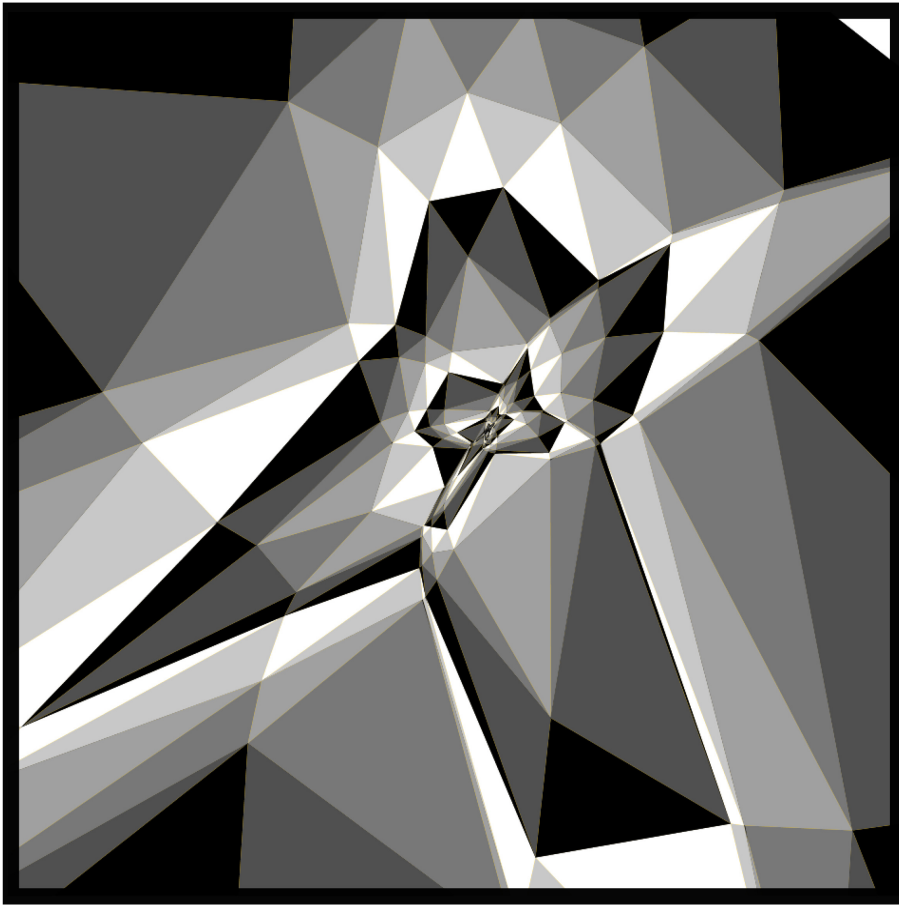


Figure 1.8: The triangulation associated to a member of  $B_{5,16}$ .

## 1.7 Paper Organization

This paper is organized as follows.

- In §2 we prove Theorem 1.2.
- In §3 we prove Statement 1 of Theorem 1.1.
- In §4 we prove Statement 2 of Theorem 1.1.
- In §5 we prove a technical result called the Degeneration Lemma, which will help with Statement 3 of Theorem 1.1.

- In §6 we prove Statement 3 of Theorem 1.1.
- In §7 we introduce the triangulations discussed above. Our Theorem 7.2 will help with the proof of Theorem 1.3.
- In §8 we prove Theorem 1.3.
- In §9, an appendix, we sketch an alternate proof of Theorem 1.2 which Anton Izosimov kindly explained. We also discuss Glick's collapse formula and star relabelings of polygons.

## 1.8 Visit the Flapping Bird Exhibit

Our results inject some more geometry into the pentagon zoo. Our results even have geometric implications for the pentagon map itself. See §9.3. There are different ways to visit the flapping bird exhibit in the zoo. You could read the proofs here, or you might just want to look at some images:

**<http://www.math.brown.edu/~reschwar/BirdGallery>**

You can also download and play with the software I wrote:

**<http://www.math.brown.edu/~reschwar/Java/Bird.TAR>**

The software has detailed instructions. You can view this paper as a justification for why the nice images actually exist.

## 2 The Energy

The purpose of this chapter is to prove Theorem 1.2. The proof, which is similar to what I do in [19], is more of a verification than a conceptual explanation. My computer program allows the reader to understand the technical details of the proof better. The reader might want to just skim this chapter on the first reading. In §9 I will sketch an alternate proof, which I learned from Anton Izosimov. Izosimov's proof also uses the first two sections of this chapter.

## 2.1 Projective Geometry

Let  $P$  denote the real projective plane. This is the space of 1-dimensional subspaces of  $\mathbf{R}^3$ . The projective plane  $P$  contains  $\mathbf{R}^2$  as the *affine patch*. Here  $\mathbf{R}^2$  corresponds to vectors of the form  $(x, y, 1)$ , which in turn define elements of  $P$ .

Let  $P^*$  denote the dual projective plane, namely the space of lines in  $P$ . The elements in  $P^*$  are naturally equivalent to 2-dimensional subspaces of  $\mathbf{R}^3$ . The line in  $P$  such a subspace  $\Pi$  defines is equal to the union of all 1-dimensional subspaces of  $\Pi$ .

Any invertible linear transformation of  $\mathbf{R}^3$  induces a *projective transformation* of  $P$ , and also of  $P^*$ . These form the *projective group*  $PSL_3(\mathbf{R})$ . Such maps preserve collinear points and coincident lines.

A *duality* from  $P$  to  $P^*$  is an analytic diffeomorphism  $P \rightarrow P^*$  which maps collinear points to coincidence lines. The classic example is the map which sends each linear subspace of  $\mathbf{R}^3$  to its orthogonal complement.

A *PolyPoint* is a cyclically ordered list of points of  $P$ . When there are  $n$  such points, we call this an  $n$ -Point. A *PolyLine* is a cyclically ordered list of lines in  $P$ , which is the same as a cyclically ordered list of points in  $P^*$ . A projective duality maps PolyLines to PolyPoints, and *vice versa*.

Each  $n$ -Point determines  $2^n$  polygons in  $P$  because, for each pair of consecutive points, we may choose one of two line segments to join them. As we mentioned in the introduction, we have a canonical choice for  $k$ -birds. Theorem 1.2 only involves PolyPoints, and our proof uses PolyPoints and PolyLines.

Given a  $n$ -Point  $P$ , we let  $P_j$  be its  $j$ th point. We make a similar definition for  $n$ -Lines. We always take indices mod  $n$ .

## 2.2 Factoring the Map

Like the pentagram map, the map  $\Delta_k$  is the product of 2 involutions. This factorization will be useful here and in later chapters.

Given a PolyPoint  $P$ , consisting of points  $P_1, \dots, P_n$ , we define  $Q = D_m(P)$  to be the

PolyLine whose successive lines are  $P_{0,m}, P_{1,m+1}$ , etc. Here  $P_{0,m}$  denotes the line through  $P_0$  and  $P_m$ , etc. We labeled the vertices so that

$$Q_{-m-i} = P_{i+m}. \quad (7)$$

This is a convenient choice. We define the action of  $D_m$  on PolyLines in the same way, switching the roles of points and lines. For PolyLines,  $P_{0,m}$  is the intersection of the line  $P_0$  with the line  $P_m$ . The map  $D_m$  is an involution which swaps PolyPoints with PolyLines. We have the compositions

$$\Delta_k = D_k \circ D_{k+1}, \quad \Delta_k^{-1} = D_{k+1} \circ D_k. \quad (8)$$

The *energy*  $\chi_k$  makes sense for  $n$ -Lines as well as for  $n$ -Points. The quantities  $\chi_k \circ D_k(P)$  and  $\chi_k \circ D_{k+1}(P)$  can be computed directly from the PolyPoint  $P$ . Figure 2.1 shows schematically the 4-tuples associated to  $\chi(0, k, Q)$  for  $Q = P$  and  $D_k(P)$  and  $D_{k+1}(P)$ . In each case,  $\chi_k(Q)$  is a product of  $n$  cross ratios like these. If we want to compute the factor of  $\chi_k(D_k(P))$  associated to index  $i$  we subtract (rather than add)  $i$  from the indices shown in the middle figure. A similar rule goes for  $D_{k+1}(P)$ .

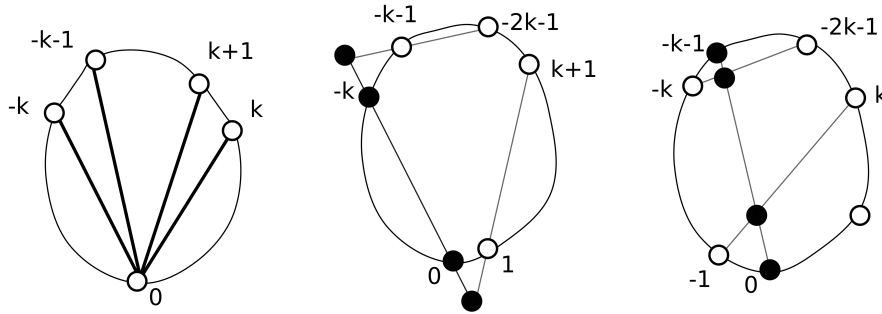


Figure 2.1: Computing the  $k$ -energy.

Theorem 1.2 follows from the next two results.

**Lemma 2.1.**  $\chi_k \circ D_k = \chi_k$ .

**Lemma 2.2.**  $\chi_k \circ D_{k+1} = \chi_k$ .

These results have almost identical proofs. We consider Lemma 2.1 in detail and then explain the small changes needed for Lemma 2.2.

## 2.3 Proof of the First Result

We study the ratio

$$R(P) = \frac{\chi_k \circ D_k(P)}{\chi_k(P)}. \quad (9)$$

We want to show that  $R(P)$  equals 1 wherever it is defined. We certainly have  $R(P) = 1$  when  $P$  is the regular  $n$ -Point.

Given a PolyPoint  $P$  we choose a pair of vertices  $a, b$  with  $|a - b| = k$ . We define  $P(t)$  to be the PolyPoint obtained by replacing  $P_a$  with

$$(1 - t)P_a + tP_b. \quad (10)$$

Figure 2.2 shows what we are talking about, in case  $k = 3$ . We have rotated the picture so that  $P_a$  and  $P_b$  both lie on the  $X$ -axis.

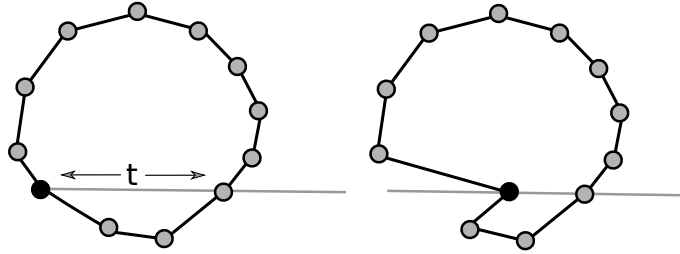


Figure 2.2: Connecting one PolyPoint to another by sliding a point.

The two functions

$$f(t) = \chi_k(P(t)), \quad g(t) = \chi_k \circ D_k(P(t)) \quad (11)$$

are each rational functions of  $t$ . Our notation does not reflect that  $f$  and  $g$  depend on  $P, a, b$ .

A *linear fractional transformation* is a map of the form

$$t \rightarrow \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbf{R}, \quad \alpha\delta - \beta\gamma \neq 0.$$

**Lemma 2.3** (Factor I). *If  $n \geq 4k + 2$  and  $P$  is a generically chosen  $n$ -Point, then  $f(t)$  and  $g(t)$  are each products of 4 linear fractional transformations. The zeros of  $f$  and  $g$  occur at the same points and the poles of  $f$  and  $g$  occur at the same points. Hence  $f/g$  is constant.*

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The only reason we choose  $n \geq 4k + 2$  in the Factor Lemma is so that the various diagonals involved in the proof do not have common endpoints. The Factor Lemma I works the same way for all  $k$  and for all choices of (large)  $n$ . We write  $P \leftrightarrow Q$  if we can choose indices  $a, b$  and some  $t \in \mathbf{R}$  such that  $Q = P(t)$ . The Factor Lemma implies that when  $P, Q$  are generic and  $P \leftrightarrow Q$  we have  $R(P) = R(Q)$ . The result for non-generic choices of  $P$  follows from continuity. Any  $n$ -Point  $Q$  can be included in a finite chain

$$P_0 \leftrightarrow P_1 \leftrightarrow \cdots \leftrightarrow P_{2n} = Q,$$

where  $P_0$  is the regular  $n$ -Point. Hence  $R(Q) = R(P_0) = 1$ . This shows that Lemma 2.1 holds for  $(k, n)$  where  $k \geq 2$  and  $n \geq 4k + 2$ . (The case  $k = 1$  is a main result of [19], and by now has many proofs.)

**Lemma 2.4.** *If Lemma 2.1 is true for all large values of  $n$ , then it is true for all values of  $n$ .*

**Proof:** If we are interested in the result for small values of  $n$ , we can replace a given PolyPoint  $P$  with its  $m$ -fold cyclic cover  $mP$ . We have  $\chi_k(mP) = \chi_k(P)^m$  and  $\chi_k(D_k(mP)) = \chi_k(D_k(P))^m$ . Thus, the result for large  $n$  implies the result for small  $n$ . ♠

In view of Equation 4 we have

$$f(t) = f_1(t) \cdots f_n(t), \quad f_j(t) = \chi(j, k, P(t)). \quad (12)$$

Thus  $f(t)$  is the product of  $n$  “local” cross ratios. We call an index  $j$  *asleep* if none of the lines involved in the cross ratio  $f_j(t)$  depend on  $t$ . In other words, the lines do not vary at all with  $t$ . Otherwise we call  $j$  *awake*.

As we vary  $t$ , only the diagonals  $P_{0,h}$  change for  $h = -k, -k - 1, k + 1, k$ . From this fact, it is not surprising that there are precisely 4 awake indices. These indices are

$$j_0 = 0, \quad j_1 = k + 1, \quad j_2 = -k - 1, \quad j_3 = -k. \quad (13)$$

The index  $k$  is not awake because the diagonal  $P_{0,k}(t)$  does not move with  $t$ .



We define a *chord* of  $P(t)$  to be a line defined by a pair of vertices of  $P(t)$ . The point  $P_0(t)$  moves at linear speed, and the 4 lines involved in the calculation of  $f_{c_j}(t)$  are distinct unless  $P_0(t)$  lies in one of the chords of  $P(t)$ . Thus  $f_{c_j}(t)$  only has zeros and poles at the corresponding values of  $t$ . It turns out that only the following chords are involved.

$$\begin{array}{cccccccc} -k & -k & -k & -k & -k-1 & -k-1 & k+1 & k+1 \\ -k-1 & k+1 & 1 & -2k-1 & -1 & -2k-1 & 1 & 2k+1 \end{array} \quad (14)$$

We call these  $c_0, \dots, c_7$ . For instance,  $c_0$  is the line through  $P_{-k}$  and  $P_{-k-1}$ . Let  $t_j$  denote the value of  $t$  such that  $P(t_j) \in c_j$ .

The PolyPoint  $Q(t) = D_k(P(t))$  has the same structure as  $P(t)$ . Up to projective transformations  $Q(t)$  is also obtained from the regular PolyPoint by moving a single vertex along one of the  $k$ -diagonals. The pattern of zeros and poles is not precisely the same because the chords of  $Q(t)$  do not correspond to the chords of  $P(t)$  in a completely straightforward way. The  $k$ -diagonals of  $Q(t)$  correspond to the vertices of  $P(t)$  and *vice versa*. The  $(k+1)$  diagonals of  $Q(t)$  correspond to the vertices of  $\Delta_k^{-1}(P(t))$ . This is what gives us our quadruples of points in the middle picture in Figure 2.1.

We now list the pattern of zeros and poles. We explain our notation by way of example. The quadruple  $(f, 2, 4, 5)$  indicates that  $f_{c_2}$  has a simple zero at  $f_4$  and a simple pole at  $t_5$ .

$$(f, 0, 0, 1), \quad (f, 1, 6, 7), \quad (f, 2, 4, 5), \quad (f, 3, 2, 3). \quad (15)$$

$$(g, 0, 6, 5), \quad (g, 1, 0, 3), \quad (g, 2, 2, 1), \quad (g, 3, 4, 7). \quad (16)$$

Since these functions have holomorphic extensions to  $\mathbb{C}$  with no other zeros and poles, these functions are linear fractional transformations. This pattern establishes the Factor Lemma I.

Checking that the pattern is correct is just a matter of inspection. We give two example checks.

