

Folding of quadrilaterals, zigzags, and Arnold-Liouville integrability

Anton Izosimov^{ID}

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Abstract: We put Darboux's porism on folding of quadrilaterals, as well as closely related Bottema's zigzag porism, in the context of Arnold-Liouville integrability.

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1 Introduction

One of the historically first manifestations of integrability is *Poncelet's porism*, also known as *Poncelet's closure theorem*. Poncelet's theorem says that if a planar n -gon is inscribed in a conic C_1 and circumscribed about another conic C_2 , then any point of C_1 is a vertex of such an n -gon, see Figure 1. The two arguably most standard proofs of this theorem are based, respectively, on complex and symplectic geometry. The complex proof goes

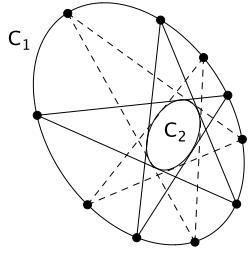


Figure 1: Every point of C_1 is a vertex of a pentagon inscribed in C_1 and circumscribed about C_2 .

roughly as follows. One can identify the space of tangents dropped from a point on C_1 to C_2 with an elliptic curve. The successive sides of a polygon inscribed in C_1 and circumscribed about C_2 are points on that curve related to each by a fixed translation. This polygon closes up if and only if the translation vector is a torsion point on the elliptic curve. Whether or not that is the case depends only on C_1 and C_2 , but not on the initial point, so all polygons inscribed in C_1 and circumscribed about C_2 will close up after the same number of steps [GH77].

The second, symplectic, proof is based on the fact that any two generic conics can be mapped, by a projective transformation, to confocal conics. In the confocal case, a polygon inscribed in C_1 and circumscribed about C_2 can be identified with a billiard trajectory in C_1 . The billiard in a conic is an integrable system, and any two polygons inscribed in C_1 and circumscribed about C_2 correspond to trajectories belonging to the same level set of the first integral. Hence, by Arnold-Liouville theorem, if one of the trajectories is periodic with period n , then so is the other one, cf. [LT07].

A lesser known relative of Poncelet's porism is *Darboux's porism on folding of quadrilaterals*. *Folding* of a vertex of a planar polygon is the reflection of that vertex in the diagonal joining its neighbors, see Figure 2. Darboux's porism says that if a sequence of alternating foldings of adjacent vertices restores, after $2n$ steps, the initial polygon, then this is the case for any polygon with the same side lengths. For example, folding any polygon with side lengths $1, 3, 3\sqrt{5}, 5$ six times, we come back to the initial polygon, see

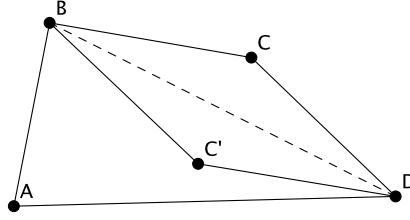


Figure 2: Folding of the vertex C of a quadrilateral $ABCD$. Its new position is C' .

[Izm23, Figure 2].

Just like Poncelet's porism, Darboux's theorem can be proved using elliptic curves. Specifically, one shows that the complexified moduli space of quadrilaterals with fixed side length is an elliptic curve. Composition of foldings at adjacent vertices amounts to a translation on that curve. Whether or not a sequence of foldings restores the initial polygon depends on whether the translation vector is a torsion point and is independent on the particular choice of a quadrilateral [Izm23].

What currently seems to be missing in the literature is a symplectic proof of Darboux's theorem. We provide such a proof in the present paper. Specifically, we show that, in an appropriate sense, Darboux's folding is Arnold-Liouville integrable, and deduce Darboux's porism.

Furthermore, we extend these results to Bottema's *zigzag porism* [Bot65], which can be stated as follows. Let C_a and C_b be two circles such that there exists a unit equilateral $2n$ -gon whose odd-indexed vertices lie on C_a and even-indexed vertices lie on C_b . Then there exist infinitely many such $2n$ -gons. The zigzag porism is equivalent to Darboux's porism when the circles are coplanar [CH00], but is in fact valid for any two circles in \mathbb{R}^3 [BHH74]. We construct the underlying Arnold-Liouville integrable system in this more general setting.

