Discrete Painlevé equations from pencils of quadrics

in $\mathbb{P}^3$ with branching generators

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Abstract: In this paper we extend the novel approach to discrete Painlevé equations initiated in our previous work [2]. A classification scheme for discrete Painlevé equations proposed by Sakai interprets them as birational isomorphisms between generalized Halphen surfaces (surfaces obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up at eight points). Sakai's classification is thus based on the classification of generalized Halphen surfaces. In our scheme, the family of generalized Halphen surfaces is replaced by a pencil of quadrics in $\mathbb{P}^3$. A discrete Painlevé equation is viewed as an autonomous transformation of $\mathbb{P}^3$ that preserves the pencil and maps each quadric of the pencil to a different one. Thus, our scheme is based on the classification of pencils of quadrics in $\mathbb{P}^3$. Compared to our previous work, here we consider a technically more

demanding case where the characteristic polynomial $\Delta(\lambda)$ of the pencil of quadrics is not a complete square. As a consequence, traversing the pencil via a 3D Painlevé map corresponds to a translation on the universal cover of the Riemann surface of $\sqrt{\Delta(\lambda)}$, rather than to a Möbius transformation of the pencil parameter λ as in [2].

1   Introduction

This paper is the second contribution to our study devoted to a novel interpretation of discrete Painlevé equations, which builds up on [2]. Discrete Painlevé equations belong to the most intriguing objects in the theory of discrete integrable systems. After some examples sporadically appeared in various applications, their systematic study started when Grammaticos, Ramani and Papageorgiou proposed the notion of "singularity confinement" as an integrability detector, and found the first examples of second order nonlinear non-autonomous difference equations with this property, which they denoted as *discrete Painlevé equations* [9, 16]. The activity of their group was summarized in [8]. A general classification scheme of discrete Painlevé equations was proposed by Sakai [18] and it is given a detailed exposition in the review paper by Kajiwara, Noumi and Yamada [11]. In the framework of Sakai's scheme, discrete Painlevé equations are birational maps between *generalized Halphen surfaces X*. The latter can be realized as $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at eight points. A monographic exposition of discrete Painlevé equations is given by Joshi [10].

Let us summarize the main ingredients and features of our alternative approach to discrete Painlevé equations, initiated in [2].

- A pencil of quadrics $\{Q_\lambda\}$ in $\mathbb{P}^3$ containing non-degenarate quadrics. Such pencils can be classified modulo projective transformations of $\mathbb{P}^3$, and they come in thirteen classes. The class of the pencil can be identified by the type of its *base curve* $Q_0 \cap$

Q_∞. This is a spatial curve of degree 4, whose type can vary from a generic one (irreducible smooth curve for a pencil of type (i)), through irreducible curves with a node (type (ii)) or with a cusp (type (iii)), to various types of reducible curves (from two non-coplanar conics intersecting at two points, type (iv), to a pair of intersecting double lines, type (xiii)).

• The second pencil of quadrics $\{P_\mu\}$ having one quadric in common with $\{Q_\lambda\}$, say $P_\infty = Q_\infty$. The base curves of both pencils intersect at eight points S_i, $i = 1, \dots, 8$.

• Given two pencils of quadrics, one can define a three-dimensional analog of a QRT map $F = i_1 \circ i_2$, where the 3D QRT involutions i_1, i_2 act along two families of generators of Q_λ, see [1]. Each involution puts into correspondence two intersection points of a generator with the quadric P_μ. By definition, such an involution, and therefore the 3D QRT map $F = i_1 \circ i_2$, leaves each quadric of two pencils invariant, and thus possesses two rational integrals of motion $\lambda = Q_0/Q_\infty$ and $\mu = P_0/P_\infty$.

• A *Painlevé deformation map* is the device which allows us to travel across the pencil $\{Q_\lambda\}$. More precisely, such a map L on $\mathbb{P}^3$ preserves the pencil, but not fiber-wise. Rather, it sends each quadric Q_λ to a different quadric $Q_{\hat\lambda}$. Moreover, L preserves the base curve of the pencil $\{Q_\lambda\}$. In the cases considered in [2], the base curve is reducible and contains straight lines. In these cases, L does not necessarily fix these straight lines point-wise. In the cases considered in the present paper, L fixes the base curve $Q_0 \cap Q_\infty$ pointwise (in particular, it fixes all eight points S_i).

• A *3D Painlevé map* is obtained by composition $\widetilde{F} = L \circ i_1 \circ L \circ i_2$, provided it possesses the singularity confinement property. It is to be stressed that the pencil $\{Q_\lambda\}$ continues to play a fundamental role in the dynamics of $\widetilde{F}$: the maps $L \circ i_1$, $L \circ i_2$ preserve the pencil and map each quadric Q_λ to $Q_{\hat\lambda}$. We do not have a straightforward description of the dynamical role of the pencil $\{P_\mu\}$, but anticipate its relation to the isomonodromic description of the discrete Painlevé equations.

One can say that in our approach the role of a family of generalized Halphen surfaces

is played by the quadrics of the pencil $\{Q_\lambda\}$ with eight distinguished points on the base curve of the pencil. The base curve itself plays the role of the unique anti-canonical divisor. Let us stress several features of our construction which are in a sharp contrast to the Sakai scheme.

- Neither the exceptional divisor nor the eight distinguished points evolve under the map $\widetilde{F}$. Their discrete time evolution is apparent and is due to their representation in the so-called *pencil-adapted coordinates*. These are coordinates $(x, y, \lambda) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ establishing an isomorphism between each quadric Q_λ of the pencil and $\mathbb{P}^1 \times \mathbb{P}^1$. The pencil-adapted coordinates of a point on the base curve do depend on λ, so traversing the pencil $\lambda \longmapsto \widehat{\lambda}$ under $\widetilde{F}$ induces an *apparent discrete time evolution* of the base curve and of the eight distinguished points.

- The shift parameter δ of discrete Painlevé equations (or its exponent $q = e^\delta$ for the q-difference equations among them) is not an intrinsic characteristic of the configuration of eight distinguished points, but is a free parameter of the construction.

One can say that our approach is a realization of the old-style idea of discrete Painlevé equations being non-autonomous versions (or modifications) of the QRT maps. This idea was instrumental in the discovery and early classification attempts of discrete Painlevé equations, summarized in [8]. A more geometric version of this procedure was proposed in the framework of the Sakai's scheme by Carstea, Dzhamay and Takenawa [5]. In their scheme, the de-autonomization of a given QRT map depends on the choice of one biquadratic curve of the pencil. In our approach, the choice of the base curve and eight distinguished point on it determines uniquely all the ingredients of the construction, starting with the two pencils of quadrics.

The structure of the paper is as follows. In Section 2, we describe the construction scheme of discrete Painlevé equations applicable to the present case and stress its distinctions from the previous paper [2]. The main distinction is that here we consider the pencils whose characteristic polynomial $\Delta(\lambda)$ is not a complete square. As a consequence,

the 3D QRT involutions i_1, i_2 and the 3D QRT map $F = i_1 \circ i_2$ are no more birational maps of $\mathbb{P}^3$. Rather, these maps become birational maps on $\mathcal{X}$, a branched double covering of $\mathbb{P}^3$, whose ramification locus is the union of the singular quadrics Q_{λ_i}, where λ_i are the branch points of the Riemann surface $\mathcal{R}$ of $\sqrt{\Delta(\lambda)}$.

In Section 3, we formulate a general recipe for the construction of the Painlevé deformation map L, responsible to the evolution $\lambda \longmapsto \widehat{\lambda}$ across the pencil of quadrics $\{Q_\lambda\}$. While in the first part [2] we had $\widehat{\lambda} = \sigma(\lambda)$, where $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ is a Möbius automorphism fixing the set $\mathrm{Sing}(Q) := \{\lambda \in \mathbb{P}^1 : Q_\lambda \text{ is degenerate}\}$, in the present paper the natural definition becomes $\widehat{\lambda} = \lambda(\widehat{\nu})$, where $\lambda = \lambda(\nu)$ is the holomorphic uniformization map for the Riemann surface $\mathcal{R}$, and $\widehat{\nu} = \nu + 2\delta$ is the translation on the universal cover $\mathbb{C}$. The recipe turns out to be applicable to all types of the pencil $\{Q_\lambda\}$ except for the generic type (i). The latter leads to the elliptic Painlevé equation, which will be treated in a separate publication.

In Section 4, we show that the so constructed L ensures the fundamental singularity confinement property for our 3D Painlevé maps.

There follow five Sections 5–9 containing a detailed elaboration of our scheme for all relevant types of the pencils except for the type (i). We recover, within our novel framework, all discrete Painlevé equations except for the elliptic one, which is left for a separate publication.

2 General scheme

We now describe the construction scheme of discrete Painlevé equations applicable to the present case and stress its distinctions from the previous paper [2]. The first steps are the same as there:

- Start with a pencil $\{C_\mu\}$ of biquadratic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ and the corresponding QRT map. Let $s_1, \dots, s_8 \in \mathbb{P}^1 \times \mathbb{P}^1$ be the base points of this pencil. Lift $\{C_\mu\}$ to a pencil of quadrics $\{P_\mu\}$ in $\mathbb{P}^3$ using the *Segre embedding* of $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$. The base curve of this

pencil passes through the lifts $S_1, \dots, S_8$ of the base points $s_1, \dots, s_8$.

- Choose one distinguished biquadratic curve C_∞ of the pencil, along with its lift to a quadric P_∞.

- Based on these data, construct the pencil of quadrics $\{Q_\lambda = Q_0 - \lambda Q_\infty\}$ in $\mathbb{P}^3$ spanned by $Q_0 = \{X_1 X_2 - X_3 X_4 = 0\}$ and $Q_\infty := P_\infty$. Recall that Q_0 is nothing but the image of $\mathbb{P}^1 \times \mathbb{P}^1$ by the Segre embedding. The base curve of the pencil $\{Q_\lambda\}$ is, by definition, the curve $Q_0 \cap Q_\infty$, which is the image of C_∞ under the Segre embedding. The intersection of this curve with the base curve of the pencil $\{P_\mu\}$ consists exactly of the points $S_1, \dots, S_8$.

The characteristic polynomial of the pencil $\{Q_\lambda\}$ is

$$\Delta(\lambda) = \det(M_\lambda) = \det(M_0 - \lambda M_\infty), \tag{1}$$

where $M_0, M_\infty \in \mathrm{Sym}_{4\times 4}(\mathbb{C})$ are symmetric matrices of the quadratic forms Q_0, Q_∞. In the present paper, we are dealing with the cases where this polynomial is *not a complete square*. According to the projective classification of pencils of quadrics, discussed in [2], these are the following six cases:

(i) *Pencil of quadrics through a non-singular spatial quartic curve.*
Segre symbol $[1,1,1,1]$; $\Delta(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$.

(ii) *Pencil of quadrics through a nodal spatial quartic curve.*
Segre symbol $[2,1,1]$; $\Delta(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)(\lambda - \lambda_3)$.

(iii) *Pencil of quadrics through a cuspidal spatial quartic curve.*
Segre symbol $[3,1]$; $\Delta(\lambda) = (\lambda - \lambda_1)^3(\lambda - \lambda_2)$.

(iv) *Pencil of quadrics through two non-coplanar conics sharing two points.*
Segre symbol $[(1,1),1,1]$; $\Delta(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)(\lambda - \lambda_3)$.

(v) *Pencil of quadrics through two non-coplanar conics touching at a point.*
Segre symbol $[(2,1),1]$; $\Delta(\lambda) = (\lambda - \lambda_1)^3(\lambda - \lambda_2)$.

(vi) *Pencil of quadrics tangent along a non-degenerate conic.*

Segre symbol $[(1,1,1),1]$; $\Delta(\lambda) = (\lambda - \lambda_1)^3(\lambda - \lambda_2)$.

As discussed in [2], for $X \in Q_\lambda$, the generators $\ell_1(X)$ and $\ell_2(X)$ are rational functions of X and of $\sqrt{\Delta(\lambda)}$. The dependence on λ can be expressed as a holomorphic dependence on the point of the Riemann surface $\mathcal{R}$ of $\sqrt{\Delta(\lambda)}$. This Riemann surface is a double cover of $\widehat{\mathbb{C}}$ branched at two or at four points. By the uniformization theorem, its universal cover is $\mathbb{C}$. We will denote the uniformizing variable $\nu \in \mathbb{C}$, so that the maps $\nu \longmapsto \lambda$ and $\nu \longmapsto \sqrt{\Delta(\lambda)}$ are holomorphic. The following three situations can be distinguished:

- case (i): four distinct branch points $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, the Riemann surface $\mathcal{R}$ is a torus, whose conformal class is determined by the cross-ratio of the branch points. This case, corresponding to the elliptic Painlevé equations, will be treated in an upcoming work;

- cases (ii), (iv): two branch points λ_2, λ_3, one of the periods of the torus becomes infinite, so that $\mathcal{R}$ is a cylinder;

- cases (iii), (v), (vi): two branch points λ_1, λ_2, both periods of the torus become infinite, so that $\mathcal{R}$ is plane.

It becomes necessary to introduce modifications in the two major ingredients of the construction in [2].

- The generators ℓ_1, ℓ_2 are not rational functions on $\mathbb{P}^3$ anymore. Rather, they become well-defined rational maps on the variety $\mathcal{X}$ which is a branched double covering of $\mathbb{P}^3$, whose ramification locus is the union of the singular quadrics Q_{λ_i}, where λ_i are the branch points of $\mathcal{R}$. The same is true for a linear projective change of variables $X = A_\nu Y$ reducing the quadratic form $Q_{\lambda(\nu)}$ to the standard form Q_0, which we now write as

$$Q_{\lambda(\nu)}(A_\nu Y) = Q_0(Y), \quad \text{or} \quad A_\nu^{\mathrm{T}} M_{\lambda(\nu)} A_\nu = M_0, \tag{2}$$

and for the *pencil-adapted coordinates*

$$
\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = A_\nu \begin{bmatrix} x \\ y \\ xy \\ 1 \end{bmatrix} =: \phi_\nu(x, y). \tag{3}
$$

Thus, ϕ_ν gives a parametrization of $Q_{\lambda(\nu)}$ by $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$, such that the generators ℓ_1, resp. ℓ_2 of Q_λ correspond to $x = $ const, resp. to $y = $ const. Interchanging two sheets of the covering corresponds to interchanging two families of generators ℓ_1, ℓ_2.