- To see why  $f_{c_2}$  has a simple zero at  $t_4$  we consider the quintuple

$$(-k-1, -2k-1, -2k-2, 0, -1).$$

## The Flapping Birds in the Pentagon Zoo

At  $t_4$  the two diagonals  $P_{-k-1,0}$  and  $P_{-k-1,-1}$  coincide. In terms of the cross ratios of the slopes we are computing  $\chi(a, b, c, d)$  with  $a = b$ . The point  $P_0(t)$  is moving with linear speed and so the zero is simple.

- To see why  $g_{c_2}$  has a simple pole at  $t_1$  we consider the 4 points

$$P_{2k+2,k+2} \cap P_{1,k+1}, \quad P_{k+1}, \quad P_1, \quad P_{1,k+1} \cap P_{-k,0}. \quad (17)$$

These are all contained in the  $k$ -diagonal  $P_{1,k+1}$ , which corresponds to the vertex  $(-k-1)$  of  $D_k(P)$ . At  $t = t_1$  the three points  $P_0(t)$  and  $P_{-k}$  and  $P_{k+1}$  are collinear. This makes the 2nd and 4th listed point coincided. In terms of our cross ratio  $\chi(a, b, c, d)$  we have  $b = d$ . This gives us a pole. The pole is simple because the points come together at linear speed.

The other explanations are similar. The reader can see graphical illustrations of these zeros and poles using our program.

### 2.4 Proof of the Second Result

The proof of Lemma 2.2 is essentially identical to the proof of Lemma 2.1. Here are the changes. The Factor Lemma II has precisely the same statement as the Factor Lemma I, except that

- When defining  $P(t)$  we use points  $P_a$  and  $P_b$  with  $|a - b| = k + 1$ .
- We are comparing  $P(t)$  with  $D_{k+1}(P(t))$ .

This changes the definition of the functions  $f$  and  $g$ . With these changes made, the Factor Lemma I is replaced by the Factor Lemma II, which has an identical statement. This time the chords involved are as follows.

$$\begin{array}{cccccccc} -k-1 & -k-1 & -k-1 & -k-1 & -k & -k & k & k \\ -k & k & -1 & -2k-1 & 1 & -2k-1 & -1 & 2k+1 \end{array} \quad (18)$$

This time the 4 awake indices are:

$$j_0 = 0, \quad j_1 = k, \quad j_2 = -k - 1, \quad j_3 = -k. \quad (19)$$

Here is the pattern of zeros and poles.

$$(f, 0, 1, 0), \quad (f, 1, 7, 6), \quad (f, 2, 3, 2), \quad (f, 3, 5, 4). \quad (20)$$

$$(g, 0, 5, 6), \quad (g, 1, 3, 0), \quad (g, 2, 7, 4), \quad (g, 3, 1, 2). \quad (21)$$

The pictures in these cases look almost identical to the previous case. The reader can see these pictures by operating my computer program. Again, the zeros of  $f$  and  $g$  are located at the same places, and likewise the poles of  $f$  and  $g$  are located at the same places. Hence  $f/g$  is constant. This completes the proof the Factor Lemma II, which implies Lemma 2.2.

### 3 The Soul of the Bird

#### 3.1 Goal of the Chapter

Given a polygon  $P \subset \mathbf{R}^2$ , let  $\hat{P}$  be the closure of the bounded components of  $\mathbf{R}^2 - P$  and let  $P^I$  be the interior of  $\hat{P}$ . (Eventually we will see that birds are embedded, so  $\hat{P}$  will be a closed topological disk and  $P^I$  will be an open topological disk.)

Suppose now that  $P(t)$  for  $t \in [0, 1]$  is a path in  $B_{n,k}$  starting at the regular  $n$ -gon  $P(0)$ . We can adjust by a continuous family of projective transformations so that  $P(t)$  is a bounded polygon in  $\mathbf{R}^2$  for all  $t \in [0, 1]$ . We orient  $P(0)$  counter-clockwise around  $P^I(0)$ . We extend this orientation choice continuously to  $P(t)$ . We let  $P_{ab}(t)$  denote the diagonal through vertices  $P_a(t)$  and  $P_b(t)$ . We orient  $P_{a,b}(t)$  so that it points from  $P_a(t)$  to  $P_b(t)$ . We take indices mod  $n$ .

We now recall a definition from the introduction: When  $P$  is embedded, we say that  $P$  is *strictly star shaped* with respect to  $x \in P^I$  if each ray emanating from  $x$  intersects  $P$  exactly once.

### The Flapping Birds in the Pentagonam Zoo

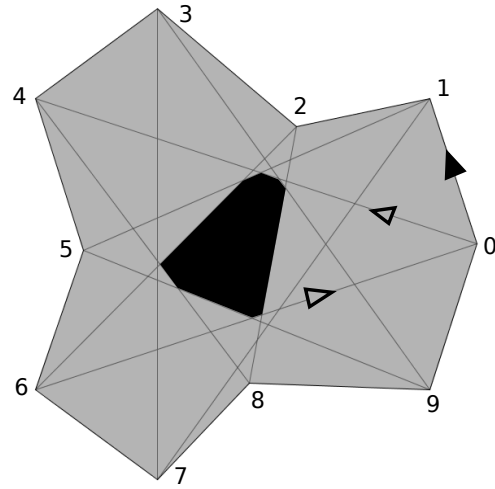


Figure 3.1: The soul of a 3-bird

Each such  $(k+1)$ -diagonal defines an oriented line that contains it, and also the (closed) *distinguished half plane* which lies to the left of the oriented line. These  $n$  half-planes vary continuously with  $t$ . The *soul* of  $P(t)$ , which we denote  $S(t)$ , is the intersection of the distinguished half-planes. Figure 3.1 shows the an example.

The goal of this chapter is to prove the following result.

**Theorem 3.1.** *Let  $P$  be a bird and let  $S$  be its soul. Then:*

1.  *$S$  has non-empty interior.*
2.  *$S \subset P^I$ .*
3.  *$P$  is strictly star-shaped with respect to any point in  $S$ .*

Theorem 3.1 immediately implies Statement 1 of Theorem 1.1.

We are going to give a homotopical proof of Theorem 3.1. We say that a value  $t \in [0, 1]$  is a *good parameter* if Theorem 3.1 holds for  $P(t)$ . All three conclusions of Theorem 3.1 are open conditions. Finally, 0 is a good parameter. For all these reasons, it suffices to prove that the set of good parameters is closed. By truncating our path at the first supposed failure, we reduce to the case when Theorem 3.1 holds for all  $t \in [0, 1)$ .

### 3.2 The Proof

For ease of notation we set  $X = X(1)$  for any object  $X$  associated to  $P(1)$ .

**Lemma 3.2.** *If  $P$  is any  $k$ -bird, then  $P_0$  and  $P_{2k+1}$  lie to the left of  $P_{k,k+1}$ . The same goes if all indices are cyclically shifted by the same amount.*

**Proof:** Consider the triangle with vertices  $P_0(t)$  and  $P_k(t)$  and  $P_{k+1}(t)$ . The  $k$ -niceness condition implies that this triangle is non-degenerate for all  $t \in [0, 1]$ . Since  $P_0(t)$  lies to the left of  $P_{k,k+1}(t)$ , the non-degeneracy implies the same result for  $t = 1$ . The same argument works for the triple  $(2k + 1, k, k + 1)$ . ♠

**Lemma 3.3.**  *$S$  is non-empty and contained in  $P^I$ .*

**Proof:** By continuity,  $S$  is nonempty and contained in  $P \cup P^I$ . By the  $k$ -niceness property and continuity,  $P_1$  lies strictly to the right of  $P_{0,k+1}$ . Hence the entire half-open edge  $[P_0, P_1)$  lies strictly to the right of  $P_{0,k+1}$ . Hence  $[P_0, P_1)$  is disjoint from  $S$ . By cyclic relabeling, the same goes for all the other half-open edges. Hence  $S \cap P = \emptyset$ . Hence  $S \subset P^I$ . ♠

**Lemma 3.4.**  *$P$  is strictly star-shaped with respect to any point of  $S$ .*

**Proof:** Since  $P(t)$  is strictly star-shaped with respect to all points of  $S(t)$  for  $t < 1$ , this lemma can only fail if there is an edge of  $P$  whose extending line contains a point  $x \in S$ . We can cyclically relabel so that the edge of  $\overline{P_0 P_1}$ .

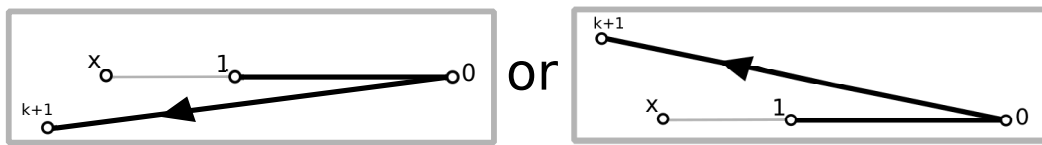


Figure 3.2: The diagonal  $P_{0,k+1}$  does not separate 1 from  $x$ .

Since  $x \notin P$ , either  $P_1$  lies between  $P_0$  and  $x$  or  $P_0$  lies in between  $x$  and  $P_1$ . In the first case, both  $P_1$  and  $x$  lie on the same side of the diagonal  $P_{0,k+1}$ . This is a contradiction:  $P_1$  is supposed to lie on the right and  $x$  is supposed to lie on the left. In the second case we get the same kind of contradiction with respect to the diagonal  $P_{-k,1}$ . ♠

We say that  $P$  has *opposing*  $(k + 1)$ -diagonals if it has a pair of  $(k + 1)$ -diagonals which lie in the same line and point in opposite directions. In this case, the two left half-spaces are on opposite sides of the common line.

**Lemma 3.5.**  $P$  does not have opposing  $(k + 1)$ -diagonals.

**Proof:** We suppose that  $P$  has opposing diagonals and we derive a contradiction. In this case  $S$ , which is the intersection of all the associated left half-planes, must be a subset of the line  $L$  containing these diagonals. But then  $P$  intersects  $L$  in at least 4 points, none of which lie in  $S$ . But then  $P$  cannot be strictly star-shaped with respect to any point of  $S$ . This is a contradiction. ♠

We call three  $(k + 1)$ -diagonals of  $P(t)$  *interlaced* if the intersection of their left half-spaces is a triangle. See Figure 3.3.

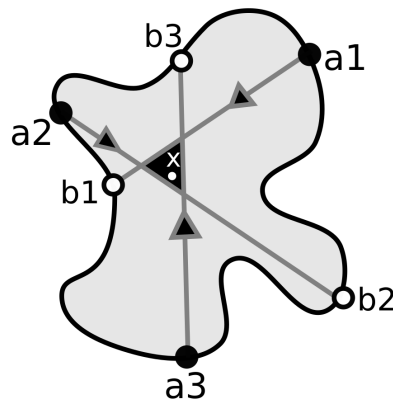


Figure 3.3: Interlaced diagonals on  $P(t)$ .

Given interlaced  $(k + 1)$ -diagonals, and a point  $x$  in the intersection, the circle of rays emanating from  $x$  encounters the endpoints of the diagonals in an alternating pattern:  $a_1, b_3, a_2, b_1, a_3, b_2$ , where  $a_1, a_2, a_3$  are the tail points and  $b_1, b_2, b_3$  are the head points. Here  $a_1$  names the vertex  $P_{a_1}(t)$ , etc.

**Lemma 3.6.**  *$P(t)$  cannot have interlaced diagonals for  $t < 1$ .*

**Proof:** Choose  $x \in S(t)$ . The  $n$ -gon  $P(t)$  is strictly star-shaped with respect to  $x$ . Hence, the vertices of  $P$  are encountered in order (mod  $n$ ) by a family of rays that emanate from  $x$  and rotates around full-circle. Given the order these vertices are encountered, we have  $a_{j+1} = a_j + \eta_j$ , where  $\eta_j \leq k$ . Here we are taking the subscripts mod 3 and the vertex values mod  $n$ . This tells us that  $n = \eta_1 + \eta_2 + \eta_3 \leq 3k$ . This contradicts the fact that  $n > 3k$ .

♠

It only remains to show that  $S$  has non-empty interior. A special case of Helly's Theorem says the following: If we have a finite number of convex subsets of  $\mathbf{R}^2$  then they all intersect provided that every 3 of them intersect. Applying Helly's Theorem to the set of interiors of our distinguished half-planes, we conclude that we can find 3 of these open half-planes whose triple intersection is empty. On the other hand, the triple intersection of the *closed* half-planes contains  $x$ . Since  $P$  has no opposing diagonals, this is only possible if the 3 associated diagonals are interlaced for  $t$  sufficiently close to 1. This contradicts Lemma 3.6. Hence  $S$  has non-empty interior.

## 4 The Feathers of the Bird

### 4.1 Goal of the Chapter

Recall that  $P^I$  is the interior of the region bounded by  $P$ . We call the union of black triangles in Figure 4.1 the *feathers* of the bird. the black region in the center is the soul.

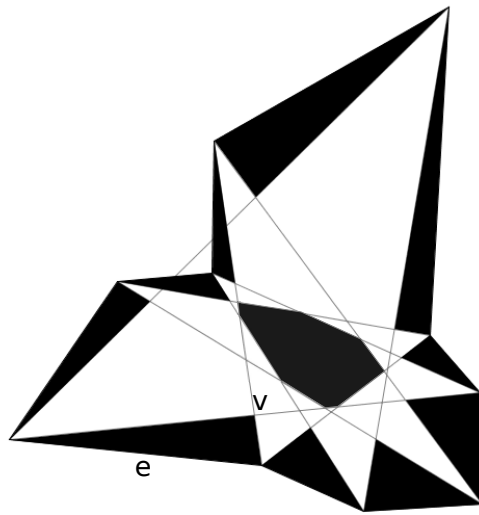


Figure 4.1 The feathers of a 3-bird.

Each feather  $F$  of a  $k$ -bird  $P$  is the convex hull of its *base*, an edge  $e$  of  $P$ , and its *tip*, a vertex of  $\Delta_k(P)$ .

The goal of this chapter is to prove the following result, which says that the simple topological picture shown in Figure 4.1 always holds.

**Theorem 4.1.** *The following is true.*

1. *Let  $F$  be an feather with base  $e$ . Then  $F - \{e\} \subset P^I$ .*
2. *Distinct feathers can only intersect at a vertex of  $P$ .*
3. *The line segment connecting two consecutive feather tips lies in  $P^I$ .*

When we apply  $\Delta_k$  to  $P$  we are just specifying the points of  $\Delta_k(P)$ . We define the *polygon*  $\Delta_k(P)$  so that the edges are the bounded segments connecting the consecutive tips of the feathers of  $P$ . With this definition, Statement 2 of Theorem 1.1 follows immediately from Theorem 4.1.



## 4.2 The Proof

There is one crucial idea in the proof of Theorem 4.1: The soul of  $P$  lies in the sector  $F^*$  opposite any of its feathers  $F$ . See Figure 4.2.

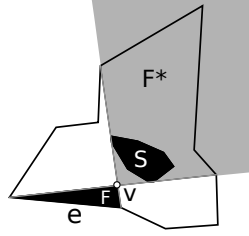


Figure 4.2 The soul lies in the sectors opposite the feathers.

We will give a homotopical proof of Theorem 4.1. By truncating our path of birds, we can assume that Theorem 4.1 holds for all  $t \in [0, 1)$ . We set  $P = P(1)$ , etc.

**Statement 1:** Figure 4.3 shows the 2 ways that Statement 1 could fail:

1. The tip  $v$  of the feather  $F$  could coincide with some  $p \in P$ .
2. Some  $p \in P$  could lie in the interior point of  $\partial F - e$ .

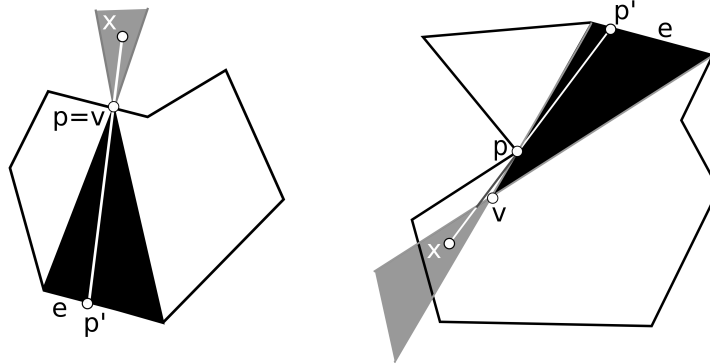


Figure 4.3: Case 1 (left) and Case 2 (right).

For all  $x \in F^*$ , the ray  $\overline{xp}$  intersects  $P$  both at  $p$  and at a point  $p' \in e$ . This contradicts the fact that for any  $x \in S \subset F^*$ , the polygon  $P$  is strictly star-shaped with respect to  $x$ .

This establishes Statement 1 of Theorem 4.1.