2 Arnold-Liouville integrability of folding

Let \mathcal{P} be the space of quadrilaterals $ABCD$ with fixed side lengths, considered up to orientation-preserving isometries. There is abundant literature on the topology of such spaces for polygons with any number of vertices, see [KM95] and references therein. The space \mathcal{P} is a smooth manifold assuming that there is no linear combination of side lengths with coefficients ± 1 which is equal to zero [KM95, Lemma 2]. In the case of quadrilaterals, this manifold is diffeomorphic to a circle or disjoint union of two circles, see [KM95, Theorem 1]. These circles are distinguished by the sign of the oriented area and are interchanged by an orientation-reversing isometry, cf. [KM95, Section 10].

Denote by $F_B : \mathcal{P} \rightarrow \mathcal{P}$ folding of the vertex B . This mapping is well-defined assuming that the vertices A and C cannot come together. This holds provided that the side lengths satisfy at least one of the following conditions: $|AB| \neq |BC|$ or $|AD| \neq |CD|$. Likewise, let $F_C : \mathcal{P} \rightarrow \mathcal{P}$ be folding of C , and let $F := F_C \circ F_B$ be the composition of the two foldings. Darboux's porism says that if $F^n(P) = P$ for some quadrilateral $P \in \mathcal{P}$, then F^n is the identity mapping on \mathcal{P} . We shall prove this by establishing Arnold-Liouville integrability of the mapping F .

Clearly, F cannot be Arnold-Liouville integrable on the space \mathcal{P} of quadrilaterals with fixed side lengths, as the latter space is one-dimensional. So, we consider a bigger space \mathcal{P}' of quadrilaterals with fixed lengths of the sides AB, BC, CD , again considered up to orientation-preserving isometries. This space is diffeomorphic to a two-dimensional torus and is parametrized by the oriented angles $\angle ABC$ and $\angle BCD$. The squared length of the side AD is a smooth function of the torus \mathcal{P}' . The space \mathcal{P} of quadrilaterals with fixed lengths of all sides is a level set of that function.

Theorem 2.1. *The folding mapping $F = F_C \circ F_B$ is Arnold-Liouville integrable on the moduli space \mathcal{P}' of quadrilaterals $ABCD$ with fixed lengths of the sides AB, BC, CD .*

Proof. Folding does not affect side lengths. In particular, $|AD|^2$ is a first integral of F .

Furthermore, the map $F : \mathcal{P}' \rightarrow \mathcal{P}'$ has an invariant symplectic structure given by

$$\Omega := d\angle ABC \wedge d\angle BCD.$$

To show invariance, consider, for instance, folding of the vertex C depicted in Figure 2. The pullback of the symplectic form Ω by this map is

$$F_C^* \Omega = d\angle ABC' \wedge d\angle BC'D = d(\angle ABC - 2\angle CBD) \wedge d(2\pi - \angle BCD) = -\Omega - 2d\angle CBD \wedge d\angle BCD.$$

Furthermore, since the side lengths $|BC|$ and $|CD|$ are fixed, the angle $\angle CBD$ is a function of the angle $\angle BCD$ and is independent of the angle $\angle ABC$. So, $d\angle CBD \wedge d\angle BCD = 0$, implying

$$F_C^* \Omega = -\Omega,$$

i.e., the form Ω is *anti-invariant* under a single folding, and hence invariant under F . \square

3 Darboux's porism

Theorem 3.1 (Darboux's porism). *Assume we are given a quadrilateral which restores its initial shape after $2n$ alternating foldings at adjacent vertices. Suppose its side lengths are such that no linear combination of them with coefficients ± 1 is equal to zero. Then any quadrilateral with the same side lengths restores its initial shape after $2n$ alternating foldings at adjacent vertices.*

Remark 3.2. The condition on linear combinations of side lengths cannot be avoided. Consider, for instance a quadrilateral with all four vertices along a line, shown in Figure 3. Here we have $|AB| = 2$, $|BC| = 1$, $|CD| = 2$, $|AD| = 3$. Clearly, this quadrilateral is invariant under any folding. However, that is not so for a generic quadrilateral with side lengths 2, 1, 2, 3.

