

Chapple porism and Tzitzeica configuration: two sides of a same coin

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Abstract: A necessary condition for a pair of circles $\{\mathcal{D}, \mathcal{I}\}$ to form a Chapple pair is that the radius of $\mathcal{D}'$ —the symmetric of $\mathcal{D}$ with respect to $\mathcal{D}$ —be half that of $\mathcal{I}$. When $\mathcal{D}$ and $\mathcal{I}$ are either nested or intersecting, this condition becomes sufficient. We establish these results and use them to derive new proofs of both the Chapple porism and Euler’s formula, based on an underlying Tzitzeica configuration of circles

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1 Introduction.

A pair of circles $\{\mathcal{D}, \mathcal{I}\}$ forms a Chapple pair if every point on $\mathcal{D}$ that lies outside $\mathcal{I}$ is a vertex of some triangle simultaneously inscribed in $\mathcal{D}$ and circumscribed about $\mathcal{I}$.

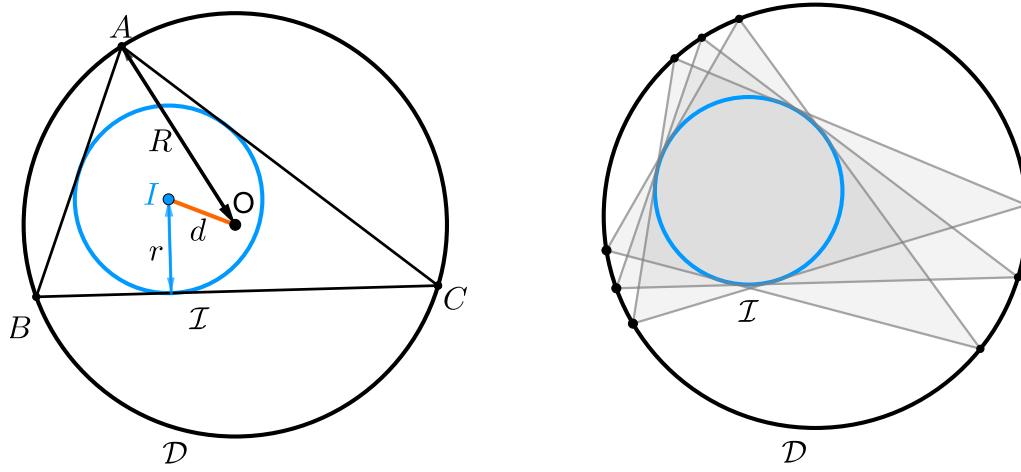


Figure 1: The Euler relations: $\frac{1}{R-d} + \frac{1}{R+d} = \frac{1}{r}$ and the gliding triangle in a Chapple pair.

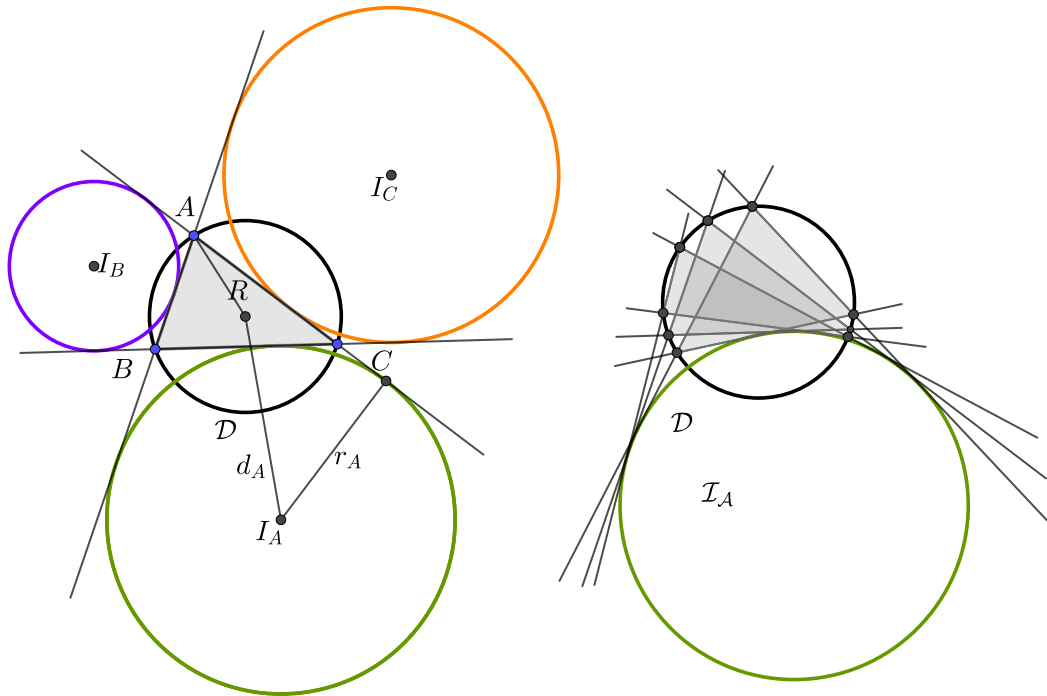


Figure 2: The Euler relations $\frac{1}{d_A-R} + \frac{1}{d_A+R} = \frac{1}{r_A}$ and an outer Chapple pair.