**Statement 2:** Let  $F_1$  and  $F_2$  be two feathers of  $P$ , having bases  $e_1$  and  $e_2$ . For Statement 2, it suffices to prove that  $F_1 - e_1$  and  $F_2 - e_2$  are disjoint.

The same homotopical argument as for Statement 1 reduces us to the case when  $F_1$  and  $F_2$  have disjoint interiors but  $\partial F_1 - e_1$  and  $\partial F_2 - e_2$  share a common point  $x$ . If  $\partial F_1$  and  $\partial F_2$  share an entire line segment then, thanks to the fact that all the feathers are oriented the same way, we would have two  $(k + 1)$  diagonals of  $P$  lying in the same line and having opposite orientation. Lemma 3.5 rules this out.

In particular  $x$  must be the tip of at least one feather. Figure 4.4 shows the case when  $x = v_1$ , the tip of  $F_1$ , but  $x \neq v_2$ . The case when  $x = v_1 = v_2$  has a similar treatment.

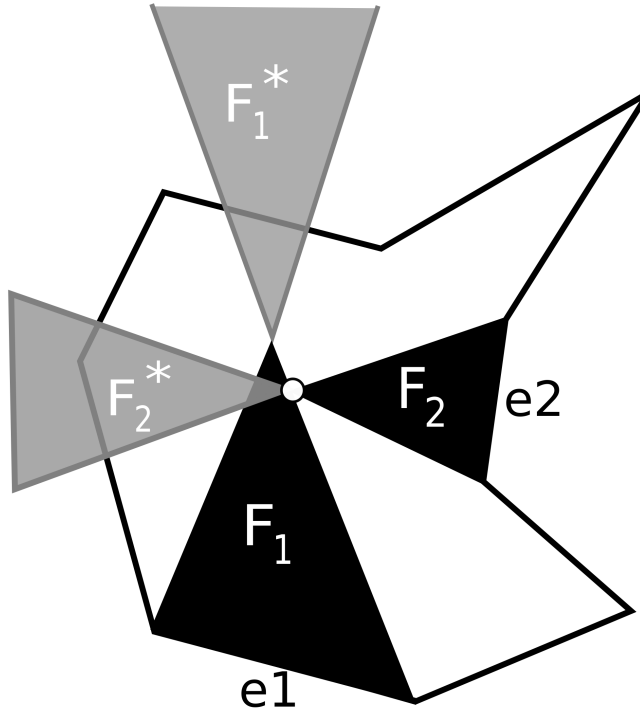


Figure 4.4: Opposing sectors are disjoint

In this case, the two sectors  $F_1^*$  and  $F_2^*$  are either disjoint or intersect in a single point.

This contradicts the fact that  $S \subset F_1^* \subset F_2^*$  has non-empty interior. This contradiction establishes Statement 2 of Theorem 4.1.

**Statement 3:** Recall that  $\hat{P} = P \cup P^I$ . Let  $F_1$  and  $F_2$  be consecutive feathers with bases  $e_1$  and  $e_2$  respectively. Let  $f$  be the edge connecting the tips of  $F_1$  and  $F_2$ . Our homotopy idea reduces us to the case when  $f \subset \hat{P}$  and  $f \cap P \neq \emptyset$ . Figure 4.5 shows the situation.

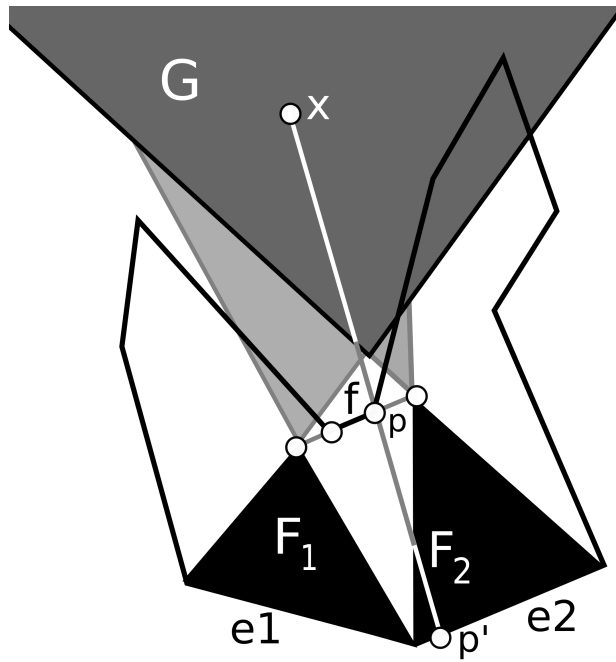


Figure 4.5: The problem a common boundary point

Note that  $f \cap P$  must be strictly contained in the interior of  $f$  because (by Statement 1 of Theorem 4.1) the endpoints of  $f$  lie in  $P^I$ . But then,  $f \cap P$  is disjoint from  $F_1^* \cap F_2^*$ , which is in turn contained in the shaded region  $G$ . For any  $x \in G$  and each vertex  $p$  of  $f$ , the ray  $\overrightarrow{xp}$  also intersects  $P$  at a point  $p' \in e_1 \cup e_2$ . This gives the same contradiction as above when we take  $x \in S \subset F_1^* \cap F_2^* \subset G$ . This completes the proof of Statement 3 of Theorem 4.1.

## 5 The Degeneration of Birds

### 5.1 Statement of Result

Let  $B_{k,n}$  denote the space of  $n$ -gons which are  $k$ -birds. Let  $\chi_k$  denote the  $k$ -energy. With the value of  $k$  fixed in the background, we say that a *degenerating path* is a path  $Q(t)$  of  $n$ -gons such that

1.  $Q(t)$  is planar for all  $t \in [0, 1]$ .
2. All vertices of  $Q(t)$  are distinct for all  $t \in [0, 1]$ .
3.  $Q(t) \in B_{k,n}$  for all  $t \in [0, 1)$  but  $Q(1) \notin B_{k,n}$ .
4.  $\chi_k(Q(t)) > \epsilon_0 > 0$  for all  $t \in [0, 1]$ .

In this chapter we will prove the following result, which will help us prove that  $\Delta_k(B_{k,n}) \subset B_{k,n}$  in the next chapter. The reader should probably just use the statement as a black box on the first reading.

**Lemma 5.1** (Degeneration). *If  $Q(\cdot)$  is a degenerating path, then all but at most one vertex of  $Q(1)$  lies in a line segment.*

**Remark:** Our proof only uses the fact that  $Q$  has nontrivial edges, nontrivial  $k$ -diagonals, and nontrivial  $(k+1)$ -diagonals. Some of the other vertices could coincide and it would not matter. Also, the same proof works if, instead of a continuous path, we have a convergent sequence  $\{Q(t_n)\}$  with  $t_n \rightarrow 1$  and a limiting polygon  $Q(1) = \lim Q(t_n)$ .

**Example:** Let us give an example for the case  $k = 1$  and  $n = 5$ . Figure 5.0 shows a picture of a pentagon  $Q(t)$  for  $t = 1 - s$ .

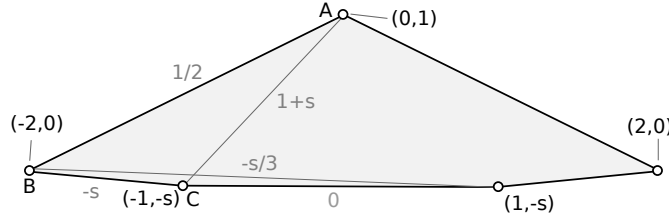


Figure 5.0: A degenerating path in the case  $k = 1$  and  $n = 5$ .

Here  $s$  ranges from 1 to 0 as  $t$  ranges from 0 to 1. We have labeled some of the slopes to facility the calculation (which we leave to the reader) that  $\chi_1(Q(t))$  remains uniformly bounded away from 0.

## 5.2 Distinguished Diagonals

We orient  $Q(t)$  so that it goes counter-clockwise around the region it bounds. We orient the diagonal  $Q_{ab}$  so that it points from  $Q_a$  to  $Q_b$ . For  $t < 1$  the vertices  $Q_1(t)$  and  $Q_k(t)$  lie to the right of the diagonal  $Q_{0,k+1}(t)$ , in the sense that a person walking along this diagonal according to its orientation would see that points in the right. This has the same proof as Lemma 3.2. The same rule holds for all cyclic relabelings of these points. The rule holds when  $t < 1$ . Taking a limit, we get a weak version of the rule: Each of  $Q_1(1)$  and  $Q_k(1)$  either lies to the right of the diagonal  $Q_{0,k+1}(1)$  or on it. The same goes for cyclic relabelings. We call this the *Right Hand Rule*.

Say that a *distinguished diagonal* of  $Q(t)$  is either a  $k$ -diagonal or a  $(k + 1)$ -diagonal. There are  $2n$  of these, and they come in a natural cyclic order:

$$Q_{0,k}(t) \quad Q_{0,k+1}(t), \quad Q_{1,k+1}(t), \quad Q_{1,k+2}(t), \dots \quad (22)$$

The pattern alternates between  $k$  and  $(k + 1)$ -diagonals. We say that a *diagonal chain* is a consecutive list of these.

We say that one oriented segment  $L_2$  lies *ahead* of another one  $L_1$  if we can rotate  $L_1$  by  $\theta \in (0, \pi)$  radians counter-clockwise to arrive at a segment parallel to  $L_2$ . In this case we write  $L_1 < L_2$ . We have

$$Q_{0,k+1}(t) < Q_{1,k+1}(t) < Q_{1,k+2}(t) < Q_{2,k+2}(t). \quad (23)$$

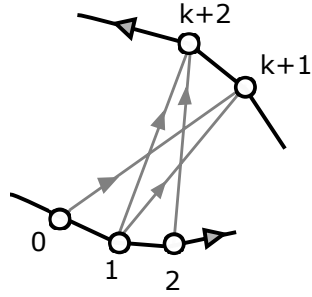


Figure 5.1: The turning rule

This certainly holds when  $t = 0$ . By continuity and the Right Hand Rule, this holds for all  $t < 1$ . Taking a limit, we see that the  $k$ -diagonals of  $Q(1)$  weakly turn counter-clockwise in the sense that either  $L_1 < L_2$  for consecutive diagonals or else  $L_1$  and  $L_2$  lie in the same line and point in the same direction. Moreover, the total turning is  $2\pi$ . If we start with one distinguished diagonal and move through the cycle then the turning angle increases by jumps in  $[0, \pi]$  until it reaches  $2\pi$ . We call these observations *the Turning Rule*.

### 5.3 Collapsed Diagonals

Figure 5.2 shows the distinguished diagonals incident to  $Q_0$ . We always take indices mod  $n$ . Thus  $-k - 1 = n - k - 1 \bmod n$ .

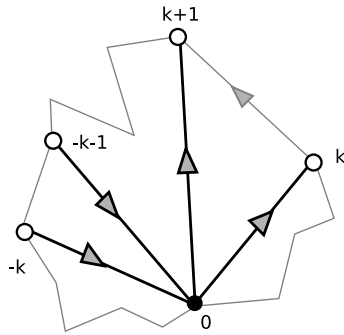


Figure 5.2: The 4 distinguished diagonals incident to  $Q_0(t)$ .

We say that  $Q$  has *collapsed diagonals* at a vertex if  $Q$  if the 4 distinguished diagonals incident to  $Q_k$  do not all lie on distinct lines. We set  $Q = Q(1)$ . We set  $X = X(1)$  for each object  $X$  associated to  $Q(1)$ .

Since  $Q$  is planar but not  $k$ -nice,  $Q$  must have collapsed diagonals at some vertex. We relabel so that the collapsed diagonals are at  $Q_0$ .

**Lemma 5.2.** *If  $Q$  has collapsed diagonals at  $Q_0$  then  $Q_{-k-1,0}$  and  $Q_{0,k+1}$  point in opposite directions or  $Q_{-k,0}$  and  $Q_{0,k}$  point in the same direction.*

**Proof:** Associated to each diagonal incident to  $Q_0$  is the ray which starts at  $Q_0$  and goes in the direction of the other endpoint of the diagonal. (Warning: The ray may have the opposite orientation than the diagonal it corresponds to.) If the angle between any of the rays tends to  $\pi$  as  $t \rightarrow 1$  then the angle between the outer two rays tends to  $\pi$ . In this case  $Q_{-k,0}$  and  $Q_{0,k}$  point in the same directions. If the angle between non-adjacent rays tends to 0 then  $Q_{-k-1,0}$  and  $Q_{0,k+1}$  are squeezed together and point in opposite directions.

Suppose that the angle between adjacent rays tends to 0. If the two adjacent rays are the middle ones, we have the case just considered. Otherwise, either the angle between the two left rays tends to 0 or the angle between the two right rays tends to 0. In either case, the uniform lower bound on the cross ratio forces a third diagonal to point either in the same or the opposite direction as these adjacent diagonals when  $t = 1$ . Any situation like this leads to a case we have already considered. ♠

## 5.4 The Case of Aligned Diagonals

We say that  $Q$  has *aligned diagonals* at the vertex  $Q_0$  if  $Q_{-k,0}$  and  $Q_{0,k}$  are parallel. This is the second option in Lemma 5.2. We make the same kind of definition at other vertices, with the indices shifted in the obvious way.

**Lemma 5.3.** *Suppose  $Q$  does not lie in a single line. Suppose also that  $Q$  has aligned diagonals at  $Q_0$ . Then the diagonals  $Q_{-k,0}, Q_{-k,1}, \dots, Q_{-1,k}, Q_{0,k}$  all are parallel and (hence) the  $2k + 1$  points  $Q_{-k}, \dots, Q_0, \dots, Q_k$  are contained in the line defined by these diagonals.*

**Proof:** These two diagonals define a short chain of diagonals, which starts with the first listed diagonal and ends with the second one. They also define a long chain, which starts

with the second and ends with the first. The total turning of the diagonals is  $2\pi$ , so one of the two chains defined by our diagonals turns  $2\pi$  and the other turns 0. Suppose first that the long chain has 0 turning. This chain involves all points of  $Q$ , and forces all points of  $Q$  to be on the same line. So, the short chain must consist of parallel diagonals. ♠

All we use in the rest of the proof is that  $Q_{-k}, \dots, Q_k$  are all contained in a line  $L$ . By shifting our indices, we can assume that  $Q_{k+1} \notin L$ . This relabeling trick comes with a cost. Now we cannot say whether the points  $Q_{-k}, \dots, Q_k$  come in order on  $L$ . We now regain this control.

**Lemma 5.4.** *The length  $2k$ -diagonal chain  $Q_{-k,0} \rightarrow \dots \rightarrow Q_{0,k}$  consists entirely of parallel diagonals. There is no turning at all.*

**Proof:** The diagonals  $Q_{-k,0}$  and  $Q_{0,k}$  are either parallel or anti-parallel. If they are anti-parallel, then the angle between the corresponding rays incident  $Q_0(t)$  tends to 0 as  $t \rightarrow 1$ . But these are the outer two rays. This forces the angle between all 4 rays incident to  $Q_0(t)$  to tend to 0. The whole picture just folds up like a fan. But one of these rays corresponds to  $Q_{0,k+1}(t)$ . This picture forces  $Q_{k+1} \in L$ . But this is not the case.

Now we know that  $Q_{-k,0}$  and  $Q_{0,k}$  are parallel. All the diagonals in our chain are either parallel or anti-parallel to the first and last ones in the chain. If we ever get an anti-parallel pair, then the diagonals in the chain must turn  $2\pi$  around. But then none of the other distinguished diagonals outside our chain turns at all. As in Lemma 5.3, this gives  $Q \subset L$ , which is false. ♠

We rotate the picture so that  $L$  coincides with the  $X$ -axis and so that  $Q_{0,k}$  points in the positive direction. Since we are already using the words *left* and *right* for another purpose, we say that  $p \in L$  is *forward of*  $q \in L$  if  $p$  has larger  $X$ -coordinate. Likewise we say that  $q$  is *backwards of*  $p$  in this situation. We say that  $Q_{0,k}$  points *forwards*. We have established that  $Q_{-k,0}, \dots, Q_{0,k}$  all point forwards.

**Lemma 5.5.**  *$Q_{k+2} \in L$  and both  $Q_{1,k+2}$  and  $Q_{2,k+2}$  point backwards.*



**Proof:** We have arranged that  $Q_{k+1} \notin L$ . Let us first justify the fact that  $Q_{k+1}$  lies above  $L$ . This follows from Right Hand Rule applied to  $Q_{0,k+1}$  and  $Q_k$  and the fact that  $Q_{0,k}$  points forwards. Since  $Q_{-k}, Q_{-k+1}, Q_1$  are collinear,  $Q$  has collapsed diagonals at  $Q_1$ . But  $Q$  cannot have aligned diagonals because  $Q_{1,k+1}$  is not parallel to  $Q_{-k,1}$ . Hence  $Q$  has folded diagonals at 1. This means that the diagonals  $Q_{-k,1}$  and  $Q_{1,k+2}$  point in opposite directions. This forces  $Q_{k+2} \in L$  and moreover we can say that  $Q_{1,k+2}$  points backwards.

