ARITHMETIC AND GEOMETRY OF MARKOV POLYNOMIALS

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ABSTRACT. Markov polynomials are the Laurent-polynomial solutions of the generalised Markov equation

$$X^2 + Y^2 + Z^2 = kXYZ, \quad k = \frac{x^2 + y^2 + z^2}{xyz}$$

which are the results of cluster mutations applied to the initial triple (x, y, z). They were first introduced and studied by Itsara, Musiker, Propp and Viana, who proved, in particular, that their coefficients are non-negative integers. We study the coefficients of Markov polynomials as functions on the corresponding Newton polygons, proposing several new conjectures. Some of these conjectures are proved for the special cases of Markov polynomials corresponding to Fibonacci and Pell numbers.

1. INTRODUCTION

One of the most remarkable Diophantine equations is the celebrated Markov equation

$$X^2 + Y^2 + Z^2 = 3XYZ.$$

Markov showed [22] that all its solutions in positive integers can be found recursively from $(1, 1, 1)$ using the Vieta involution

$$(1) \qquad (X, Y, Z) \mapsto \left(X, Y, Z'\right), \qquad Z' = \frac{X^2 + Y^2}{Z},$$

and permutation of variables.

Following Itsara, Musiker, Propp and Viana [17, 27], who were inspired by the earlier work of Fomin and Zelevinsky [15], we consider the following generalisation of the Markov equation

$$(2) \qquad X^2 + Y^2 + Z^2 = k(x, y, z)XYZ, \quad k(x, y, z) = \frac{x^2 + y^2 + z^2}{xyz}.$$

with the solutions being rational functions of the parameters x, y, z. Starting with the initial solution $(X, Y, Z) = (x, y, z)$ we can apply the Vieta involutions (1) and permutations to arrive at solutions

$$X = X(x, y, z), \ Y = Y(x, y, z), \ Z = Z(x, y, z)$$

of the equation (2), certain Laurent polynomials of x, y, z, which we call *Markov polynomials.* They correspond to the cluster algebra related to the Markov quiver (see below).

The Laurent property follows from general cluster algebra framework of Fomin and Zelevinsky [15], but in this case also from the alternative form of the Vieta involution

$$(3) \qquad\qquad Z' = k(x, y, z)XY - Z.$$

In [27] it was shown that the coefficients of Markov polynomials are non-negative. When we specialise all the parameters x, y, z to 1, we get the celebrated Markov numbers, playing seminal role in many areas of mathematics [1]. In this sense Markov polynomials can be viewed as a 3-parameter quantisation of Markov numbers.

The aim of this paper is to study these polynomials in more detail. We will view their coefficients as the functions on the corresponding Newton polygons, which we will

describe explicitly. We prove some new results and state several conjectures about the coefficients of Markov polynomials, some of which are proved for the Markov polynomials corresponding to Fibonacci and Pell numbers. We discuss the continuum limit, introducing the corresponding entropy function, which is conjectured to be concave. We also provide a description of the Markov polynomials *via* a suitable generalisation of Cohn matrices.

2. MARKOV POLYNOMIALS ON THE CONWAY TOPOGRAPH

To represent Markov numbers it is now customary (see e.g. [29]) to use the Conway topograph introduced by J.H. Conway [7] to vizualise the values of quadratic forms.

The Conway topograph consists of the planar domains which are connected components of the complement to the trivalent tree imbedded in the plane. These domains were originally labelled by superbases in the integer lattice $\mathbb{Z}^2$, but can be also labelled by the rational numbers using the Farey mediant $\frac{p_1}{q_1} \oplus \frac{p_2}{q_2} = \frac{p_1+p_2}{q_1+q_2}$. We will only be interested in the rationals in $[0,1]$, leading to the rational Conway topograph given in Figure 1. Neighbouring regions are occupied by Farey neighbours, i.e. $\frac{a}{b}, \frac{c}{d}$ where $|ad - bc| = 1$.
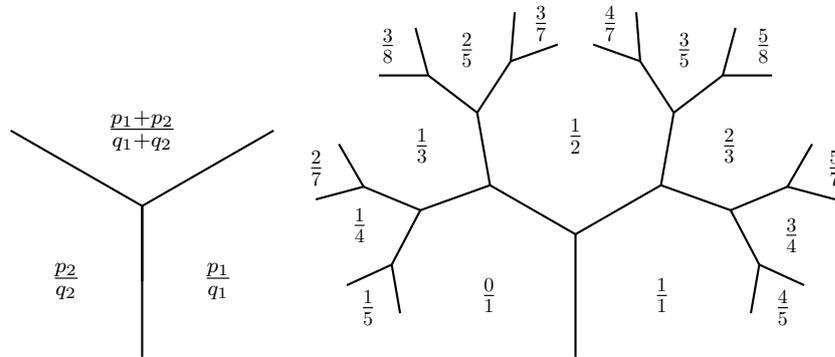

FIGURE 1. Rationals in $[0,1]$ represented on the Conway topograph.

To represent Markov numbers on the Conway topograph we iterate *via* the Vieta formula from the Markov equation, as shown on the left of Figure 2, starting with the initial state $X = Y = Z = 1$. Markov triples can then be seen by looking at the three domains adjacent to a vertex. The first few iterations of this process can also be seen in Figure 2.

Juxtaposition with the rational Conway topograph establishes the parametrization of the Markov numbers by rational numbers $\rho \in [0,1]$, a way of parametrizing Markov numbers which goes back to Frobenius [16].
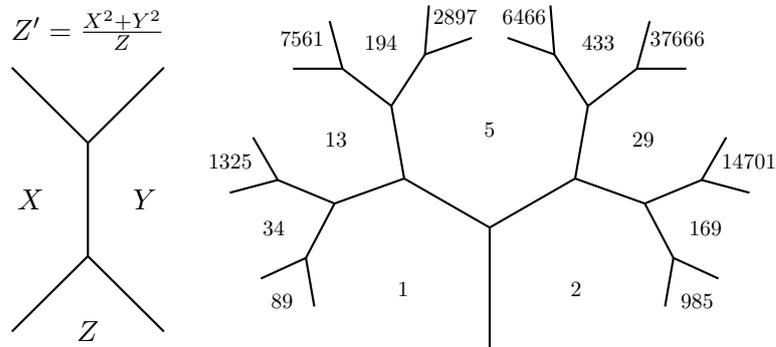

FIGURE 2. Markov numbers represented on the Conway topograph.

If instead of $(1,1,1)$ on the Conway topograph we start with the triple (x,y,z) and apply the same Vieta formula we get Laurent polynomial solutions of the equation (2).

$M_{3/7}$ $\quad$ $M_{4/7}$

$M_{3/8}$ $\quad$ $M_{2/5}$ $\quad\quad\quad$ $M_{3/5}$ $\quad$ $M_{5/8}$

$M_{1/2} =$
$$\frac{x^4+2x^2y^2+y^4+x^2z^2}{yz^2}$$

$M_{1/3}$ $\quad\quad\quad\quad\quad$ $M_{2/3}$

$M_{2/7}$ $\quad\quad\quad\quad\quad\quad\quad\quad$ $M_{5/7}$

$M_{1/4}$ $\quad\quad\quad\quad\quad\quad$ $M_{3/4}$

$M_{2/7}$

$M_{0/1} = x$ $\quad\quad$ $M_{1/1} = \frac{x^2+y^2}{z}$

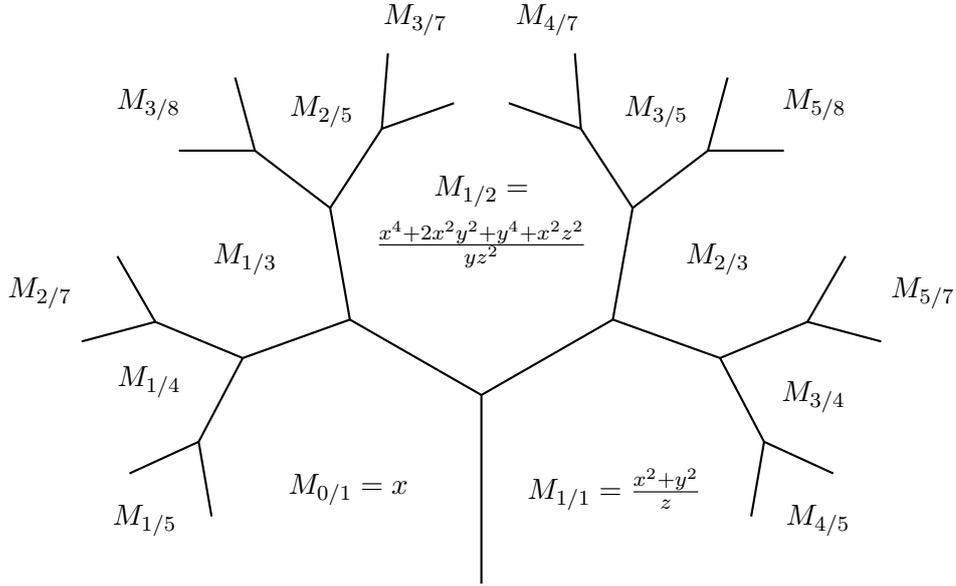$M_{1/5}$ $\quad\quad\quad\quad\quad\quad\quad\quad$ $M_{4/5}$

FIGURE 3. Markov polynomials on the Conway topograph.

We will use the notation $M_\rho = M_\rho(x,y,z)$ to represent the Markov polynomial occupying the same region as the rational ρ on their respective Conway topographs. Formally, this Frobenius parametrization is a mapping

(4) $$F : \rho \in [0,1] \cap \mathbb{Q} \mapsto M_\rho(x,y,z).$$

Setting $x = y = z = 1$ in Markov polynomials gives the corresponding Markov number: $M_\rho(1,1,1) = m_\rho$.

We have the correspondence between Markov triples of numbers and the triples of Markov polynomials M_ρ with $\rho \in [0,1] \cap \mathbb{Q}$. In particular, the first Markov triples $(1,1,1)$, $(1,1,2)$, $(1,2,5)$ correspond to the triples

$$(x,y,z), \ \left(x,y,\frac{x^2+y^2}{z}\right), \ \left(x,\frac{x^2+y^2}{z},\frac{x^4+2x^2y^2+y^4+x^2z^2}{yz^2}\right)$$

respectively. Thus we have the first few Markov polynomials

(5) $$M_{0/1} = x, \qquad M_{1/1} = \frac{x^2+y^2}{z}, \qquad M_{1/2} = \frac{(x^2+y^2)^2+x^2z^2}{yz^2}, \ldots$$

Many of our proofs will proceed by induction on the Conway topograph, and the solutions (5) will provide the base cases for the induction.