It is clear that the circumcircle and incircle, or the circumcircle and any excscribed circle of some triangle, are natural candidates for Chapple pairs. As a matter of fact, these are the only one!

This remarkable fact is known as the Chapple Porism - named after William Chapple (1718-1781), who first stated it explicitly in *An Essay on the Properties of Triangles Inscribed in, and Circumscribed about Two Given Circles*, published in 1749 in *The Gentleman's Magazine* [3].

In the same paper, Chapple also correctly stated the metric conditions relating the inscribed and excscribed circles of a triangle, as follows. A circle $\mathcal{D}$ of radius R , together with two circles $\mathcal{I}$ and $\mathcal{I}_A$ of radii r and r_A , whose centers are at distances d and d_A from the center of $\mathcal{D}$, are the circumcircle and the inscribed or an excscribed circle of a triangle, as shown in [Figures 1](#) and [2](#), if and only if

$$\frac{1}{R-d} + \frac{1}{R+d} = \frac{1}{r}, \quad \text{or} \quad \frac{1}{d_A-R} + \frac{1}{d_A+R} = \frac{1}{r_A}. \quad (1)$$

Chapple proved only a particular case of [Equation \(1\)](#), when the triangle is isosceles. The general case was proved by Leonard Euler (1707-1783) in *Problematvm qvorvmdam geometricorvm difficillimorvm*, published in 1765 in *Novi commentarii Academiae Scientiarum Imperialis Petropolitanae* [6]. For this reason, [Equation \(1\)](#) are usually called *the Euler relations*.

The fact that any pair of circles verifying the relation [Equation \(1\)](#) form a Chapple pair was originally proved in [12]; for more details, see [5]. Other proofs can be found in [2, 10].

In [7] a Chapple pair is proved to be projectively equivalent to the pair formed by a conic and its director circle. The latter porism is then established using the optical property of conics.

Here, the starting point was a naive question: if, in a Chapple pair, the triangle *glides* between the annular circles, why not do it *on wheels*?

As it turns out, the wheels were just a circle-inversion apart! [Figure 3](#) reveals the core geometric mechanism of the porism: a Tzitzeica configuration of circles.

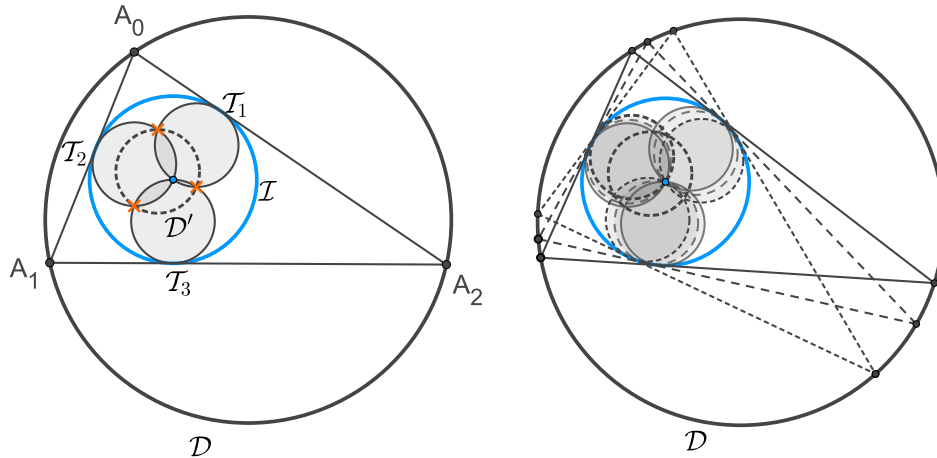


Figure 3: Chapple pair on wheels.

Main results.

The main result, [Theorem 3.1](#), led to a new synthetic proof of the Chapple porism. The argument is based on circle inversion, and the key step is an elementary fact from triangle geometry—the Tzitzeica configuration [Theorem 2.1](#). In [Theorem 4.1](#), we give a new, inversive proof of the Euler relations and show that any pair of circles satisfying them forms a Chapple pair.

2 Preliminary results.

2.1 Tzitzeica configuration

The following fact, known as *the three-coin problem*, was proposed as a challenging problem in a 1908 student contest in *Gazeta Matematica* by the Romanian mathematician G. Tzitzeica. Rediscovered in [9], it was designated as the *Tzitzeica configuration* in [14], a name we adopt here; see [Figure 4](#).