We have  $Q_2 \in L$  because  $2 \leq k$ . We want to see that  $Q_{2,k+2}$  points forwards and they suppose not. We consider the 3 distinguished diagonals

$$Q_{0,k}, \quad Q_{1,k+2}, \quad Q_{2,k+2}.$$

These diagonals respectively point forwards, backwards, forwards and they all point one direction or the other along  $L$ . But then, in going from  $Q_{0,k}$  to  $Q_{2,k+2}$ , the diagonals have already turned  $2\pi$ . Since the total turn is  $2\pi$ , the diagonals  $Q_{2,k+2}, Q_{3,k+3}, \dots, Q_{n,n+k}$  are all parallel. But then  $Q_2, \dots, Q_n \in L$ . This contradicts the fact that  $Q_{k+1} \notin L$ . ♠

**Lemma 5.6.** *For at least one of the two indices  $j \in \{2k+2, 2k+3\}$  we have  $Q_j \in L$  and  $Q_{k+2,j}$  points forwards.*

**Proof:** Since  $Q_1, Q_2, Q_{k+2}$  are collinear,  $Q$  has collapsed diagonals at  $Q_{k+2}$ . So, by Lemma 5.2, we either have folded diagonals at  $Q_{k+2}$  or aligned diagonals at  $Q_{k+2}$ . The aligned case gives  $Q_{2k+2} \in L$  and the folded case gives  $Q_{2k+3} \in L$ . We need to work out the direction of pointing in each case.

Consider the aligned case. Suppose  $Q_{k+2,2k+2}$  points backwards, as shown in Figure 5.3.

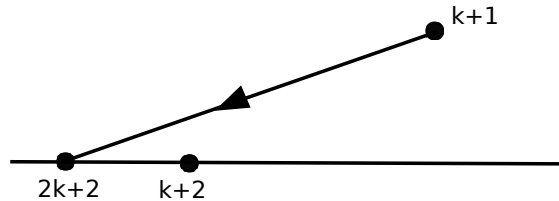


Figure 5.3: Violation of the Right Hand Rule

## The Flapping Birds in the Pentagon Zoo

This violates the Right Hand Rule for  $Q_{k+2}$  and  $Q_{k+1,2k+2}$  because  $Q_{k+1}$  lies above  $L$ .

Consider the folded case. Since  $Q_{k+2,2k+3}$  and  $Q_{1,k+2}$  point in opposite directions, and  $Q_{1,k+2}$  points backwards (by the previous lemma),  $Q_{k+2,2k+3}$  points forwards. ♠

Let  $j \in \{2k+2, 2k+3\}$  be the index from Lemma 5.6. Consider the 3 diagonals

$$Q_{0,k}, \quad Q_{1,k+1}, \quad Q_{k+2,j}.$$

These diagonals are all parallel to  $L$  and respectively point in the forwards, backwards, forwards direction. This means that the diagonals in the chain  $Q_{0,k} \rightarrow \dots \rightarrow Q_{k+2,j}$  have already turned  $2\pi$  radians. But this means that the diagonals

$$Q_{k+2,2k+3}, \quad Q_{k+3,2k+3}, \quad Q_{k+3,2k+4}, \quad \dots \quad Q_{0,k} = Q_{n,n+k}$$

are all parallel and point forwards along  $L$ . Hence  $Q_{k+2}, Q_{k+3}, \dots, Q_n \in L$ . Hence all points but  $Q_{k+1}$  lie in  $L$ .

### 5.5 The Case of Double Folded Diagonals

We fix a value of  $k$ . Say that two indices  $a, b \in \mathbb{Z}/n$  are *far* if their distance is at least  $k$  in  $\mathbb{Z}/n$ . We say that  $Q$  has *far folded diagonals* if  $Q$  has folded diagonals at  $Q_a$  and  $Q$  has folded diagonals at  $b$  and  $a, b$  are far.

In this case we have two parallel diagonals  $Q_{a,a+k+1}$  and  $Q_{b,b+k+1}$ . As in the proof of Lemma 5.3, one of the two diagonal chains defined by these diagonals consists of parallel diagonals. The far condition guarantees that at least  $2k+1$  consecutive points are involved in each chain. But then we get  $2k+1$  collinear points. So, if  $Q$  has far folded diagonals, then the same proof as in the previous section shows that the conclusion of the Degeneration Lemma holds for  $Q$ .

### 5.6 Good Folded Diagonals

We say that the folded diagonals  $Q_{-k-1,0}$  and  $Q_{0,k+1}$  are *good* if all the points  $Q_{k+1}, Q_{k+2}, \dots, Q_{n-k-1}$  are collinear. This notion is empty when  $k=2$  and  $n=7$  but otherwise it has content. In

this section we treat the case when  $Q$  has a pair of good folded diagonals. We start by discussing an auxiliary notion.

We say that  $Q$  has *backtracked edges* at  $Q_a$  if the angle between the edges  $Q_{a,a+1}$  and  $Q_{a,a-1}$  is either 0 or  $2\pi$ .

**Lemma 5.7.** *If  $Q$  has backtracked edges at  $Q_a$  then  $Q$  has folded diagonals at  $Q_a$ .*

**Proof:** For  $t \in [0, 1)$ , the edges of  $Q$  emanating from  $a$  divide the plane into 4 sectors, and one of these sectors,  $C(t)$  contains all the distinguished diagonals emanating from  $Q_a(t)$ . The sector  $C(t)$  is the one which locally intersects  $Q(t)$  near  $Q_a(t)$ . The angle of  $C(t)$  tends to 0 as  $t \rightarrow 1$ , forcing all the distinguished diagonals emanating from  $Q_a(t)$  to squeeze down as  $t \rightarrow 1$ . This gives us the folded diagonals. ♠

We will use Lemma 5.7 in our analysis of good folded edges. Now we get to it. We rotate so that our two diagonals are in the line  $L$ , which is the  $X$ -axis. We normalize so that  $Q_0$  is the origin, and  $Q_{k+1}$  and  $Q_{-k-1}$  are forward of  $Q_0$ .

**Lemma 5.8.** *If  $n > 3k+1$  and  $Q_{-k-1,0}, Q_{0,k+1}$  are good folded diagonals, then the Degeneration Lemma is true for  $Q$ .*

**Proof:** Suppose first that  $Q_1 \in L$ . Then  $Q$  has folded diagonals at  $Q_{k+1}$ . When  $n > 3k+1$  the indices  $(k+1)$  and  $(-k-1)$  are  $k$ -far. This gives  $Q$  far folded diagonals, a case we have already treated.

To finish our proof, we show that  $Q_1 \in L$ . We explore some of the other points. We know that  $Q_{k+1}, \dots, Q_{n-k-1} \in L$ . We can relabel dihedrally so that  $Q_{n-k-1}$  is forwards of  $Q_{k+1}$ . We claim that  $Q_{k+2}$  is forwards of  $Q_{k+1}$ . Suppose not. Then there is some index  $a \in \{k+2, \dots, -k-2\}$  such that  $Q_a$  is backwards of  $Q_{a\pm 1}$ . What is going on is that our points would start by moving backwards on  $L$  and eventually they have to turn around. The index  $a$  is the turn-around index. This gives us backtracked edges at  $Q_a$ . By Lemma 5.7, we have folded diagonals at  $Q_a$ . But  $a$  and 0 are  $k$ -far indices. This gives  $Q$  far-folded diagonals.

The only way out of the contradiction is that  $Q_{k+2}$  is forwards of  $Q_{k+1}$ .

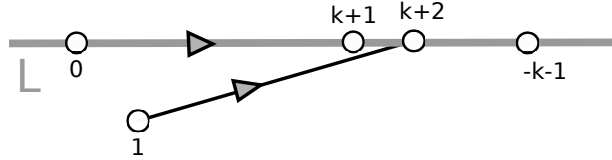


Figure 5.4: A contradiction involving  $Q_1$ .

Suppose  $Q_1 \notin L$ . by the Right Hand Rule applied to the diagonal  $Q_{0,k+1}$ , the point  $Q_1$  lies beneath  $L$ , as shown in Figure 5.4. But then  $Q_{k+1}$  lies to the left of the diagonal  $Q_{1,k+2}$ . This violates the Right Hand Rule. Now we know that  $Q_1 \in L$ . ♠

**Lemma 5.9.** *Suppose  $n = 3k + 1$  and  $k > 2$ . If  $Q_{-k-1,0}, Q_{0,k+1}$  are good folded diagonals, then the Degeneration Lemma is true for  $Q$ .*

**Proof:** The same argument as in Lemma 5.8 establishes the key containment  $Q_1 \in L$ . (We need  $k > 2$  for this.) From here, as in Lemma 5.8, we deduce that  $Q_{-k-1,0}$  and  $Q_{k+1,2k+2}$  are parallel. This time the conclusion we get from this is not as good. We get a run of  $k$  points in  $L$ , and then a point not necessarily in  $L$ , and then a run of  $k$  additional points in  $L$ .

The points are  $Q_{k+1}, \dots, Q_{2k+1}, \dots, Q_0$  with the point  $Q_{-k}$  omitted. But then  $Q$  has folded diagonals at each of these points except the outer two,  $Q_{k+1}$  and  $Q_0$ . But then For each such index  $h$ , we see that both  $Q_{h \pm (k+1)}$  belong to  $L$ . This gives us all but one point in  $L$ .

It is instructive to consider an example, say  $k = 4$  and  $n = 13$ . In this case, our initial run of points in  $L$  is  $Q_5, Q_6, Q_7, Q_8, Q_{10}, Q_{11}, Q_{12}, Q_{13}$ . The folded diagonals at  $Q_6, Q_7, Q_8$  respectively give  $Q_1, Q_2, Q_3 \in L$ . The folded diagonals at  $Q_{10}, Q_{11}, Q_{12}$  respectively give  $Q_2, Q_3, Q_4 \in L$ . ♠

Finally we consider the case  $k = 2$  and  $n = 7$ . In this case all we know is that  $Q_0, Q_3, Q_4 \in L$  with  $Q_3, Q_4$  forwards of  $Q_0$ . We can dihedrally relabel to that  $Q_4$  is forwards of  $Q_3$ . Here  $Q_3 = Q_{k+1}$  and  $Q_4 = Q_{k+2}$ . So, now we can run the same argument as in Lemma 5.9 to conclude that  $Q_1 \in L$ . Now we proceed as in the proof of Lemma 5.9.

## 5.7 Properties of the Soul

We define  $S = S(1)$  to be the set of all accumulation points of sequences  $\{p(t_n)\}$  where  $p(t_n) \in S(t_n)$  and  $t_n \rightarrow 1$ . Here  $S(t_n)$  is the soul of  $P(t_n)$ . We have one more case to analyze, namely ungood folded diagonals. To make our argument go smoothly, we first prove some properties about  $S$ .

**Lemma 5.10.** *Suppose that  $Q$  has folded diagonals at  $Q_0$ . If the Degeneration Lemma is false for  $Q$ , then  $S$  is contained in the line segment joining  $Q_0$  to  $Q_{k+1}$*

**Proof:** Here is a general statement about  $S$ . Since  $S(t)$  is non-empty and closed for all  $t < 1$ , we see by compactness that  $S$  is also a non-empty closed subset of the closed region bounded by  $Q$ . By continuity  $S$  lies to the left of all the closed half-planes defined by the oriented  $(k + 1)$  diagonals (or in their boundaries). Since  $S$  lies to the left of (or on) each  $(k + 1)$  diagonal,  $S$  is a subset of the line  $L$  common to the folded diagonals and indeed  $S$  lies to one side of the fold point  $Q_0$ . From the way we have normalized,  $S$  lies in the  $X$ -axis forward of  $Q_0$ . (The point  $Q_0$  might be an endpoint of  $S$ .)

If  $S$  contains points of  $L$  that lie forward of  $Q_{k+1}$  then either the diagonal  $Q_{k+1,2k+2}$  points along the positive  $X$ -axis or into the lower half-plane. In the former cases, the diagonals  $Q_{0,k+1}, Q_{k+1,2k+2}$  are parallel and we get at least  $2k + 1$  collinear points and so the Degeneration Lemma holds for  $Q$ .

If  $Q_{k+1,2k+2}$  points into the negative half-plane, then the diagonal  $Q_{0,k+1}$  turns more than  $\pi$  degrees before reaching  $Q_{k+1,2k+2}$ . But then the diagonals in the chain  $Q_{-k-1,0} \rightarrow \dots \rightarrow Q_{0,k+1} \dots \rightarrow Q_{k+1,2k+2}$  turn more than  $2\pi$  degrees. This is a contradiction. ♠

**Remark:** The same argument works with  $Q_{-k-1}$  in place of  $Q_{k+1}$ .

**Lemma 5.11.** *If the Degeneration Lemma is false for  $Q$  then  $S$  cannot intersect  $Q$  in the interior of an edge of  $Q$ .*

**Proof:** Suppose this happens. We relabel so that the edge is  $Q_{0,1}$ . By the Right Hand Rule, the point  $Q_1$  is not on the left of the diagonal  $Q_{0,k+1}$ . At the same time,  $S$  is not on the right

of the diagonal. The only possibility is that  $Q_1, Q_0, Q_{k+1}$  are collinear. Likewise  $Q_{-k}, Q_0, Q_1$  are collinear. Furthermore, the  $(k+1)$ -diagonals  $Q_{-k,1}$  and  $Q_{0,k+1}$  are parallel. Figure 5.5 shows the situation for  $Q(t)$  and  $S(t)$  when  $t$  is very near 1.

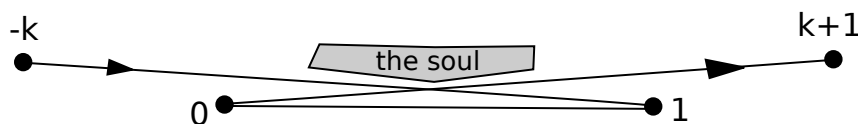


Figure 5.5: The relevant points and lines.

But now we have two  $(k+1)$ -diagonals that are parallel and which start at indices that are  $k$  apart in  $\mathbb{Z}/n$ . This gives us  $2k+1$  consecutive collinear points on the line containing our edge. We know how to finish the Degeneration Lemma in this case. The only way out is that  $S$  cannot intersect  $Q$  in the interior of an edge of  $Q$ . ♠

**Lemma 5.12.** *If the Degeneration Lemma is false for  $Q$ , then  $S$  cannot contain a vertex of  $Q$ .*

**Proof:** We relabel so that  $Q_0 \in S$ . The same analysis as in the previous lemma shows that  $Q_1, Q_0, Q_{-k}$  are collinear. Figure 5.6 shows the situation for  $t$  near 1. At the same time, the points  $Q_{-1}, Q_0, Q_k$  are collinear.

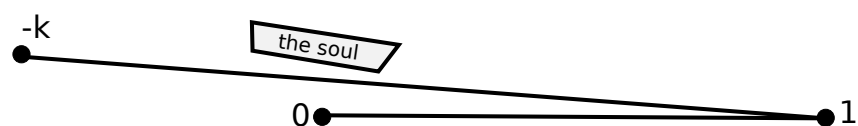


Figure 5.6: The relevant points and lines

To avoid a case of the Degeneration Lemma we have already done,  $Q$  must have folded diagonals at  $Q_{-k}$ . Likewise  $Q$  must have folded diagonals at  $Q_k$ . But then  $Q$  has far folded diagonals, and the Degeneration Lemma holds for  $Q$ . ♠

Now let us bring back our assumptions:  $Q$  has folded diagonals at  $Q_0$  and the points  $Q_0, Q_{k+1}, Q_{-k-1}$  all lie in the  $X$ -axis in the forward order listed.

**Corollary 5.13.** *If the Degeneration Lemma is false for  $Q$  then  $S$  lies in the open interval bounded by  $Q_0$  and  $Q_{k+1}$  and no point of  $S$  lies in  $Q$ . In particular,  $S$  contains a point  $x$ , forwards of  $Q_0$  and backwards of both  $Q_{k+1}$  and  $Q_{-k-1}$ , that is disjoint from  $Q$ .*

## 5.8 Ungood Folded Diagonals

The only case left is when  $Q$  does not have  $2k + 1$  consecutive collinear points, and when all folded diagonals of  $Q$  are ungood. Without loss of generality, we will consider the case when  $Q$  has ungood folded diagonals at  $Q_0$ . We normalize as in the previous section, so that  $Q_0, Q_{k+1}, Q_{-k-1}$  lie in forward order on  $L$ , which is the  $X$ -axis. Let  $x$  be a point from Corollary 5.13.