- Also the *3D QRT involutions* i_1, i_2 for the pencil $\{Q_\lambda\}$, defined by intersections of its generators ℓ_1, ℓ_2 with the quadrics P_μ (see [1]), are not birational maps of $\mathbb{P}^3$ anymore, and the same is true for the *3D QRT map* $F = i_1 \circ i_2$. Rather, these maps become birational maps on $\mathcal{X}$.

The next main deviation from the construction of [2] is that it becomes unnatural to consider Painlevé deformation maps L as birational maps $\mathbb{P}^3$ preserving the pencil $\{Q_\lambda\}$ and sending each Q_λ to $Q_{\sigma(\lambda)}$, where $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ is a Möbius automorphism fixing the set $\mathrm{Sing}(Q) := \{\lambda \in \mathbb{P}^1 : Q_\lambda \text{ is degenerate}\}$. Instead, in the present context we formulate the following requirement.

- A Painlevé deformation map L is a birational map on $\mathcal{X}$ preserving the pencil $\{Q_\lambda\}$ and sending $Q_{\lambda(\nu)}$ to $Q_{\lambda(\hat{\nu})}$, where $\nu \longmapsto \hat{\nu} = \nu + 2\delta$ is a translation on the universal cover of $\mathcal{R}$.

As compared with [2], our construction will involve some additional ingredients, required to establish the relation to the form of discrete Painlevé equations known from the literature. The Painlevé deformation map L is decomposed in two factors, each one depending only on one of the variables x, y, and shifting the variable ν by δ. This can be done in two ways:

$$
L = L_1 \circ R_2, \quad \text{where} \quad L_1 : (x, y, \nu) \longmapsto (x, \widetilde{y}, \nu + \delta), \quad R_2 : (x, y, \nu) \longmapsto (\widetilde{x}, y, \nu + \delta), \tag{4}
$$

resp.

$$L = L_2 \circ R_1, \quad \text{where} \quad L_2 : (x, y, \nu) \longmapsto (\widetilde{x}, y, \nu + \delta), \quad R_1 : (x, y, \nu) \longmapsto (x, \widetilde{y}, \nu + \delta). \tag{5}$$

(The indices $1, 2$ refer to the variables which *do not* change under the map, like for i_1, i_2.) Each one of L_1, L_2, R_1, R_2 maps $Q_{\lambda(\nu)}$ to $Q_{\lambda(\nu+\delta)}$. We set

$$\nu_n = \nu_0 + 2n\delta \quad \text{for} \quad n \in \frac{1}{2}\mathbb{Z},$$

so that $\nu_{n+1/2} = \nu_n + \delta$. The variables associated to the discrete Painlevé equations known from the literature, parametrize in our formulation the quadrics with half-integer indices, namely

$$(x_n, y_n, \nu_{2n-1/2}) \in Q_{\lambda(\nu_{2n-1/2})}, \quad (x_{n+1}, y_n, \nu_{2n+1/2}) \in Q_{\lambda(\nu_{2n+1/2})}.$$

Definition 1. *A 3D Painlevé map is given by*

$$\widetilde{F} = \widetilde{i_1} \circ \widetilde{i_2}, \quad \text{where} \quad \widetilde{i_1} = R_1 \circ i_1 \circ L_1, \quad \widetilde{i_2} = R_2 \circ i_2 \circ L_2, \tag{6}$$

or, in coordinates,

$$(x_n, y_n, \nu_{2n-1/2}) \overset{L_2}{\to} (x, y_n, \nu_{2n}) \overset{i_2}{\to} (\widetilde{x}, y_n, \nu_{2n}) \overset{R_2}{\to} (x_{n+1}, y_n, \nu_{2n+1/2}) \tag{7}$$

$$\overset{L_1}{\to} (x_{n+1}, y, \nu_{2n+1}) \overset{i_1}{\to} (x_{n+1}, \widetilde{y}, \nu_{2n+1}) \overset{R_1}{\to} (x_{n+1}, y_{n+1}, \nu_{2n+3/2}). \tag{8}$$

The map $\widetilde{F}$ is conjugate to $L \circ i_1 \circ L \circ i_2$; note that the latter map acts between the quadrics with integer indices.

Our last requirement repeats the one in [2]:

• The singularity confinement properties of $\widetilde{i}_1, \widetilde{i}_2$ are the same as that of i_1, i_2.

Reduction to the symmetric case. If the eight points s_i are symmetric with respect to the symmetry switch $\sigma : (x, y) \longmapsto (y, x)$, we can define a 2D QRT root $f = i_1 \circ \sigma = \sigma \circ i_2$ such that $F = f \circ f$. In this case, the map L in the pencil-adapted coordinates satisfies $L = \sigma \circ L \circ \sigma$, and therefore its decomposition factors satisfy

$$L_2 = \sigma \circ L_1 \circ \sigma, \quad R_2 = \sigma \circ R_1 \circ \sigma.$$

The 3D Painlevé map F can be written as

$$
\begin{aligned}
\widetilde{F} &= R_1 \circ i_1 \circ L_1 \circ R_2 \circ i_2 \circ L_2 = R_1 \circ i_1 \circ L_1 \circ \sigma \circ R_1 \circ \sigma \circ i_2 \circ \sigma \circ L_1 \\
&= R_1 \circ i_1 \circ \sigma \circ L_2 \circ R_1 \circ \sigma \circ i_2 \circ L_2 \\
&= (R_1 \circ f \circ L_2)^2.
\end{aligned}
$$

Therefore, one can define the Painlevé deformed QRT root as $\widetilde{f} = R_1 \circ f \circ L_2$, then the discrete Painlevé map decomposes as $\widetilde{F} = \widetilde{f} \circ \widetilde{f}$.

3 Construction of the Painlevé deformation map

The desired properties of the Painlevé deformation map L are ensured by the following construction.

Theorem 1. *If the polynomial Q_∞ does not depend on X_3, define the map $L : [X_1 : X_2 : X_3 : X_4] \mapsto [\widehat{X}_1 : \widehat{X}_2 : \widehat{X}_3 : \widehat{X}_4]$ by requiring that, for $X \in Q_{\lambda(\nu)}$, there holds*

$$
\begin{cases}
\widehat{X}_1 = X_1 X_4, \\
\widehat{X}_2 = X_2 X_4, \\
\widehat{X}_3 = X_3 X_4 - \big(\lambda(\widehat{\nu}) - \lambda(\nu)\big) Q_\infty(X), \\
\widehat{X}_4 = X_4^2,
\end{cases}
\tag{9}
$$

where $\widehat{\nu} = \nu + 2\delta$. If Q_∞ does not depend on X_1, define

$$
\begin{cases}
\widehat{X}_1 = X_1 X_2 + \big(\lambda(\widehat{\nu}) - \lambda(\nu)\big) Q_\infty(X), \\
\widehat{X}_2 = X_2^2, \\
\widehat{X}_3 = X_2 X_3, \\
\widehat{X}_4 = X_2 X_4.
\end{cases}
\tag{10}
$$

Then L sends each $Q_{\lambda(\nu)}$ to $Q_{\lambda(\widehat{\nu})}$ and fixes all points of the base curve of the pencil $\{Q_\lambda\}$ not belonging to $\{X_4 = 0\}$ (resp. to $\{X_2 = 0\}$), including all eight base points S_i, $i = 1, \dots, 8$.

Proof. It follows by a simple computation. For instance, for the case (9):

$$
\widehat{X}_1 \widehat{X}_2 - \widehat{X}_3 \widehat{X}_4 - \lambda(\widehat{\nu}) Q_\infty(\widehat{X}) = X_4^2 \big(X_1 X_2 - X_3 X_4 - \lambda(\nu) Q_\infty(X)\big).
$$

Futher, if $Q_\infty(X) = 0$ and $X_4 \neq 0$, then $[\widehat{X}_1 : \widehat{X}_2 : \widehat{X}_3 : \widehat{X}_4] = [X_1 : X_2 : X_3 : X_4]$. $\blacksquare$

The recipe of Theorem 1 covers all cases treated in the present paper (pencils of the types (ii)-(vi)). In retrospect, we notice that, with a natural modification (replace $\widehat{\lambda} - \lambda = \lambda(\widehat{\nu}) - \lambda(\nu)$ by $\sigma(\lambda) - \lambda$), this recipe covers also the cases considered in the first part of this study [2]. For pencils of the type (i) the quadric Q_∞ is non-degenerate, so a modification of the recipe is required.

4   Singularity confinement

Our case-by-case computations reveal the following observation. In all examples of the present paper, the eight points $s_1, \dots, s_8$ in $\mathbb{P}^1 \times \mathbb{P}^1$ serve as the indeterminacy set for the 2D QRT involutions i_1, i_2. The singularity confinement structure can be summarised as:

$$\{x = a_i\} \xrightarrow{i_1} s_i \xrightarrow{i_2} \{y = b_i\}, \quad i = 1, \dots, 8. \tag{11}$$

In the pencil-adapted coordinates, the 3D QRT involutions restricted to $Q_{\lambda(\nu)}$ are given by the same formulas as the original 2D QRT involutions, with the points s_i replaced by their deformations $s_i(\nu)$. The latter still support a pencil of biquadratic curves, which are the pre-images under ϕ_ν of the intersection curves $Q_{\lambda(\nu)} \cap P_\mu$. Therefore, for the 3D QRT involutions i_1 and i_2, we have

$$\{x = a_i(\nu)\} \xrightarrow{i_1} s_i(\nu) \xrightarrow{i_2} \{y = b_i(\nu)\}. \tag{12}$$

Let $\Phi_i \subset \mathbb{P}^3$ be the ruled surface comprised of the lines on $Q_{\lambda(\nu)}$ given, in the pencil-adapted coordinates ϕ_ν, by the equations $\{x = a_i(\nu)\}$. Likewise, let $\Psi_i \subset \mathbb{P}^3$ be the ruled surface comprised of the lines on $Q_{\lambda(\nu)}$ given in the coordinates ϕ_ν by the equations $\{y = b_i(\nu)\}$. Then, in view of (12), we obtain the following singularity confinement patterns for i_1, i_2:

$$\Phi_i \xrightarrow{i_1} S_i \xrightarrow{i_2} \Psi_i. \tag{13}$$

We emphasize that the surfaces Φ_i are blown down to points (rather than curves), and these points are blown up to surfaces Ψ_i again.

Theorem 2. *Suppose that the involutions $i_1, i_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ have a singularity confinement pattern of the type* (13), *and L satisfies*

$$L(S_i) = S_i. \tag{14}$$

Then for the deformed maps $\widetilde{i}_1 = R_1 \circ i_1 \circ L_1, \widetilde{i}_2 = R_2 \circ i_2 \circ L_2$ we have:

$$L_1^{-1}(\Phi_i) \xrightarrow{\widetilde{i}_1} R_1(S_i) \xrightarrow{\widetilde{i}_2} R_2(\Psi_i), \tag{15}$$

which implies for $\widetilde{F} = \widetilde{i}_1 \circ \widetilde{i}_2$ the singularity confinement pattern

$$(L_1 \circ \widetilde{i}_2)^{-1}(\Phi_i) \xrightarrow{\widetilde{F}} R_1(S_i) \xrightarrow{\widetilde{F}} (\widetilde{i}_1 \circ R_2)(\Psi_i). \tag{16}$$

An important observation is that the eight points $R_1(S_i)$ participating in these singularity confinement patterns do not support a net of quadrics.

5 From a pencil of type (v) to the d-Painlevé equation of the surface type $A_1^{(1)}$

2D QRT map. We consider the QRT map corresponding to the pencil of biquadratic curves $\{C_\mu\}$ through eight points $s_i = (a_i, b_i)$, $i = 1, \ldots, 8$, where

$$b_i = -a_i, \quad i = 1, \ldots, 4, \quad \text{and} \quad b_i = 1 - a_i, \quad i = 5, \ldots, 8. \tag{17}$$

These eight points support a pencil of biquadratic curves if they satisfy the condition

$$a_1 + a_2 + a_3 + a_4 - a_5 - a_6 - a_7 - a_8 = -2. \tag{18}$$

This pencil contains a reducible curve, consisting of two (1,1)-curves:

$$C_\infty = \{(x + y)(x + y - 1) = 0\}. \tag{19}$$

The vertical involution i_1 for this pencil can be described by the following equation:

$$i_1(x, y) = (x, \widetilde{y}), \quad \frac{(\widetilde{y} + x)(x + y)}{(\widetilde{y} + x - 1)(x + y - 1)} = \frac{\prod_{i=1}^{4}(x - a_i)}{\prod_{i=5}^{8}(x - a_i)}. \tag{20}$$
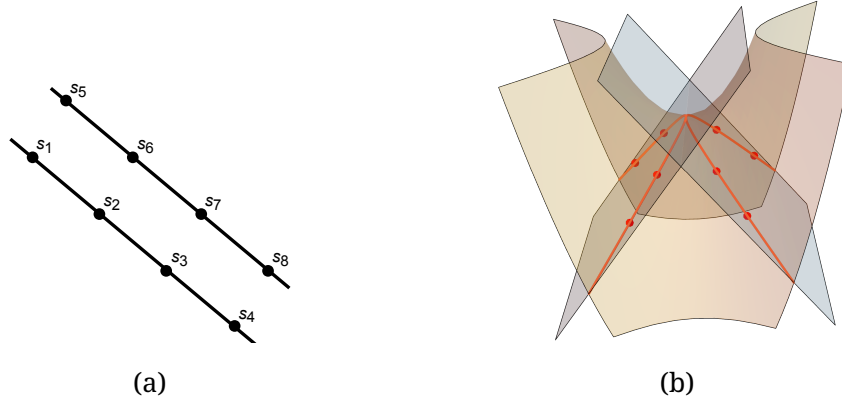
Figure 1: (a) Base set of the surface type $A_1^{(1)}$: two quadruples of points on two touching (1,1)-curves in $\mathbb{P}^1 \times \mathbb{P}^1$. (b) Pencil of quadrics through two touching non-coplanar conics

Similarly, the horizontal involution i_2 can be described by the following equation:

$$i_2(x,y) = (\widetilde{x}, y), \quad \frac{(\widetilde{x}+y)(x+y)}{(\widetilde{x}+y-1)(x+y-1)} = \frac{\prod_{i=1}^{4}(y+a_i)}{\prod_{i=5}^{8}(y+a_i-1)}. \tag{21}$$

The QRT map is the composition of these two involutions, $F = i_1 \circ i_2$. The singularity confinement structure of the involutions i_1, i_2 is as in (11). The symmetric case corresponds to

$$a_{2i} = -a_{2i-1}, \; i = 1,2, \qquad a_{2i} = 1 - a_{2i-1}, \; i = 3,4.$$

In this case, $F = f \circ f$, with $f = i_1 \circ \sigma = \sigma \circ i_2$ being the QRT root (here $\sigma(x,y) = (y,x)$).