To extend the Frobenius correspondence (4) to all rationals we can use the action of the symmetric group $S_3 \cong \mathrm{SL}_2(\mathbb{Z}_2)$:

(6) $$M_\rho(x,y,z) = M_{\frac{1}{\rho}}(y,x,z), \quad M_\rho(x,y,z) = M_{-\frac{1}{1+\rho}}(z,x,y).$$

We believe that in this way we get all such solutions. More precisely, we have the following
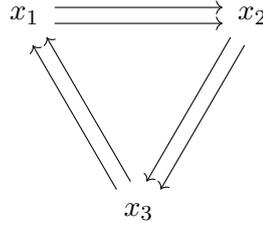
FIGURE 4. Markov quiver

Conjecture 2.1. *All the solutions $X, Y, Z \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ of equation (2) in integer Laurent polynomials, up to a change of sign of any two of them, are given by triples of Markov polynomials.*

From now on we will focus only on the Markov polynomials corresponding to rationals in $[0, 1]$ as all others can be obtained by the above symmetries.

From the point of view of the theory of cluster algebras Markov polynomials correspond to the special quiver (called the *Markov quiver*) shown in Figure 4.

By the general procedure of Fomin and Zelevinsky [14] the mutation of a quiver Q with vertices labelled by $1, 2, \ldots, N$ at a vertex k consists of three steps:-

(1) For each instance of a path $j \to k \to l$ introduce a new edge $j \to l$;
(2) Reverse the direction of all of the edges incident to k;
(3) Remove oriented 2-cycles.

The Markov quiver is special since it remains invariant (up to orientation) under all possible mutations.

Let $x_1, \ldots, x_N$ be the initial (seed) variables assigned to the corresponding vertices. The seed mutation at a vertex k changes the cluster $(x_1, \ldots, x_N)$ to $(x'_1, \ldots, x'_N)$ with $x'_j = x_j$ for all $j \neq k$ and x_k, x'_k related by the exchange relation

$$x_k x'_k = \prod_{j \to k} x_j + \prod_{k \to l} x_l.$$

In the case of the Markov quiver the mutation at vertex 3 leads to the exchange relation

$$x_3 x'_3 = x_1^2 + x_2^2,$$

corresponding to the Vieta involution (1). Starting with the seeds $x_1 = x, x_2 = y, x_3 = z$ and performing mutations we come to the Markov polynomials.

3. STRUCTURE OF MARKOV POLYNOMIALS

The general structure of the Markov polynomial $M_{a/b}(x, y, z)$ can be described as follows.

Theorem 3.1. *Markov polynomials $M_{a/b}(x, y, z)$ with coprime $a, b > 0$ have the following general form*

(7)
$$M_{a/b}(x, y, z) = \frac{P_{a/b}(x^2, y^2, z^2)}{x^{a-1} y^{b-1} z^{a+b-1}},$$

where $P_{a/b}(u, v, w)$ is a homogeneous polynomial of degree $a + b - 1$ with non-negative integer coefficients, indivisible by x, y or z.

We set here $P_{0/1} \equiv 1$ to have the correct formula for $M_{0/1}(x, y, z) = x$.

Proof. The proof is by induction using the Vieta formula in the form (3).

$$M_{\rho_{Z'}}, \ \rho_{Z'} = \frac{a+2c}{b+2d}$$

$$M_{\rho_X}, \ \rho_X = \frac{c}{d} \qquad M_{\rho_Y}, \ \rho_Y = \frac{a+c}{b+d}$$
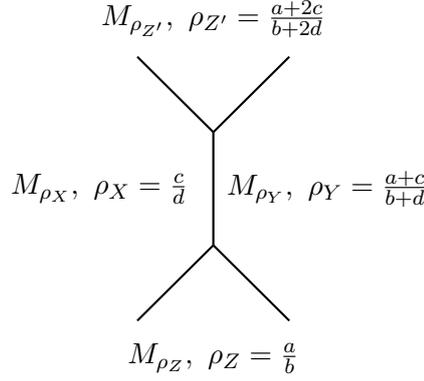
$$M_{\rho_Z}, \ \rho_Z = \frac{a}{b}$$

FIGURE 5. Section of the Conway topograph showing correspondence.

Consider the part of the Conway topograph shown in Figure 5. We assume that the statement holds for M_{ρ_X}, M_{ρ_Y} and M_{ρ_Z}. Then, according to (3),

$$M_{\rho_{Z'}} = k(x,y,z)M_{\rho_X}M_{\rho_Y} - M_{\rho_Z}$$
$$= \frac{(x^2+y^2+z^2)P_{c/d}(x^2,y^2,z^2)P_{(a+c)/(b+d)}(x^2,y^2,z^2) - x^{2c}y^{2d}z^{2c+2d}P_{a/b}(x^2,y^2,z^2)}{x^{a+2c-1}y^{b+2d-1}z^{a+b+2c+2d-1}},$$

where the denominator is of the required form. We need only to be sure that there is no cancellation of factors from the denominator, but for that to happen, one of $P_{c/d}$ or $P_{(a+b)/(c+d)}$ would have to be divisible by x, y or z, which cannot happen. We further see that the numerator satisfies the required homogeneity property. The base cases (5) satisfy (7) so the proof is complete.

The non-negativity of the coefficients of Markov polynomials was known before [27], but we will provide an independent proof based on quantised Cohn matrices (see Corollary 10.2 below). □

As a consequence of the proof, we obtain a formula for the evolution of the numerators along the Conway topograph:

$$(8) \qquad P_{\frac{a+2c}{b+2d}} = (u+v+w)P_{\frac{c}{d}}(u,v,w)P_{\frac{a+c}{b+d}}(u,v,w) - u^c v^d w^{c+d} P_{\frac{a}{b}}(u,v,w).$$

3.1. **Newton polygon of the numerator.** To study the numerator $P_\rho(u,v,w)$ of the Markov polynomial $M_\rho(x,y,z)$, $\rho = a/b$, it is natural to look first at the corresponding Newton polygon Δ_ρ, which can be considered as tropicalization of $P_\rho(u,v,w)$.

Whilst our polynomials consist of three variables u,v,w, by noting homogeneity of terms—each term has total order in the numerator of $a+b-1$—we can simplify the problem (see Theorem 3.1). Indeed we can write P_ρ as the following sum

$$(9) \qquad P_\rho(u,v,w) = \sum A_{ij} u^i v^j w^{a+b-1-i-j}.$$

As a result of this homogeneity we can consider the plane projections $w = 1$, looking at the Newton polygon in the two variables u,v, and losing no additional information in the process. Note that this is now precisely the affine version ($w = 1$) of the Newton polygon for the polynomial $P_\rho(u,v,w)$.

For example, when $\rho = \frac{2}{3}, m_\rho = 29$ we have

$$P_\rho(u,v,w) = u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + v^4 + 2u^3w + 5u^2vw + 4uv^2w + v^3w + u^2w^2,$$

and setting $w = 1$ we obtain

$$P_\rho(u, v, w) = u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + v^4 + 2u^3 + 5u^2v + 4uv^2 + v^3 + u^2.$$

We define the Newton polygon of $M_\rho(x, y, z)$ as the convex hull

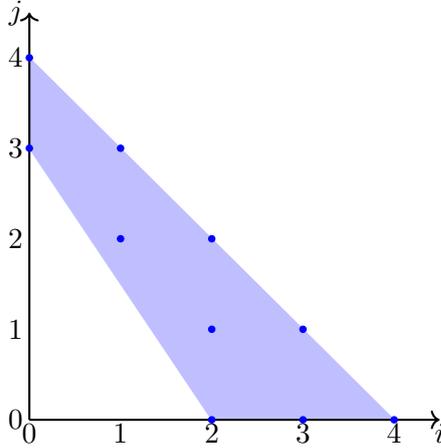(10) $$\Delta_\rho = \Delta(P_\rho) := \mathrm{Conv}\{(i, j) : A_{ij} \neq 0\} \subset \mathbb{Z}^2.$$

FIGURE 6. Newton polygon Δ_ρ, of the Markov polynomial $M_\rho, \rho = \frac{2}{3}$.

We have the following explicit description[1] of the Newton polygons of the numerators of Markov polynomials in general case.[2]

Theorem 3.2. *Given a rational $\rho = \frac{a}{b}$, the Newton polygon $\Delta_{a/b}$ of the numerator of Markov polynomial $M_\rho(x, y, z)$ is the area on the ij-plane with $i, j \geq 0$ satisfying the conditions*

(11) $$\begin{cases} \dfrac{i}{a} + \dfrac{j}{b} \geq 1 \\ i + j \leq a + b - 1. \end{cases}$$

The proof is by induction. The first few cases are easily verifiable by direct computations. For the inductive step, we use the Vieta formula $ZZ' = X^2 + Y^2$ and verify that the Minkowski sum of the Newton polygons on the LHS coincides with the convex hull of the union of the scaled (due to squared terms) polygons on the RHS, taking care with the effect of the denominators on the subsequent numerator during the addition on the RHS. This is straightforward but a bit technical, so we omit the details.

We know due to [27] that Markov polynomials have non-negative coefficients, but some of the terms corresponding to the lattice points in the Newton polygon Δ_ρ, in principle, could be zero. We conjecture that this is not the case and all the corresponding coefficients are positive.

Conjecture 3.3. *(Saturation conjecture) The terms that appear in the numerator of a Markov polynomial M_ρ are precisely those corresponding to the integer lattice points in the Newton polygon Δ_ρ.*

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[1]We are very grateful to Alexey Ustinov for the stimulating discussions of the Newton polygons for Markov polynomials, which led to formula (11).

[2]As we were informed by Ralf Schiffler, this result can be derived from the general description of Newton polygons for rank 3 cluster algebras [21].

Note that in the original (x, y, z) variables the Newton polygon is twice larger and saturation does not hold.

We will prove Conjecture 3.3 in certain special cases of Markov polynomials (see Section 5).

We should also mention in this relation an important result by Fei [11], who proved saturation for the F-polynomials of a rigid representation of any finite-dimensional basic algebra.

4. COEFFICIENTS OF MARKOV POLYNOMIALS

We can view the coefficients A_{ij} of Markov polynomial

$$M_{a/b}(x, y, z) = \frac{1}{x^{a-1} y^{b-1} z^{a+b-1}} \sum_{i+j \leq a+b-1, \frac{i}{a} + \frac{j}{b} \geq 1} A_{ij} x^{2i} y^{2j} z^{2(a+b-1-i-j)}$$

as the weights on the lattice points of Newton polygon.