Theorem 2.1. *Let three circles of equal radius intersect at a single point. Then the circum-*

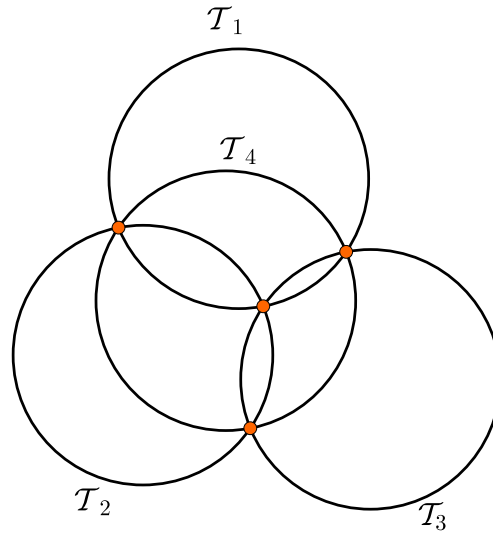


Figure 4: A Tzitzeica configuration: four circles and four points, such that through any point pass exactly three circles. If three of the circles are congruent, so is the fourth.

circle of the triangle whose vertices are the intersections of pairs of these circles has the same radius as the three original circles.

There are too many proofs for this beautiful result; see e.g. [14, 9, 11]. We therefore omit it, trusting the reader may wish to find one on his own.

2.2 Some facts on circle inversion.

Apart from the Tzitzeica configuration, we shall consistently use circle inversion.

Definition 2.2. Let $\mathcal{I}$ be a circle centered at I with radius r , called the inversion circle. For any point $P \neq I$, its inverse with respect to $\mathcal{I}$ is the point P' such that: i) P' lies on the ray from I through P ; ii) $IP \cdot IP' = r^2$.

The literature on circle inversion is extensive; for definitions and basic results, see, e.g., [4, 2, 10]. The most remarkable propriety of circle inversion is that it sends circles

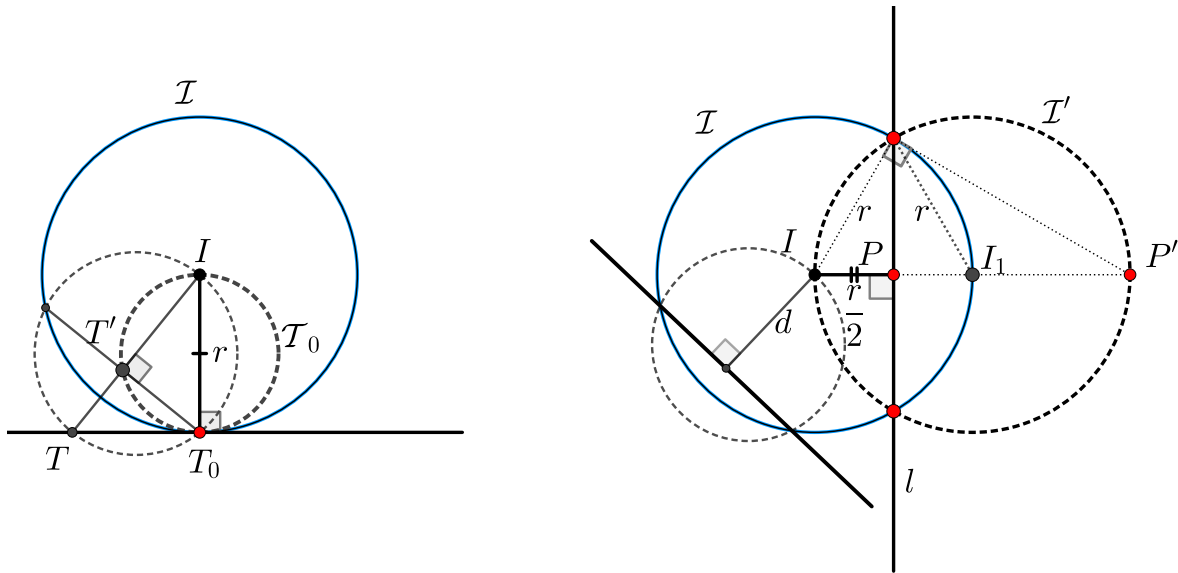


Figure 5: Illustration for [Lemma 2.3](#)

and lines to circles or lines. More precisely, if a circle or line passes through the center of inversion, it is transformed into a line, whereas if it does not pass through the center, it is transformed into a circle.

Proofs of these generic proprieties are standard and are therefore omitted. We shall instead prove a specific result, repeatedly need in the sequel.

Lemma 2.3. *Let $\mathcal{J}$ an inversion circle and l a line, which does not pass through I . Then, the inverse of l with respect to $\mathcal{J}$ is:*

1. *a circle congruent to $\mathcal{J}$, if and only if the distance from the center of $\mathcal{J}$ to the line is half the radius of $\mathcal{J}$.*
2. *a circle of diameter r passing through I , if and only if l and $\mathcal{J}$ are tangent.*

Proof. Refer to [Figure 5](#). Let I and r be the center and the radius of $\mathcal{J}$. The inverse of a line l with respect to $\mathcal{J}$ is a circle if and only if the line l does not pass through the center of $\mathcal{J}$. Its diameter is IP' , where P' is the inverse of P , the projection of I onto l . Since $IP \cdot IP' = r^2$, then $IP' = \frac{r^2}{d}$.