3D Painlevé map. We consider the pencil of quadrics $\{P_\mu\}$ in $\mathbb{P}^3$, the Segre lift of the pencil of curves $\{C_\mu\}$. The pencil $\{Q_\lambda\}$ is spanned by $Q_0 = \{X_1 X_2 - X_3 X_4 = 0\}$ and $Q_\infty = P_\infty = \{(X_1 + X_2)(X_1 + X_2 - X_4) = 0\}$:

$$Q_\lambda = \left\{ X_1 X_2 - X_3 X_4 - \lambda(X_1 + X_2)(X_1 + X_2 - X_4) = 0 \right\}. \tag{22}$$

The base set of the pencil Q_λ consists of the two conics, $\{X_1 X_2 - X_3 X_4 = 0, \; X_1 + X_2 = 0\}$ and $\{X_1 X_2 - X_3 X_4 = 0, \; X_1 + X_2 = X_4\}$, which have one common (touching) point $[0 : 0 : 1 : 0]$.

This is a pencil of type (v). The intersection of this base set with the base set of the pencil $\{P_\mu\}$ consists of eight points

$$S_i = [a_i : b_i : a_i b_i : 1], \quad i = 1, \dots, 8,$$

which are nothing but the lifts of the points s_i under the Segre embedding.

The matrix M_λ of the quadratic form Q_λ:

$$M_\lambda = \begin{pmatrix} -2\lambda & 1 - 2\lambda & 0 & -\lambda \\ 1 - 2\lambda & -2\lambda & 0 & -\lambda \\ 0 & 0 & 0 & -1 \\ -\lambda & -\lambda & -1 & 0 \end{pmatrix}. \tag{23}$$

The characteristic polynomial of the pencil $\{Q_\lambda\}$ is: $\Delta(\lambda) = \det(M_\lambda) = 1 - 4\lambda$, which is not a complete square, and $\mathrm{Sing}(Q_\lambda) = \{\frac{1}{4}, \infty\}$. We uniformize the Riemann surface of $\sqrt{\Delta(\lambda)}$ via

$$\lambda = \frac{1 - \nu^2}{4}, \quad \sqrt{\Delta(\lambda)} = \nu. \tag{24}$$

Thus, $\lambda(-\nu) = \lambda(\nu)$, while $\sqrt{\Delta(\lambda)}$ changes its sign as $\nu \to -\nu$. This gives us a double cover of the original pencil branched at $\lambda = 1/4$ (corresponding to $\nu = 0$), and at $\lambda = \infty$ (corresponding to $\nu = \infty$). The normalizing transformation of $Q_{\lambda(\nu)}$ to the canonical form Q_0 can be found as follows:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = A_\nu \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}, \tag{25}$$

where

$$A_\nu = \begin{pmatrix} \frac{1}{2\nu}(1 + \nu) & \frac{1}{2\nu}(1 - \nu) & 0 & 0 \\ \frac{1}{2\nu}(1 - \nu) & \frac{1}{2\nu}(1 + \nu) & 0 & 0 \\ \frac{1}{4\nu}(1 - \nu^2) & \frac{1}{4\nu}(1 - \nu^2) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

Indeed, one immediately verifies that

$$A_\nu^{\mathrm{T}} M_{\lambda(\nu)} A_\nu = M_0.$$

Now, we are in the position to derive a parametrization of the quadric Q_λ:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = A_\nu \begin{bmatrix} x \\ y \\ xy \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\nu}((1+\nu)x + (1-\nu)y) \\ \frac{1}{2\nu}((1-\nu)x + (1+\nu)y) \\ xy + \frac{1-\nu^2}{4\nu}(x+y) \\ 1 \end{bmatrix} =: \phi_\nu(x, y). \tag{27}$$

Observe that this parametrization is neither valid for $\nu = 0$ nor for $\nu = \infty$. The pencil-adapted coordinates (x, y, ν) on (the double cover of) $\mathbb{P}^3$ are:

$$x = \frac{(1+\nu)X_1 - (1-\nu)X_2}{2X_4}, \quad y = \frac{(1+\nu)X_2 - (1-\nu)X_1}{2X_4}, \tag{28}$$

which have to be supplemented with

$$\lambda = \frac{1-\nu^2}{4} = \frac{X_1 X_2 - X_3 X_4}{(X_1 + X_2)(X_1 + X_2 - X_4)}. \tag{29}$$

Theorem 3. *For any $\delta \in \mathbb{C} \setminus \{0\}$, define the Painlevé deformation map corresponding to the translation $\nu \longmapsto \hat{\nu} = \nu + 2\delta$ by*

$$L : \begin{cases} \hat{X}_1 &= X_1 X_4, \\ \hat{X}_2 &= X_2 X_4, \\ \hat{X}_3 &= X_3 X_4 - \big(\lambda(\hat{\nu}) - \lambda(\nu)\big) Q_\infty(X) \\ &= X_3 X_4 + \delta(\nu + \delta)(X_1 + X_2)(X_1 + X_2 - X_4), \\ \hat{X}_4 &= X_4^2. \end{cases}$$

Then, in pencil-adapted coordinates, the map L acts as follows:

$$L : (x, y, \nu) \longmapsto (\hat{x}, \hat{y}, \hat{\nu}), \quad \hat{x} = x + \frac{\delta}{\nu}(x+y), \quad \hat{y} = y + \frac{\delta}{\nu}(x+y), \quad \hat{\nu} = \nu + 2\delta. \tag{30}$$

For the latter map, the factorizations (4), (5) *are given by*

$$L_1 = R_1 : (x, y, \nu) \longmapsto (x, \tilde{y}, \nu + \delta), \quad L_2 = R_2 : (x, y, \nu) \longmapsto (\tilde{x}, y, \nu + \delta),$$

where

$$\widetilde{y} = y + \frac{\delta}{\nu}(x + y) \quad \Leftrightarrow \quad \frac{\widetilde{y} + x}{\widetilde{y} + x - \nu - \delta} = \frac{y + x}{y + x - \nu}, \tag{31}$$

$$\widetilde{x} = x + \frac{\delta}{\nu}(x + y) \quad \Leftrightarrow \quad \frac{\widetilde{x} + y}{\widetilde{x} + y - \nu - \delta} = \frac{x + y}{x + y - \nu}. \tag{32}$$

Relation to the d-Painlevé equation of the surface type $A_1^{(1)}$. We now compute the 3D Painlevé map $\widetilde{F} = R_1 \circ i_1 \circ L_1 \circ R_2 \circ i_2 \circ L_2$ in the pencil-adapted coordinates (x, y, ν). For each fixed ν, the intersection curves $Q_{\lambda(\nu)} \cap P_\mu$ form a pencil through eight points

$$s_i(\nu) = (a_i, -a_i), \quad i = 1, \dots, 4, \tag{33}$$

$$s_i(\nu) = \left(\frac{\nu - 1}{2} + a_i, \frac{1 + \nu}{2} - a_i \right), \quad i = 5, \dots, 8, \tag{34}$$

which are just the points $S_1, \dots, S_8$ (which are, recall, independent of ν) expressed in the pencil-adapted coordinates on $Q_{\lambda(\nu)}$. The curve $C_\infty(\nu)$, which is the image of the base curve of the pencil $\{Q_\lambda\}$ in the pencil-adapted coordinates on $Q_{\lambda(\nu)}$, is given by the equation

$$C_\infty(\nu) = \{(x + y)(x + y - \nu) = 0\}. \tag{35}$$

The map L sends $C_\infty(\nu)$ to $C_\infty(\nu + 2\delta)$, while the maps $L_1 = R_1$ and $L_2 = R_2$ send $C_\infty(\nu)$ to $C_\infty(\nu + \delta)$. We observe that the map L fixes the (x, y) coordinates of the points of the component $\{x + y = 0\}$ of $C_\infty(\nu)$, and acts as $(x, y) \longmapsto (x + \delta, y + \delta)$ on the component $\{x + y = \nu\}$. This "shift" under the map L is, however, only apparent, as this map fixes the curve $Q_0 \cap Q_\infty$ pointwise. Similarly, the map $L_1 = R_1$ acts on the second component as $(x, y) \longmapsto (x, y + \delta)$, while $L_2 = R_2$ acts as $(x, y) \longmapsto (x + \delta, y)$. These actions are non-trivial in homogeneous coordinates X.

The formulas for the 3D QRT involutions i_1, i_2 restricted to $Q_{\lambda(\nu)}$ coincide, in the pencil-adapted coordinates, with the original QRT involutions (20) and (21), upon replacing s_i by $s_i(\nu)$:

$$i_1(x, y) = (x, \widetilde{y}), \quad \frac{(\widetilde{y} + x)(x + y)}{(\widetilde{y} + x - \nu)(x + y - \nu)} = \frac{\prod_{i=1}^{4}(x - a_i)}{\prod_{i=5}^{8}(x - a_i - \frac{\nu - 1}{2})} =: \psi_1(x, \nu), \tag{36}$$

$$i_2(x,y) = (\widetilde{x}, y), \quad \frac{(\widetilde{x} + y)(x + y)}{(\widetilde{x} + y - \nu)(x + y - \nu)} = \frac{\prod_{i=1}^{4}(y + a_i)}{\prod_{i=5}^{8}(y + a_i - \frac{1+\nu}{2})} =: \psi_2(y, \nu). \tag{37}$$

In the notation of the equations (7), (8), we have:

$$\frac{(\widetilde{x} + y_n)(y_n + x)}{(\widetilde{x} + y_n - \nu_{2n})(y_n + x - \nu_{2n})} = \psi_2(y_n, \nu_{2n}), \tag{38}$$

$$\frac{(\widetilde{y} + x_{n+1})(x_{n+1} + y)}{(\widetilde{y} + x_{n+1} - \nu_{2n+1})(x_{n+1} + y - \nu_{2n+1})} = \psi_1(x_{n+1}, \nu_{2n+1}). \tag{39}$$

It remains to express $x, y, \widetilde{x}, \widetilde{y}$ in these formulas in terms of x_n, y_n. According to (7), we have:

$$L_2 : (x_n, y_n, \nu_{2n-1/2}) \longmapsto (x, y_n, \nu_{2n}) \quad \text{and} \quad R_2 : (\widetilde{x}, y_n, \nu_{2n}) \longmapsto (x_{n+1}, y_n, \nu_{2n+1/2}),$$

and with expressions (32) for the maps L_2, R_2, we find:

$$\frac{x + y_n}{x + y_n - \nu_{2n}} = \frac{x_n + y_n}{x_n + y_n - \nu_{2n-1/2}}, \tag{40}$$

$$\frac{\widetilde{x} + y_n}{\widetilde{x} + y_n - \nu_{2n}} = \frac{x_{n+1} + y_n}{x_{n+1} + y_n - \nu_{2n+1/2}}. \tag{41}$$

Similarly, according to (8), we have:

$$L_1 : (x_{n+1}, y_n, \nu_{2n+1/2}) \longmapsto (x_{n+1}, y, \nu_{2n+1}) \quad \text{and} \quad R_1 : (x_{n+1}, \widetilde{y}, \nu_{2n+1}) \longmapsto (x_{n+1}, y_{n+1}, \nu_{2n+3/2}),$$

and with expressions (31) for the maps L_1, R_1, we find:

$$\frac{y + x_{n+1}}{y + x_{n+1} - \nu_{2n+1}} = \frac{x_{n+1} + y_n}{x_{n+1} + y_n - \nu_{2n+1/2}}, \tag{42}$$

$$\frac{x_{n+1} + \widetilde{y}}{x_{n+1} + \widetilde{y} - \nu_{2n+1}} = \frac{x_{n+1} + y_{n+1}}{x_{n+1} + y_{n+1} - \nu_{2n+3/2}}. \tag{43}$$

Combining equations (38), (39) with (40)–(43) results in the following non-autonomous system:

$$\frac{(x_{n+1} + y_n)(x_n + y_n)}{(x_{n+1} + y_n - \nu_{2n+1/2})(x_n + y_n - \nu_{2n-1/2})} = \psi_2(y_n, \nu_{2n}), \tag{44}$$

$$\frac{(x_{n+1} + y_{n+1})(x_{n+1} + y_n)}{(x_{n+1} + y_{n+1} - \nu_{2n+3/2})(x_{n+1} + y_n - \nu_{2n+1/2})} = \psi_1(x_{n+1}, \nu_{2n+1}). \tag{45}$$

This is nothing but the d-Painlevé equation of the surface type $A_1^{(1)}$, as given in [11].