For example, for $\rho = \frac{2}{3}$ the 'weighted' polygon, Δ_ρ is given in Figure 7.
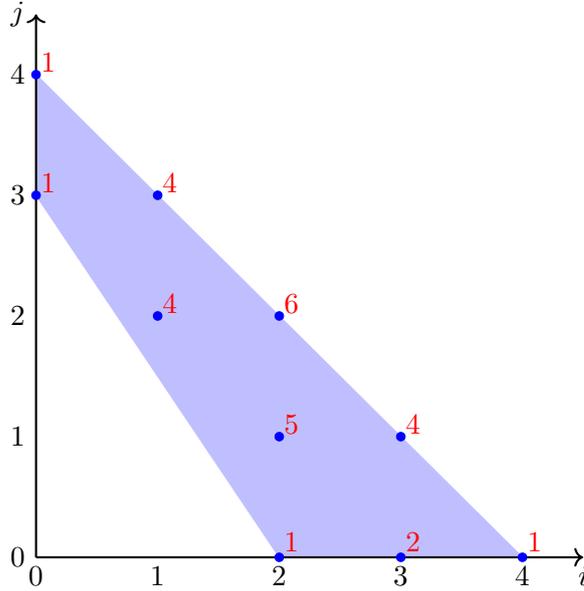

FIGURE 7. 'Weighted' Newton polygon of the Markov polynomial $M_{2/3}$.

Looking at the coefficients that appear on the lines $i = 0, j = 0$ and $i + j = a + b - 1 = 4$, we notice that they are binomial coefficients. This is no coincidence as we prove below.

Expanding the Markov polynomials as Laurent series in z, y, x leads respectively to

$$M_{a/b}(x, y, z) = \frac{1}{x^{a-1} y^{b-1}} \left[\frac{T_0(x^2, y^2)}{z^{a+b-1}} + \frac{T_1(x^2, y^2)}{z^{a+b-3}} + \cdots + \frac{T_k(x^2, y^2)}{z^{a+b-1-2k}} + \cdots \right]$$

$$= \frac{1}{x^{a-1} z^{a+b-1}} \left[\frac{R_0(x^2, z^2)}{y^{b-1}} + \frac{R_1(x^2, z^2)}{z^{b-3}} + \cdots + \frac{R_k(x^2, z^2)}{y^{b-1-2k}} + \cdots \right]$$

$$= \frac{1}{y^{b-1} z^{a+b-1}} \left[\frac{S_0(y^2, z^2)}{x^{a-1}} + \frac{S_1(y^2, z^2)}{x^{a-3}} + \cdots + \frac{S_k(y^2, z^2)}{x^{a-1-2k}} + \cdots \right]$$

for certain polynomials T_i, R_i, S_i. These polynomials with $i = 0$ contain information about the coefficients on the boundary of the Newton polygon (S_0 on the vertical, R_0 on the horizontal and T_0 on the upper diagonal line), while $i = 1, 2, \ldots$ correspond to the neighbouring parallel lines.

Theorem 4.1. *In the following cases the polynomials S_i, R_i, T_i can be written explicitly as:*

- $S_0(v, w) = v^b(v + w)^{a-1}$

- $R_0(u, w) = u^a(u + w)^{b-1}$

- $R_1(u, w) = u^a(3a - 1)(u + w)^{b-2} + u^{a+1}(b - 2a)(u + w)^{b-3}$

- $T_0(u, v) = (u + v)^{a+b-1}$

- $T_1(u, v) = (a - 1)(u + v)^{a+b-2} + u(b - a)(u + v)^{a+b-3}$

- $T_2(u, v) = \frac{(a-1)(a-2)}{2}(u + v)^{a+b-3} + u[a(b - a) - a](u + v)^{a+b-4}$
  $+ \frac{1}{2}u^2[(b - a)^2 + 5a - 3b](u + v)^{a+b-5}$.

Proof. We will prove only the formula for T_0; all other cases are similar.

It is clear from (5) that $P_{0/1}, P_{1/1}, P_{1/2}$ all have the required form, so let us consider again the section of the Conway topograph in Figure 5, and suppose that

$$P_{\rho_X}(u, v, w) = T_{0,X}(u, v) + O(w) = (u + v)^{c+d-1} + O(w)$$
$$P_{\rho_Y}(u, v, w) = T_{0,Y}(u, v) + O(w) = (u + v)^{a+b+c+d-1} + O(w)$$
$$P_{\rho_Z}(u, v, w) = T_{0,Z}(u, v) + O(w) = (u + v)^{a+b-1} + O(w),$$

where $O(w)$ denotes a polynomial function of u, v, w with w as a factor.

From (8) we have

$$P_{\rho_{Z'}}(u, v, w) = (u + v + w)P_{\rho_X}(u, v, w)P_{\rho_Y}(u, v, w) - u^c v^d w^{c+d} P_{\rho_Z}(u, v, w)$$
$$= (u + v)(u + v)^{c+d-1}(u + v)^{a+b+c+d-1} + O(w)$$
$$= (u + v)^{a+b+2c+2d-1} + O(w),$$

giving us the expected form for $P_{\rho_{Z'}}(u, v, w)$ with $\rho_{Z'} = (a + 2c)/(b + 2d)$. $\qquad\square$

As a corollary we have an explicit form of the following coefficients (see Figure 8).

Theorem 4.2. *For the Markov polynomial $M_{a/b}(x, y, z)$ we know the coefficients A_{ij} in the following cases:*

- $A_{0,j} = \dbinom{a - 1}{j - b}$

- $A_{i,0} = \dbinom{b - 1}{i - a}$

- $A_{i,1} = (3a - 1)\dbinom{b - 2}{i - a} + (b - 2)\dbinom{b - 3}{i - a - 1}$

- *If $i + j = a + b - 1$ then $A_{i,j} = \dbinom{a + b - 1}{i}$,*

- *If $i + j = a + b - 2$ then*

$$A_{i,j} = (a - 1)\dbinom{a + b - 2}{i} + (b - a)\dbinom{a + b - 3}{i - 1}$$

- *If $i + j = a + b - 3$ then*

$$A_{i,j} = \frac{(a-1)(a-2)}{2} \binom{a+b-3}{i} + [a(b-a) - a] \binom{a+b-4}{i-1}$$
$$+ \frac{1}{2}[(b-a)^2 + 5a - 3b] \binom{a+b-5}{i-2}$$

In particular, the boundary coefficients are binomials and we have an explicit expression for some of the interior coefficients as sums of binomial coefficients.
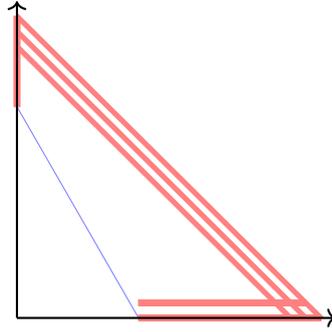

FIGURE 8. Lines on the Newton polygon where the coefficients are explicitly known.

For the rationals of the form $\rho = \frac{1}{n}$ and $\rho = \frac{2}{2n-1}$ we can add also the coefficients on one more vertical line with $i = 1$.

Theorem 4.3. *For the Markov polynomials $M_{1/n}$ and $M_{2/(2n-1)}$ we have respectively*

$$S_1(y^2, z^2) = \sum_{k=1}^{n} k y^{2(k-1)} z^{2(n-k)}$$

and

$$S_1(y^2, z^2) = 2n y^{4n-2} + \sum_{k=1}^{n-1} 4k y^{2(n+k-1)} z^{2(n-k)}.$$

The proof of these results is similar to that of Theorems 3.1 and 4.1 and is based on relation (8).

In particular, on the line $i = 1$ the Markov polynomial $M_{1/n}(x, y, z)$ has the coefficients $1, 2, \ldots, n$ (see Figure 9 for the example with $n = 5$, corresponding to the Markov number 89).
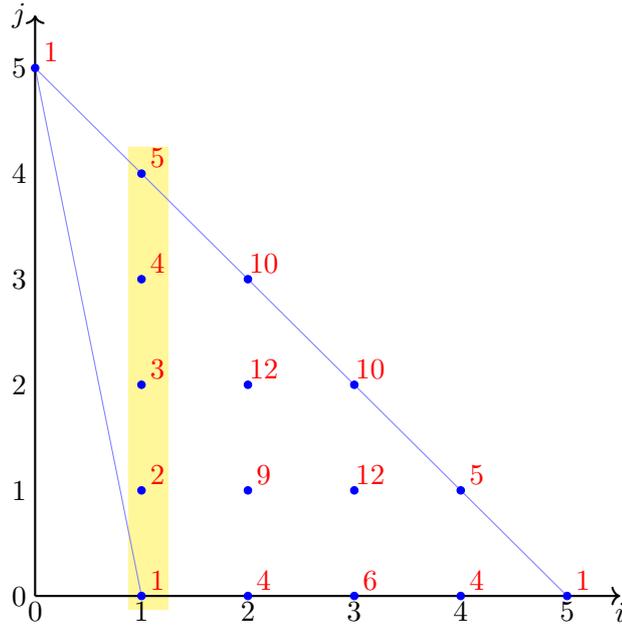
FIGURE 9. Weighted Newton polygon of $M_{1/5}$ with the highlighted coefficients on the line $i = 1$.

In the next two sections we consider in more detail two special series of Markov polynomials M_ρ with $\rho = 1/n$ and $\rho = n/(n+1)$.

5. MARKOV POLYNOMIALS CORRESPONDING TO FIBONACCI AND PELL NUMBERS

5.1. Markov-Fibonacci polynomials. Fibonacci numbers satisfy the following recurrence

$$(12) \qquad F_{n+1} = F_n + F_{n-1},$$

subject to the intial conditions $F_0 = 0, F_1 = 1$ (sequence A000045 of [25]). The first few terms are:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots.$$

It is known that the odd-indexed Fibonacci numbers F_{2k+1} are also Markov numbers [1]. Indeed, the odd-indexed Fibonacci numbers satisfy

$$F_{2k+1} = 3F_{2k-1} - F_{2k-3},$$

which means that $Y_k = F_{2k-1}$ satisfy the Vieta recursion $Y_{k+1} = 3Y_k - Y_{k-1}$ for the Markov triples $(1, Y_{k-1}, Y_k)$.

We will call the Markov polynomials $M_{1/n}(x, y, z)$ corresponding to the Fibonacci numbers F_{2n-1} the *Markov-Fibonacci polynomials.* They can be viewed as 3-parameter quantisations of the odd-indexed Fibonacci numbers.

We should note that a closely related notion was introduced by Caldero and Zelevinsky [4] as cluster variables of the $A_1^{(1)}$ cluster algebra *via* the recurrence relation

$$(13) \qquad f_{m+1} = \frac{f_m^2 + 1}{f_{m-1}}.$$

The *Caldero-Zelevinsky Fibonacci polynomial* f_m is defined as the corresponding $f_m(x_1, x_2)$ considered as the function of the two initial cluster variables $f_1 = x_1, f_2 = x_2$.

Theorem 5.1 ([4], [32]). *For every $n \geq 0$ there are the explicit formulæ*

$$f_{2n} = \frac{1}{x_1^n x_2^n} \sum_{q+r \leq n} \binom{n-r}{q}\binom{n-q}{r} x_1^{2q} x_2^{2r},$$

$$f_{2n+1} = \frac{1}{x_1^{n+1} x_2^n} \left(x_2^{2(n+1)} + \sum_{q+r \leq n} \binom{n-r}{q}\binom{n+1-q}{r} x_1^{2q} x_2^{2r} \right).$$

In particular f_m are Laurent polynomials in x_1, x_2 with positive coefficients.