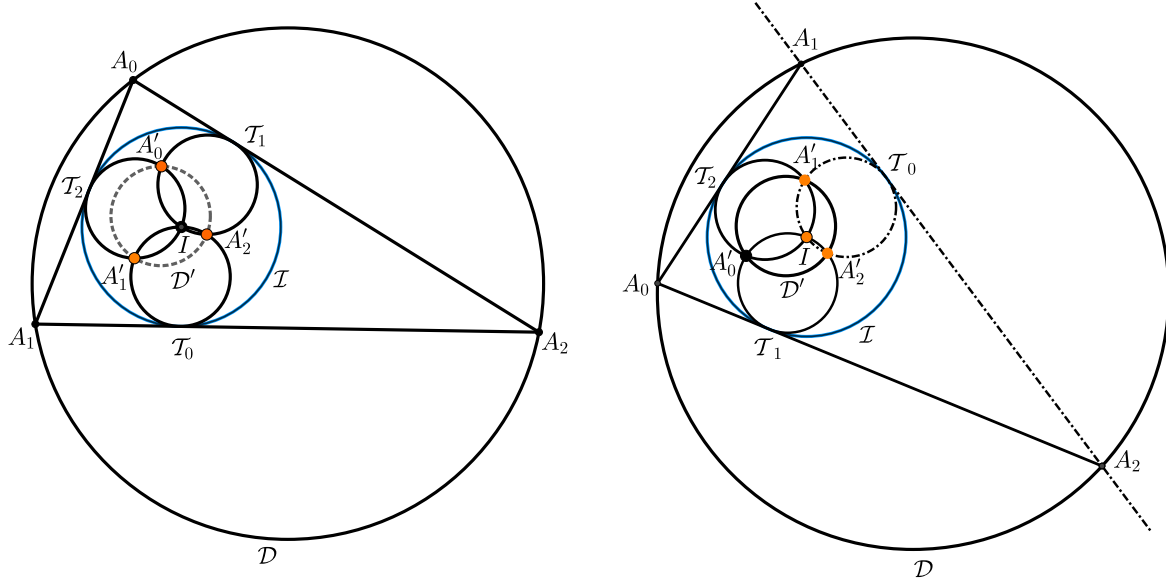


Figure 6: **Left:** The inverses of $\triangle A_0A_1A_2$ sides with respect to $\mathcal{I}$ are three congruent circles passing through I . The fourth congruent circle is $\mathcal{D}'$ the inverse of $\mathcal{D}$ with respect to $\mathcal{I}$. **Right:** If A_0A_1 and A_0A_2 are the tangents from A_0 to $\mathcal{I}$, then $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{D}'\}$ are three congruent circles, with common point A'_0 the inverse of A_0 . The fourth congruent circle is $\mathcal{T}_0$, passing through I, A'_1, A'_2 . It is the inverse of A_1A_2 with respect to $\mathcal{I}$.

Therefore $IP' = 2r$ if and only if $d = \frac{r}{2}$, proving the first statement. Similarly, $IP' = r$ if and only if $d = r$, finishing the proof. $\square$

3 Main result.

Our main result is an inversive characterization of Chapple pairs.

Theorem 3.1. *Let $\{\mathcal{D}, \mathcal{I}\}$ be any pair of circles and denote by $\mathcal{D}'$ the inverse of $\mathcal{D}$ with respect to $\mathcal{I}$. The following statements are equivalent:*

1. *There exist a triangle inscribed in $\mathcal{D}$ and circumscribed about $\mathcal{I}$.*