Remark. The symmetric case can be characterized by $\psi_1(x, v) = \psi_2(x, v)$. In this case the latter equations become two instances of

$$\frac{(u_{n+1} + u_n)(u_n + u_{n-1})}{(u_{n+1} + u_n - v_{n+1/2})(u_n + u_{n-1} - v_{n-1/2})} = \psi_1(u_n, v_n), \tag{46}$$

if we set $u_{2n-1} = x_n$, $u_{2n} := y_n$.

6 From a pencil of type (vi) to the d-Painlevé equation of the surface type $D_4^{(1)}$

By a simple limiting procedure, the results of the previous section lead to similar results for the d-Painlevé equation of the surface type $D_4^{(1)}$. We refrain from giving complete details here, and restrict ourselves only to the main results.

2D QRT map. We consider the QRT map corresponding to the pencil of biquadratic curves $\{C_\mu\}$ through eight points

$$s_i = (a_i, -a_i), \quad s_{i+4} = (a_i + \epsilon, -a_i + \epsilon), \quad i = 1, \dots, 4, \tag{47}$$

where the points $s_5, \dots, s_8$ are infinitely near to $s_1, \dots, s_4$, respectively. This pencil contains a reducible curve:

$$C_\infty = \{(x + y)^2 = 0\}. \tag{48}$$

The vertical involution i_1 and the horizontal involution i_2 for this pencil can be described by the following equations:

$$i_1(x, y) = (x, \tilde{y}), \quad \frac{1}{\tilde{y} + x} + \frac{1}{x + y} = \frac{1}{2} \sum_{i=1}^{4} \frac{1}{x - a_i}, \tag{49}$$

$$i_2(x, y) = (\tilde{x}, y), \quad \frac{1}{\tilde{x} + y} + \frac{1}{x + y} = \frac{1}{2} \sum_{i=1}^{4} \frac{1}{y + a_i}. \tag{50}$$
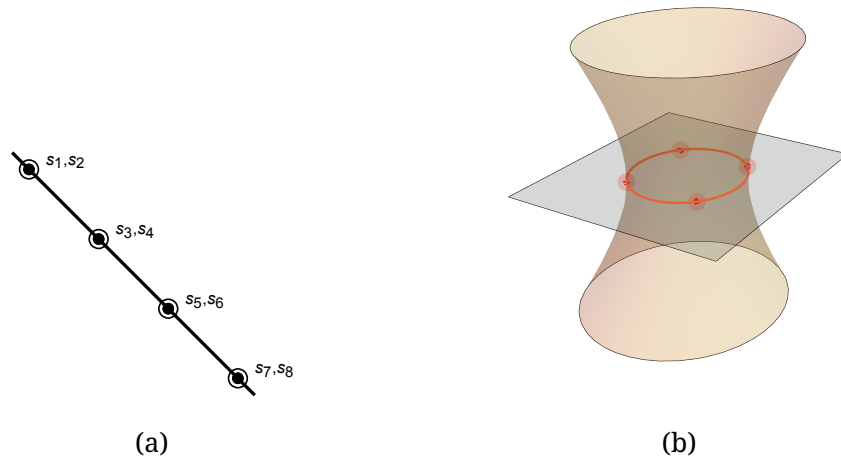
Figure 2: (a) Base set of the surface type $D_4^{(1)}$: four double points on a double (1,1)-curve in $\mathbb{P}^1 \times \mathbb{P}^1$. (b) Pencil of quadrics touching along a conic

3D Painlevé map. We consider the pencil of quadrics $\{P_\mu\}$ in $\mathbb{P}^3$ obtained as the Segre lift of the pencil of curves $\{C_\mu\}$. The pencil $\{Q_\lambda\}$ is spanned by $Q_0 = \{X_1 X_2 - X_3 X_4 = 0\}$ and $Q_\infty = P_\infty = \{(X_1 + X_2)^2 = 0\}$:

$$Q_\lambda = \left\{ X_1 X_2 - X_3 X_4 - \lambda (X_1 + X_2)^2 = 0 \right\}. \tag{51}$$

The base set of the pencil Q_λ is the double conic $\{X_1 X_2 - X_3 X_4 = 0,\ X_1 + X_2 = 0\}$. This is a pencil of type (vi). The matrix M_λ of the quadratic form Q_λ is:

$$M_\lambda = \begin{pmatrix} -2\lambda & 1 - 2\lambda & 0 & 0 \\ 1 - 2\lambda & -2\lambda & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{52}$$

The characteristic polynomial of the pencil $\{Q_\lambda\}$ is: $\Delta(\lambda) = \det(M_\lambda) = 1 - 4\lambda$, the same as in Section 5. The normalizing transformation of $Q_{\lambda(\nu)}$ to the canonical form Q_0 reads:

$$
\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = A_\nu \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix},
\tag{53}
$$

where

$$
A_\nu = \begin{pmatrix}
\frac{1}{2\nu}(1+\nu) & \frac{1}{2\nu}(1-\nu) & 0 & 0 \\
\frac{1}{2\nu}(1-\nu) & \frac{1}{2\nu}(1+\nu) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\tag{54}
$$

A parametrization of the quadric $Q_{\lambda(\nu)}$ is given by:

$$
\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = A_\nu \begin{bmatrix} x \\ y \\ xy \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\nu}((1+\nu)x + (1-\nu)y) \\ \frac{1}{2\nu}((1-\nu)x + (1+\nu)y) \\ xy \\ 1 \end{bmatrix} =: \phi_\nu(x,y).
\tag{55}
$$

The pencil-adapted coordinates (x, y, ν) on (the double cover of) $\mathbb{P}^3$ are:

$$
x = \frac{(1+\nu)X_1 - (1-\nu)X_2}{2X_4}, \quad y = \frac{(1+\nu)X_2 - (1-\nu)X_1}{2X_4},
\tag{56}
$$

which have to be supplemented with

$$
\lambda = \frac{1-\nu^2}{4} = \frac{X_1 X_2 - X_3 X_4}{(X_1 + X_2)^2}.
\tag{57}
$$

Theorem 4. *For any $\delta \in \mathbb{C} \setminus \{0\}$, define the Painlevé deformation map corresponding to the*

translation $\nu \longmapsto \widehat{\nu} = \nu + 2\delta$ *by*

$$
L : \begin{cases}
\widehat{X}_1 &= X_1 X_4, \\
\widehat{X}_2 &= X_2 X_4, \\
\widehat{X}_3 &= X_3 X_4 - \big(\lambda(\widehat{\nu}) - \lambda(\nu)\big)Q_\infty(X) \\
&= X_3 X_4 + \delta(\nu + \delta)(X_1 + X_2)^2, \\
\widehat{X}_4 &= X_4^2.
\end{cases}
$$

Then, in pencil-adapted coordinates, the map L acts as follows:

$$
L : (x, y, \nu) \longmapsto (\widehat{x}, \widehat{y}, \widehat{\nu}), \quad \widehat{x} = x + \frac{\delta}{\nu}(x + y), \quad \widehat{y} = y + \frac{\delta}{\nu}(x + y), \quad \widehat{\nu} = \nu + 2\delta. \tag{58}
$$

For the latter map, the factorizations (4), (5) *are given by*

$$
L_1 = R_1 : (x, y, \nu) \longmapsto (x, \widetilde{y}, \nu + \delta), \quad L_2 = R_2 : (x, y, \nu) \longmapsto (\widetilde{x}, y, \nu + \delta),
$$

where

$$
\widetilde{y} = y + \frac{\delta}{\nu}(x + y) \quad \Leftrightarrow \quad \frac{\nu + \delta}{\widetilde{y} + x} = \frac{\nu}{y + x}, \tag{59}
$$

$$
\widetilde{x} = x + \frac{\delta}{\nu}(x + y) \quad \Leftrightarrow \quad \frac{\nu + \delta}{\widetilde{x} + y} = \frac{\nu}{x + y}. \tag{60}
$$

Computing the 3D Painlevé map $\widetilde{F} = R_1 \circ i_1 \circ L_1 \circ R_2 \circ i_2 \circ L_2$ in the pencil-adapted coordinates (x, y, ν), we come to the following non-autonomous system:

$$
\frac{\nu_{2n+1/2}}{x_{n+1} + y_n} + \frac{\nu_{2n-1/2}}{x_n + y_n} = \frac{\nu_{2n}}{2} \sum_{i=1}^{4} \frac{1}{y_n + a_i}, \tag{61}
$$

$$
\frac{\nu_{2n+3/2}}{x_{n+1} + y_{n+1}} + \frac{\nu_{2n+1/2}}{x_{n+1} + y_n} = \frac{\nu_{2n+1}}{2} \sum_{i=1}^{4} \frac{1}{x_{n+1} - a_i}. \tag{62}
$$

This can be considered as a *d*-Painlevé equation of the surface type $D_4^{(1)}$, in a realization different from that in [11]. We remark here that the latter equation was put into our scheme in [2, sect. 9], however in the framework of pencils of quadrics with rational (non-branching) generators. There is no obvious relation between these two systems, and it would be desirable to clarify this point.

The symmetric case is characterised by $a_{2i} = -a_{2i-1}$, $i = 1, 2$. In this case the latter equations become two instances of

$$\frac{v_{n+1/2}}{u_{n+1} + u_n} + \frac{v_{n-1/2}}{u_n + u_{n-1}} = v_n \left(\frac{u_n}{u_n^2 - a_1^2} + \frac{u_n}{u_n^2 - a_3^2} \right), \tag{63}$$

if we set $u_{2n-1} = x_n$, $u_{2n} := y_n$.

7 From a pencil of type (iv) to the q-Painlevé equation of the surface type $A_1^{(1)}$

2D QRT map. Consider the QRT map corresponding to the pencil of biquadratic curves through eight points

$$s_i = (a_i, b_i) = \left(\kappa c_i, \kappa c_i^{-1} \right), \quad i = 1, \dots, 4, \tag{64}$$

$$s_i = (a_i, b_i) = \left(c_i, c_i^{-1} \right), \quad i = 5, \dots, 8, \tag{65}$$

with $\kappa \neq 0, 1$. These eight points support a pencil of biquadratic curves if they satisfy the condition

$$\frac{\prod_{i=1}^{4} c_i}{\prod_{i=5}^{8} c_i} = 1 \quad \Leftrightarrow \quad \frac{\prod_{i=1}^{4} a_i}{\prod_{i=5}^{8} a_i} = \kappa^4 \quad \Leftrightarrow \quad \frac{\prod_{i=1}^{4} b_i}{\prod_{i=5}^{8} b_i} = \kappa^4. \tag{66}$$

They are symmetric with respect to $\sigma(x, y) = (y, x)$ if $c_{2i} = c_{2i-1}^{-1}$, $i = 1, \dots, 4$. See Fig. 3 (a).

This pencil contains a reducible curve consisting of two $(1,1)$-curves:

$$C_\infty = \left\{ (xy - 1)(xy - \kappa^2) = 0 \right\}. \tag{67}$$

The vertical involution i_1 can be described by the following equation:

$$i_1(x, y) = (x, \widetilde{y}), \quad \frac{(x\widetilde{y} - \kappa^2)(xy - \kappa^2)}{(x\widetilde{y} - 1)(xy - 1)} = \frac{\prod_{i=1}^{4}(x - \kappa c_i)}{\prod_{i=5}^{8}(x - c_i)}. \tag{68}$$

Similarly, the horizontal involution i_2 can be described by the following equation:

$$i_2(x, y) = (\widetilde{x}, y), \quad \frac{(\widetilde{x}y - \kappa^2)(xy - \kappa^2)}{(\widetilde{x}y - 1)(xy - 1)} = \frac{\prod_{i=1}^{4}(y - \kappa c_i^{-1})}{\prod_{i=5}^{8}(y - c_i^{-1})}. \tag{69}$$
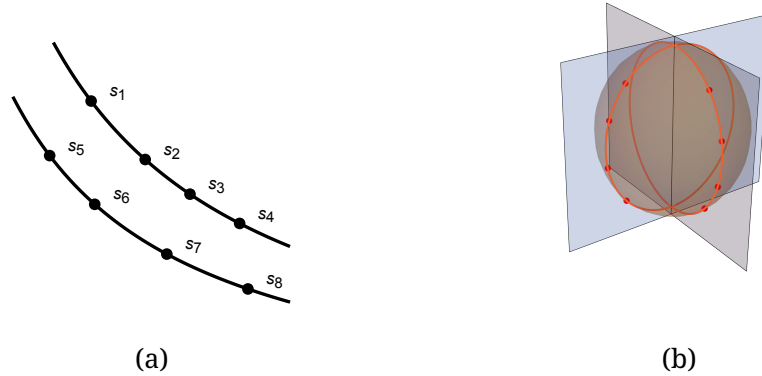
Figure 3: (a) Base set of the surface type $A_1^{(1)}$: two quadruples of points on two (1,1)-curves (hyperbolas) in $\mathbb{P}^1 \times \mathbb{P}^1$ intersecting at two points $(\infty, 0)$ and $(0, \infty)$. (b) Pencil of quadrics through two non-coplanar conics intersecting at two points

The QRT map F is the composition of these two involutions, $F = i_1 \circ i_2$. The singularity confinement structure of the QRT involutions is as in (11). In the symmetric case we have $F = f^2$, with $f = i_1 \circ \sigma = \sigma \circ i_2$ being the QRT root.