Comparing Eqs. (13) with the Vieta recurrence for Markov polynomials

$$(14) \qquad\qquad M_{1/(k+1)} = \frac{M_{1/k} + x^2}{M_{1/(k-1)}}$$

we see that $M_{1/k}$'s are a homogeneous version of the cluster variables f_m:

$$(15) \qquad\qquad f_m(x_1, x_2) = M_{1/m}(1, x_2, x_1).$$

Using Theorem 5.1, we have

Theorem 5.2. *The Markov-Fibonacci polynomials $M_{1/(n+1)}(x, y, z)$ have coefficients*

$$(16) \qquad\qquad A_{ij} = \binom{n-j}{n+1-i-j}\binom{i+j}{j},$$

where A_{ij} is the coefficient of the numerator monomial $x^{2i} y^{2j} z^{2(n+1-i-j)}$.

Corollary 5.3. *Markov-Fibonacci polynomials satisfy the Saturation Conjecture 3.3.*

Another simple consequence of this result is the first part of Theorem 4.3 since setting $i = 1$ leads to

$$A_{1j} = \binom{n-j}{n-j}\binom{1+j}{j} = j + 1.$$

It is interesting to note[3] that our Markov-Fibonacci polynomials $M_{1/n}(u, v, w)$ are related to Fibonacci polynomials $\mathcal{F}_n(q)$ from the theory of q-rational numbers developed by Morier-Genoud and Ovsienko [23].

The Fibonacci polynomials $\mathcal{F}_n(q)$ are the denominators of the q-deformed rationals $\left[\frac{F_{n+1}}{F_n}\right]_q$, where F_n are the Fibonacci numbers [20]. They can be determined by the recurrence relation

$$(17) \qquad\qquad \mathcal{F}_{n+2}(q) = [3]_q \mathcal{F}_n(q) - q^2 \mathcal{F}_{n-2}(q),$$

where $[3]_q = 1 + q + q^2$, and the initial conditions

$$\mathcal{F}_0(q) = 1, \ \mathcal{F}_2(q) = 1 + q \qquad \text{and} \qquad \mathcal{F}_1(q) = 1, \ \mathcal{F}_3(q) = 1 + q + q^2.$$

Proposition 5.4. *The Fibonacci polynomials $\mathcal{F}_n(q)$ are the following specialisations*

$$(18) \qquad\qquad \mathcal{F}_{2n}(q) = P_{1/n}(q, 1, q^2)$$

of the numerators $P_{1/n}(u, v, w)$ of the Markov polynomials $M_{1/n}(u, v, w)$.

Proof. This follows from the comparison of the recurrence relations (8) and (17), and $P_{1/0}(u, v, w) = 1$, $P_{1/1}(u, v, w) = u + v$. $\qquad\square$

---

[3]This was observed by Sam Evans after discussions with Valentin Ovsienko.

5.2. **Markov-Pell Polynomials.** We now turn to the Markov polynomials corresponding to Pell numbers, satisfying the recurrence

$$P_{n+1} = 2P_n + P_{n-1}, \tag{19}$$

subject to the intial conditions $P_0 = 0, P_1 = 1$ (sequence A000129 of [25]). The first few Pell numbers are $0, 1, 2, 5, 12, 29, 70, 169, \ldots$.

It is known [1] that the odd-indexed Pell numbers P_{2k+1} are also Markov numbers, since they satisfy the recurrence

$$P_{2k+1} = 6P_{2k-1} - P_{2k-3},$$

equivalent to Vieta recursion for Markov triples $(2, Y_{k-1}, Y_k)$, $Y_k = P_{2k-1}$.

The Markov polynomials $M_{k/(k+1)}$ can be viewed as a quantization of the Pell numbers P_{2k+1}.

We introduce now the *Markov-Pell polynomials via* the following recurrences, (which are quantized versions of (19))

$$\begin{aligned}
R_{2k+1} &= (x^2 + y^2)R_{2k} + x^2 z^2 R_{2k-1}, \\
R_{2k} &= (x^2 + y^2)R_{2k-1} + y^2 z^2 R_{2k-2},
\end{aligned} \tag{20}$$

with $R_0 = 0$, $R_1 = 1$, $R_2 = x^2 + y^2$, $R_3 = x^4 + 2x^2 y^2 + y^4 + x^2 z^2$.

The odd-indexed Markov-Pell polynomials satisfy the recurrence

$$R_{2k+1} = (x^2 + y^2)(x^2 + y^2 + z^2)R_{2k-1} - x^2 y^2 z^4 R_{2k-3}, \tag{21}$$

which is precisely the recurrence for the numerators of the Markov polynomials $M_{k/(k+1)}$ (and because of the special initial data they coincide with these numerators).

From the equations (20) we can deduce

Corollary 5.5. *Markov-Pell polynomials satisfy the Saturation Conjecture 3.3.*

The equations (21) imply the recurrence for the corresponding coefficient $A_{i,j}^{(2k+1)}$ at the monomial $x^{2i} y^{2j} z^{2(k-i-j)}$ in R_{2k+1} :

$$A_{i,j}^{(2k+1)} = A_{i-2,j}^{(2k-1)} + 2A_{i-1,j-1}^{(2k-1)} + A_{i,j-2}^{(2k-1)} + A_{i-1,j}^{(2k-1)} + A_{i,j-1}^{(2k-1)} - A_{i-1,j-1}^{(2k-3)}. \tag{22}$$

We can also produce the Binet-type formula for the corresponding Markov-Pell polynomials, similar to the classical Binet formula for the Fibonacci numbers

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Indeed, we can rewrite Eqs. (20) in matrix form as

$$\begin{pmatrix} R_{2k+1} \\ R_{2k} \end{pmatrix} = \begin{pmatrix} (x^2+y^2)^2 + x^2 z^2 & (x^2+y^2)x^2 z^2 \\ (x^2+y^2) & y^2 z^2 \end{pmatrix} \begin{pmatrix} R_{2k-1} \\ R_{2k-2} \end{pmatrix}$$

The characteristic equation of the corresponding matrix is

$$\lambda^2 - (x^2 + y^2)(x^2 + y^2 + z^2)\lambda + x^2 y^2 z^4 = 0 \tag{23}$$

giving the eigenvalues

$$\lambda_{\pm}(x, y, z) = \frac{1}{2}\left[(x^2 + y^2)(x^2 + y^2 + z^2) \pm \sqrt{\mathcal{D}}\right]$$

where

$$\mathcal{D} = (x^2 + y^2)^4 + 2z^2(x^2 + y^2)^3 + z^4(x^2 - y^2)^2$$

is the discriminant of (23).

The standard procedure leads now to the following Binet-type formula for the Markov-Pell polynomials:

(24)
$$R_{2k+1} = \frac{1}{\sqrt{\mathcal{D}}} \left((\lambda_+ - y^2 z^2)\lambda_+^k - (\lambda_- - y^2 z^2)\lambda_-^k \right).$$

6. Log-Concavity

A large number of naturally-occurring combinatorial sequences exhibit the *logarithmically concave* (log-concave) property [30]. Further to saturation, experimental data that we have looked-at so far has shown that coefficients of the Markov polynomials are log-concave in all directions.

Definition 6.1. A sequence $x = (x_0, x_1, \ldots, x_n)$ is said to be *log-concave* if it satisfies the property

(25)
$$x_k^2 \geq x_{k-1} x_{k+1},$$

for $k \in \{1, 2, \ldots, n-1\}$.

A well-known example is the binomial sequence $x_k = \binom{n}{k}$, $k = 0, \ldots, n$:

$$\frac{x_k^2}{x_{k-1} x_{k+1}} = \binom{n}{k}^2 \Big/ \binom{n}{k-1}\binom{n}{k+1} = \frac{k+1}{k}\frac{n-k+1}{n-k} > 1.$$

Definition 6.2. We say that the weights on the Newton polygon satisfy the *weak log-concavity* property if their sequences in all principal directions (horizontal, vertical and diagonal with $i + j = $ constant) are log-concave.

Conjecture 6.3. *(Log-concavity conjecture) The coefficients of Markov polynomials are weakly log-concave on the corresponding Newton polygon.*

Note that the Conjecture 6.3 implies the Saturation Conjecture 3.3, claiming the positivity of all the coefficients inside the Newton polygon. Indeed, we know that the coefficients on the boundary are positive, so the appearance of internal zero coefficients will contradict the log-concavity property (25).

Theorem 6.4. *Conjecture 6.3 holds for the Markov-Fibonacci polynomials.*

Proof. We need to check the log-concavity condition in all three principal directions. Using (16), in the horizontal direction we have

$$A_{i,j}^2 = \binom{n-j}{n+1-i-j}^2 \binom{i+j}{j}^2 = \left(\frac{(n-j)!^2}{(n+1-i-j)!^2 \, (i-1)!^2} \right) \left(\frac{(i+j)!^2}{i!^2 \, j!^2} \right),$$

$$A_{i-1,j} A_{i+1,j} = \binom{n-j}{n+2-i-j}\binom{n-j}{n-i-j}\binom{i+j-1}{j}\binom{i+j+1}{j}$$

$$= \left(\frac{(n-j)!^2}{(n+2-i-j)! \, (i-2)! \, (n-i-j)! \, i!} \right) \left(\frac{(i-1+j)! \, (i+1+j)!}{j!^2 \, (i-1)! \, (i+1)!} \right).$$

This implies that

$$\frac{A_{i,j}^2}{A_{i-1,j} A_{i+1,j}} = \left(\frac{n+2-i-j}{n+1-i-j} \right) \left(\frac{i}{i-1} \right) \left(\frac{i+j}{i+j+1} \right) \left(\frac{i+1}{i} \right)$$

$$= \left(\frac{n+2-i-j}{n+1-i-j} \right) \left(\frac{i}{i-1} \right) \left(\frac{i(i+1) + j(i+1)}{i(i+1) + ji} \right).$$

We see that all three terms here are greater than 1 and so we have the required inequality $A_{i,j}^2 > A_{i-1,j} A_{i+1,j}$.

Similarly in the vertical case we have

$$
\frac{A_{i,j}^2}{A_{i,j-1} A_{i,j+1}} = \left[\left(\frac{n-j}{n+1-j} \right) \left(\frac{n+2-i-j}{n+1-i-j} \right) \right] \left[\left(\frac{i+j}{i+j+1} \right) \left(\frac{j+1}{j} \right) \right]
$$

$$
= \left[\frac{(n+1-i-j)(n-j) + (n-j)}{(n+1-i-j)(n-j) + (n+1-i-j)} \right] \left[\frac{j(i+j) + (i+j)}{j(i+j) + j} \right],
$$

with both of the final fractions being greater than 1 (the first requires that $i \geq 1$ but the vertical line for $i = 0$ consists of only a single point, so can be ignored).