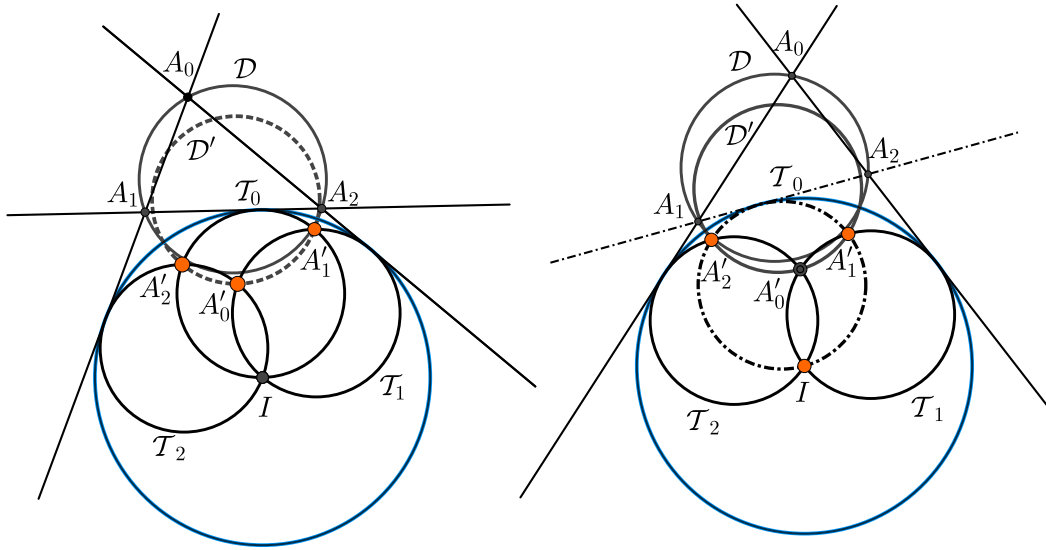


Figure 7: **Left:** The inverses of the triangle's $\triangle A_0A_1A_2$ sides with respect to (one of) the excscribed circle, $\mathcal{J}$, are three congruent circles passing through its center, I . The fourth congruent circle is $\mathcal{D}'$ the inverse of $\mathcal{D}$ with respect to $\mathcal{J}$. **Right:** If A_0A_1 and A_0A_2 are two the tangents from A_0 to $\mathcal{J}$, then $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{D}'\}$ are three congruent circles, with common point A'_0 , the inverse of A_0 with respect to $\mathcal{J}$. The fourth congruent circle is $\mathcal{T}_0$, passing through I, A'_1, A'_2 . It is the inverse of A_1A_2 with respect to $\mathcal{J}$.

2. The interiors of $\mathcal{D}$ and $\mathcal{J}$ have not empty intersection and the diameter of $\mathcal{D}'$ equals the radius of $\mathcal{J}$.

Proof. $1 \Rightarrow 2$ Let $\triangle A_0A_1A_2$ be inscribed in $\mathcal{D}$ and circumscribed about to $\mathcal{J}$.

Regardless of whether $\mathcal{J}$ is the incircle or an excircle of $\triangle A_0B_0C_0$, by [Corollary 5.2](#), the interiors of $\mathcal{D}$ and $\mathcal{J}$ necessarily intersect and I is contained in $\mathcal{D}$, as shown in [Figure 8](#).

To show that the diameter of $\mathcal{D}'$ equals the radius of $\mathcal{J}$, we use [Lemma 2.3](#), according to which the inverses of the sides of $\triangle A_0B_0C_0$ with respect to $\mathcal{J}$ are three circles $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2$ passing through I and internally tangent to $\mathcal{J}$. Consequently, these are three congruent circles, whose diameters equal the radius of $\mathcal{J}$, shown in [Figures 6 and 7](#).

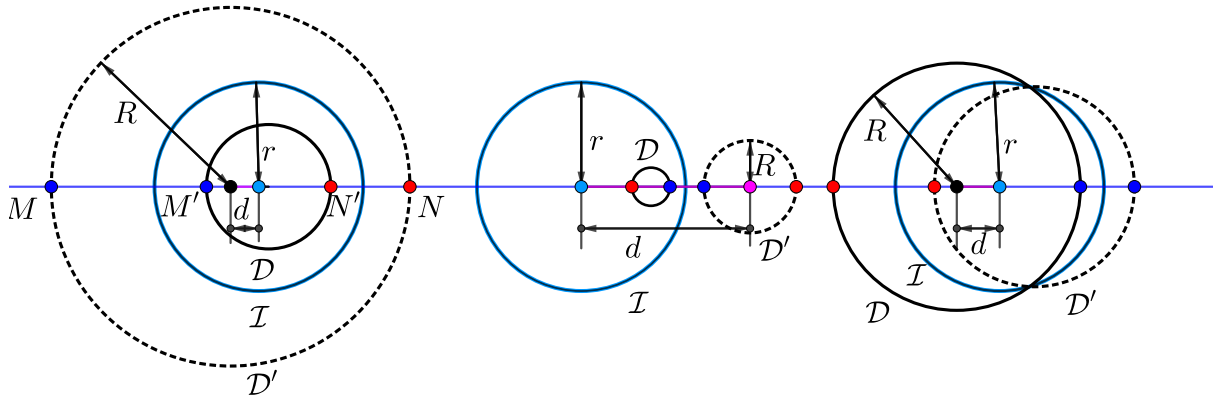


Figure 8: Illustration for [Theorem 3.1](#).

These circles have a common point in I , the center of $\mathcal{I}$, and they pairwise intersect in A'_0, A'_1, A'_2 , the inverses of A_0, A_1, A_2 with respect to $\mathcal{I}$. This ensures that $\mathcal{D}'$, the circle that pass through A'_1, A'_2 , and A'_3 coincide with the inverse of $\mathcal{D}$ with respect to $\mathcal{I}$.