3D Painlevé map.   As usual, we identify $\mathbb{P}^1 \times \mathbb{P}^1$ with the quadric

$$Q_0 = \{X_1 X_2 - X_3 X_4 = 0\} \subset \mathbb{P}^3,$$

via $[X_1 : X_2 : X_3 : X_4] = [x : y : xy : 1]$. The points s_i are lifted to

$$S_i = [a_i : b_i : a_i b_i : 1] = \begin{cases} \left[\kappa c_i : \kappa c_i^{-1} : \kappa^2 : 1 \right], \ i = 1, \dots, 4, \\[2mm] \left[c_i : c_i^{-1} : 1 : 1 \right], \ i = 5, \dots, 8. \end{cases} \tag{70}$$

We declare Q_λ to be spanned by Q_0 and $Q_\infty = P_\infty = (X_3 - \kappa^2 X_4)(X_3 - X_4)$:

$$Q_\lambda = \left\{ X_1 X_2 - X_3 X_4 - \lambda (X_3 - \kappa^2 X_4)(X_3 - X_4) = 0 \right\}. \tag{71}$$

The base set of the pencil Q_λ consists of two conics, $\{X_1 X_2 - X_3 X_4 = 0, \ X_3 - \kappa^2 X_4 = 0\}$ and $\{X_1 X_2 - X_3 X_4 = 0, \ X_3 - X_4 = 0\}$, which intersect at two points $[0 : 1 : 0 : 0]$ and $[1 : 0 : 0 : 0]$. This is a pencil of type (iv).

The matrix M_λ of the quadratic form Q_λ:

$$M_\lambda = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2\lambda & -1 + (1 + \kappa^2)\lambda \\ 0 & 0 & -1 + (1 + \kappa^2)\lambda & -2\kappa^2\lambda \end{pmatrix} \tag{72}$$

The characteristic polynomial of the pencil $\{Q_\lambda\}$ is:

$$\Delta(\lambda) = \det(M_\lambda) = \left(1 - (1 + \kappa^2)\lambda\right)^2 - 4\kappa^2\lambda^2 = \left(1 - (1 + \kappa)^2\lambda\right)\left(1 - (1 - \kappa)^2\lambda\right),$$

so that $\text{Sing}(Q_\lambda) = \{(1+\kappa)^{-2}, (1-\kappa)^{-2}, \infty\}$. This polynomial is not a complete square, and we have to uniformize $\sqrt{\Delta(\lambda)}$. The uniformizing variable is $\nu \in \mathbb{C}$. However, in the present situation it will be convenient to use $w = e^\nu$ instead, with $w \in \mathbb{C} \setminus \{0\}$. The shift $\nu \longmapsto \nu + \delta$ will be replaced by $w \longmapsto qw$ with $q = e^\delta$. We set

$$\lambda = \lambda(w) = -\frac{(\kappa - w)(1 - \kappa w)}{(1 - \kappa^2)^2 w}. \tag{73}$$

Then $\Delta(\lambda)$ becomes a square:

$$\Delta(\lambda) = \frac{\kappa^2(1 - w^2)^2}{w^2(1 - \kappa^2)^2} \quad \Rightarrow \quad \sqrt{\Delta(\lambda)} = \frac{\kappa(1 - w^2)}{w(1 - \kappa^2)}.$$

Observe that $\lambda(w) = \lambda(w^{-1})$, while $\sqrt{\Delta(\lambda)}$ changes its sign under $w \longmapsto w^{-1}$. This gives us a double cover of the original pencil branched at $\lambda = (1+\kappa)^{-2}$ (corresponding to $w = 1$), and at $\lambda = (1 - \kappa)^{-2}$ (corresponding to $w = -1$). The point $\lambda = \infty$ (corresponding to $w = 0, \infty$) is not a branch point. The normalizing transformation of $Q_\lambda(X)$ to the canonical form $Q_0(Y) = Y_1 Y_2 - Y_3 Y_4$ is achieved by the transformation

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = A_w \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}, \tag{74}$$

where one can take

$$
A_w = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \dfrac{1 - \kappa w}{1 - w^2} & \dfrac{w(\kappa - w)}{1 - w^2} \\
0 & 0 & \dfrac{\kappa - w}{\kappa(1 - w^2)} & \dfrac{w(1 - \kappa w)}{\kappa(1 - w^2)}
\end{pmatrix}.
\tag{75}
$$

Indeed, one immediately verifies that

$$
A_w^{\mathrm{T}} M_{\lambda(w)} A_w = M_0.
$$

Now, we are in the position to derive a parametrization of the quadric Q_λ:

$$
\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = A_w \begin{bmatrix} x \\ y \\ xy \\ 1 \end{bmatrix} =: \phi_w(x, y).
\tag{76}
$$

Observe that this parametrization is neither valid for $w = 0$ nor for $w = \infty$. The pencil-adapted coordinates (x, y, w) on (the double cover of) $\mathbb{P}^3$ are:

$$
x = \frac{(1 - \kappa w)X_3 - \kappa(\kappa - w)X_4}{(1 - \kappa^2)X_2} = \frac{w(1 - \kappa^2)X_1}{\kappa(1 - \kappa w)X_4 - (\kappa - w)X_3},
\tag{77}
$$

$$
y = \frac{(1 - \kappa w)X_3 - \kappa(\kappa - w)X_4}{(1 - \kappa^2)X_1} = \frac{w(1 - \kappa^2)X_2}{\kappa(1 - \kappa w)X_4 - (\kappa - w)X_4},
\tag{78}
$$

which have to be supplemented with

$$
\lambda = -\frac{(\kappa - w)(1 - \kappa w)}{(\kappa^2 - 1)^2 w} = \frac{X_1 X_2 - X_3 X_4}{(X_3 - \kappa^2 X_4)(X_3 - X_4)}.
\tag{79}
$$

Theorem 5. *For any $q \neq \pm 1$, define the Painlevé deformation map corresponding to the translation $w \longmapsto \widehat{w} = q^2 w$ by*

$$
L : \begin{cases}
\widehat{X}_1 = X_1 X_2 + \big(\lambda(\widehat{w}) - \lambda(w)\big)(X_3 - X_4)(X_3 - \kappa^2 X_4), \\
\widehat{X}_2 = X_2^2, \\
\widehat{X}_3 = X_2 X_3, \\
\widehat{X}_4 = X_2 X_4,
\end{cases}
\tag{80}
$$

where $\lambda = \lambda(w)$ is given by (73). *Then, in pencil-adapted coordinates, the map L acts as follows:*

$$L: \quad \hat{x} = \frac{q^2 w^2 - 1}{w^2 - 1} x - \frac{(q^2 - 1)w^2}{w^2 - 1} y^{-1}, \quad \hat{y}^{-1} = \frac{q^2 w^2 - 1}{q^2(w^2 - 1)} y^{-1} - \frac{(q^2 - 1)}{q^2(w^2 - 1)} x, \quad \hat{w} = q^2 w. \tag{81}$$

For the latter map, the factorizations (4), (5) *are given by*

$$L_1 = R_1 : (x, y, w) \longmapsto (x, \tilde{y}, qw), \quad L_2 = R_2 : (x, y, w) \longmapsto (\tilde{x}, y, qw)$$

where

$$\tilde{y}^{-1} = \frac{q^2 w^2 - 1}{q^2(w^2 - 1)} y^{-1} - \frac{(q^2 - 1)}{q^2(w^2 - 1)} x \quad \Leftrightarrow \quad \frac{\tilde{y}x - q^2 w^2}{\tilde{y}x - 1} = q^2 \frac{yx - w^2}{yx - 1}, \tag{82}$$

and

$$\tilde{x} = \frac{q^2 w^2 - 1}{w^2 - 1} x - \frac{(q^2 - 1)w^2}{w^2 - 1} y^{-1} \quad \Leftrightarrow \quad \frac{\tilde{x}y - q^2 w^2}{\tilde{x}y - 1} = \frac{xy - w^2}{xy - 1}. \tag{83}$$

Relation to the q-Painlevé equation of the surface type $A_1^{(1)}$. We now compute the 3D Painlevé map $\widetilde{F} = R_1 \circ i_1 \circ L_1 \circ R_2 \circ i_2 \circ L_2$ in the pencil-adapted coordinates (x, y, w). For each fixed w, the intersection curves $Q_{\lambda(w)} \cap P_\mu$ form a pencil through eight points

$$s_i(w) = \left(wc_i, wc_i^{-1} \right), \quad i = 1, \dots, 4, \tag{84}$$

$$s_i(w) = \left(c_i, c_i^{-1} \right), \quad i = 5, \dots, 8, \tag{85}$$

which are just the points $S_1, \dots, S_8$ expressed in the pencil-adapted coordinates on $Q_{\lambda(w)}$. The formulas for the 3D QRT involutions i_1, i_2 restricted to $Q_{\lambda(w)}$ coincide, in the pencil-adapted coordinates, with the original QRT involutions (68) and (69), upon replacing κ by w, and s_i by $s_i(w)$:

$$i_1(x, y) = (x, \tilde{y}), \quad \frac{(x\tilde{y} - w^2)(xy - w^2)}{(x\tilde{y} - 1)(xy - 1)} = \frac{\prod_{i=1}^{4}(x - wc_i)}{\prod_{i=5}^{8}(x - c_i)}, \tag{86}$$

$$i_2(x, y) = (\tilde{x}, y), \quad \frac{(\tilde{x}y - w^2)(xy - w^2)}{(\tilde{x}y - 1)(xy - 1)} = \frac{\prod_{i=1}^{4}(y - wc_i^{-1})}{\prod_{i=5}^{8}(y - c_i^{-1})}. \tag{87}$$

In the notation of the equations (7), (8), the latter two equations read:

$$\frac{(\widetilde{x}y_n - w_{2n}^2)(xy_n - w_{2n}^2)}{(\widetilde{x}y_n - 1)(xy_n - 1)} = \frac{\prod_{i=1}^4 (y_n - w_{2n}c_i^{-1})}{\prod_{i=5}^8 (y_n - c_i^{-1})}, \tag{88}$$

$$\frac{(\widetilde{y}x_{n+1} - w_{2n+1}^2)(yx_{n+1} - w_{2n+1}^2)}{(\widetilde{y}x_{n+1} - 1)(yx_{n+1} - 1)} = \frac{\prod_{i=1}^4 (x_{n+1} - w_{2n+1}c_i)}{\prod_{i=5}^8 (x_{n+1} - c_i)}, \tag{89}$$

where

$$w_{2n-1/2} = q^{-1}w_{2n}, \quad w_{2n+1/2} = qw_{2n}. \tag{90}$$

According to (7), we have:

$$L_2 : (x_n, y_n, w_{2n-1/2}) \longmapsto (x, y_n, w_{2n}) \quad \text{and} \quad R_2 : (\widetilde{x}, y_n, w_{2n}) \longmapsto (x_{n+1}, y_n, w_{2n+1/2}).$$

With expressions (83) for the maps L_2, R_2, we find:

$$\frac{xy_n - w_{2n}^2}{xy_n - 1} = \frac{x_n y_n - w_{2n}w_{2n-1}}{x_n y_n - 1}, \tag{91}$$

$$\frac{\widetilde{x}y_n - w_{2n}^2}{\widetilde{x}y_n - 1} = \frac{x_{n+1}y_n - w_{2n+1}w_{2n}}{x_{n+1}y_n - 1}. \tag{92}$$

Similarly, according to (8), we have:

$$L_1 : (x_{n+1}, y_n, w_{2n+1/2}) \longmapsto (x_{n+1}, y, w_{2n+1}) \quad \text{and} \quad R_1 : (x_{n+1}, \widetilde{y}, w_{2n+1}) \longmapsto (x_{n+1}, y_{n+1}, w_{2n+3/2}),$$

and with expressions (82) for the maps L_1, R_1, we find:

$$q^{-2}\frac{x_{n+1}y - w_{2n+1}^2}{x_{n+1}y - 1} = \frac{x_{n+1}y_n - w_{2n+1}w_{2n}}{x_{n+1}y_n - 1}, \tag{93}$$

$$q^2 \frac{\widetilde{y}x_{n+1} - w_{2n+1}^2}{\widetilde{y}x_{n+1} - 1} = \frac{x_{n+1}y_{n+1} - w_{2n+2}w_{2n+1}}{x_{n+1}y_{n+1} - 1}. \tag{94}$$

Combining equations (88), (89) with (91)–(94) results in the following non-autonomous system:

$$\frac{(x_{n+1}y_n - w_{2n+1}w_{2n})(x_n y_n - w_{2n}w_{2n-1})}{(x_{n+1}y_n - 1)(x_n y_n - 1)} = \frac{\prod_{i=1}^4 (y_n - w_{2n}c_i^{-1})}{\prod_{i=5}^8 (y_n - c_i^{-1})}, \tag{95}$$

$$\frac{(y_{n+1}x_{n+1} - w_{2n+2}w_{2n+1})(y_n x_{n+1} - w_{2n+1}w_{2n})}{(y_{n+1}x_{n+1} - 1)(y_n x_{n+1} - 1)} = \frac{\prod_{i=1}^{4}(x_{n+1} - w_{2n+1}c_i)}{\prod_{i=5}^{8}(x_{n+1} - c_i)}. \tag{96}$$

This is the q-Painlevé equation of the surface type $A_1^{(1)}$, as given in [11]. In the symmetric case, if $c_{2i} = c_{2i-1}^{-1}$, $i = 1, \dots, 4$, these equations become two instances of

$$\frac{(u_{n+1}u_n - w_{n+1}w_n)(u_n u_{n-1} - w_n w_{n-1})}{(u_{n+1}u_n - 1)(u_n u_{n-1} - 1)} = \frac{\prod_{i=1}^{4}(u_n - w_n c_i)}{\prod_{i=5}^{8}(u_n - c_i)}. \tag{97}$$

8 From a pencil of type (iii) to the d-Painlevé equation of the surface type $A_0^{(1)}$

2D QRT map. We consider the QRT map corresponding to the pencil of biquadratic curves through eight points $s_i = (a_i, b_i)$, $i = 1, \dots, 8$, where

$$a_i = z_i(z_i - \kappa_1), \quad b_i = z_i(z_i - \kappa_2).$$

These eight points support a pencil of biquadratic curves if they satisfy the condition

$$\sum_{i=1}^{8} z_i = 2(\kappa_1 + \kappa_2).$$

They belong to the curve with the equation

$$(x - y)^2 = (\kappa_2 - \kappa_1)(\kappa_2 x - \kappa_1 y).$$

This is a biquadratic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ with a cusp point at (∞, ∞), see Fig. 4 (a).