We call an edge of  $Q$  *escaping* if  $e \cap L$  is a single point. We call two different edges of  $Q$ , in the labeled sense, *twinned* if they are both escaping and if they intersect in an open interval. Even if two distinctly labeled edges of  $Q$  coincide, we consider them different as labeled edges.

**Lemma 5.14.**  *$Q$  cannot have twinned escaping edges.*

**Proof:** Consider  $Q(t)$  for  $t$  near 1. This polygon is strictly star shaped with respect to a point  $x(t)$  near  $x$ .

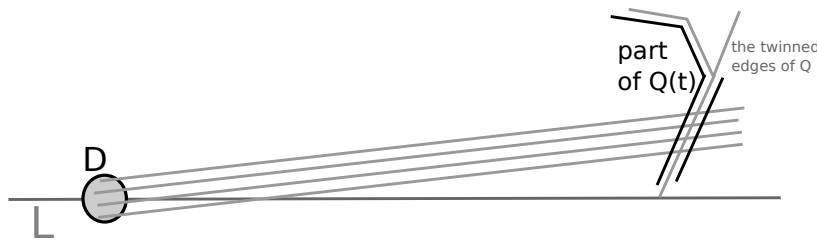


Figure 5.7: Rays intersecting the twinned segments

There is a disk  $D$  about  $x$  such that every  $p \in D$  contains a ray which intersects the twinned edges in the middle third portion of their intersection. Figure 5.7 shows what we mean. Once  $t$  is sufficiently near 1, the soul  $S(t)$  will intersect  $D$ , and for all points  $p \in D$  there will be a ray which intersects  $Q(t)$  twice. This contradicts the fact that  $Q(t)$  is

strictly star-shaped with respect to all points of  $S(t)$ . ♠

We say that an escape edge *rises above*  $L$  if it intersects the upper half plane in a segment.

**Lemma 5.15.**  *$Q$  cannot have two escape edges which rise above  $L$  and intersect  $Q$  on the same side of the point  $x$ .*

**Proof:** This situation is similar to the previous proof. In this case, there is a small disk  $D$  about  $x$  such that every point  $p \in D$  has a ray which intersects both rising escape edges transversely, and in the middle third of each of the two subsegments of these escape edges that lie above  $L$ . Figure 5.8 shows this situation.

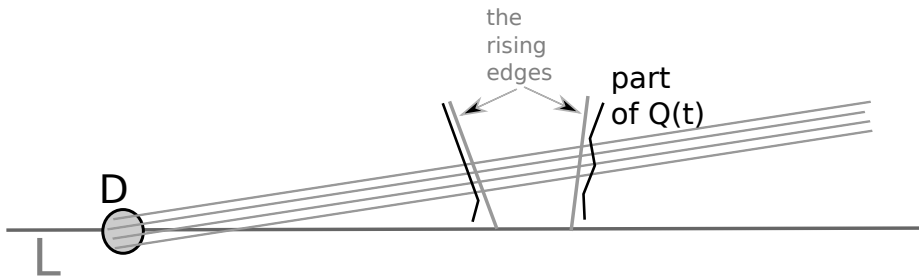


Figure 5.8: Rays intersecting the rising segments.

In this case, some part of  $Q(t)$  closely shadows our two escape edges for  $t$  near 1. But then, once  $t$  is sufficiently near 1, each ray we have been talking about intersects  $Q(t)$  at least twice, once by each escaping edge. This gives the same contradiction as in the previous lemma. ♠

We define falling escape segments the same way. The same statement as in Lemma 5.15 works for falling escape segments. Since  $x \notin Q$  we conclude that  $Q$  can have at most 4 escaping segments total.

But  $Q = Q_+ \cup Q_-$ , where  $Q_{\pm}$  is an arc of  $Q$  that starts at  $Q_{k+1}$  and ends at  $Q_{-k-1}$ . Since both these arcs start and end on  $L$ , and since both do not remain entirely on  $L$ , we see



that each arc has at least 2 escape edges, and none of these are twinned. This means that both  $Q_+$  and  $Q_-$  have exactly two escape edges.

Now for the moment of truth: Consider  $Q_+$ . Since  $Q_+$  just has 2 escape edges, they both have to be either rising or falling. Also, since  $Q_+$  starts and ends on the same side of  $x$ , and cannot intersect  $x$ , both the escape edges for  $Q_+$  are on the same side of  $x$ . This is a contradiction. The same argument would work for  $Q_-$  but we don't need to make it.

## 6 The Persistence of Birds

In this chapter we prove Statement 3 of Theorem 1.1, namely the fact that  $\Delta_k(B_{n,k}) = B_{n,k}$ . First we use the Degeneration Lemma to prove that  $\Delta_k(B_{n,k}) \subset B_{n,k}$ . Then we deduce the opposite containment from projective duality and from the factoring of  $\Delta_k$  given in §2.2.

### 6.1 Containment

Suppose for the sake of contradiction that there is some  $P \in B_{k,n}$  such that  $\Delta(P) \notin B_{k,n}$ . Recall that there is a continuous path  $P(t)$  for  $t \in [0, 1]$  such that  $P(0)$  is the regular  $n$ -gon.

Define  $Q(t) = \Delta_k(P(t))$ . There is some  $t_0 \in [0, 1]$  such that  $Q(t_0) \notin B_{k,n}$ . We can truncate our path so that  $t_0 = 1$ . In other words,  $Q(t) \in B_{n,k}$  for  $t \in [0, 1)$  but  $Q(1) \notin B_{k,n}$ .

**Lemma 6.1.**  *$Q(\cdot)$  is a degenerating path.*

**Proof:** Note that  $Q(\cdot)$  is planar and hence satisfies Property 1 for degenerating paths. Let  $P = P(1)$  and  $Q = Q(1)$ . If  $Q$  does not have all distinct vertices then two different feathers of  $P$  intersect at a point which (by Statement 2 of Theorem 1.1) lies in  $P^I$ . This contradicts Statement 2 of Theorem 4.1. Hence  $Q(\cdot)$  satisfies Property 2 for degenerating paths. By construction,  $Q(t) \in B_{n,k}$  for all  $t \in [0, 1)$ . Hence  $Q(\cdot)$  satisfies Property 3. The energy  $\chi_k$  is well-defined and continuous on  $B_{k,n}$ . Hence, by compactness,  $\chi_k(P(t)) > \epsilon_0$  for some  $\epsilon_0 > 0$  and all  $t \in [0, 1]$ . Now for the crucial step: We have already proved that  $\chi_k \circ \Delta_k = \chi_k$ . Hence  $\chi_k(Q(t)) > \epsilon_0$  for all  $t \in [0, 1]$ . That is,  $Q(\cdot)$  satisfies Property 4 for degenerating

paths. ♠

Now we apply the Degeneration Lemma to  $Q(\cdot)$ . We conclude that all but at most 1 vertex of  $Q(1)$  lies in a line  $L$ . Stating this in terms of  $P(1)$ , we can say that all but at most one of the feathers of  $P(1)$  have their tips in a single line  $L$ . Call an edge of  $P(1)$  *ordinary* if the feather associated to it has its tip in  $L$ . We call the remaining edge, if there is one, *special*. Thus, all but at most one edge of  $P$  is ordinary.

Let  $S(t)$  be the soul of  $P(t)$ . We know that  $S(1)$  has non-empty interior by Theorem 3.1. For ease of notation we set  $P = P(1)$  and  $S = S(1)$ .

**Lemma 6.2.**  *$P$  cannot have ordinary edges  $e_1$  and  $e_2$  that lie on opposite sides of  $L$  and are disjoint from  $L$ .*

**Proof:** Suppose this happens. Figure 6.1 shows the situation.

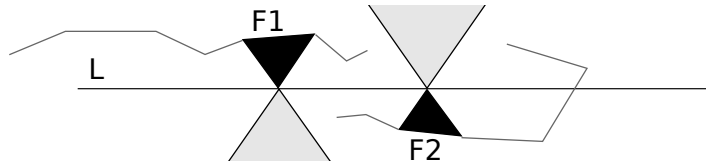


Figure 6.1: Two feathers on opposite sides of  $L$ .

Let  $F_1$  and  $F_2$  be the two associated feathers. Then the opposite sector  $F_1^*$  lies above  $L$ , and the opposite sector  $F_2^*$  lies below  $L$  and the two tips are distinct. But then  $S(1)$ , which must lie in the intersection of these sectors, is empty. ♠

**Lemma 6.3.**  *$P$  cannot have more than 2 ordinary edges which intersect  $L$ .*

**Proof:** Note that an ordinary edge cannot lie in  $L$  because then the tip would not. So, an ordinary edge that intersects  $L$  does so either at a single vertex or at an interior point. As we trace along  $L$  in one direction or the other we encounter the first intersecting edge and then the last one and then some other intersecting edge. Let  $F_1.F_2.F_3$  be the two feathers,

as shown in Figure 6.3. Let  $e_j$  be the edge of  $F_j$  that belongs to  $P$ . Let  $v_j$  be the tip of  $F_j$ . (Figure 6.3 shows the case when  $e_j \cap L$  is an interior point of  $e_j$  for each  $j = 1, 2, 3$ , but the same argument would work if some of these intersection points were vertices.)

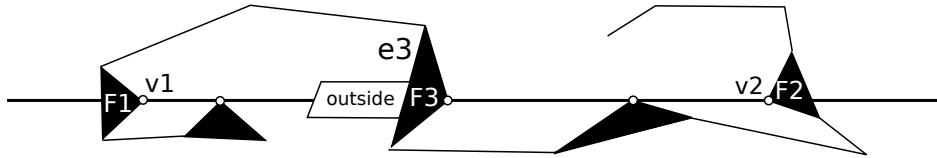


Figure 6.2: Three or more crossing edges

One of the two arcs  $\alpha$  of  $Q$  joining  $v_1$  to  $v_2$  stays in  $L$ , namely the one avoiding the one point of  $Q$  not on  $L$ . However,  $\alpha$  passes right through  $F_3$  and in particular crosses  $e_3$  transversely. However, one side of  $F_3$  is outside  $P$ . Hence  $\alpha$  is not contained in  $P^I$ , the interior of the region bounded by  $P$ . This contradicts Statement 2 of Theorem 1.1, which says that  $Q \subset P^I$ . ♠

The line  $L$  divides the plane into two open half-planes, which we call the *sides* of  $L$ . Lemma 6.2 says that  $P$  cannot have ordinary edges contained in opposite sides of  $L$ . Lemma 6.3 says that at most 2 ordinary edges can intersect  $L$ . Hence, all but at most 2 of the ordinary edges of  $P$  lie on one side of  $L$ . Call this the *abundant side* of  $L$ . Call the other side the *barren side*. The barren side contains no ordinary edges at all, and perhaps the special edge. In particular, at most two vertices of  $P$  lie in the barren side.

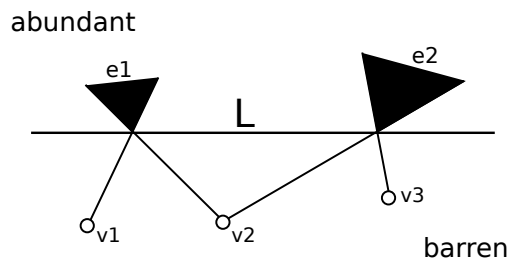


Figure 6.3: Following the diagonals bounding a feather

At the same time, each ordinary edge on the abundant side contributes two vertices to the barren side: We just follow the diagonals comprising the corresponding feather. These

diagonals cross  $L$  from the abundant side into the barren side. Two different ordinary edges contribute at least 3 distinct vertices to the barren side. This is a contradiction.

We have ruled out all possible behavior for  $P = P(1)$  assuming that  $Q = Q(1)$  is degenerate. Hence,  $Q(1)$  is not degenerate. This means that  $Q(1)$  is a bird. This completes the proof that

$$\Delta_k(B_{k,n}) \subset B_{k,n}. \quad (24)$$

## 6.2 Equality

We use the notation from §2.2. Equation 8 implies that

$$\Delta_k^{-1} = D_{k+1} \circ \Delta_k \circ D_{k+1}. \quad (25)$$

So far, Equation 25 makes sense in terms of PolyPoints and PolyLines.

Below we will explain how to interpret  $D_{k+1}$  as a map from polygons in  $\mathbf{P}$  to polygons in  $\mathbf{P}^*$  and also as a map from polygons in  $\mathbf{P}^*$  to polygons in  $\mathbf{P}$ . Since the dual projective plane  $\mathbf{P}^*$  is an isomorphic copy of  $\mathbf{P}$ , it makes sense to define  $B_{k,n}^*$ . This space is just the image of  $B_{k,n}$  under any projective duality. Below we will prove

**Theorem 6.4.**  $D_{k+1}(B_{k,n}) \subset B_{k,n}^*$ .

It then follows from projective duality that  $D_{k+1}(B_{k,n}^*) \subset B_{k,n}$ . Combining these equations with Equation 25 we see that  $\Delta_k^{-1}(B_{n,k}) \subset B_{n,k}$ . This combines with Equation 24 to finish the proof of Theorem 1.1.

Now we prove Theorem 6.4.

**Lemma 6.5.** *If  $P \in B_{k,n}$ , then we can enhance  $D_{k+1}(P)$  in such a way that  $D_{k+1}(P)$  is a planar polygon in  $\mathbf{P}^*$ . The enhancement varies continuously.*

**Proof:** A polygon is a PolyPoint together with additional data specifying an edge in  $\mathbf{P}$  joining each consecutive pair of points. Dually, we get a polygon in  $\mathbf{P}^*$  from a PolyLine by specifying, for each pair of consecutive lines  $L_j, L_{j+1}$ , an arc of the pencil of lines through the intersection point which connects  $L_j$  to  $L_{j+1}$ .

Specifying an enhancement of  $D_{k+1}(P)$  is the same as specifying, for each consecutive pair  $L_1, L_2$  of  $(k + 1)$  diagonals of  $P$ , an arc of the pencil through their intersection that connects  $L_1, L_2$ . There are two possible arcs. One of them avoids the interior of the soul of  $P$  and the other one sweeps through the soul of  $P$ . We choose the arc that avoids the soul interior. Figure 6.4 shows that we mean for a concrete example.

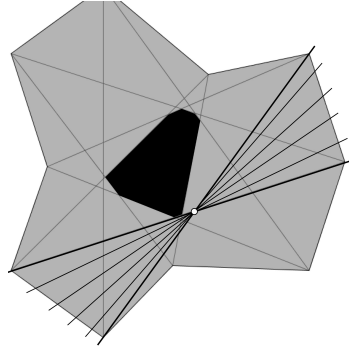


Figure 6.4: Enhancing a PolyLine to a polygon: Avoid the soul.

Since the soul of  $P$  has non-empty interior, there exists a point  $x \in P$  which is disjoint from all these pencil-arcs. Applying duality, this exactly says that there is some line in  $\mathbf{P}^*$  which is disjoint from all the edges of our enhanced  $D_{k+1}(P)$ . Hence, this enhancement makes  $D_{k+1}(P)$  planar. Our choice also varies continuously on  $B_{n,k}$ . ♠

**Lemma 6.6.**  $D_{k+1}$  maps a member of  $B_{k,n}$  to an  $n$ -gon which is  $k$ -nice.

**Proof:** Let  $Q = D_{k+1}(P)$ . A  $(k + 1)$ -diagonal of  $Q$  is just a vertex of  $P$ . A  $k$  diagonal of  $Q$  is a vertex of  $\Delta_k(p)$ . Thus, to check the  $k$ -nice property for  $Q$  we need to take  $n$ -collections of 4-tuples of points and check that they are distinct. In each case, the points are collinear because the lines of  $Q$  are coincident.

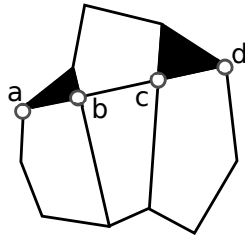


Figure 6.5 One of the  $n$  different 4-tuples we need to check.

Once we make this specification, there is really combinatorially only possibility for which collections we need to check. Figure 6.5 shows one such 4-tuple,  $a, b, c, d$ . The shaded triangles are the two feathers of  $P$  whose tips are  $b, c$ . But  $a, b, c, d$  are distinct vertices of  $P \cup \Delta_k(P)$  and so they are distinct. That is all there is to it. ♠

To show that  $Q = D_{k+1}(P)$  is a  $k$ -bird, we consider a continuous path  $P(t)$  from the regular  $n$ -gon  $P(0)$  to  $P = P(1)$ . We set  $Q(t) = P(t)$ . By construction,  $Q(0)$  is a copy of the regular  $n$ -gon in  $P^*$ , and  $Q(t)$  is  $k$ -nice for all  $t \in [0, 1]$ , and  $Q(t)$  is a planar polygon for all  $t \in [0, 1]$ . By definition  $Q = Q(1)$  is a  $k$ -bird. This completes the proof of Theorem 6.4.