Proof of Theorem 3.1. The assumption on linear combinations of side lengths ensures that the moduli space \mathcal{P} of quadrilaterals with such side lengths is a regular level set of the function $|AD|^2$ on the moduli space \mathcal{P}' of polygons with fixed lengths of AB, BC, CD .



Figure 3: A degenerate polygon.

We are given that there is a quadrilateral $P \in \mathcal{P}$ on that level set such that $F^n(P) = P$. So, by Arnold-Liouville integrability of F , we have that F^n is the identity on the connected component of \mathcal{P} containing P . Moreover, since there are at most two components, and they are interchanged by an orientation-reversing isometry which commutes with foldings, we must have that F^n is the identity of the whole of \mathcal{P} , as desired. \square

4 A remark on polygons with more vertices

The F -invariant symplectic form on the moduli space \mathcal{P}' of quadrilaterals with fixed lengths of the sides AB, BC, CD induces an F -invariant non-vanishing 1-form on any non-singular level set of the first integral $|AD|^2$, i.e., on the moduli space \mathcal{P} of quadrilaterals with fixed side lengths. The existence of this 1-form is at heart of Arnold-Liouville theorem. It can be shown that, up to a constant factor, this form is given by

$$\frac{d\angle ABC}{\text{area of } \triangle ACD}.$$

This expression is invariant under cyclic permutation of vertices, up to sign. Likewise, the expression

$$\frac{d\phi_{i+2} \wedge \cdots \wedge d\phi_{i-2}}{\text{area of the triangle formed by vertices } i-1, i, i+1},$$

where ϕ_i is the angle subdued at i th vertex (the indices are understood cyclically, modulo n), gives a volume form on the moduli space of n -gons with fixed side lengths which is anti-invariant under each folding and hence invariant under an even number of foldings. However, for $n > 4$, this does not imply any kind of integrable behavior. Moreover, already for pentagons a random sequence of foldings has dense orbits on the moduli space \mathcal{P} [CD23].

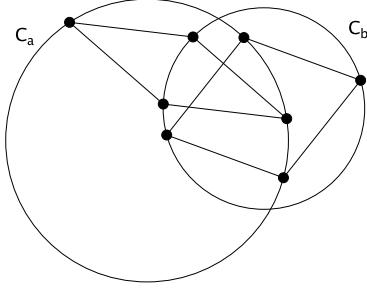


Figure 4: The zigzag porism: all zigzags with the same edge length close after the same number of steps.

5 The zigzag porism

Let C_a and C_b be two circles in \mathbb{R}^3 . A *zigzag* between C_a and C_b is an equilateral polygon whose odd-indexed vertices lie on C_a and even-indexed vertices lie on C_b . The zigzag porism says that if there exists a closed $2n$ -gonal zigzag between C_a and C_b , then any zigzag between C_a and C_b with the same edge length is also a closed $2n$ -gon [Bot65, BHH74], see Figure 4.

A zigzag between two circles C_a, C_b may be built by iterating the *zigzag map* $Z : C_a \times C_b \rightarrow C_a \times C_b$ which sends a pair $A \in C_a, B \in C_b$ to a pair $A' \in C_a, B' \in C_b$ such that $|A'B'| = |A'B| = |AB|$, see Figure 5. This map is a composition of two involutions, namely $(A, B) \mapsto (A', B)$, where $|A'B| = |AB|$, and $(A', B) \mapsto (A', B')$, where $|A'B'| = |A'B|$. Observe

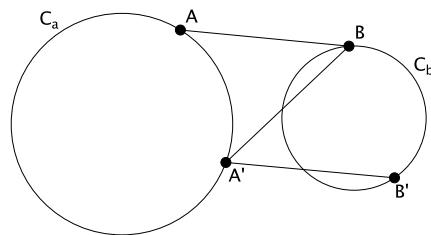


Figure 5: The zigzag map $Z : (A, B) \mapsto (A', B')$.