$\{\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2\}$ forms a Tzitzeica configuration, in which $\mathcal{D}'$ acts as *the fourth* circle, hence its diameter equals the radius of $\mathcal{I}$, as stated.

$2 \Rightarrow 1$. If $\mathcal{D}$ and $\mathcal{I}$ are two circles whose interiors have nonempty intersection, there are three possibilities:

1. $\{\mathcal{D}, \mathcal{I}\}$ are secant;
2. $\{\mathcal{D}, \mathcal{I}\}$ are nested, the latter being contained in the former;
3. $\{\mathcal{I}, \mathcal{D}\}$ are nested, the latter being contained in the former.

First note that the third case cannot occur. If $\mathcal{D}$ were contained in $\mathcal{I}$ and contained its center I , as in [Figure 8](#) (left), then its inverse $\mathcal{D}'$ would contain the inversion circle $\mathcal{I}$, which contradicts the assumptions on their radii.

If, on the other hand, $\mathcal{D}$ were contained in $\mathcal{I}$ but did not contain its center, as in [Figure 8](#) (middle), then the interiors of $\mathcal{D}'$ and $\mathcal{I}$ would be disjoint, which is ruled out by the hypothesis.

This guarantees that only the first two cases can occur. Thus, $\mathcal{A}$, the set of all points on $\mathcal{D}$ that lie neither inside nor on $\mathcal{J}$, is nonempty.

Let A_0 be any point in $\mathcal{A}$, and let A_0A_1 and A_0A_2 , with A_1, A_2 in $\mathcal{D}$, be the two tangents from A_0 to $\mathcal{J}$, as in [Figure 7](#). We need to prove that the line A_1A_2 is also tangent to $\mathcal{J}$.

The inverses of the lines A_0A_1 and A_0A_2 with respect to $\mathcal{J}$ are two circles, $\mathcal{T}_2$ and $\mathcal{T}_1$, internally tangent to $\mathcal{J}$ and passing through its center, I ; as such, their diameters equal the radius of $\mathcal{J}$. On the other hand, by hypothesis, the diameter of $\mathcal{D}'$ also equals the radius of $\mathcal{J}$.

Since A_0 is a point on $\mathcal{D}$, the lines A_0A_1 , A_0A_2 and the circle $\mathcal{D}$ have one common point: A_0 . Their inverses, the circles $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{D}'$ also have a common point: A'_0 , the inverse of A_0 , and thus $\{\mathcal{D}', \mathcal{T}_1, \mathcal{T}_2\}$ forms a Tzitzeica configuration.

Apart from their common point A'_0 , the three congruent circles $\mathcal{D}'$, $\mathcal{T}_1$, and $\mathcal{T}_2$ intersect pairwise at I , A'_1 , and A'_2 , where the latter two are the inverses of A_1 and A_2 with respect to $\mathcal{J}$. Therefore, $\mathcal{T}_0$, the circle through I , A'_1 , and A'_2 , is congruent to $\mathcal{D}'$.

On the other hand, by [Lemma 2.3](#), since the circle $\mathcal{T}_0$ passes through I and its radius is half that of $\mathcal{J}$, then $\mathcal{T}_0$ is internally tangent to $\mathcal{J}$.

The inverse of $\mathcal{T}_0$ with respect to $\mathcal{J}$ is therefore a line passing through A_1 and A_2 —the inverses of A'_1 and A'_2 . Since the circle $\mathcal{T}_0$ is tangent to $\mathcal{J}$, its inverse, the line A_1A_2 is tangent to $\mathcal{J}$, as well, proving the claim.

□

[Theorem 3.1](#) imply the Chapple porism in its classic form [\[3, 8, 10\]](#).

Corollary 3.2. *If there exists one triangle inscribed into a circle $\mathcal{D}$ and circumscribed with respect to a circle $\mathcal{J}$, then from any point A_0 of $\mathcal{D}$, which lie outside $\mathcal{J}$, there exists a triangle with a vertex in A_0 , inscribed in $\mathcal{D}$ and circumscribed with respect to $\mathcal{J}$.*