## 7 The Triangulation

### 7.1 Basic Definition

In this section we gather together the results we have proved so far and explain how we construct the triangulation  $\tau_P$  associated to a bird  $P \in B_{k,n}$ .

Since  $\Delta_k(B_{k,n}) \subset B_{k,n}$ , we know that  $\Delta_k(P)$  is also a  $k$ -bird. Combining this with Theorem 3.1 and Theorem 4.1 we can say that  $\Delta_k(P)$  is one embedded  $n$ -gon contained in  $P^I$ , the interior of the region bounded by the embedded  $P$ . The region between  $P$  and  $\Delta_k(P)$  is a topological annulus. Moreover,  $\Delta_k(P)$  is obtained from  $P$  by connecting the tips of the feathers of  $P$ . The left side Figure 7.1 shows how this region is triangulated. The black triangles are the feathers of  $P$  and each of the white triangles is made from an edge of  $\Delta_k(P)$  and two edges of adjacent feathers.

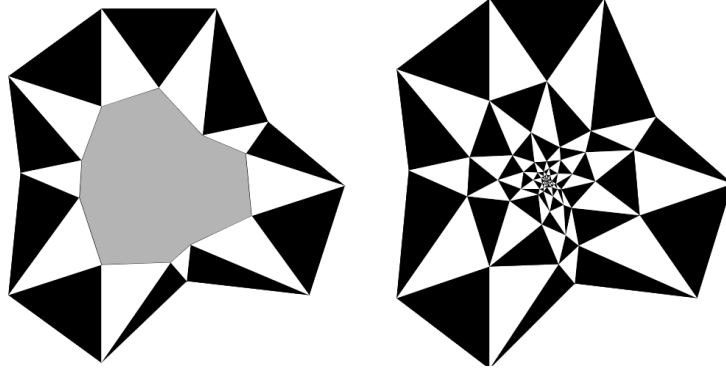


Figure 7.1: The triangulation of the annulus

**Lemma 7.1.** *For every member  $P \in B_{k,n}$ , the associated  $2n$  triangles have pairwise disjoint interiors, and thus triangulate the annular region between  $P$  and  $\Delta_k(P)$ .*

**Proof:** As usual, we make a homotopical argument. If this result is false for some  $P$ , then we can look at path which starts at the regular  $n$ -gon (for which it is true) and stop at the first place where it fails. Theorem 4.1 tells us that nothing goes wrong with the feathers of  $P$ . The only thing that can go wrong is  $\Delta_k(P)$  fails to be an embedded polygon. Since this does not happen, we see that in fact there is no counter-example at all. ♠

We can now iterate, and produce  $2n$  triangles between  $\Delta_k(P)$  and  $\Delta_k^2(P)$ , etc. The right side of Figure 7.1 shows the result of doing this many times. The fact that  $\Delta_k(B_{k,n}) = B_{k,n}$  allows us to extend outward as well. When we iterate forever in both directions, we get an infinite triangulation of a (topological) cylinder that has degree 6 everywhere. This is what Figure 1.6 is showing. We call this bi-infinite triangulation  $\tau_P$ .

## 7.2 Some Structural Results

The following result will help with the proof of Theorem 1.3.

**Theorem 7.2.** *Let  $P \in B_{n,k}$ . Let  $S$  be the soul of  $B$ . Then for  $\ell \geq n$  we have  $\Delta_k^\ell(P) \subset S$ .*

**Proof:** We first note the existence of certain infinite polygonal arcs in  $\tau_P$ . We start at a vertex of  $P$  and then move inward to a vertex of  $\Delta_k(P)$  along one of the edges. We then continue through this vertex so that 3 triangles are on our left and 3 on our right. Figure 7.2 below shows the two paths like this that emanate from the same vertex of  $P$ .

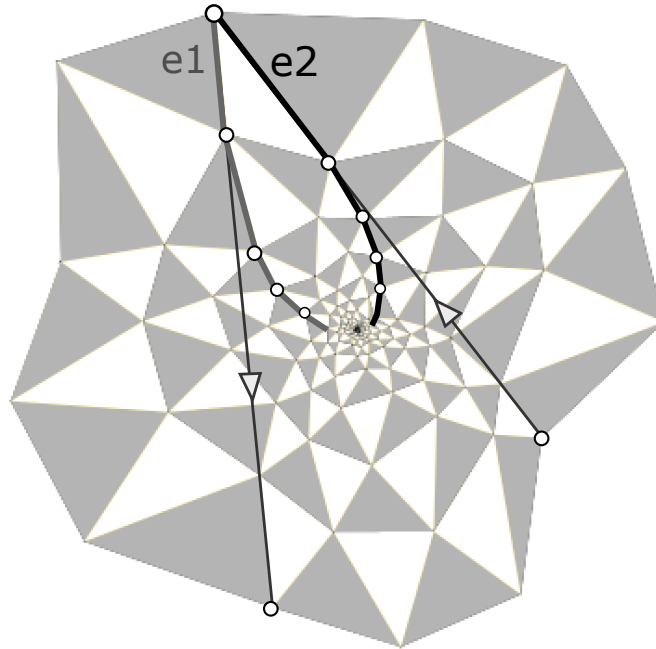


Figure 7.2: The spiral paths.

The usual homotopical argument establishes the fact that the spiral paths are locally convex. One can understand their combinatorics, and how they relate to the polygons in the orbit, just by looking at the case of the regular  $n$ -gon. We call the two spiral paths in Figure 7.2 *partners*. In the regular  $n$ -gon the partners intersect infinitely often. So this is true in general. Each spiral path has an initial segment joining the initial endpoint on  $P$  to the first intersection point with the partner. We define a *petal* to be the region bounded by the initial paths of the two partners.

It is convenient to write  $P^\ell = \Delta_k^\ell(P)$ . In the regular case,  $P^\ell$  is contained in the petal for  $\ell > n - 1$ . Hence, the same goes in the general case. Because the initial segments are locally convex, the petal lies to the left of the lines extending the edges  $e_1$  and  $e_2$  when



these edges are oriented according to the  $(k + 1)$ -diagonals of  $P$ . But this argument works for every pair of partner spiral paths which start at a vertex of  $P$ . We conclude that for  $\ell \geq n$ , the polygon  $P^\ell$  lies to the left of all the  $(k + 1)$ -diagonals of  $P$ . But the soul of  $P$  is exactly the intersection of all these left half planes. ♠

Theorem 7.2 in turn gives us information about the nesting properties of birds within an orbit. Let  $S_\ell$  denote the soul of  $P^\ell$ . Let

$$S_\infty = \bigcap_{\ell \in \mathbf{Z}} S_\ell, \quad S_{-\infty} = \bigcup_{\ell \in \mathbf{Z}} S_\ell. \quad (26)$$

It follows from Theorem 7.2 that  $\hat{P}_\infty = S_\infty$  and  $\hat{P}_{-\infty} = S_{-\infty}$ , because

$$S_{\ell+n} \subset P^{\ell+n} \subset S_\ell \subset P^\ell. \quad (27)$$

Hence these sets are all convex subsets of an affine plane.

**Corollary 7.3.** *Any  $P \in B_{k,n}$  is strictly star-shaped with respect to all points in the convex hull of  $\Delta_k^n(P)$ .*

**Proof:** Since  $P^{\ell+n} \subset S_\ell$ , and  $P^\ell$  is strictly star shaped with respect to all points of  $S^\ell$ , we see that  $P^\ell$  is strictly star shaped with respect to all points of  $P^{\ell+n}$ . Since  $S_\ell$  is convex, we can say more strongly that  $P^\ell$  is strictly star-shaped with respect to all points of the convex hull of  $P^{\ell+n}$ . Now we just set  $\ell = 0$  and recall the meaning of our notation, we get the exact statement of the result. ♠

An immediate corollary is that  $P$  is strictly star-shaped with respect to  $\hat{P}_\infty$ . (Theorem 1.3 says that this is a single point.)

## 8 Nesting Properties of Birds

### 8.1 Duality

In this chapter we prove Theorem 1.3. In this first section we show how Statement 1 of Theorem 1.3 implies Statement 2. We want to prove that the “backwards union”  $\hat{P}_{-\infty}$  is an affine plane. Here  $P \in B_{n,k}$  is a  $k$ -bird.

We take  $\ell \geq 0$  and consider  $P^{-\ell} = \Delta_k^{-\ell}(P)$ . Since  $P^{-\ell}$  is planar, there is a closed set  $\Lambda_\ell$  of lines in  $P$  which miss  $P^{-\ell}$ . These sets of lines are nested:  $\Lambda_1 \supset \Lambda_2 \supset \Lambda_3 \dots$ . The intersection is non-empty and contains some line  $L$ . We can normalize so that  $L$  is the line at infinity. Thus all  $P^{-\ell}$  lie in  $\mathbf{R}^2$ . We want to see that  $\hat{P}_{-\infty} = \mathbf{R}^2$ .

Let  $D_{k+1}$  be the map from §2.2 and §6.2. From Equation 8 we see that  $D_{k+1}$  conjugates  $\Delta_k$  to  $\Delta_k^{-1}$ . With Theorem 6.4 in mind, define the following “dual”  $k$ -birds:

$$\Pi^\ell = \Delta_k^\ell(D_{k+1}(P)) = D_{k+1}(P^{-\ell}). \quad (28)$$

From Statement 1 of Theorem 1.3, the sequence of  $k$ -birds  $\{\Pi^\ell\}$  shrinks to a point in the dual plane  $P^*$ . The vertices of  $\Pi^\ell$  are the  $(k+1)$ -diagonals of  $P^{-\ell}$ . Because the vertices of  $\Pi^\ell$  shrink to a single point, all the  $(k+1)$ -diagonals of  $P^{-\ell}$  converge to a single line  $L'$ .

**Lemma 8.1.**  *$L'$  is the line at infinity.*

**Proof:** Suppose not. When  $\ell$  is large, all the  $(k+1)$ -diagonals point nearly in the same direction as  $L'$ . In particular, this is true of the subset of these diagonals which define the soul  $S^{-\ell}$ . But these special diagonals turn monotonically and by less than  $\pi$  radians as we move from one to the next. Hence, some of these diagonals nearly point in one direction along  $L'$  and some point nearly in the opposite direction. But then  $S^{-\ell}$  converges to a subset of  $L'$ . This is a contradiction, ♠

The soul  $S^{-\ell}$  is a convex set, containing the origin, and is bounded by some of the  $(k+1)$  diagonals. If  $S^{-\ell}$  does not converge to the whole plane, then some  $(k+1)$ -diagonal

intersects a uniformly bounded region in  $\mathbf{R}^2$  for each  $\ell$ . But this produces a sequence of  $(k + 1)$ -diagonals that does not converge to the line at infinity. This is a contradiction. Hence  $S^{-\ell}$  converges to all of  $\mathbf{R}^2$ . But then so does  $P^{-\ell}$ .

## 8.2 The Pre-Compact Case

The rest of the chapter is devoted to proving Statment 1 of Theorem 1.3. Let  $P \in B_{n,k}$  and let  $P^\ell = \Delta^\ell(P)$ . We take  $\ell = 0, 1, 2, 3, \dots$

**Conjecture 8.2.** *The sequence  $\{P^\ell\}$  is pre-compact modulo affine transformations. That is, this sequence has a convergent subsequence which converges to another element of  $B_{n,k}$ .*

In this section I will prove the  $\hat{P}_\infty$  is a single point under the assumption that  $\{P^\ell\}$  is pre-compact.

We would like to see that the diameter of  $P^\ell$  steadily shrinks, but the notion of diameter is not affinely natural. We first develop a notion of affinely natural diameter. For each direction  $v$  in the plane, we let  $\|S\|_v$  denote the maximum length of  $L \cap S$  where  $L$  is a straight line parallel to  $v$ . We then define

$$\delta(S_1, S_2) = \sup_v \frac{\|S_1\|_v}{\|S_2\|_v} \in [0, 1]. \quad (29)$$

The quantity  $\delta(S_1, S_2)$  is affine invariant, and (choosing a direction  $\mu$  which realizes the diameter of  $S_1$ ) we have

$$\frac{\text{diam}(S_1)}{\text{diam}(S_2)} \leq \frac{\|S_1\|_\mu}{\|S_2\|_\mu} \leq \delta(S_1, S_2). \quad (30)$$

Let  $S^\ell$  be the soul of  $P^\ell$ . By Theorem 5.11 we have  $S^{\ell+n} \subset S^\ell$ . More precisely, the former set is contained in the interior of the latter set. Under the pre-compactness assumption, there are infinitely many indices  $\ell_j$  and some  $\epsilon > 0$  such that

$$\delta(S^{\ell_j+n}, S^{\ell_j}) < 1 - \epsilon. \quad (31)$$

But then

$$\frac{\text{diam}(S^{\ell_j+n})}{\text{diam}(S^{\ell_j})} < 1 - \epsilon \quad (32)$$

infinitely often. This forces  $\text{diam}(S^\ell) \rightarrow 0$ . But  $\hat{P}_\infty$  is contained in this nested intersection and hence is a point.

If we knew the truth of Conjecture 8.2 then our proof of Theorem 1.3 would be done. Since we don't know this, we have to work much harder to prove Statement 1 in general.

### 8.3 Normalizing by Affine Transformations

Henceforth we assume that the forward orbit  $\{P^\ell\}$  of  $P$  under  $\Delta_k$  is not pre-compact modulo affine transformations.

**Lemma 8.3.** *There is a sequence  $\{T_\ell\}$  of affine transformations such that*

1.  $T_\ell(P^\ell)$  has (the same) 3 vertices which make a fixed equilateral triangle.
2.  $T_\ell$  expands distances on  $P^\ell$  for all  $\ell$ .
3.  $T_\ell(P^\ell)$  is contained in a uniformly bounded subset of  $\mathbf{R}^2$ .

**Proof:** To  $P^\ell$  we associate the triangle  $\tau_\ell$  made from 3 vertices of  $P^\ell$  and having maximal area. The diameter of  $\tau_\ell$  is uniformly small, so we can find a single equilateral triangle  $T$  and an expanding affine map  $T_\ell : \tau_\ell \rightarrow T$ . Let  $d$  be the side length of  $T$ . Every vertex of  $T_\ell(P^\ell)$  is within  $d$  of all the sides of  $T$ , because otherwise we'd have a triangle of larger area. The sequence  $\{T_\ell\}$  has the advertised properties. ♠

Let  $Q^\ell = T_\ell(P^\ell)$ . By compactness we can pass to a subsequence so that the limit polygon  $Q$  exists, in the sense that the vertices and the edges converge. Let  $Q_0, Q_1$ , etc. be the vertices of  $Q$ . Perhaps some of these coincide. Each distinguished diagonal of  $Q^\ell$  defines the unit vector which is parallel to it. Thus  $Q^\ell$  defines a certain list of  $2n$  unit vectors. We can pass to a subsequence so that all these unit vectors converge. Thus  $Q$  still has well defined distinguished diagonals even when the relevant points coincide.

We now define the “limiting soul”. Let  $S^\ell = S(Q^\ell)$ , the soul of  $Q^\ell$ . As in §5.7. let  $S$  be the set of accumulation points of sequences  $\{p^\ell\}$  with  $p^\ell \in S^\ell$ . Since  $S^\ell \subset Q^\ell$  for all  $\ell$  we

have  $S \subset Q$ . Now we define a related object. We have a left half-plane associated to each diagonal of  $Q$ . We define  $\Sigma$  to be the intersection of all these half-planes. We will use the set  $\Sigma$  at various places below to get control over the set  $S$ .

**Lemma 8.4.**  $S \subset \Sigma$ .

**Proof:** Fix  $\epsilon > 0$ . If this is not the case, then by compactness we can find a convergent sequence  $\{p^\ell\}$ , with  $p^\ell \in S^\ell$ , which does not converge to a point of  $\Sigma$ . But  $p^\ell$  lies in every left half plane associated to  $Q^\ell$ . But then, by continuity, the accumulation point  $p$  lies in every left half plane associated to  $Q$ . Hence  $p \in \Sigma$ . ♠

## 8.4 Structure of the Normalized Limits

We work under the assumption that  $\hat{P}_\infty$  is not a single point. The goal of this section is to establish several structural properties about the sets  $S$  and  $Q$ . Our first property guarantees that there is a chord  $S^*$  of  $S$  connecting vertices of  $Q$ . Once we establish this, we show that  $Q$  is a union of two “monotone” arcs joining the endpoints of  $S^*$ . These structural properties will be used repeatedly in subsequent sections of this chapter.