Folding of quadrilaterals

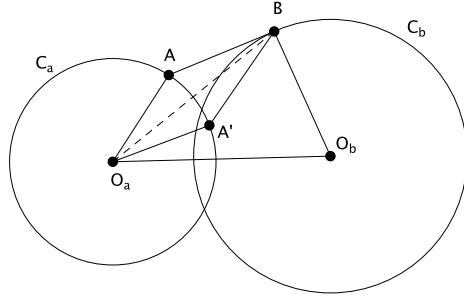


Figure 6: Two successive legs AB, BA' of a zigzag are related by folding.

that, in the case when the circles C_a, C_b are coplanar, these involutions are just foldings of the quadrilateral O_aABO_b , where O_a, O_b are centers of C_a, C_b , at A and B respectively, see Figure 6. So, the planar case of the zigzag porism is equivalent to Darboux's porism [CH00]. Here we show that the integrability result carries over to the spatial situation:

Theorem 5.1. *The zigzag map Z is Arnold-Liouville integrable for any circles C_a, C_b in \mathbb{R}^3 .*

Proof. By definition, the map $Z : (A, B) \mapsto (A', B')$ preserves the squared distance between A and B . So, it suffices to find an area form on $C_a \times C_b$ invariant under Z . Let $\phi_a, \phi_b \in \mathbb{R}/2\pi\mathbb{Z}$ be standard angular parameters on C_a, C_b . We will prove that the form $d\phi_a \wedge d\phi_b$ on $C_a \times C_b$ is preserved by Z . To that end, it suffices to establish anti-invariance of that form with respect to the involutions whose composition gives Z . Furthermore, since those involutions are related to each other by interchanging the roles of the circles

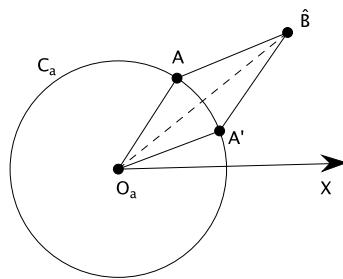


Figure 7: The involution $(A, B) \mapsto (A', B)$ takes the form $d\phi_a \wedge d\phi_b$ to $-d\phi_a \wedge d\phi_b$.

C_a, C_b , it is sufficient to consider the involution $(A, B) \mapsto (A', B)$ defined by the condition $|A'B| = |AB|$, where $A, A' \in C_a$. Let \hat{B} be the orthogonal projection of B onto the plane containing C_a . Then $|A\hat{B}| = |A'\hat{B}|$, see Figure 7. Here O_aX is the reference direction used to define the angular coordinate ϕ_a on C_a . We have

$$\angle XO_aA + \angle XO_aA' = 2\angle XO_a\hat{B}.$$

So, the sum on the left only depends on the position of the point B but not A . Therefore, in coordinates ϕ_a, ϕ_b , the involution $(A, B) \mapsto (A', B)$ has the form

$$(\phi_a, \phi_b) \mapsto (f(\phi_b) - \phi_a, \phi_b)$$

for a certain smooth function f . So, the form $d\phi_a \wedge d\phi_b$ is indeed anti-invariant under this involution. \square

In terms of the map Z , the zigzag porism says that if an orbit of $(A, B) \in C_a \times C_b$ under Z is n -periodic, then the same holds for any $(A', B') \in C_a \times C_b$ with $|A'B'| = |AB|$. This is derived from Theorem 5.1 in the same way as Darboux's porism is obtained from Theorem 2.1.

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AUTHOR

Anton Izosimov
Department of Mathematics,
University of Arizona
and
School of Mathematics & Statistics,
University of Glasgow;
email: Anton.Izosimov@glasgow.ac.uk