In the diagonal case the second binomials in $A_{i-1,j+1}, A_{i,j}, A_{i+1,j-1}$ are respectively $\binom{i+j}{j+1}$, $\binom{i+j}{j}$, $\binom{i+j}{j-1}$, which are consecutive binomial coefficients satisfying the log-concavity property. As a result we have

$$
\frac{A_{i,j}^2}{A_{i-1,j+1} A_{i+1,j-1}} \geq \left(\frac{(n-j)!^2}{(n-j+1)! \, (n-j-1)!} \right) \left(\frac{(n-i-j)! \, (n+2-i-j)!}{(n+1-i-j)!^2} \right)
$$

$$
= \left(\frac{n-j}{n-j+1} \right) \left(\frac{n+2-i-j}{n+1-i-j} \right) = \frac{(n-j)(n+1-i-j) + (n-j)}{(n-j)(n+1-i-j) + (n+1-i-j)},
$$

which is greater than or equal to 1 as required. □

The Conjecture 6.3 holds along the boundaries of the polygon, since these coefficients are precisely binomial coefficients, which are known to be log-concave. We will show now that this holds also on the 2nd horizontal line and upper diagonal of the Newton polygon.

Lemma 6.5. *Sequences of the form*

$$
x_k = A \binom{c}{k} + B \binom{c}{k-1}, \quad k = 0, 1, \ldots, c
$$

for positive A, B, c are always log-concave.

Proof. We have

$$
x_k^2 = A^2 \binom{c}{k}^2 + 2AB \binom{c}{k} \binom{c}{k-1} + B^2 \binom{c}{k-1}^2,
$$

$$
x_{k-1} x_{k+1} = A^2 \binom{c}{k-1} \binom{c}{k+1} + AB \left[\binom{c}{k-1} \binom{c}{k} + \binom{c}{k-2} \binom{c}{k+1} \right]
$$

$$
+ B^2 \binom{c}{k-2} \binom{c}{k}.
$$

It is clear that the coefficients of A^2 and B^2 are greater in the case of x_k^2 due to the log-concavity of the binomial coefficients. So if we can show that

$$
\binom{c}{k-1} \binom{c}{k} \geq \binom{c}{k-2} \binom{c}{k+1},
$$

then the result will follow. Since

$$
\frac{\binom{c}{k-1} \binom{c}{k}}{\binom{c}{k-2} \binom{c}{k+1}} = \frac{k+1}{k-1} \frac{c-k+2}{c-k} > 1,
$$

this completes the proof. □

Theorem 6.6. *The coefficients of Markov polynomial M_ρ that appear on the lines with $j = 1$ and $i + j = a + b - 2$ of its Newton polygon are strictly log-concave.*

Proof. The proof follows from Theorem 4.2 and the previous lemma. $\square$

As for the 3rd diagonal we are only able to prove the log-concavity in the case where $\rho \leq 3/5$.

Theorem 6.7. *The coefficients of Markov polynomials M_ρ that appear on the line with $i + j = a + b - 3$ of its Newton polygon are log-concave for $\frac{a}{b} \leq \frac{3}{5}$.*

Proof. We will use the following result quoted by Stanley [30] (but which dates back to Newton [24]).

Theorem 6.8 (Stanley [30]). *Let $p(X) = \sum_{k=0}^{n} \alpha_k X^k$. If the polynomial $p(X)$ has only real zeros, then the sequence α_k is log-concave.*

Let α_k be the coefficients on the diagonal $i + j - 3$ stated in Theorem 4.2:

$$\alpha_k = E\binom{a + b - 3}{k} + F\binom{a + b - 4}{k - 1} + G\binom{a + b - 5}{k - 2},$$

$$E = \frac{(a - 1)(a - 2)}{2}, \ F = a(b - a - 1), \ G = \frac{1}{2}\left[(b - a)^2 + 5a - 3b\right].$$

Let

$$p(X) = \sum_{k=0}^{a+b-3} \alpha_k X^k,$$

then we have

$$p(X) = E(1 + X)^{a+b-3} + FX(1 + X)^{a+b-4} + GX^2(1 + X)^{a+b-5}$$
$$= (1 + X)^{a+b-5}\left[E(1 + X)^2 + FX(1 + X) + GX^2\right] = (1 + X)^{a+b-5}q(X),$$

where

$$q(X) := (E + F + G)X^2 + (2E + F)X + E$$
$$= \frac{(b - 1)(b - 2)}{2}X^2 + (ab - 4a + 2)X + \frac{(a - 1)(a - 2)}{2}$$

Since $(1 + X)^{a+b-5}$ has clearly real roots, to apply Theorem 6.8 we need to show that $q(X)$ has only real roots. The discriminant Δ of $q(X)$ is

$$\Delta = (ab - 4a + 2)^2 - 4\frac{(b - 1)(b - 2)}{2}\frac{(a - 1)(a - 2)}{2}$$
$$= (ab - 4a + 2)^2 - (ab - 2a - b + 2)(ab - a - 2b + 2).$$

If $a/b \leq 3/5$ then $a + 2b \geq 4a + \frac{a}{3}$ and $2a + b \geq 4a - \frac{a}{3}$, so

$$\Delta \geq (ab - 4a + 2)^2 - \left(ab - 4a + 2 + \frac{a}{3}\right)\left(ab - 4a + 2 - \frac{a}{3}\right) = \frac{a^2}{9} \geq 0.$$

$\square$

7. Continuum limit and entropy function

Consider the sequence of fractions $\rho_n = a_n/b_n$ tending to some $\alpha \in [0, 1]$ when $n \to \infty$. Let (i_n, j_n) be the sequence of points in the Newton polygons

$$(26) \qquad \Delta_{\rho_n} = \{i, j \in \mathbb{Z}_{\geq 0} : i/a_n + j/b_n \geq 1, \quad i + j \leq a_n + b_n - 1\}$$

such that the corresponding points $(\xi_n = i_n/b_n, \eta_n = j_n/b_n) \to (\xi, \eta) \in \mathcal{N}_\alpha$ as $n \to \infty$, where $\mathcal{N}_\alpha$ is the scaled Newton polygon

$$(27) \qquad \mathcal{N}_\alpha := \{\xi, \eta \in \mathbb{R} : \xi, \eta > 0, \quad \xi + \alpha\eta > \alpha, \quad \xi + \eta < \alpha + 1.\}$$

Define the *entropy function* of this sequence as

$$(28) \qquad \mathcal{H} = \limsup_{n \to \infty} \frac{1}{b_n} \ln A_{i_n, j_n}(\rho_n),$$

where $A_{i,j}(\rho_n)$ are the coefficients of the Markov polynomial M_{ρ_n}.

Proposition 7.1. *The corresponding limit superior does exist for any such sequences.*

The proof follows from the estimate $A_{i,j}(\rho_n) < m_{\rho_n}$, where m_{ρ_n} is the corresponding Markov number, and the results of Fock [13] (see also [28]).

We have two natural questions:

Q.1 *Does the entropy $\mathcal{H}$ depend only on ξ, η and α, but not on the choice of the sequences?*

We conjecture that at least for irrational α this is the case.

Q.2 *What are the analytic properties of the entropy function $\mathcal{H}_\alpha(\xi, \eta)$ on the corresponding Newton polygon $\mathcal{N}_\alpha$?*

We conjecture that the entropy function $\mathcal{H}_\alpha(\xi, \eta)$ is strictly concave on the Newton polygon $\mathcal{N}_\alpha$ (and, in particular, it is continuous there).

We prove this now for the sequence $\rho_n = 1/n$ corresponding to Fibonacci polynomials. In that case the scaled Newton polygon is the triangle with vertices at $(0, 0), (1, 0), (0, 1)$.

Let us start with a well-known fact about binomial coefficients.

Let m_n be a sequence of positive integers such that $m_n/n \to p \in [0, 1]$ when $n \to \infty$.

Proposition 7.2 (Entropy Function for Binomial). *For any such sequence*

$$\lim_{n \to \infty} \frac{1}{n} \ln \binom{n}{m_n} = H(p),$$

$$(29) \qquad H(p) := -p \ln p - (1 - p) \ln(1 - p).$$

Proof. Recall the Stirling formula

$$n! \sim \sqrt{2\pi n}(n/e)^n,$$

implying that

$$(30) \qquad \ln n! = n \ln n - n + O(\ln n).$$

Using this we have

$$\frac{1}{n} \ln \binom{n}{m_n} = \frac{1}{n} \left[-m_n \ln \frac{m_n}{n} - (n - m_n) \ln \frac{n - m_n}{n} + O(\ln n) \right],$$

which turns to $H(p)$ when n goes to infinity. $\qquad \square$

To compute the entropy of the Fibonacci polynomials $M_{1/n}$ we can use the explicit formula (16) for the corresponding coefficients $A_{i,j}(1/n)$.

Proposition 7.3 (Entropy Function for Fibonacci Polynomials)**.**

$$\lim_{n\to\infty} \frac{1}{n} \ln A_{i_n,j_n}(1/n) = (1-\eta)H\left(\frac{\xi}{1-\eta}\right) + (\xi+\eta)H\left(\frac{\xi}{\xi+\eta}\right),$$

where $\xi = \lim_{n\to\infty} \frac{i_n}{n}, \eta = \lim_{n\to\infty} \frac{j_n}{n}$ *and* H *is given by* *(29)*.

Proof. Substituting $i_n = n\xi_n, j_n = n\eta_n$ into Eq. *(16)* we have

$$A_{i_n,j_n} = \binom{n(1-\eta_n)}{n(1-\xi_n-\eta_n)}\binom{n(\xi_n+\eta_n)}{n\eta_n}$$

and therefore

$$\frac{1}{n}\ln A_{i_n,j_n} = \frac{1}{n}\ln\binom{n(1-\eta_n)}{n(1-\xi_n-\eta_n)} + \frac{1}{n}\ln\binom{n(\xi_n+\eta_n)}{n\eta_n}.$$

Applying again formula *(30)* and using the fact that $\xi_n \to \xi, \eta_n \to \eta$ we have

$$\lim_{n\to\infty}\frac{1}{n}\ln A_{i_n,j_n} = (1-\eta)H\left(\frac{\xi}{1-\eta}\right) + (\xi+\eta)H\left(\frac{\xi}{\xi+\eta}\right),$$

with $H(p) = -p\ln p - (1-p)\ln(1-p) = H(1-p)$. □

Theorem 7.4. *The entropy function of Markov-Fibonacci polynomials*

$$(31) \qquad \mathcal{F}(\xi,\eta) := (1-\eta)H\left(\frac{\xi}{1-\eta}\right) + (\xi+\eta)H\left(\frac{\xi}{\xi+\eta}\right)$$

with H *given by* *(29)*, *is strictly concave on the region* $\xi, \eta > 0, \xi + \eta < 1$. *It is invariant under the change* $(\xi,\eta) \to (\xi, 1-\xi-\eta)$.