The vertical involution i_1 can be described by the following equation:

$$i_1(x, y) = (x, \widetilde{y}),$$

$$\frac{(\widetilde{y} - \xi(\xi - \kappa_2))(y - \xi(\xi - \kappa_2))}{(\widetilde{y} - (\xi - \kappa_1)(\xi - \kappa_1 + \kappa_2))(y - (\xi - \kappa_1)(\xi - \kappa_1 + \kappa_2))} = \frac{U(\xi)}{U(\kappa_1 - \xi)}, \quad x = \xi(\xi - \kappa_1). \tag{98}$$

Here we use the abbreviation

$$U(z) = \prod_{i=1}^{8}(z - z_i). \tag{99}$$
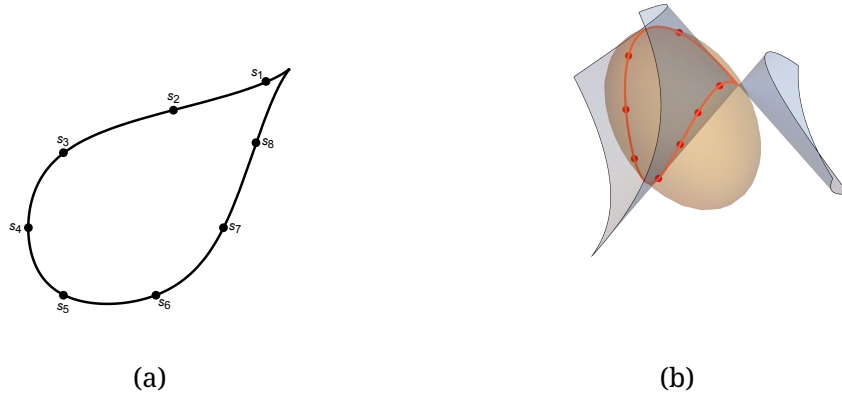
(a) (b)

Figure 4: (a) Base set of the surface type $A_0^{(1)}$: eight points on a cuspidal (2,2)-curve in $\mathbb{P}^1 \times \mathbb{P}^1$. (b) Pencil of quadrics through cuspidal spatial quartic in $\mathbb{P}^3$

Formula (98) is understood as follows. Written as a polynomial in ξ, it is anti-symmetric with respect to $\xi \leftrightarrow \kappa_1 - \xi$. Upon division by $\xi - 2\kappa_1$, the resulting polynomial is symmetric and therefore it can be actually expressed as a polynomial in $x = \xi(\xi - \kappa_1)$. This defines i_1 as a birational involution (its symmetry w.r.t. $y \leftrightarrow \widetilde{y}$ is obvious).

Similarly, the horizontal involution i_2 can be described by the following equation:

$$i_2(x, y) = (\widetilde{x}, y),$$

$$\frac{\left(\widetilde{x} - \eta(\eta - \kappa_1)\right)\left(x - \eta(\eta - \kappa_1)\right)}{\left(\widetilde{x} - (\eta - \kappa_2)(\eta - \kappa_2 + \kappa_1)\right)\left(x - (\eta - \kappa_2)(\eta - \kappa_2 + \kappa_1)\right)} = \frac{U(\eta)}{U(\kappa_2 - \eta)}, \quad y = \eta(\eta - \kappa_2). \quad (100)$$

The QRT map F is the composition of these two involutions, $F = i_1 \circ i_2$. The eight points $s_1, \dots, s_8$ in $\mathbb{P}^1 \times \mathbb{P}^1$ serve as the indeterminacy set for i_1 and for i_2. The singularity confinement structure is as in (11).

Remark. In what follows, we restrict ourselves to the case $\kappa_1 + \kappa_2 = 0$. This restriction is not essential, but will allow us to shorten some of the formulas. Thus, from now on we set

$$\kappa_1 = -\kappa, \quad \kappa_2 = \kappa. \quad (101)$$

If, additionally, the points z_i satisfy the condition

$$z_{i+4} = -z_i, \quad i = 1, \dots, 4, \tag{102}$$

then the QRT involutions admits a symmetry $i_1 = \sigma \circ i_2 \circ \sigma$, where $\sigma(x, y) = (y, x)$, so that one can introduce the QRT root $f = i_1 \circ \sigma = \sigma \circ i_2$, such that $F = f \circ f$.

3D Painlevé map. As usual, we identify $\mathbb{P}^1 \times \mathbb{P}^1$ with the quadric $Q_0 = \{X_1 X_2 - X_3 X_4 = 0\} \subset \mathbb{P}^3$ via $[X_1 : X_2 : X_3 : X_4] = [x : y : xy : 1]$. The points s_i are lifted to

$$S_i = [a_i : b_i : a_i b_i : 1].$$

We declare Q_λ to be spanned by Q_0 and $Q_\infty = P_\infty = (X_1 - X_2)^2 - 2\kappa^2(X_1 + X_2)X_4$:

$$Q_\lambda = \left\{ X_1 X_2 - X_3 X_4 - \lambda\left((X_1 - X_2)^2 - 2\kappa^2(X_1 + X_2)X_4\right) = 0 \right\}. \tag{103}$$

The base set of the pencil Q_λ is a cuspidal space curve of degree 4, $\{X_1 X_2 - X_3 X_4 = 0, (X_1 - X_2)^2 - 2\kappa^2(X_1 + X_2)X_4 = 0\}$, with the cusp at $[0 : 0 : 1 : 0]$. This is a pencil of type (iii).

The matrix M_λ of the quadratic form Q_λ:

$$M_\lambda = \begin{pmatrix} -2\lambda & 1 + 2\lambda & 0 & 2\lambda\kappa^2 \\ 1 + 2\lambda & -2\lambda & 0 & 2\lambda\kappa^2 \\ 0 & 0 & 0 & -1 \\ 2\lambda\kappa^2 & 2\lambda\kappa^2 & -1 & 0 \end{pmatrix} \tag{104}$$

The characteristic polynomial of the pencil $\{Q_\lambda\}$ is: $\Delta(\lambda) = \det(M_\lambda) = 1 + 4\lambda$, so that $\mathrm{Sing}(Q_\lambda) = \{-\frac{1}{4}, \infty\}$. We set

$$\lambda = \frac{\nu^2 - 1}{4}, \quad \sqrt{\Delta(\lambda)} = \nu. \tag{105}$$

Thus, $\lambda(\nu) = \lambda(-\nu)$, while $\sqrt{\Delta(\lambda)}$ changes its sign as $\nu \longmapsto -\nu$. This gives us a double cover of the original pencil branched at $\nu = 0$, corresponding to $\lambda = -1/4$, and at $\nu = \infty$, corresponding to $\lambda = \infty$. The normalizing transformation of Q_λ to the canonical form Q_0

can be found as follows:

$$
\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = A_\nu \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}, \tag{106}
$$

where

$$
A_\nu = \begin{pmatrix} \frac{1}{2\nu}(\nu+1) & \frac{1}{2\nu}(\nu-1) & 0 & 0 \\ \frac{1}{2\nu}(\nu-1) & \frac{1}{2\nu}(\nu+1) & 0 & 0 \\ \frac{\kappa^2}{2}(\nu^2-1) & \frac{\kappa^2}{2}(\nu^2-1) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{107}
$$

Indeed, one immediately verifies that

$$
A_\nu^{\mathrm{T}} M_{\lambda(\nu)} A_\nu = M_0.
$$

Now, we are in the position to derive a parametrization of the quadric $Q_{\lambda(\nu)}$:

$$
\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = A_\nu \begin{bmatrix} x \\ y \\ xy \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\nu}((\nu+1)x+(\nu-1)y) \\ \frac{1}{2\nu}((\nu-1)x+(\nu+1)y) \\ xy+\frac{\kappa^2}{2}(\nu^2-1)(x+y) \\ 1 \end{bmatrix} =: \phi_\nu(x,y). \tag{108}
$$

Observe that this parametrization is neither valid for $\nu = 0$ nor for $\nu = \infty$. The pencil-adapted coordinates (x, y, ν) on (the double cover of) $\mathbb{P}^3$ are:

$$
x = \frac{(\nu+1)X_1 - (\nu-1)X_2}{2X_4}, \quad y = \frac{(\nu+1)X_2 - (\nu-1)X_1}{2X_4}, \tag{109}
$$

which have to be supplemented with

$$
\lambda = \frac{\nu^2-1}{4} = \frac{X_1 X_2 - X_3 X_4}{(X_1-X_2)^2 - 2\kappa^2(X_1+X_2)X_4}. \tag{110}
$$

The degenerate quadrics for $\nu = \infty$ and for $\nu = 0$ are cones.

Theorem 6. *For any $\delta \in \mathbb{C} \setminus \{0\}$, define the Painlevé deformation map corresponding to the translation $\nu \longmapsto \widehat{\nu} = \nu + 2\delta$ by*

$$
L : \begin{cases}
\widehat{X}_1 &= X_1 X_4, \\
\widehat{X}_2 &= X_2 X_4, \\
\widehat{X}_3 &= X_3 X_4 - \big(\lambda(\widehat{\nu}) - \lambda(\nu)\big) Q_\infty(X) \\
&= X_3 X_4 - \beta(\nu + \beta)(X_1 - X_2)^2 + 2\kappa^2 \beta(\nu + \beta)(X_1 + X_2) X_4, \\
\widehat{X}_4 &= X_4^2.
\end{cases}
$$

Then, in pencil-adapted coordinates, the map L acts as follows:

$$
L : (x, y, \nu) \longmapsto (\widehat{x}, \widehat{y}, \widehat{\nu}), \quad \widehat{x} = x + \frac{\delta(x - y)}{\nu}, \quad \widehat{y} = y + \frac{\delta(y - x)}{\nu}, \quad \widehat{\nu} = \nu + 2\delta. \tag{111}
$$

For the latter map, the factorizations (4), (5) *are given by*

$$
L_1 = R_1 : (x, y, \nu) \longmapsto (x, \widetilde{y}, \nu + \delta), \quad \widetilde{y} = y + \frac{\delta}{\nu}(y - x), \tag{112}
$$

$$
L_2 = R_2 : (x, y, \nu) \longmapsto (\widetilde{x}, y, \nu + \delta), \quad \widetilde{x} = x + \frac{\delta}{\nu}(x - y). \tag{113}
$$

Relation to the d-Painlevé equation of the surface type $A_0^{(1)}$. In the pencil-adapted coordinates (x, y, ν), for each fixed ν, the intersection curves $Q_{\lambda(\nu)} \cup P_\mu$ form the pencil through the points

$$
s_i(\nu) = (a_i(\nu), b_i(\nu)) = \big(z_i(z_i + \kappa \nu), z_i(z_i - \kappa \nu)\big), \quad i = 1, \dots, 8, \tag{114}
$$

which are just the points S_i expressed in the pencil-adapted coordinates on $Q_{\lambda(\nu)}$. Thus, the 3D QRT involutions i_1, i_2 act on each quadric $Q_{\lambda(\nu)}$ in the pencil-adapted coordinates via formulas which are obtained from the corresponding 2D formulas by replacing κ by $\kappa\nu$:

$$
i_1(x, y) = (x, \widetilde{y}),
$$

$$
\frac{\big(\widetilde{y} - \xi(\xi - \kappa\nu)\big)\big(y - \xi(\xi - \kappa\nu)\big)}{\big(\widetilde{y} - (\xi + \kappa\nu)(\xi + 2\kappa\nu)\big)\big(y - (\xi + \kappa\nu)(\xi + 2\kappa\nu)\big)} = \frac{U(\xi)}{U(-\kappa\nu - \xi)}, \quad x = \xi(\xi + \kappa\nu), \tag{115}
$$

$$i_2(x, y) = (\widetilde{x}, y),$$

$$\frac{(\widetilde{x} - \eta(\eta + \kappa\nu))(x - \eta(\eta + \kappa\nu))}{(\widetilde{x} - (\eta - \kappa\nu)(\eta - 2\kappa\nu))(x - (\eta - \kappa\nu)(\eta - 2\kappa\nu))} = \frac{U(\eta)}{U(\kappa\nu - \eta)}, \quad y = \eta(\eta - \kappa\nu). \tag{116}$$

In notations of (7), (8), the latter two equations take the following form:

$$\frac{(\widetilde{x} - \eta(\eta + \kappa\nu_{2n}))(x - \eta(\eta + \kappa\nu_{2n}))}{(\widetilde{x} - (\eta - \kappa\nu_{2n})(\eta - 2\kappa\nu_{2n}))(x - (\eta - \kappa\nu_{2n})(\eta - 2\kappa\nu_{2n}))} = \frac{U(\eta)}{U(\kappa\nu_{2n} - \eta)},$$

$$y_n = \eta(\eta - \kappa\nu_{2n}), \tag{117}$$

$$\frac{(\widetilde{y} - \xi(\xi - \kappa\nu_{2n+1}))(y - \xi(\xi - \kappa\nu_{2n+1}))}{(\widetilde{y} - (\xi + \kappa\nu_{2n+1})(\xi + 2\kappa\nu_{2n+1}))(y - (\xi + \kappa\nu_{2n+1})(\xi + 2\kappa\nu_{2n+1}))} = \frac{U(\xi)}{U(-\kappa\nu_{2n+1} - \xi)},$$

$$x_{n+1} = \xi(\xi + \kappa\nu_{2n+1}). \tag{118}$$

Recall that here

$$\nu_{2n+1} = \nu_{2n+1/2} + \delta = \nu_{2n} + 2\delta.$$

To express in (117) the variables x, $\widetilde{x}$ through x_n, y_n, we observe that

$$L_2 : (x_n, y_n, \nu_{2n-1/2}) \longmapsto (x, y_n, \nu_{2n}), \quad R_2 : (\widetilde{x}, y_n, \nu_{2n}) \longmapsto (x_{n+1}, y_n, \nu_{2n+1/2})$$

can be written, according to (113), as follows:

$$x = x_n + \frac{\delta}{\nu_{2n-1/2}}(x_n - y_n), \quad \text{resp.} \quad x_{n+1} = \widetilde{x} + \frac{\delta}{\nu_{2n}}(\widetilde{x} - y_n).$$