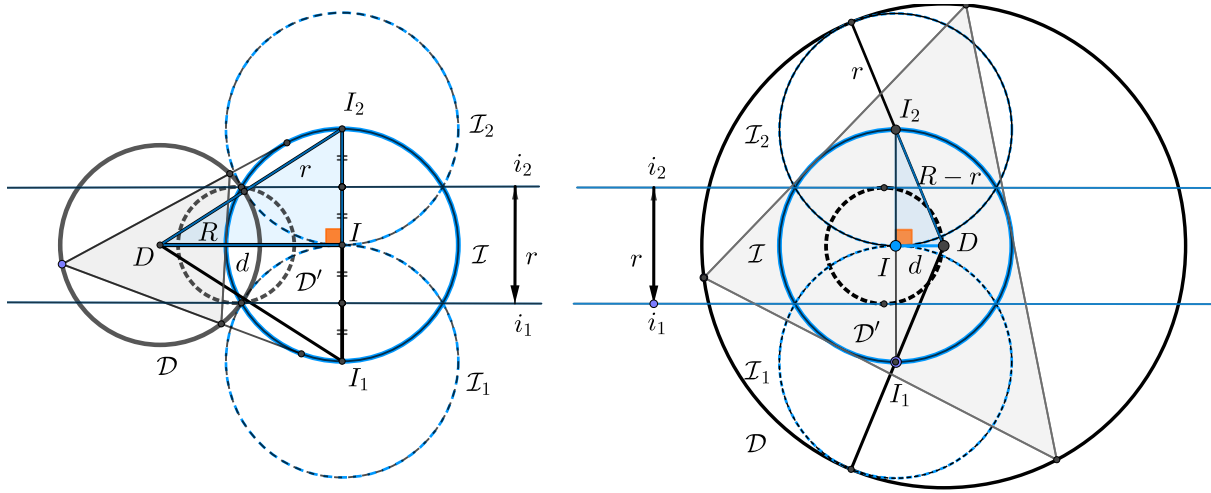


Figure 9: Illustration for [Theorem 4.1](#)

4 Euler formulas: inversive proofs.

The ideas developed earlier lead naturally to new inversive proofs of the Euler relations; an alternative proof is in [\[7\]](#).

Theorem 4.1. (see [\[6\]](#)) *Let $\mathcal{D}$ and $\mathcal{I}$ be two circles of radii R and r , respectively; let d the distance between their centers. Then:*

1. $\{\mathcal{D}, \mathcal{I}\}$ is a Chapple pair, in which $\mathcal{D}$ is the circumcircle and $\mathcal{I}$ is the incircle if and only if

$$(R - r)^2 = r^2 + d^2. \quad (2)$$

2. $\{\mathcal{D}, \mathcal{I}\}$ is a Chapple pair, in which $\mathcal{D}$ is the circumcircle and $\mathcal{I}$ is the excircle if and only if

$$(R + r)^2 = r^2 + d^2. \quad (3)$$

Proof. First assume that the circles $\mathcal{D}$ and $\mathcal{I}$ are the circumcircle and the excircle of some triangle; these are two circles that intersect and the center of $\mathcal{I}$ lies outside $\mathcal{D}$; see

Corollary 5.2. Let $\mathcal{D}'$ be the inverse of $\mathcal{D}$ with respect to $\mathcal{J}$, and let i_1 and i_2 be the two tangents at $\mathcal{D}'$, which are parallel to DI , the line joining the centers of $\mathcal{D}$ and $\mathcal{J}$, as in Figure 9.

The distance between these two lines equals the diameter of $\mathcal{D}'$, which by Theorem 3.1 equals the radius of $\mathcal{J}$; hence the distance from I to either i_1 or i_2 is $r/2$. By Lemma 2.3, the inverses of i_1 and i_2 with respect to $\mathcal{J}$, denoted $\mathcal{J}_1$ and $\mathcal{J}_2$, are two circles congruent to $\mathcal{J}$.

The point I lies inside the strip delimited by i_1 and i_2 ; therefore $\mathcal{J}_1, \mathcal{J}_2$ are mutually tangent at I and externally tangent to $\mathcal{D}$.

Let I_1 and I_2 be their centers and let $d = DI$. In $\triangle DII_1$, which is right-angled in I , the condition for tangency between the circles $\mathcal{J}_1$ and $\mathcal{D}$ can be expressed as Equation (3).

To prove the converse, assume that $\{\mathcal{D}, \mathcal{J}\}$ satisfies Equation (3). This ensures that $d^2 = R^2 + 2rR > R^2$, hence I lie outside $\mathcal{D}$.

Let I_1I_2 be the diameter of $\mathcal{J}$ which is perpendicular to DI , and let $\mathcal{J}_1$ and $\mathcal{J}_2$ be the circles centered at I_1 and I_2 , passing through I . Then $\mathcal{J}_1, \mathcal{J}_2$, and $\mathcal{J}$ are three congruent circles; furthermore, $\mathcal{J}_1$ and $\mathcal{J}_2$ are externally tangent and both tangent to the line DI at I .

Let i_1 and i_2 be the inverses of these circles with respect to $\mathcal{J}$; then i_1 and i_2 are two parallel lines, both tangent to $\mathcal{D}'$, the inverse of $\mathcal{D}$ with respect to $\mathcal{J}$. By Lemma 2.3, the distance from I to i_1 is $r/2$, hence the distance between the lines i_1 and i_2 is r .

By Pythagoras' theorem in $\triangle DII_2$, $DI_2^2 = r^2 + d^2$; this relation, together with Equation (3), gives $DI_2 = R + r$, and guarantees that the circles $\mathcal{D}$ and $\mathcal{J}_2$ are externally tangent. Similarly, $\mathcal{D}$ and $\mathcal{J}_1$ are externally tangent. The circle $\mathcal{D}'$ is therefore tangent to the pair of parallel lines i_1 and i_2 , which means that the diameter of $\mathcal{D}'$ is r , the distance between the two lines.