Let  $H_Q$  denote the convex hull of  $Q$ . Note that  $S \subset Q \subset H_Q$ .

**Corollary 8.5.** *Suppose that  $\hat{P}_\infty$  is not a single point. Then  $\delta(S, H_Q) = 1$ .*

**Proof:** Suppose not. Note that  $H_{Q^\ell} \subset S^{\ell-n}$  by Theorem 7.2 and convexity. Then for  $\ell$  large we have

$$\delta(Q^{\ell-n}) = \delta(S^\ell, S^{\ell-n}) \leq \delta(S^\ell, H_{Q^\ell}) < \delta(S, H_Q) + \epsilon,$$

and we can make  $\epsilon$  as small as we like. This gives us a uniform  $\delta < 1$  such that  $\delta(Q^\ell) < \delta$  once  $\ell$  is large enough. The argument in the compact case now shows that  $\hat{P}_\infty$  is a single point. ♠

Corollary 8.5 says that  $S$  and  $Q$  have the same diameter. Hence there is a chord  $S^* \subset S$  which has the same diameter as  $Q$ . Since  $Q$  is a polygon, this means that  $Q$  must have vertices at either endpoint of  $S^*$ . We normalize so that  $S^*$  is the unit segment joining  $(0, 0)$  to  $(1, 0)$ .

**Lemma 8.6.** *Let  $Q' \subset Q$  be an arc of  $Q$  that joins  $(0, 0)$  to  $(1, 0)$ .*

1. *The vertices of  $Q'$  must have non-decreasing  $x$ -coordinates.*
2. *If consecutive vertices of  $Q'$  have the same  $x$ -coordinate, they coincide.*
3. *Either  $Q' \subset S^*$  or  $Q'$  intersects  $S^*$  only at  $(0, 0)$  and  $(1, 0)$ .*

**Proof:** Suppose the Statement 1 is false. Then we can find a vertical line  $\Lambda$  which intersects  $S^*$  at a relative interior point and which intersects  $Q'$  transversely at 3 points. But then once  $\ell$  is sufficiently large,  $Q^\ell$  will intersect all vertical lines sufficiently close to  $\Lambda$  in at least 3 points and moreover some of these lines will contain points of  $S^\ell$ . This contradicts the fact that  $Q^\ell$  is strictly star-shaped with respect to all points of  $Q^\ell$ .

For Statement 2, we observe that  $Q'$  does not contain any point of the form  $(0, y)$  or  $(1, y)$  for  $y \neq 0$ . Otherwise  $Q$  has larger diameter than 1. This is to say that once  $Q'$  leaves  $(0, 0)$  it immediately moves forward in the  $X$ -direction. Likewise, once  $Q'$  (traced out the other way) leaves  $(1, 0)$  it immediately moves backward in the  $X$ -direction. If Statement 2 is false, then we can find a non-horizontal line  $\Lambda'$  which intersects  $S^*$  in a relative interior point and which intersects  $Q'$  transversely at 3 points. The slope of  $\Lambda'$  depends on which of the two vertices of  $Q'$  lies above the other. Once we have  $\Lambda'$  we play the same game as for the first statement, and get the same kind of contradiction.

Suppose Statement 3 is false. We use the kind of argument we had in §5.8. By Statements 1 and 2 together,  $Q'$  must have an escape edge which touches  $S^*$  in a relative interior point. Moreover, this one escape edge is paired with another escape edge. Thus we can find a point  $x \in S^*$  which strictly lies on the same side of both of these same-type escape edges. The argument in §5.8 now shows that  $Q^\ell$  is not strictly star-shaped with

respect to points of  $S^\ell$  very near  $x$ . ♠

**Corollary 8.7.** *Suppose  $0 \leq a < b < n$  and  $Q_a = Q_b$ . Then either we have  $Q_a = Q_{a+1} = \dots = Q_b$  or else we have  $Q_b = Q_{b+1} = \dots = Q_{a+n}$ .*

**Proof:** In view of Lemma 8.6 it suffices to show that our two monotone arcs comprising  $Q$  are disjoint except at their endpoints.

Let  $U$  denote the open upper halfplane, bounded by the  $X$ -axis. After reflecting in the  $X$ -axis we can guarantee that one of our monotone arcs  $\alpha$  has a point in  $U$ . But then, by Lemma 8.6, all of  $\alpha$  lies in  $U$  except for its endpoints. If the other monotone arc  $\beta$  intersects  $\alpha$  away from the endpoints, then  $\beta$  has a point in  $U$ , but then, by Lemma 8.6, all of  $\beta$  lies in  $U$  except for the endpoints. But then  $S$  lies in  $U$ , except for the points  $(0, 0)$  and  $(1, 0)$ . This contradicts the fact that  $S^* \subset S$ . ♠

Our argument shows in particular that  $Q$  is embedded, up to adding repeated vertices. However, we will not directly use this property in our proof below.

## 8.5 The Triangular Case

We continue with the assumption that  $\hat{P}$  is not a single point. Here we pick off a special case:

- There is a line  $L$  such that  $Q_0 \notin L$ .
- $Q_k, Q_{k+1}, \dots, Q_{n-k-1}, Q_{n-k} \in L$  and
- $Q_k \neq Q_{n-k}$ .

Figure 8.1 shows the situation. As always, the notation  $Q_{-k}$  and  $Q_{n-k}$  names the same point. All but  $2k - 1$  points are on  $L$ , and except for  $Q_0$  we don't know where these other  $2k - 1$  points are.

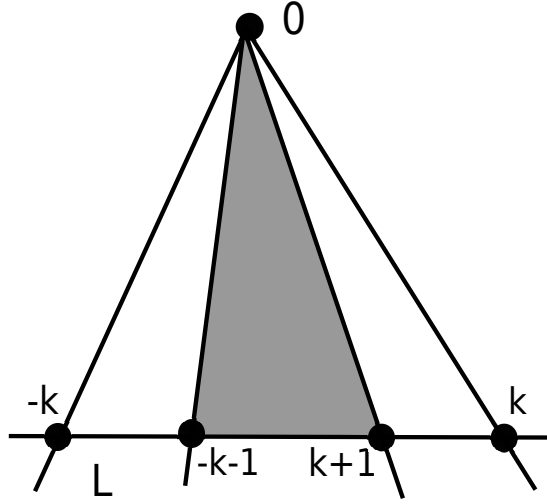


Figure 8.1: The triangular limit  $Q$ .

Given the constant energy of our orbit, the cross ratio of the lines

$$Q_{0,k}, Q_{0,k+1}, Q_{n-k-1,0}, Q_{n-k,0}$$

is at least  $\epsilon_0$ . Also, these lines are cyclically ordered about 0 as indicated in Figure 8.1, thanks to the  $k$ -niceness property and continuity. Also, the two lines containing  $Q_{0,k}$  and  $Q_{-k,0}$  are not parallel because  $Q_0 \notin L$ . Hence  $S$  is contained in the shaded region in Figure 8.1, namely the triangle with vertices  $Q_0$  and  $Q_{\pm(k+1)}$ . But this shaded region has diameter strictly smaller than the triangle  $\tau$  with vertices  $Q_0$  and  $Q_{\pm k}$ . Hence  $\text{diam}(S) < \text{diam}(\tau) \leq \text{diam}(Q)$ . This contradicts Corollary 8.5 which says, in particular, that  $S$  and  $Q$  have the same diameter.

## 8.6 The Case of No Folded Diagonals

We work under the assumption that  $\hat{P}_\infty$  is not a single point. The notions of collapsed diagonals, folded diagonals, and aligned diagonals from §5 make sense for  $Q$  because the concepts just involve the directions of the diagonals. The proof of Lemma 5.3 also works the same way.



**Lemma 8.8.**  *$Q$  must either have a trivial edge, a trivial distinguished diagonal, or collapsed diagonals,*

**Proof:** As remarked in §5, the proof of the Degeneration Lemma works for sequences as well as paths, and only uses the fact that the limiting polygon has nontrivial edges and nontrivial distinguished diagonals. So, if  $Q$  has no trivial edges and no trivial distinguished diagonals, then all but one vertex of  $Q$  lies in a single line. But then  $Q$  has collapsed diagonals. ♠

**Remark:** Here is a second, more direct proof. If Lemma 8.8 is false then we have a picture as in the left side of Figure 7.1. The feathers defined in §4.1 would be all non-degenerate and the segments joining the tips of consecutive feathers would be nontrivial. This would force  $S$  to lie in the interior of  $Q$ . But then  $\text{diam}(S) < \text{diam}(Q)$ , contradicting Corollary 8.5.

If  $Q$  has a trivial distinguished diagonal, then by Lemma 8.7, we see that  $Q$  also has a trivial edge. If  $Q$  has a trivial edge, say  $Q_{-1} = Q_0$ , then the diagonals at  $Q$  are collapsed at  $Q_k$ . So, in all cases,  $Q$  has collapsed diagonals. We assume in this section that  $Q$  has no folded diagonals anywhere. This means that  $Q$  has aligned diagonals, say at  $Q_k$ . Thus  $Q_{0,k}$  and  $Q_{k,2k}$  are parallel. Since  $Q$  does not lie in a line, Lemma 5.3 tells us that the chain of  $2k + 1$  parallel distinguished diagonals:

$$Q_{0,k}, Q_{0,k+1}, Q_{1,k+1}, Q_{1,k+2}, \dots, Q_{k-1,2k}, Q_{k,2k} \quad (33)$$

Now we have a “runaway situation”. The two diagonals  $Q_{2k,k}$  and  $Q_{2k,k-1}$  (which are just the reversals of the last two in Equation 33) are parallel. Thus  $Q$  has collapsed diagonals at  $Q_{2k}$ . Since  $Q$  has no folded diagonals,  $Q$  has aligned diagonals at  $Q_{2k}$ . But then, applying Lemma 5.3 again, we can extend that chain in Equation 33 so that it continues as  $\dots, Q_{2k-1,3k}, Q_{2k,3k}$ . But now  $Q$  has collapsed diagonals at  $Q_{3k}$ . And so on. Continuing this way, we end up with all points on  $Q$ . This is a contradiction.

The only way out is that  $Q$  must have folded diagonals somewhere

## 8.7 The Case of Folded Diagonals

We continue to work under the assumption that  $\hat{P}_\infty$  is not a single point. Now we consider the case when  $Q$  has folded diagonals at, say,  $Q_0$ . What this means that the diagonals  $Q_{0,k+1}$ ,  $Q_{0,-k-1}$  are parallel. (Again, these diagonals are well defined even when their endpoints coincide; we are just using a notational convention to name them here.) But then the corresponding half planes intersect along a single line  $L$ , forcing  $\Sigma \subset L$ . By Lemma 8.4, the soul  $S$  is contained in  $\Sigma$ . Hence,  $S \subset L$ . Letting  $S^*$  be the chord from §8.4, we also have  $S = S^*$ . This is because  $S$  and  $S^*$  are segments of the same diagonal and in the same line. We will use  $S$  and  $S^*$  interchangeably below.

We normalize so that  $S$  is the line segment connecting  $(0, 0)$  to  $(1, 0)$ . As in §8.4, both these points are vertices of  $Q$ . The folding condition forces  $\Sigma$  (and hence  $S$ ) to lie to one side of these points. Hence, we have either  $Q_0 = (0, 0)$  or  $Q_0 = (1, 0)$ . Without loss of generality we consider the case when  $Q_0 = (0, 0)$ . Note that points of  $Q - S$  do not belong to  $L$ , because  $Q$  and  $S$  have the same diameter. We break the analysis down into cases.

**Case 1:** Suppose that  $Q_{k+1}$  is not an endpoint of  $S^*$  and  $Q_{n-k-1} \neq (1, 0)$ . Consider the arc  $Q'$  given by  $Q_0 \rightarrow \dots \rightarrow Q_{k+1} \rightarrow \dots \rightarrow Q_\beta = (1, 0)$ . Here  $\beta$  is some index we do not know explicitly, but we take  $\beta$  as large as possible, in the sense that  $Q_{\beta+1} \neq (1, 0)$ . The arc  $Q'$  connects  $(0, 0)$  to  $(1, 0)$  and intersects  $S^*$  at  $Q_{k+1}$ , a point which is neither  $(0, 0)$  or  $(1, 0)$ . By Lemma 8.6, we have  $Q' \subset S^*$ . We conclude that  $Q_0, \dots, Q_\beta \subset S^*$ .

If  $\beta$  does not lie in the index interval  $(k + 1, n - k - 1)$  then we have just shown that  $Q_{k+1}, \dots, Q_{n-k-1} \in S^*$ . If  $\beta = n - k - 1$  we have the same result. Here is what we do if  $\beta$  does lie in  $(k + 1, n - k - 1)$ . We apply our same argument as in the previous paragraph to the arc  $Q_\beta \rightarrow \dots \rightarrow Q_{n-k-1}$ , and see that  $Q_\beta, \dots, Q_{n-k-1} \in S$ . So, in all cases, we see that  $Q_{k+1}, \dots, Q_{n-k-1} \in S$ .

In short,  $Q_j \in L$  unless  $j \in \{-k, \dots, -1\}$ . All but  $k$  vertices belong to  $L$ . In particular, we have an index  $h \in \{-k, \dots, -1\}$  such that  $Q_h \notin L$  but  $Q_{h+k}, Q_{h+k+1}, \dots, Q_{h+n-k-1}, Q_{h+n-k} \in L$ . Now we are close to the Triangular case from §8.5 except that all the indices are shifted

by  $h$ . If it happens that  $Q_{h+k} \neq Q_{h+n-k}$  then we have the Triangular Case and we are done.

The other possibility is that  $Q_{h+k} = Q_{h+n-k}$ . In this case, Lemma 8.7 gives us  $Q_{h+k} = Q_{h+k+1} = Q_{h+n-k-1} = Q_{h+n-k}$ . In particular, the diagonals  $Q_{h,h+k+1}$  and  $Q_{h,h+n-k-1}$  are folded at  $Q_h$ . Since  $Q_h \notin L$  this means that there is some other line  $L'$  such that  $S \subset L'$ . This is a contradiction.

**Case 2:** Suppose  $Q_{-k-1} = Q_{k+1} = (1, 0)$ . Before analyzing this case, we remember a lesson from the end of Case 1: It is not possible for  $Q$  to have folded diagonals at a point not on  $S$ .

Corollary 8.7 says that  $Q_{k+1} = \dots = Q_{n-k-1} = (1, 0)$ . This is a run of  $k + \beta$  points, where  $\beta = n - (3k + 1) \geq 0$ . There is some index  $h \in \{\pm 1, \dots, \pm k\}$  such that  $Q_h \notin L$ . Without loss of generality we will take  $h \in \{1, \dots, k\}$ .

Suppose first that  $n > 3k + 1$ . Then there are at least  $k + 1$  consecutive vertices sitting at  $(1, 0)$  and so both diagonals  $Q_{h,k+h}$  and  $Q_{h,k+h+1}$  point from  $Q_h$  to  $(1, 0) \neq Q_h$ . This means that  $Q$  has collapsed diagonals at  $Q_h$ . Remembering our lesson, we know that  $Q$  does not have folded diagonals at  $Q_h$ . Hence  $Q$  has aligned diagonals at  $Q_h$ .

Now we have the same runaway situation we had in §8.6. The diagonals in the chain  $Q_{h-k,h} \dots Q_{h,h+k}$  point are all pointing along the line connecting  $(1, 0)$  to  $Q_h$ , and they are pointing away from  $(1, 0)$ . This gives us collapsed diagonals at  $Q_{h+k}$ . Remembering our lesson, we see that  $Q$  has aligned diagonals at  $Q_{h+k}$ . And so on. All the points after  $Q_h$  get stuck on  $L'$  and we have a contradiction.

If  $n = 3k + 1$ , then the same argument works as long as  $h \neq \pm k$ . So, we just have to worry about the case when all points of  $Q$  belong to  $S$  except for  $Q_k$  and  $Q_{-k}$ , which do not belong to  $S$ . Applying Lemma 8.6 to the arc  $Q_0 \rightarrow Q_1 \rightarrow \dots \rightarrow Q_k \rightarrow (1, 0)$  we conclude that  $Q_0 = \dots = Q_{k-1} = (0, 0)$ . Applying Lemma 8.6 to the arc  $Q_0 \rightarrow Q_{-1} \rightarrow \dots \rightarrow Q_{-k} \rightarrow (1, 0)$  we conclude that  $Q_0 = \dots = Q_{k-1} = (0, 0)$ . But now we have a run of  $2k - 1 \geq k + 1$  points sitting at  $(0, 0)$  and we can run the same argument as in the case  $n > 3k + 1$ , with  $(0, 0)$  in place of  $(1, 0)$ .