Proof. We will prove that the Hessian matrix

$$\mathrm{Hess}(\mathcal{F}) = \begin{pmatrix} \mathcal{F}_{\xi\xi} & \mathcal{F}_{\xi\eta} \\ \mathcal{F}_{\eta\xi} & \mathcal{F}_{\eta\eta} \end{pmatrix}$$

is negative definite. By Sylvester's criterion it is enough to show that $\mathcal{F}_{\xi\xi} < 0$ and $\det \mathrm{Hess}(\mathcal{F}) > 0$. Straightforward calculations lead to

$$\mathrm{Hess}(\mathcal{F}) = \begin{pmatrix} \frac{1}{\xi+\eta-1} - \frac{2}{\xi} + \frac{1}{\xi+\eta} & \frac{1}{\xi+\eta-1} + \frac{1}{\xi+\eta} \\ \frac{1}{\xi+\eta-1} + \frac{1}{\xi+\eta} & \frac{1}{\xi+\eta-1} + \frac{1}{1-\eta} - \frac{1}{\eta} + \frac{1}{\xi+\eta} \end{pmatrix}.$$

Clearly $\mathcal{F}_{\xi\xi} < 0$ since $\frac{1}{\xi+\eta-1} < 0$ and $\frac{1}{\xi} > \frac{1}{\xi+\eta} > 0$. Computing the determinant and simplifying we find

$$\det \mathrm{Hess}(\mathcal{F}) = \frac{1}{\eta(\xi+\eta)(1-\eta)(1-\xi-\eta)},$$

where all terms in the denominator are positive in the specified region and hence the determinant is positive as required. Therefore the function is strictly concave.

The symmetry $(\xi,\eta) \to (\xi, 1-\xi-\eta)$ is evident from formula *(31)*. □

The graph of the corresponding function $\mathcal{F}(\xi,\eta)$ is shown in Figure 10. The maximum value is $2\ln\frac{1+\sqrt{5}}{2}$ achieved at $\xi = \frac{1}{\sqrt{5}}$, $\eta = \frac{5-\sqrt{5}}{10}$.
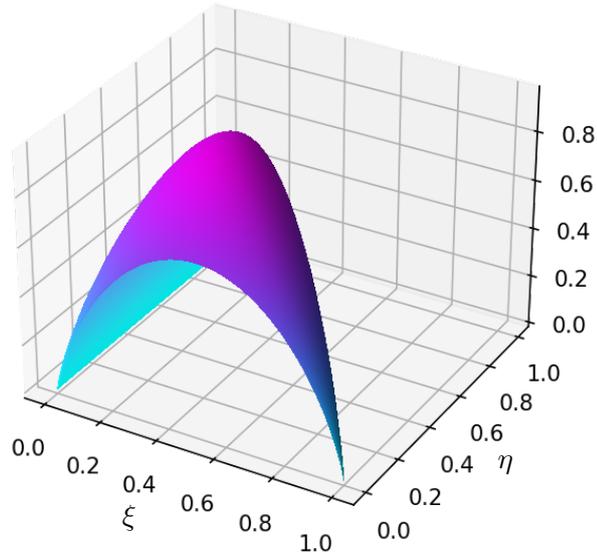
FIGURE 10. Graph of the entropy for Fibonacci polynomials.

8. CRITICAL TRIANGLE AND MARKOV SAILS

We can split up the Newton polygon by drawing 'critical lines' $i = a, j = b$ as shown in Figure 11. The set of lattice points with

$$i < a, \ j < b, \ \frac{i}{a} + \frac{j}{b} > 1$$
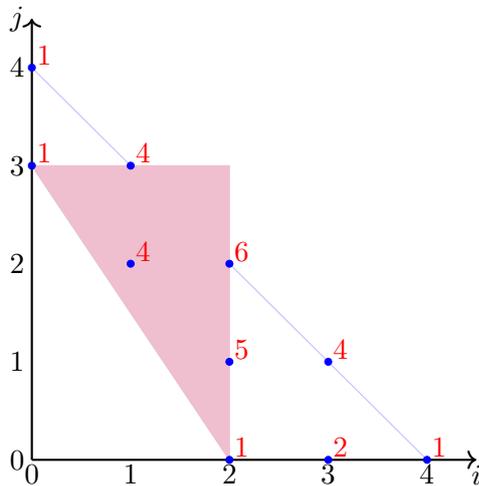
we will refer to as the *critical triangle*.

FIGURE 11. Critical triangle of the weighted Newton polygon Δ_{29}. Notice that per our definition, the lines $i = 2$ and $j = 3$ themselves are not considered as part of the critical triangle.

In this section we will look at Markov coefficients inside the critical triangle in more detail. We begin with the following conjecture based on the analysis of experimental data.

Conjecture 8.1. *(Factor 4 Conjecture) The coefficients of Markov polynomials strictly within the critical triangle are divisible by 4.*

We are going to be more specific about the coefficients on the lower boundary of the critical triangle, using the following geometric interpretation of continued fractions proposed by Felix Klein.

Take the region $[0, a] \times [0, b]$ for some positive integers a, b and divide it into two parts with the line $y = \frac{b}{a}x$. Then consider the convex hull of integer points in each of the two parts, the boundary either side of the line $y = \frac{b}{a}x$ will be a broken line that is called, following Arnold, the *sail* (see [2, 18]). More precisely,

- The vertices on the sail containing the point $(1, 0)$ are defined to be $A_0, A_1, \ldots, A_n$ (where $A_0 = (1, 0)$). This is the lower sail (i.e., below the line).
- The vertices on the sail containing the point $(0, 1)$ are defined to be $B_0, B_1, \ldots, B_n$ (where $B_0 = (0, 1)$). This is the upper sail (i.e., above the line).

One can prove that these vertices on the sails can be found as the convergents of the continued fraction for $\frac{b}{a}$ (see for example [18]).

In particular, we have the following procedure to find the Klein diagram. Plot points (q_k, p_k) to represent the continuants C_k alongside the line $y = \frac{b}{a}x$, then connect with straight lines the the points (q_k, p_k) and (q_{k+2}, p_{k+2}). Note that the points representing even-indexed continuants will all lie above the line $y = \frac{b}{a}x$ whilst the odd-indexed ones will lie below.

Example 8.2. Consider the rational $\frac{5}{3} = [1, 1, 2]$, its continuants are given in the following table.

			1	1	2
p_k	0	1	1	2	5
q_k	1	0	1	1	3

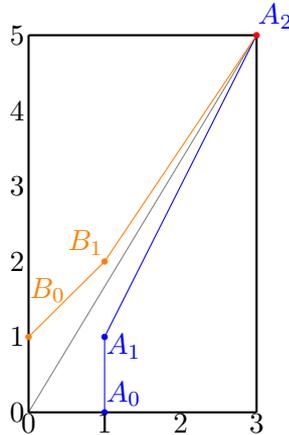$(C_{-1} = \frac{0}{1}, C_0 = \frac{1}{0}, C_1 = \frac{1}{1}, C_2 = \frac{2}{1}, C_3 = \frac{5}{3})$.

FIGURE 12. Klein Diagram for the rational $\frac{5}{3}$

More formally we can define A_i's and B_i's as follows,

$$(32) \qquad A_i = (q_{2i-1}, p_{2i-1}), \qquad B_i = (q_{2i}, p_{2i}),$$

where $\frac{p_k}{q_k} = [a_1, a_2, \ldots, a_k]$ is the kth convergent. Note that for rationals we always consider the final continuant (i.e., the point (a, b)) to be part of both sails.

There may be more integer points lying on the sail, indeed in the Example 8.2 the point $(2, 3)$ lies on the line between A_1 and A_2. These additional integer points can actually be found immediately if we instead look at the convergents of the Hirzebruch (negative) continued fraction instead, as it was shown by Ustinov in [31].

From the theory of continued fractions, we know

$$p_k = a_k p_{k-1} + p_{k-2}, \qquad q_k = a_k q_{k-1} + p_{k-2}.$$

This means that the line joining the second-last convergent's coordinates (p_{k-2}, q_{k-2}) to the corner will have gradient equal to the penultimate convergent. This means that if we flip one of the sails, so that they are both on the same side of the diagonal, they will intersect at the point corresponding to the final convergent. In this way we can combine the sails onto one side. More explicitly, if we consider the coordinate system of the lower sail (blue) to have origin at (a, b) and directions reversed from normal then we can connect the sails. In practise, continuing on from Example 8.2 this is shown on the left in Figure 13.
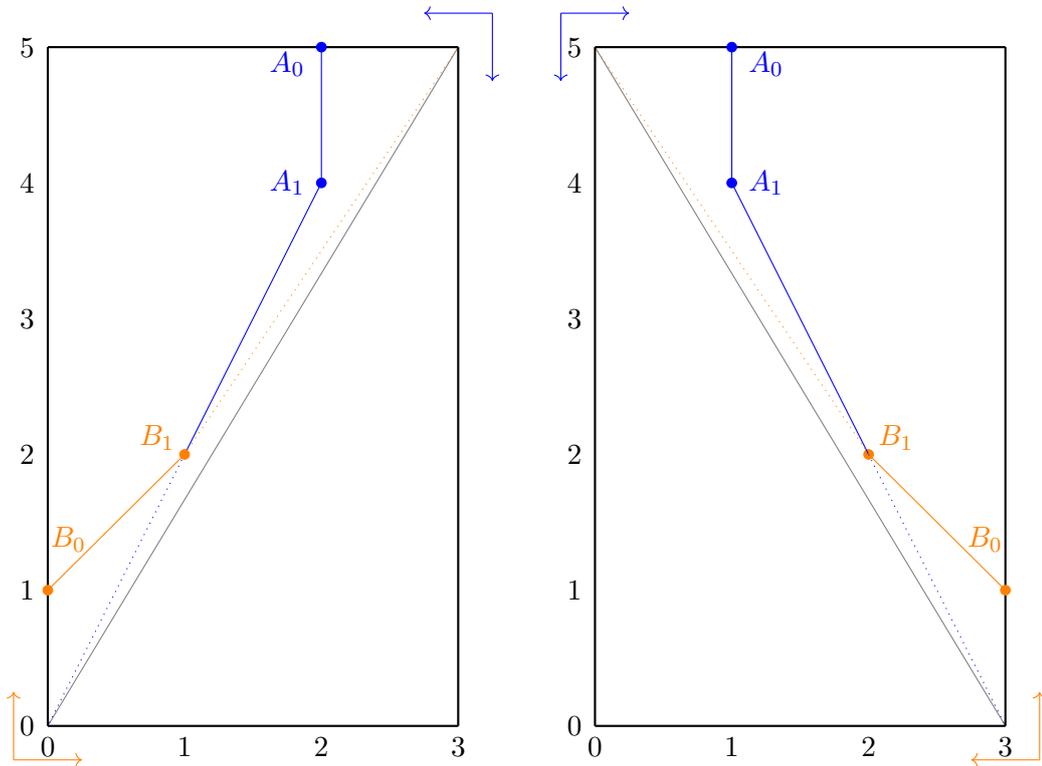

FIGURE 13. (Left) Combined sail for the rational $\frac{5}{3}$; (Right) the reflection in the line $x = a/2$.