Finally, by Equation (3), $d^2 = (R + r)^2 - r^2 < (R + r)^2$, hence $\mathcal{D}$ and $\mathcal{J}$ are secant circles. Thus, by Theorem 3.1 $\{\mathcal{D}, \mathcal{J}\}$ form a Chapple pair.

The proof for the first statement is similar. In this case, $d < R$, the center I lie inside $\mathcal{D}$ and the circles $\mathcal{J}_1$ and $\mathcal{J}_2$ are internally tangent to $\mathcal{D}$, as shown in the right-side of Figure 9. The remaining proof just follows the previous case and we omit. $\square$

Theorem 4.1 implicitly proved the Euler relations.

Corollary 4.2. (see [6]) *In any triangle*

$$R^2 - 2Rr = d^2, \text{ and } R^2 + 2Rr' = d'^2,$$

where R is the circumradius, r the inradius, r' an exradius, and d, d' are the distances from the circumcenter to the incenter and any excenter.

5 Appendix.

The incenter-excenter lemma.

The relative positions of the incircle, the excircles, and the circumcircle are well known; nevertheless, standard proofs (see, e.g., [10, p. 186]) rely on Euler's relations. So as not to fall into a vicious circle, we present a proof based on a classical observation, known as the *trident lemma* or *incenter-excenter lemma*; see, e.g., [13, 15, 1].

Lemma 5.1. *In any triangle, the midpoint of the segment joining each excenter to the incenter lies on the circumcircle.*

Proof. Let A_m be the midpoint of the arc BC opposite the vertex A . Then points A, I, A_m , and I_A are collinear.

Since BI and BI_A are the internal and external bisectors of $\angle ABC$, we have $\angle IBI_A = 90^\circ$; similarly, $\angle ICI_A = 90^\circ$. Therefore, the quadrilateral IBI_AC is inscribed in the circle with diameter II_A .

Its circumcenter is the intersection of II_A with the perpendicular bisector of BC , which is DA_m . Thus, the circumcenter is A_m itself, so A_m is the midpoint of II_A , as claimed.

□

Corollary 5.2. *In any triangle:*

1. *The circumcircle contains the incircle.*

2. The excenters lie outside the circumcircle.

3. Each excircle intersect the circumcircle.

Proof. Refer to Figure 10. I , the incenter, is the common point of the internal bisectors. Each of these bisectors is contained within its corresponding angle; therefore, the point I lies inside $\triangle ABC$, which, by convexity, is contained in $\mathcal{D}$. Also by convexity, the incircle $\mathcal{I}$ is contained in $\triangle ABC$, proving the first statement.

By Lemma 5.1, I_A is the reflection of I about A_m ; since I is inside $\mathcal{D}$ and A_m is on $\mathcal{D}$, the I_A is outside $\mathcal{D}$, proving the second statement.

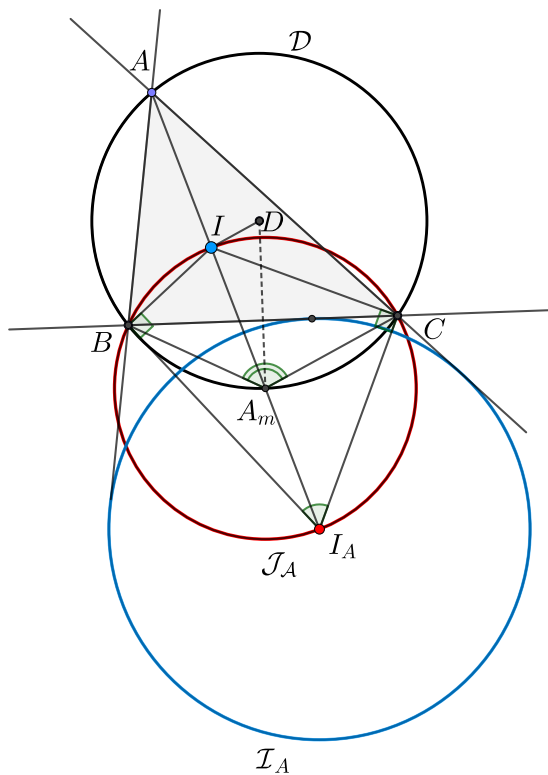


Figure 10: Illustration for Lemma 5.1.

The excircle $\mathcal{I}_A$ centered in I_A is tangent to the side BC and to the prolongations of AB and AC , as shown in Figure 10.

The excircle $\mathcal{I}_A$ is centered at I_A and is tangent to the segment BC , which, by convexity, is contained in $\mathcal{D}$. Consequently, the interiors of $\mathcal{D}$ and $\mathcal{I}_A$ intersect.

□

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