**Case 3:** The only cases left to consider is when one or both of  $Q_{\pm(k+1)}$  equals  $(0, 0)$ . We suppose without loss of generality that  $Q_{-k-1} = (0, 0)$ . Since we also have  $Q_0 = (0, 0)$ , Lemma 8.7 gives  $Q_{-k-1} = \dots = Q_0 = (0, 0)$ . This is a run of  $k + 2$  consecutive points sitting at  $(0, 0)$ .

There is some smallest  $h > 0$  so that  $Q_h \notin S$ . Applying Lemma 8.6 to the arc  $Q_0 \rightarrow \dots \rightarrow Q_k \rightarrow \dots \rightarrow (1, 0)$ , we conclude that  $Q_{h-1} = \dots = Q_1 = (0, 0)$ . (Otherwise Lemma 8.6 would force  $Q_h \in S$ .)

Now we know that  $Q$  has collapsed diagonals at  $Q_h \notin L$ . We now get a contradiction from the same runaway situation as in Case 2.

## 9 Appendix

### 9.1 The Energy Invariance Revisited

In this section we sketch Anton Izosimov's proof that  $\chi_k \circ \Delta_k = \chi_k$ . This proof requires the machinery from [6]. (The perspective comes from [8], but the needed result for  $\Delta_k$  is in the follow-up paper [6].)

Let  $P$  be an  $n$ -gon. We let  $V_1, \dots, V_n$  be points in  $\mathbf{R}^3$  representing the consecutive vertices of  $P$ . Thus the vertex  $P_j$  is the equivalence class of  $V_j$ . We can choose periodic sequences  $\{a_i\}, \{b_i\}, \{c_i\}, \{d_i\}$  such that

$$a_i V_i + b_i V_{i+k} + c_i V_{i+k+1} + d_i V_{i+2k+1} = 0, \quad \forall i. \quad (34)$$

Recall from §2.2 that  $\Delta_k = D_k \circ D_{k+1}$ .

**Lemma 9.1.** *One of the cross ratio factors of  $\chi_k \circ D_{k+1}$  is  $(a_0 d_{-k}) / (c_0 b_{-k})$ .*

**Proof:** One of the factors is the cross ratio of  $P_0, y, x, P_{k+1}$ , where

$$x = P_{0,k+1} \cap P_{k,2k+1}, \quad y = P_{-k,1} \cap P_{0,k+1}.$$

(Compare the right side of Figure 2.1, shifting all the indices there by  $k + 1$ .)

The points  $x$  and  $y$  respectively are represented by vectors

$$X = a_0 V_0 + c_0 V_{k+1} = -b_0 V_k - d_0 V_{2k+1},$$

$$Y = -a_{-k} V_{-k} - c_{-k} V_1 = b_{-k} V_0 + d_{-k} V_{k+1}.$$

The point here is that the vector  $X$  lies in the span of  $\{V_0, V_{k+1}\}$  and in the span of  $\{V_k, V_{2k+1}\}$  and projectively this is exactly what is required. A similar remark applies to  $Y$ .

Setting  $\Omega = V_0 \times V_{k+1}$ , we compute the relevant cross ratio as

$$\frac{V_0 \times Y}{V_0 \times X} \cdot \frac{X \times V_{k+1}}{Y \times V_{k+1}} = \frac{d_{-k} \Omega}{c_0 \Omega} \times \frac{a_0 \Omega}{b_{-k} \Omega} = \frac{d_{-k} a_0}{b_{-k} c_0}, \quad (35)$$

which is just a rearrangement of the claimed term. ♠

The other cross ratio factors are obtained by shifting the indices in an obvious way. As an immediate corollary, we see that

$$\chi_k(D_{k+1}(P)) = \prod_{i=1}^n \frac{a_i d_i}{b_i c_i}. \quad (36)$$

Let us call this quantity  $\mu_k(P)$ .

**Lemma 9.2.** *If  $\mu_k \circ \Delta_k = \mu_k$  then  $\chi_k \circ \Delta_k = \chi_k$ .*

**Proof:** If  $\mu_k \circ \Delta_k = \mu_k$  then  $\mu_k \circ \Delta_k^{-1} = \mu_k$ . Equation 36 says that

$$\chi_k \circ D_{k+1} = \mu_k, \quad \mu_k \circ D_{k+1} = \chi_k. \quad (37)$$

The first equation implies the second because  $D_{k+1}$  is an involution. Since  $D_{k+1}$  conjugates  $\Delta_k$  to  $\Delta_k^{-1}$  we have

$$\chi_k \circ \Delta_k = \chi_k \circ D_{k+1} \circ \Delta_k^{-1} \circ D_{k+1} = \mu_k \circ \Delta_k^{-1} \circ D_{k+1} = \mu_k \circ D_{k+1} = \chi_k.$$

This completes the proof. ♠

Let  $\tilde{P} = \Delta_k(P)$ . Let  $\{\tilde{a}_i\}$ , etc., be the sequences associated to  $\tilde{P}$ . We want to show that

$$\prod_{i=1}^n \frac{a_i d_i}{b_i c_i} = \prod_{i=1}^n \frac{\tilde{a}_i \tilde{d}_i}{\tilde{b}_i \tilde{c}_i}. \quad (38)$$

This is just a restatement of the equation  $\mu_k \circ \Delta_k = \mu_k$ .

Now we use the formalism from [6] to establish Equation 38. We associate to our polygon  $P$  operator  $D$  on the space  $\mathcal{V}$  of bi-infinite sequences  $\{V_i\}$  of vectors in  $\mathbf{R}^3$ . The definition of  $D$  is given coordinate-wise as

$$D(V_i) = a_i V_i + b_i T^k(V_i) + c_i T^{k+1}(V_i) + d_i T^{2k+1}(V_i). \quad (39)$$

Here  $T$  is the shift operator, whose action is  $T(V_i) = V_{i+1}$ . If we take  $\{V_i\}$  to be a periodic bi-infinite sequence of vectors corresponding to our polygon  $P$ , then  $D$  maps  $\{V_i\}$  to the 0-sequence.

Next, we write  $D = D_+ + D_-$  where coordinate-wise

$$D_+(V_i) = a_i V_i + c_i T^{k+1}(V_i), \quad D_-(V_i) = b_i T^k(V_i) + d_i T^{2k+1}(V_i). \quad (40)$$

The pair  $(D_+, D_-)$  is associated to the polygon  $P$ .

Let  $\tilde{D}$  and  $(\tilde{D}_+, \tilde{D}_-)$  be the corresponding operators associated to  $\tilde{P}$ . One of the main results of [6] is that the various choices can be made so that

$$\tilde{D}_+ D_- = \tilde{D}_- D_+. \quad (41)$$

This is called *refactorization*. Equating the lowest (respectively highest) terms of the relation in Equation 41 gives us the identity  $\tilde{a}_i b_i = \tilde{b}_i a_{i+k}$  (respectively  $\tilde{c}_i d_{i+k+1} = \tilde{d}_i c_{i+2k+1}$ .) These relations hold for all  $i$  and together imply Equation 38.

## 9.2 Extensions of Glick's Formula

Theorem 1.1 in [3] says that the coordinates for the collapse point of the pentagon map  $\Delta_1$  are algebraic functions of the coordinates of the initial polygon. In Equation 1.1 of [3], Glick goes further and gives a formula for the collapse point. I will explain his

formula. Let  $(x^*, y^*)$  denote the accumulation point of the forward iterates of  $P$  under  $\Delta_1$ . Let  $\hat{P}_\infty = (x^*, y^*, 1)$  be the collapse point. In somewhat different notation, Glick introduces the operator

$$T_P = nI_3 - G_P, \quad G_P(v) = \sum_{i=1}^n \frac{|P_{i-1}, v, P_{i+1}|}{|P_{i-1}, P_i, P_{i+1}|} P_i. \quad (42)$$

Here  $|a, b, c|$  denotes the determinant of the matrix with rows  $a, b, c$  and  $I_3$  is the  $3 \times 3$  identity matrix. It turns out  $T_P$  is a  $\Delta_1$ -invariant operator, in the sense that  $T_{\Delta_0(P)} = T_P$ . Moreover  $P_\infty$  is an eigenvector of  $T_P$ . This is Glick's formula for  $\hat{P}_\infty$ . Actually, one can say more simply that  $G_P$  is a  $\Delta_0$ -invariant operator and that  $\hat{P}_\infty$  is a fixed point of the projective action of  $G_P$ . This means that the vectors representing these points in  $\mathbf{R}^3$  are eigenvectors for the operator. The reason Glick uses the more complicated expression  $nI_3 - G_P$  is that geometrically it is easier to work with.

Define  $G_{P,a,b}$  by the formula

$$G_{P,a,b}(v) = \sum_{i=1}^n \frac{|P_{i-a}, v, P_{i+b}|}{|P_{i-a}, P_i, P_{i+b}|} P_i. \quad (43)$$

Let  $\hat{P}_{\infty,k}$  be the limit point of the forward iterates of  $P$  under  $\Delta_k$ .

A lot of experimental evidence suggests the following conjecture.

**Conjecture 9.3.** *Let  $k \geq 2$ . If  $n = 3k + 1$  the point  $\hat{P}_\infty$  is a fixed point for the projective action of  $G_{P,k,k}$ . If  $n = 3k + 2$  the point  $\hat{P}_\infty$  is a fixed point for the projective action of  $G_{P,k+1,k+1}$ . In particular, in these cases the coordinates of  $\hat{P}_\infty$  are algebraic functions of the vertices of  $P$ .*

Anton Izosimov kindly explained the following lemma, which seems like a big step in proving the conjecture. (I am still missing the geometric side of Glick's argument in this new setting.)

**Lemma 9.4.** *When  $n = 3k + 1$  the operator  $G_{P,k,k}$  is invariant under  $\Delta_k$ . When  $n = 3k + 2$  the operator  $G_{P,k+1,k+1}$  is invariant under  $\Delta_k$ .*

**Proof:** These operators are Glick's operator in disguise. When  $n = 3k + 1$  we can relabel our  $n$ -gons in a way that converts  $\Delta_k$  to the pentagram map. The corresponding space of

birds  $B_{n,k}$  corresponds to some strange set of “reabeled  $k$ -birds”. This relabeling converts  $G_{P,k,k}$  respectively to Glick’s original operator. This proves the invariance of  $G_{P,k,k}$  under  $\Delta_k$  when  $n = 3k + 1$ . A similar thing works for  $n = 3k + 2$ , but this time the relabeling converts  $\Delta_k$  to the inverse of the pentagram map. ♠

I was not able to find any similar formulas when  $n > 3k + 2$ .

**Question 9.5.** *When  $n > 3k + 2$  and  $P$  is a  $k$ -bird, are the coordinates of the collapse point  $\hat{P}_\infty$  algebraic functions of the vertices of  $P$ ?*

Here is one more thing I have wondered about. Suppose that  $n$  is very large and  $P$  is a convex  $n$ -gon. Then  $P$  can be considered as a  $k$ -bird for all  $k = 1, 2, \dots, \beta$ , where  $\beta$  is the largest integer such that  $n \geq 3\beta + 1$ . When we apply the map  $\Delta_k$  for these various values of  $k$  we get potentially  $\beta$  different collapse points. All I can say, based on experiments, is that these points are not generally collinear.

**Question 9.6.** *Does the collection of  $\beta$  collapse points in this situation have any special meaning?*

### 9.3 Star Relabelings

Let us further take up the theme in the proof of Lemma 9.4. Given an  $n$ -gon  $P$  and some integer  $r$  relatively prime to  $n$ , we define a new  $n$ -gon  $P^{*r}$  by the formula

$$P_j^{*r} = P_{rj}. \quad (44)$$

Figure 1.5 shows the  $P^{*(-3)}$  when  $P$  is the regular 10-gon.

As we have already mentioned, the action of  $\Delta_1$  on the  $P^{*(-k)}$  is the same as the action of  $\Delta_k$  on  $P$  when  $n = 3k + 1$ . So, when  $n = 3k + 1$ , the pentagram map has another nice invariant set (apart from the set of convex  $n$ -gons), namely

$$B_{k,n}^{*(-k)} = \{P^{*(-k)} \mid P \in B_{k,n}\}.$$



The action of the pentagram map on this set is geometrically nice. If we suitably star-relabel, we get star-shaped (and hence embedded) polygons. A similar thing works when  $n = 3k + 2$ .

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## References

- [1] Q. Aboud and A. Izosimov, *The Limit Point of the Pentagram Map and Infinitesimal Monodromy*, I.M.R.N., Vol 7, (2022) [11](#)
- [2] M. Gekhtman, M. Shapiro, S. Tabachnikov, A. Vainshtein, *Integrable cluster dynamics of directed networks and pentagram maps*, Adv. Math. 300 (2016), pp 390-450 [12](#)
- [3] M. Glick, *The Limit Point of the Pentagram Map*, International Mathematics Research Notices 9 (2020) pp. 2818–2831 [11](#), [71](#)
- [4] M. Glick, *The pentagram map and Y-patterns*, Adv. Math. **227**, 2012, pp. 1019–1045. [12](#)
- [5] A. B. Goncharov and R. Kenyon, *Dimers and Cluster Integrable Systems*, Ann. Sc. Ec. Norm. Super. (4) 46 (2013) no. 5 pp 747–813 [12](#)
- [6] A. Izosimov and B. Khesin, *Long Diagonal Pentagram Maps*, Bulletin of the L.M.S., vol. 55, no. 3, (2023) pp. 1-15 [12](#), [13](#), [69](#), [71](#)

- [7] A. Izosimov, *The pentagram map, Poncelet polygons, and commuting difference operators*, Compos. Math. 158 (2022) pp 1084-1124
- [8] A. Izosimov, *Pentagram maps and refactorization in Poisson-Lie groups*, Advances in Mathematics, vol. 404 (2022) [12](#), [69](#)
- [9] A. Izosimov, *Intersecting the Sides of the Polygon*, Proc. A.M.S. **150** (2022) 639-649. [11](#)
- [10] B. Khesin, F. Soloviev *Integrability of higher pentagram maps*, Mathem. Annalen. Vol. 357 no. 3 (2013) pp. 1005–1047 [12](#)
- [11] B. Khesin, F. Soloviev *The geometry of dented pentagram maps*, J. European Math. Soc. Vol 18 (2016) pp. 147 – 179 [12](#)
- [12] G. Mari Beffa, *On Generalizations of the Pentagonagram Map: Discretizations of AGD Flows*, Journal of Nonlinear Science, Vol 23, Issue 2 (2013) pp. 304–334 [12](#)
- [13] G. Mari Beffa, *On integrable generalizations of the pentagram map* Int. Math. Res. Notices (2015) (12) pp. 3669-3693 [12](#)
- [14] R Felipe and G. Mari-Beffa, *The pentagram map on Grassmannians*, Ann. Inst. Fourier (Grenoble) 69 (2019) no. 1 pp 421–456 [12](#)
- [15] Th. Motzkin, *The pentagon in the projective plane, with a comment on Napier’s rule*, Bull. Amer. Math. Soc. **52**, 1945, pp. 985–989. [11](#)
- [16] N. Ovenhouse, *The Non-Commutative Integrability of the Grassman Pentagonagram Map*, arXiv 1810.11742 (2019) [12](#)
- [17] V. Ovsienko, R. E. Schwartz, S. Tabachnikov, *The pentagram map: A discrete integrable system*, Comm. Math. Phys. **299**, 2010, pp. 409–446. [11](#), [17](#)

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- [18] V. Ovsienko, R. E. Schwartz, S. Tabachnikov, *Liouville-Arnold integrability of the pentagram map on closed polygons*, Duke Math. J. Vol 162 No. 12 (2012) pp. 2149–2196 [11](#), [17](#)
- [19] R. E. Schwartz, *The pentagram map*, Exper. Math. **1**, 1992, pp. 71–81. [11](#), [17](#), [21](#), [25](#)
- [20] R. E. Schwartz, *Discrete monodromy, pentagrams, and the method of condensation*, J. of Fixed Point Theory and Appl. **3**, 2008, pp. 379–409. [12](#), [17](#)
- [21] R. E. Schwartz, *A Textbook Case of Pentagon Rigidity*, arXiv 2108-07604 preprint (2021)[ [13](#)
- [22] R. E. Schwartz, *Pentagon Rigidity for Centrally Symmetric Octagons*, I.M.R.N. (2024) pp 9535–9561 [13](#)
- [23] F. Soloviev *Integrability of the Pentagon Map*, Duke Math J. Vol 162. No. 15, (2012) pp. 2815 – 2853 [11](#), [12](#)
- [24] M. Weinreich, *The Algebraic Dynamics of the Pentagon Map*, Ergodic Theory and Dynamical Systems 43 (2023) no. 10, pp. 3460 – 3505 [12](#)

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