This is precisely the convex hull of the integer lattice above the line $y = \frac{b}{a}x$ (with endpoints $(0, 0), (3, 5)$ omitted).

From Eq. (11) it is clear to see that the critical triangle of the Newton polygon consists of a region $[0, a] \times [0, b]$ split *via* a central diagonal, which is comparable to that of the Klein diagram. Furthermore, we are again only interested in integral points.

Indeed, reflecting the leftmost picture in Figure 13 in the line $x = \frac{a}{2}$ we obtain the critical triangle, where coordinate frames have been shown in their corresponding colours.

The integer points on this combined sail all refer to a monomial in the Markov polynomial and as such they each have their own coefficient. We will refer to this combined sail, together with the coefficients at each point as the *Markov sail*.

With our change of coordinate frames, we reformulate the definition of points from Eq. (32) to

$$\widetilde{A_i} = (q_{2i-1}, b - p_{2i-1}), \qquad \widetilde{B_i} = (a - q_{2i}, p_{2i}),$$

but going forward we will omit the tilde's for simplicity.

8.1. **Markov Sail Duality.** There is an important *Edge-Angle duality* between the two sails, saying that the length of a unbroken line on one sail is equal to the 'index' at a vertex/rational angle on the opposite sail, see Karpenkov [18].

Here the *integer length* (denoted $\mathrm{l}\ell(AB)$) of a line segment AB is defined to be the number of integer points on the its interior plus 1, and the *index* (denoted $\mathrm{l}\alpha(\angle BAC)$) of a rational angle $\angle BAC$ is defined to be the index of the sublattice generated by the integer vectors of the lines AB and AC in the integer lattice, see [18].

Formally, Karpenkov stated this duality as follows

$$\mathrm{l}\alpha(\angle A_i A_{i+1} A_{i+2}) = \mathrm{l}\ell(B_i B_{i+1}) = a_{2i+2},$$
$$\mathrm{l}\alpha(\angle B_i B_{i+1} B_{i+2}) = \mathrm{l}\ell(A_{i+1} A_{i+2}) = a_{2i+3},$$

where a_i's are the partial quotients of the continued fraction $\frac{b}{a} = [a_1, a_2, \dots]$.

We conjecture that in our case (for the so called Markov sails) we have a sort of duality between the 'weights' of these vertices/lines on the sail. Denote $\mathcal{M}(C)$ to be the coefficient of the monomial represented at the point C on the Newton polygon, we will refer to these as $\mathcal{M}$-*values* and define this only for integer points in the interior of the critical triangle (i.e., not on the boundary).

It appears as though, on unbroken lines of the sail, the coefficients differ according to an arithmetic progression. Denote $d(C_i C_{i+1})$ to be the common difference of $\mathcal{M}$-values of integer points along an unbroken line segment of the Markov sail between vertices C_i and C_{i+1}. We conjecture the following extended duality.

Conjecture 8.3. *(Markov Sail Duality) The coefficients of Markov polynomials satisfy Markov sail duality*

$$d(B_i B_{i+1}) = -\mathcal{M}(A_{i+1})$$
$$d(A_{i+1} A_{i+2}) = -\mathcal{M}(B_{i+1})$$

This conjecture will provide us with a method for determining almost all $\mathcal{M}$-values on the sail if we having some starting value of which to work from. This is because we known how many integer points lie on each unbroken line from the duality, Corollary 3.1 in [18], so can compute $\mathcal{M}$-values using the equations from this conjecture from some initial condition.

With regard to this initial starting value, we claim that the $\mathcal{M}$-value corresponding to the point of the penultimate continuant is always 4.

Conjecture 8.4. *(Location of 4) Consider the continued fraction $\frac{b}{a} = [a_1, a_2, \dots, a_n]$. If $n = 2m + 1$ (odd) then $\mathcal{M}(B_m) = 4$. If $n = 2m$ (even) then $\mathcal{M}(A_m) = 4$.*

From this value we would then be able to work backwards (alternating between each side of the sail) to recover the $\mathcal{M}$-values on most of the sail (all but on line $A_0 A_1$ if there are any integer points on its interior). To see this, consider the following example.

Example 8.5. $\frac{b}{a} = \frac{18}{13} = [1, 2, 1, 1, 2]$

			1	2	1	1	2
p_k	0	1	1	3	4	7	18
q_k	1	0	1	2	3	5	13

According to Eq. (8) we therefore have $A_0 = (1, 18)$, $A_1 = (1, 17)$, $A_2 = (3, 14)$ and $B_0 = (13, 1)$, $B_1 = (11, 3)$, $B_2 = (8, 7)$.
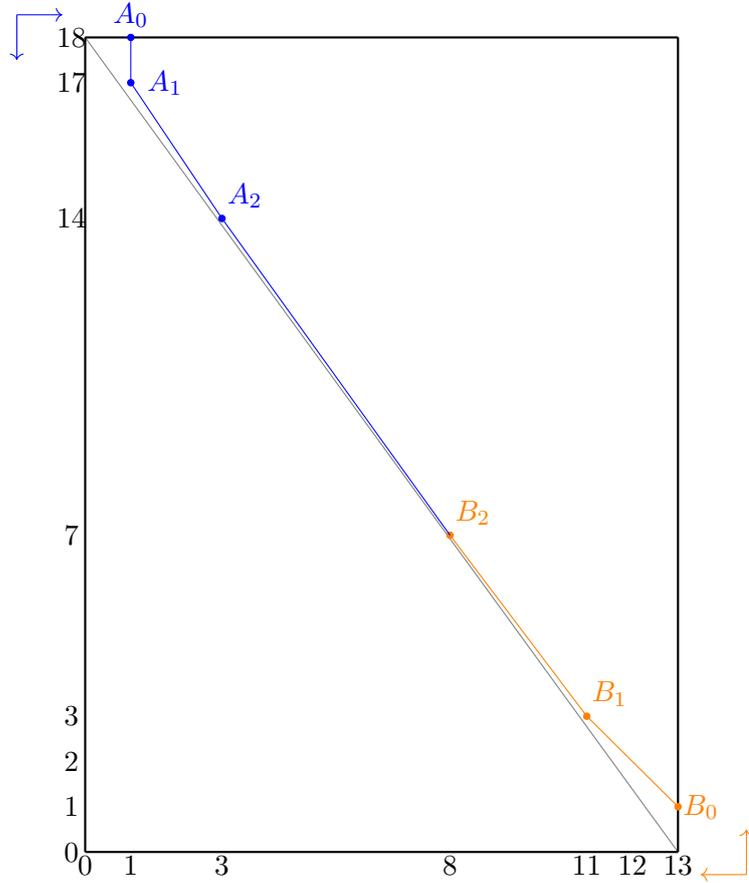

FIGURE 14. Markov sail for Example 8.5.

According to Conjecture 8.4, since the continued fraction has odd length ($n = 5, m = 2$) we should have $m(B_2) = 4$. Now using Conjecture 8.3, we should have

$$d(A_2 A_3) = -\mathcal{M}(B_2)$$
$$d(B_1 B_2) = -\mathcal{M}(A_2)$$
$$d(A_1 A_2) = -\mathcal{M}(B_1)$$
$$d(B_0 B_1) = -\mathcal{M}(A_1).$$

Note the integer lengths

$$l\ell(A_2 A_3) = a_5 = 2$$
$$l\ell(B_1 B_2) = a_4 = 1,$$
$$l\ell(A_1 A_2) = a_3 = 1,$$
$$l\ell(B_0 B_1) = a_2 = 2.$$

Noting that B_m is the penultimate integer point before A_{m+1} on the line $A_m A_{m+1}$ for n odd (this will always be the case), we have

$$\mathcal{M}(A_2) = \mathcal{M}(B_2) + (a_5 - 1)\mathcal{M}(B_2) = 8,$$
$$\mathcal{M}(B_1) = \mathcal{M}(B_2) + a_4\mathcal{M}(A_2) = 12,$$
$$\mathcal{M}(A_1) = \mathcal{M}(A_2) + a_3\mathcal{M}(B_1) = 20.$$

Since B_0 is not in the interior of the critical triangle we cannot apply to conjecture to find its $\mathcal{M}$-value. However, since the integer length of $B_0 B_1$ is 2 there will be an integer point on the interior of this line (which will lie in the interior of the critical triangle). This interior point however can be found from the conjecture and should have $\mathcal{M}$-value equal to

$$\mathcal{M}(B_1) + \mathcal{M}(A_1) = 32.$$

Checking numerically, all of the $\mathcal{M}$-values calculated in this example agree with the actual $\mathcal{M}$-values, supporting the conjectures in this case.

9. Sails of Pell polynomials

Returning to our 'simplest' cases of Markov polynomials, we observe that the critical triangle (and hence sail) is empty for Fibonacci Markov polynomials (as $a = 1$). However, in the case of Markov polynomials corresponding to Pell numbers we can prove some of our earlier conjectures using Eq. (22).

The first of which is the presence and location of the coefficient 4. According to our conjecture the coefficient 4 should lie at coordinates corresponding to the penultimate convergent of $\frac{b}{a}$. Here

$$\frac{b}{a} = \frac{n+1}{n} = [1, n]$$

So the penultimate convergent is $[1] = \frac{1}{1}$ and we expect the location of the 4 to be at $(1, n)$ (1 in each direction from corner $(0, n+1)$ since continued fraction is even length).

Theorem 9.1. *For Markov polynomials M_ρ, where $\rho = \frac{n}{n+1}$, the coefficient at the coordinate $(1, n)$ in the Newton polygon is 4.*

Proof. Looking at $i = 1, j = n$ in Eq. (22), we find

$$(33) \qquad A_{1,n}^{(2n+1)} = \left[A_{-1,n}^{(2n-1)} + 2A_{0,n-1}^{(2n-1)} + A_{1,n-2}^{(2n-1)} \right] + \left[A_{0,n}^{(2n-1)} + A_{1,n-1}^{(2n-1)} \right] - A_{0,n-1}^{(2n-3)}.$$

Clearly if i or j is negative the coefficient will be 0 however this is not the only case where the coefficient is 0. The lower vertices of the Newton polygon for R_{2n-1} are at $(0, n)$ and $(n - 1, 0)$ therefore $A_{0,n-1}^{(2n-1)}, A_{1,n-2}^{(2n-1)}$ will also be 0 (they are outside the boundaries of the Newton polygon - can be seen rigorously by showing $\frac{i}{n-1} + \frac{j}{n} \le 1$). This leaves the final 3 terms; $A_{0,n}^{(2n-1)}$ and $A_{0,n-1}^{(2n-3)}$ are vertices of their respective Newton polygons and are known to be 1 (and cancel each other out) so we are left with $A_{1,n}^{(2n+1)} = A_{1,n-1}^{(2n-1)}$. So all we need to do is prove that it holds in the base case ($n = 2$, since critical triangle is empty in $n = 1$ case). Direct computation of $M_{2/3}$ (corresponding to the Markov number 29) shows that this is indeed the case (see Figure 11). $\qquad \square$

Combining our conjectures regarding coefficients on the sail, in the case $M_{n/n+1}$ the integer points on the sail should have coefficients following an arithmetic progression with common difference 4.