A simple computation confirms that these relations are equivalent to

$$\frac{x - \eta(\eta + \kappa\nu_{2n})}{x - (\eta - \kappa\nu_{2n})(\eta - 2\kappa\nu_{2n})} = \frac{x_n - \eta(\eta + \kappa\nu_{2n-1})}{x_n - (\eta - \kappa\nu_{2n})(\eta - \kappa\nu_{2n} - \kappa\nu_{2n-1})}, \quad y_n = \eta(\eta - \kappa\nu_{2n}), \tag{119}$$

$$\frac{\widetilde{x} - \eta(\eta + \kappa\nu_{2n})}{\widetilde{x} - (\eta - \kappa\nu_{2n})(\eta - 2\kappa\nu_{2n})} = \frac{x_{n+1} - \eta(\eta + \kappa\nu_{2n+1})}{x_{n+1} - (\eta - \kappa\nu_{2n})(\eta - \kappa\nu_{2n+1} - \kappa\nu_{2n})}, \quad y_n = \eta(\eta - \kappa\nu_{2n}). \tag{120}$$

Similarly, to express in (118) the variables y, $\widetilde{y}$ through x_{n+1}, y_n, we observe that

$$L_1 : (x_{n+1}, y_n, \nu_{2n+1/2}) \longmapsto (x_{n+1}, y, \nu_{2n+1}), \quad R_1 : (x_{n+1}, \widetilde{y}, \nu_{2n+1}) \longmapsto (x_{n+1}, y_{n+1}, \nu_{2n+3/2}),$$

which, according to (112), can be put as follows:

$$y = y_n + \frac{\delta}{\nu_{2n+1/2}}(y_n - x_{n+1}), \quad y_{n+1} = \widetilde{y} + \frac{\delta}{\nu_{2n+1}}(\widetilde{y} - x_{n+1}).$$

Again, these relations are equivalent to

$$\frac{y - \xi(\xi - \kappa\nu_{2n+1})}{y - (\xi + \kappa\nu_{2n+1})(\xi + 2\kappa\nu_{2n+1})} = \frac{y_n - \xi(\xi - \kappa\nu_{2n})}{y_n - (\xi + \kappa\nu_{2n+1})(\xi + \kappa\nu_{2n+1} + \kappa\nu_{2n})},$$
$$x_{n+1} = \xi(\xi + \kappa\nu_{2n+1}) \qquad (121)$$

$$\frac{\widetilde{y} - \xi(\xi - \kappa\nu_{2n+1})}{\widetilde{y} - (\xi + \kappa\nu_{2n+1})(\xi + 2\kappa\nu_{2n+1})} = \frac{y_{n+1} - \xi(\xi - \kappa\nu_{2n+2})}{y_{n+1} - (\xi + \kappa\nu_{2n+1})(\xi + \kappa\nu_{2n+2} + \kappa\nu_{2n+1})},$$
$$x_{n+1} = \xi(\xi + \kappa\nu_{2n+1}). \qquad (122)$$

Substituting (119)–(122) into (117), (118), we arrive at the following system of non-autonomous difference equations for the variables x_n, y_n :

$$\frac{\big(x_{n+1} - \eta(\eta + \kappa\nu_{2n+1})\big)\big(x_n - \eta(\eta + \kappa\nu_{2n-1})\big)}{\big(x_{n+1} - (\eta - \kappa\nu_{2n})(\eta - \kappa\nu_{2n+1} - \kappa\nu_{2n})\big)\big(x_n - (\eta - \kappa\nu_{2n})(\eta - \kappa\nu_{2n} - \kappa\nu_{2n-1})\big)}$$
$$= \frac{U(\eta)}{U(\kappa\nu_{2n} - \eta)}, \quad y_n = \eta(\eta - \kappa\nu_{2n}), \qquad (123)$$

$$\frac{\big(y_{n+1} - \xi(\xi - \kappa\nu_{2n+2})\big)\big(y_n - \xi(\xi - \kappa\nu_{2n})\big)}{\big(y_{n+1} - (\xi + \kappa\nu_{2n+1})(\xi + \kappa\nu_{2n+2} + \kappa\nu_{2n+1})\big)\big(y_n - (\xi + \kappa\nu_{2n+1})(\xi + \kappa\nu_{2n+1} + \kappa\nu_{2n})\big)}$$
$$= \frac{U(\xi)}{U(-\kappa\nu_{2n+1} - \xi)}, \quad x_{n+1} = \xi(\xi + \kappa\nu_{2n+1}). \qquad (124)$$

This is the d-Painlevé equation of the surface type $A_0^{(1)}$, as given in [20], [11].

Remark. In the symmetric situation, when $U(z) = U(-z)$, the system (123), (124) can be interpreted as a one-field second order difference equation, with $x_n = u_{2n-1}$ and $y_n = u_{2n}$. To see this, one should make the change $\xi \longmapsto -\xi$ in equation (124), after which it matches (123).

9 From a pencil of type (ii) to the q-Painlevé equation of the surface type $A_0^{(1)}$

2D QRT map. We consider the QRT map corresponding to the pencil of biquadratic curves through eight points $s_i = (a_i, b_i)$, where

$$a_i = z_i + \frac{\kappa_1}{z_i}, \quad b_i = \frac{1}{z_i} + \frac{z_i}{\kappa_2}, \quad i = 1, \dots, 8.$$

These eight points support a pencil of biquadratic curves if they satisfy the condition

$$\prod_{i=1}^{8} z_i = \kappa_1^2 \kappa_2^2.$$

They belong to the curve with the equation

$$(x - \kappa_2 y)(y - \kappa_1^{-1} x) = (\kappa_1 \kappa_2)^{-1}(\kappa_1 - \kappa_2)^2.$$

This is a biquadratic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ with a simple node at (∞, ∞), see Fig. 5 (a).

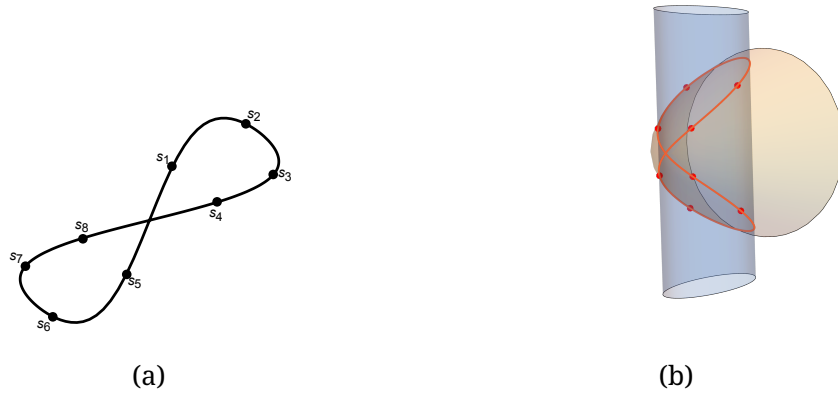(a)   (b)

Figure 5: (a) Base set of the surface type $A_0^{(1)}$: eight points on a nodal (2,2)-curve in $\mathbb{P}^1 \times \mathbb{P}^1$. (b) Pencil of quadrics through a nodal spatial quartic in $\mathbb{P}^3$

The vertical involution i_1 can be described by the following equation:

$$i_1(x,y) = (x, \widetilde{y}), \quad \frac{\left(\widetilde{y} - \dfrac{1}{\xi} - \dfrac{\xi}{\kappa_2}\right)\left(y - \dfrac{1}{\xi} - \dfrac{\xi}{\kappa_2}\right)}{\left(\widetilde{y} - \dfrac{\xi}{\kappa_1} - \dfrac{\kappa_1}{\kappa_2 \xi}\right)\left(y - \dfrac{\xi}{\kappa_1} - \dfrac{\kappa_1}{\kappa_2 \xi}\right)} = \frac{U(\xi)}{U\left(\dfrac{\kappa_1}{\xi}\right)}, \quad x = \xi + \frac{\kappa_1}{\xi}. \tag{125}$$

Here we use the abbreviation

$$U(z) = z^{-4} \prod_{i=1}^{8} (z - z_i). \tag{126}$$

Formula (125) is understood as follows. Written as a Laurent polynomial in ξ, it is anti-symmetric with respect to $\xi \leftrightarrow \kappa_1/\xi$. Upon division by $\xi - \kappa_1/\xi$, the resulting Laurent

polynomial is symmetric and therefore it can be actually expressed as a polynomial in $x = \xi + \kappa_1/\xi$. This defines i_1 as a birational involution (its symmetry w.r.t. $y \leftrightarrow \widetilde{y}$ is obvious).

Similarly, the horizontal involution i_2 can be described by the following equation:

$$i_2(x, y) = (\widetilde{x}, y), \quad \frac{\left(\widetilde{x} - \eta - \frac{\kappa_1}{\eta}\right)\left(x - \eta - \frac{\kappa_1}{\eta}\right)}{\left(\widetilde{x} - \frac{\kappa_2}{\eta} - \frac{\kappa_1\eta}{\kappa_2}\right)\left(x - \frac{\kappa_2}{\eta} - \frac{\kappa_1\eta}{\kappa_2}\right)} = \frac{U(\eta)}{U\left(\frac{\kappa_2}{\eta}\right)}, \quad y = \frac{1}{\eta} + \frac{\eta}{\kappa_2}. \tag{127}$$

The eight points $s_1, \dots, s_8$ in $\mathbb{P}^1 \times \mathbb{P}^1$ serve as the indeterminacy set for i_1 and for i_2. The singularity confinement structure is as in (11). The QRT map F is the composition of these two involutions, $F = i_1 \circ i_2$.

Remark. In what follows, we restrict ourselves to the case $\kappa_1 \kappa_2 = 1$. This restriction is not essential, but will allow us to shorten some of the formulas. Thus, from now on we set in this section

$$\kappa_1 = \frac{1}{\kappa}, \quad \kappa_2 = \kappa. \tag{128}$$

If, additionally, the points z_i satisfy the condition

$$z_{i+4} = z_i^{-1}, \quad i = 1, \dots, 4, \tag{129}$$

then the QRT involutions admits a symmetry $i_1 = \sigma \circ i_2 \circ \sigma$, where $\sigma(x, y) = (y, x)$, so that one can introduce the QRT root $f = i_1 \circ \sigma = \sigma \circ i_2$, such that $F = f \circ f$.

3D Painlevé map. As usual, we identify $\mathbb{P}^1 \times \mathbb{P}^1$ with the quadric $Q_0 = \{X_1 X_2 - X_3 X_4 = 0\} \subset \mathbb{P}^3$ via $[X_1 : X_2 : X_3 : X_4] = [x : y : xy : 1]$. The points s_i are lifted to

$$S_i = [a_i : b_i : a_i b_i : 1].$$

We declare Q_λ to be spanned by Q_0 and

$$Q_\infty = P_\infty = \kappa(X_1^2 + X_2^2) - (1 + \kappa^2)X_1 X_2 + (\kappa - \kappa^{-1})^2 X_4^2. \tag{130}$$

The base set of the pencil Q_λ is a nodal space curve $\{Q_0 = 0, P_\infty = 0\}$ of degree 4, with the node at $[0 : 0 : 1 : 0]$. This is a pencil of type (ii).

The matrix M_λ of the quadratic form Q_λ:

$$M_\lambda = \begin{pmatrix} -2\kappa\lambda & 1+(1+\kappa^2)\lambda & 0 & 0 \\ 1+(1+\kappa^2)\lambda & -2\kappa\lambda & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -2(\kappa-\kappa^{-1})^2\lambda \end{pmatrix}. \tag{131}$$

The characteristic polynomial of the pencil $\{Q_\lambda\}$ is:

$$\Delta(\lambda) = \det(M_\lambda) = \left(1+(1+\kappa^2)\lambda\right)^2 - 4\kappa^2\lambda^2 = \left(1+(1+\kappa)^2\lambda\right)\left(1+(1-\kappa)^2\lambda\right),$$

so that $\mathrm{Sing}(Q_\lambda) = \{-(1+\kappa)^{-2}, -(1-\kappa)^{-2}, \infty\}$. This polynomial is not a complete square, and we have to uniformize $\sqrt{\Delta(\lambda)}$. The uniformizing variable is $\nu \in \mathbb{C}$. As in Sect. 7, it will be convenient to use $w = e^\nu$ instead, with $w \in \mathbb{C} \setminus \{0\}$. We set

$$\lambda = \lambda(w) = \frac{(\kappa-w)(1-\kappa w)}{(1-\kappa^2)^2 w}. \tag{132}$$

Then $\Delta(\lambda)$ becomes a square:

$$\Delta(\lambda) = \frac{\kappa^2(1-w^2)^2}{w^2(1-\kappa^2)^2} \quad \Rightarrow \quad \sqrt{\Delta(\lambda)} = \frac{\kappa(1-w^2)}{w(1-\kappa^2)}.$$

Observe that $\lambda(w) = \lambda(w^{-1})$, while $\sqrt{\Delta(\lambda)}$ changes its sign under $w \mapsto w^{-1}$. This gives us a double cover of the original pencil branched at $\lambda = -(1+\kappa)^{-2}$ (corresponding to $w = 1$), and at $\lambda = -(1-\kappa)^{-2}$ (corresponding to $w = -1$). The point $\lambda = \infty$ is not a branch point (it corresponds to $w = 0, \infty$). The normalizing transformation of $Q_\lambda(X)$ to the canonical form $Q_0(Y) = Y_1 Y_2 - Y_3 Y_4$ is achieved by the transformation

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = A_w \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}, \tag{133}$$

where one can take

$$
A_w = \begin{pmatrix}
\dfrac{w(1-\kappa w)}{\kappa(1-w^2)} & \dfrac{w(\kappa-w)}{\kappa(1-w^2)} & 0 & 0 \\[2ex]
\dfrac{w(\kappa-w)}{\kappa(1-w^2)} & \dfrac{w(1-\kappa w)}{\kappa(1-w^2)} & 0 & 0 \\[2ex]
0 & 0 & \dfrac{w}{\kappa} & -\dfrac{(1-\kappa w)(\kappa-w)}{\kappa^2 w} \\[2ex]
0 & 0 & 0 & 1
\end{pmatrix}.
\tag{134}
$$

Indeed, one immediately verifies that

$$
A_w^{\mathrm{T}} M_{\lambda(w)} A_w = \frac{w}{\kappa} M_0.
$$

There follows a parametrization of the quadric $Q_{\lambda(w)}$:

$$
\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = A_w \begin{bmatrix} x \\ y \\ xy \\ 1 \end{bmatrix} =: \phi_w(x,y).
\tag{135}
$$

This parametrization is neither valid for $w = 0$ nor for $w = \infty$. The pencil-adapted coordinates (x,y,w) on (the double cover of) $\mathbb{P}^3$ are:

$$
x = \frac{\kappa}{w} \cdot \frac{(1-\kappa w)X_1 - (\kappa-w)X_2}{(1-\kappa^2)X_4}, \quad y = \frac{\kappa}{w} \cdot \frac{(1-\kappa w)X_2 - (\kappa-w)X_1}{(1-\kappa^2)X_4},
\tag{136}
$$

which have to be supplemented with

$$
\lambda = \frac{(\kappa-w)(1-\kappa w)}{(\kappa^2-1)^2 w} = \frac{X_1 X_2 - X_3 X_4}{\kappa X_1^2 + \kappa X_2^2 - (1+\kappa^2)X_1 X_2 + (\kappa - \kappa^{-1})^2 X_4^2}.
\tag{137}
$$

Theorem 7. *For any $q \neq \pm 1$, define the Painlevé deformation map corresponding to the translation $w \mapsto \hat{w} = q^2 w$ by*

$$
L : \begin{cases}
\hat{X}_1 = X_1 X_4, \\
\hat{X}_2 = X_2 X_4, \\
\hat{X}_3 = X_3 X_4 - \big(\lambda(\hat{w}) - \lambda(w)\big) Q_\infty(X), \\
\hat{X}_4 = X_4^2,
\end{cases}
\tag{138}
$$

where $\lambda = \lambda(w)$ is given by (132), *and $Q_\infty(X)$ is given in* (130). *Then, in pencil-adapted coordinates, the map L acts as follows:*

$$L: \quad \widehat{x} = x + \frac{1 - q^{-2}}{w^2 - 1}(x - wy), \quad \widehat{y} = y + \frac{1 - q^{-2}}{w^2 - 1}(y - wx), \quad \widehat{w} = q^2 w. \tag{139}$$

For the latter map, the factorizations (4), (5) *are given by*

$$L_1 : (x, y, w) \longmapsto (x, \widetilde{y}, qw), \quad \widetilde{y} = y + \frac{1 - q^{-2}}{w^2 - 1}(y - qwx), \tag{140}$$

$$R_1 : (x, y, w) \longmapsto (x, \widetilde{y}, qw), \quad \widetilde{y} = y + \frac{1 - q^{-2}}{w^2 - 1}(y - wx), \tag{141}$$

$$L_2 : (x, y, w) \longmapsto (\widetilde{x}, y, qw), \quad \widetilde{x} = x + \frac{1 - q^{-2}}{w^2 - 1}(x - qwy), \tag{142}$$

$$R_2 : (x, y, w) \longmapsto (\widetilde{x}, y, qw), \quad \widetilde{x} = x + \frac{1 - q^{-2}}{w^2 - 1}(x - wy). \tag{143}$$

Relation to the q-Painlevé equation of the surface type $A_0^{(1)}$. In the pencil-adapted coordinates (x, y, w), for each fixed w, the intersection curves $Q_{\lambda(w)} \cup P_\mu$ form the pencil through the points

$$s_i(w) = (a_i(w), b_i(w)) = \left(z_i + \frac{1}{w z_i}, \frac{1}{z_i} + \frac{z_i}{w}\right), \quad i = 1, \ldots, 8, \tag{144}$$

which are just the points S_i expressed in the pencil-adapted coordinates on $Q_{\lambda(w)}$. Thus, the 3D QRT involutions i_1, i_2 act on each quadric $Q_{\lambda(w)}$ in the pencil-adapted coordinates via formulas which are obtained from the corresponding 2D formulas by replacing κ by w:

$$i_1(x, y) = (x, \widetilde{y}), \quad \frac{\left(\widetilde{y} - \frac{1}{\xi} - \frac{\xi}{w}\right)\left(y - \frac{1}{\xi} - \frac{\xi}{w}\right)}{\left(\widetilde{y} - w\xi - \frac{1}{w^2\xi}\right)\left(y - w\xi - \frac{1}{w^2\xi}\right)} = \frac{U(\xi)}{U\left(\frac{1}{w\xi}\right)}, \quad x = \xi + \frac{1}{w\xi}, \tag{145}$$

$$i_2(x, y) = (\widetilde{x}, y), \quad \frac{\left(\widetilde{x} - \eta - \frac{1}{w\eta}\right)\left(x - \eta - \frac{1}{w\eta}\right)}{\left(\widetilde{x} - \frac{w}{\eta} - \frac{\eta}{w^2}\right)\left(x - \frac{w}{\eta} - \frac{\eta}{w^2}\right)} = \frac{U(\eta)}{U\left(\frac{w}{\eta}\right)}, \quad y = \frac{1}{\eta} + \frac{\eta}{w}. \tag{146}$$

In notations of (7), (8), this takes the form

$$\frac{\left(\widetilde{x} - \eta - \dfrac{1}{w_{2n}\eta}\right)\left(x - \eta - \dfrac{1}{w_{2n}\eta}\right)}{\left(\widetilde{x} - \dfrac{w_{2n}}{\eta} - \dfrac{\eta}{w_{2n}^2}\right)\left(x - \dfrac{w_{2n}}{\eta} - \dfrac{\eta}{w_{2n}^2}\right)} = \frac{U(\eta)}{U\left(\dfrac{w_{2n}}{\eta}\right)}, \quad y_n = \frac{1}{\eta} + \frac{\eta}{w_{2n}}, \tag{147}$$

$$\frac{\left(\widetilde{y} - \dfrac{1}{\xi} - \dfrac{\xi}{w_{2n+1}}\right)\left(y - \dfrac{1}{\xi} - \dfrac{\xi}{w_{2n+1}}\right)}{\left(\widetilde{y} - w_{2n+1}\xi - \dfrac{1}{w_{2n+1}^2\xi}\right)\left(y - w_{2n+1}\xi - \dfrac{1}{w_{2n+1}^2\xi}\right)} = \frac{U(\xi)}{U\left(\dfrac{1}{w_{2n+1}\xi}\right)}, \quad x_{n+1} = \xi + \frac{1}{w_{2n+1}\xi}. \tag{148}$$

Here, recall,

$$w_{2n+1} = qw_{2n+1/2} = q^2 w_{2n}. \tag{149}$$

To express in (147) the variables x, $\widetilde{x}$ through x_n, y_n, we observe that

$$L_2 : (x_n, y_n, w_{2n-1/2}) \longmapsto (x, y_n, w_{2n}), \quad R_2 : (\widetilde{x}, y_n, w_{2n}) \longmapsto (x_{n+1}, y_n, w_{2n+1/2}).$$

According to (142), (143), we find:

$$x = x_n + \frac{1 - q^{-2}}{w_{2n-1/2}^2 - 1}(x_n - qw_{2n-1/2}y_n), \quad x_{n+1} = \widetilde{x} + \frac{1 - q^{-2}}{w_{2n}^2 - 1}(\widetilde{x} - w_{2n}y_n).$$

A straightforward computation confirms that these equations are equivalent to

$$\frac{x - \eta - \dfrac{1}{w_{2n}\eta}}{x - \dfrac{w_{2n}}{\eta} - \dfrac{\eta}{w_{2n}^2}} = \frac{x_n - \eta - \dfrac{1}{w_{2n-1}\eta}}{x_n - \dfrac{w_{2n}}{\eta} - \dfrac{\eta}{w_{2n}w_{2n-1}}}, \quad y_n = \frac{1}{\eta} + \frac{\eta}{w_{2n}}, \tag{150}$$

$$\frac{\widetilde{x} - \eta - \dfrac{1}{w_{2n}\eta}}{\widetilde{x} - \dfrac{w_{2n}}{\eta} - \dfrac{\eta}{w_{2n}^2}} = \frac{x_{n+1} - \eta - \dfrac{1}{w_{2n+1}\eta}}{x_{n+1} - \dfrac{w_{2n}}{\eta} - \dfrac{\eta}{w_{2n}w_{2n+1}}}, \quad y_n = \frac{1}{\eta} + \frac{\eta}{w_{2n}}. \tag{151}$$

Similarly, to express in (148) the variables y, $\widetilde{y}$ through x_{n+1}, y_n, we observe that

$$L_1 : (x_{n+1}, y_n, w_{2n+1/2}) \longmapsto (x_{n+1}, y, w_{2n+1}), \quad R_1 : (x_{n+1}, \widetilde{y}, w_{2n+1}) \longmapsto (x_{n+1}, y_{n+1}, w_{2n+3/2}).$$

According to (140), (141), we find:

$$y = y_n + \frac{1 - q^{-2}}{w_{2n+1/2}^2 - 1}(y_n - qw_{2n+1/2}x_{n+1}), \quad y_{n+1} = \widetilde{y} + \frac{1 - q^{-2}}{w_{2n+1}^2 - 1}(\widetilde{y} - w_{2n+1}x_{n+1}).$$

These equations are equivalent to

$$\frac{y - \dfrac{1}{\xi} - \dfrac{\xi}{w_{2n+1}}}{y - w_{2n+1}\xi - \dfrac{1}{w_{2n+1}^2\xi}} = \frac{y_n - \dfrac{1}{\xi} - \dfrac{\xi}{w_{2n}}}{y_n - w_{2n+1}\xi - \dfrac{1}{w_{2n+1}w_{2n}\xi}}, \quad x_{n+1} = \xi + \frac{1}{w_{2n+1}\xi}, \tag{152}$$

$$\frac{\widetilde{y} - \dfrac{1}{\xi} - \dfrac{\xi}{w_{2n+1}}}{\widetilde{y} - w_{2n+1}\xi - \dfrac{1}{w_{2n+1}^2\xi}} = \frac{y_{n+1} - \dfrac{1}{\xi} - \dfrac{\xi}{w_{2n+2}}}{y_{n+1} - w_{2n+1}\xi - \dfrac{1}{w_{2n+2}w_{2n+1}\xi}}, \quad x_{n+1} = \xi + \frac{1}{w_{2n+1}\xi}. \tag{153}$$

Substitute (150)–(153) into (147), (148). This results in the following system of non-autonomous difference equations for the variables x_n, y_n:

$$\frac{\left(x_{n+1} - \eta - \dfrac{1}{w_{2n+1}\eta}\right)\left(x_n - \eta - \dfrac{1}{w_{2n-1}\eta}\right)}{\left(x_{n+1} - \dfrac{w_{2n}}{\eta} - \dfrac{\eta}{w_{2n}w_{2n+1}}\right)\left(x_n - \dfrac{w_{2n}}{\eta} - \dfrac{\eta}{w_{2n}w_{2n-1}}\right)} = \frac{U(\eta)}{U\left(\dfrac{w_{2n}}{\eta}\right)}, \quad y_n = \frac{1}{\eta} + \frac{\eta}{w_{2n}}, \tag{154}$$

$$\frac{\left(y_{n+1} - \dfrac{1}{\xi} - \dfrac{\xi}{w_{2n+2}}\right)\left(y_n - \dfrac{1}{\xi} - \dfrac{\xi}{w_{2n}}\right)}{\left(y_{n+1} - w_{2n+1}\xi - \dfrac{1}{w_{2n+2}w_{2n+1}\xi}\right)\left(y_n - w_{2n+1}\xi - \dfrac{1}{w_{2n+1}w_{2n}\xi}\right)} = \frac{U(\xi)}{U\left(\dfrac{1}{w_{2n+1}\xi}\right)},$$

$$x_{n+1} = \xi + \frac{1}{w_{2n+1}\xi}. \tag{155}$$

This is the q-Painlevé equation of the surface type $A_0^{(1)}$, as given in [20], [11].

Remark. In the symmetric situation, when $U(z) = U(z^{-1})$, the system (154), (155) can be interpreted as a one-field second order difference equation, with $x_n = u_{2n-1}$ and $y_n = u_{2n}$. To see this, one should make in equation (155) the change $\xi \longmapsto \xi^{-1}$, after which it matches (154).

10   Conclusions

In this paper, we carried out the largest part of the task left open in [2], namely extended our novel approach to the pencils for which the generators through a point $X \in \mathbb{P}^3$

depend on X in a non-rational (branching) way. The only case left open for a further investigation is the pencil of the generic type (i), associated (in our scheme) with the elliptic Painlevé equation. Also the problem of an interpretation of the isomonodromic property of discrete Painlevé equations within our scheme remains open and is left for the future research. Finally, it will be important to extend the scheme of the present paper to discrete Painlevé equations corresponding to further translations in the corresponding affine Weyl symmetry groups. A path to this goal (via additional geometric involutions related to pencils and nets of quadrics) was sketched in the concluding remarks of [2]. The first step towards this goal (in the two-dimensional framework) has been performed in [3].

Acknowledgments

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Discrete Painlevé equations

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