Theorem 9.2. *For Markov polynomials M_ρ where $\rho = \frac{n}{n+1}$, the coefficients at the coordinates $(m, n + 1 - m)$ in the Newton polygon are $4m$, for $m = 1, 2, \ldots, n - 1$.*

Proof. For $m = 1$ the result is proven - it is precisely Theorem 9.1.

To complete the proof we need to consider two seperate cases; $m = 2, \ldots, n - 2$ and $m = n - 1$. In both cases, Eq. (22) becomes

$$A_{m,n+1-m}^{(2k+1)} = \left[A_{m-2,n+1-m}^{(2k-1)} + 2A_{m-1,n-m}^{(2k-1)} + A_{m,n-m-1}^{(2k-1)} \right]$$
$$+ \left[A_{m-1,n+1-m}^{(2k-1)} + A_{m,n-m}^{(2k-1)} \right] - A_{m-1,n-m}^{(2k-3)}.$$

For $m = 1, 2, \ldots, n - 1$, the first 3 terms on the RHS are 0 (outside the Newton polygon), leaving

$$A_{m,n+1-m}^{(2k+1)} = \left[A_{m-1,n+1-m}^{(2k-1)} + A_{m,n-m}^{(2k-1)} \right] - A_{m-1,n-m}^{(2k-3)}.$$

By induction this becomes $A_{m,n+1-m}^{(2k+1)} = \left[4(m-1) + 4m \right] - 4(m-1) = 4m$, where we can check the first two (relevant) cases directly $k = 2, 3$.

For $m = n - 1$, the term $A_{m,n-m-1}^{(2k-1)} = A_{n-1,0}^{(2k-1)}$ does not disappear, in fact it corresponds to a lower boundary point and equals 1 and also the coefficients $A_{m,n-m}^{(2k-1)} = A_{n-1,1}^{(2k-1)}$ and $A_{m-1,n-m}^{(2k-3)} = A_{n-2,1}^{(2k-3)}$ lie on the boundary of the critical triangle (not in the interior so we can use the same inductive argument).

However, Theorem 4.2 tells us that the first point on the first horizontal ($j = 1$) has coefficient $A_{n-1,1}^{(2k-1)} = 3(n-1) - 1$ and $A_{n-2,1}^{(2k-3)} = 3(n-2) - 1$ and so Eq. (22) becomes

$$A_{n-1,2}^{(2k+1)} = A_{n-1,0}^{(2k-1)} + \left[A_{n-2,2}^{(2k-1)} + A_{n-1,1}^{(2k-1)} \right] - A_{n-2,1}^{(2k-3)}$$
$$= 1 + \left[4(m-1) + (3(n-1) - 1) \right] - (3(n-2) - 1) = 4m$$

as required. $\square$

So now we have proven the coefficients on the whole of the interior of the sail. As mentioned we also know one of the boundary coefficients of the sail, namely $A_{n,1}^{(2k+1)} = 3n - 1$. If we can determine the other boundary $A_{1,n+1}^{(2k+1)}$ then we will know the sail coefficients in their entirety.

Again using Eq. (22),

$$A_{1,n+1}^{(2k+1)} = \left[A_{-1,n+1}^{(2k-1)} + 2A_{0,n}^{(2k-1)} + A_{1,n-1}^{(2k-1)} \right] + \left[A_{0,n+1}^{(2k-1)} + A_{1,n}^{(2k-1)} \right] - A_{0,n}^{(2k-3)}.$$

Since

$$A_{-1,n+1}^{(2k-1)} = 0, \; A_{0,n}^{(2k-1)} = 1, \; A_{1,n-1}^{(2k-1)} = 4, \; A_{0,n+1}^{(2k-1)} = n - 2, \; A_{0,n}^{(2k-3)} = n - 3,$$

we have

$$A_{1,n+1}^{(2k+1)} = \left[0 + 2(1) + 4 \right] + \left[n - 2 + A_{1,n}^{(2k-1)} \right] - (n - 3) = A_{1,n}^{(2k-1)} + 7.$$

Computing $A_{1,3}^{(3)} = 4$, the solution to this recursion is $A_{1,n+1}^{(2k+1)} = 7n - 10$, so now we have the weights on the sail in its entirety: from top to bottom $(7n - 10, \; 4, \; 8, \; \ldots, \; 4n - 4, \; 3n - 1)$.

10. Markov polynomials via generalised Cohn matrices

Harvey Cohn [6] proposed the following remarkable way to compute Markov numbers. He suggested to replace the addition of integer numbers in $u + v = w$ in the Euclid tree by multiplication of 2×2 matrices, starting from a special pair.

More precisely, using the Conway topograph we define the *Cohn tree* as in Figure 15,
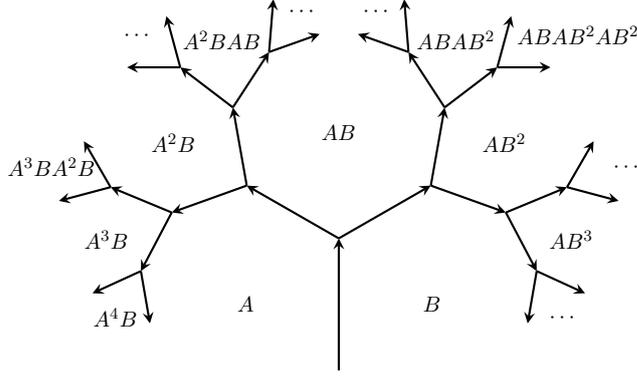
FIGURE 15. The Cohn tree.

starting with the *Cohn matrices*

$$(34) \qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}.$$

The relation between Cohn and Markov trees is given by the trace map (see [1, 29]):

$$C \mapsto m = \frac{1}{3} \operatorname{tr} C,$$

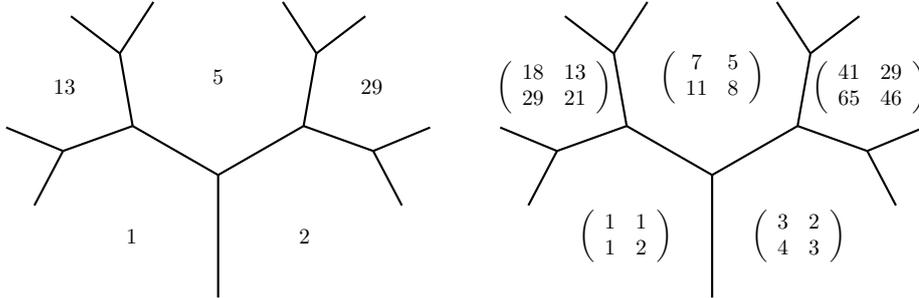or, by taking the off-diagonal element $m = C_{12}$ (see Fig. 16).

FIGURE 16. Cohn and Markov trees related by trace map

Define now the following *generalised Cohn matrices* as
$$(35)$$
$$A = C_{0/1} := \begin{pmatrix} y^2 & x \\ xy^2 & x^2 + z^2 \end{pmatrix}, \ B = C_{1/1} := \begin{pmatrix} x^2y^2 + y^4 + y^2z^2 & x^3 + xy^2 \\ x^3y^2 + xy^4 + 2xy^2z^2 & x^4 + x^2y^2 + x^2z^2 \end{pmatrix},$$

and consider the corresponding generalisation of Cohn tree (15). We claim that the analogue of Cohn's construction leads to the Markov polynomials.

Theorem 10.1. *The numerator $P_{a/b}(x^2, y^2, z^2)$ of the corresponding Markov polynomial (7) can be found from Cohn matrices as*

$$(36) \qquad P_{a/b}(x^2, y^2, z^2) = \frac{\operatorname{tr} C_{a/b}}{x^2 + y^2 + z^2},$$

or, as the upper right entry of $C_{a/b}$ divided by x.

The proof is a modification of the arguments from Chapter 4 of Aigner's book—see [9] for the details.

Note in the special case $u = q, v = 1, w = q^2$ our Cohn matrices are conjugated to Cohn matrices introduced in [?]. Deformations of Cohn matrices from [10, 19] lead to different modifications of the Markov equation.

Since all the coefficients of the entries of the Cohn matrices are positive, as a corollary we have an independent proof of the non-negativity of Markov coefficients [27].

Corollary 10.2. *All the coefficients of the Markov polynomials are non-negative.*

It would be interesting to extend this to the proof of the saturation conjecture.

The usual Cohn matrices A and B are known to generate the commutator subgroup of the modular group $\mathrm{PSL}(2, \mathbb{Z})$ (see [1]).

A natural question is if the same is true for the Laurent analogue of the modular group $\Gamma_L = \mathrm{PSL}(2, \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}])$, which is the quotient of the group $\mathrm{SL}(2, \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}])$ consisting of the matrices

$$g = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A_{ij} \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}], \quad \det g = x^i y^j z^k, \, i, j, k \in \mathbb{Z}$$

by the monomial subgroup consisting of $\pm x^i y^j z^k I$, $i, j, k \in \mathbb{Z}$.

Consider the following deformation of the standard generators of the modular group

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

defined by

$$(37) \qquad \widetilde{L} = \begin{pmatrix} y & 0 \\ xy & z \end{pmatrix}, \quad \widetilde{R} = \begin{pmatrix} z & 1 \\ 0 & x \end{pmatrix}, \quad \widetilde{S} = \begin{pmatrix} 0 & 1 \\ -y^2 & 0 \end{pmatrix}.$$

Their choice is motivated by the desire to keep the commutator relations between the Cohn matrices and the generators L, R, S of the modular group:

$$A = [L, S], \quad B = [L, S][R, S],$$

where $[x, y] := xyx^{-1}y^{-1}$ is the group commutator. Note that $\widetilde{L}, \widetilde{R}, \widetilde{S}$ have determinants yz, xz, y^2 respectively and thus do not generate the whole group Γ_L. It would be interesting to find a good set of generators of Γ_L.

Proposition 10.3. *Generalised Cohn matrices (36) belong to the commutator of the Laurent modular group Γ_L and satisfy the same relations as in the classical case:*

$$A = \left[\widetilde{L}, \widetilde{S} \right], \quad B = \left[\widetilde{L}, \widetilde{S} \right] \left[\widetilde{R}, \widetilde{S} \right],$$

The proof follows from the matrix equalities

$$\left[\widetilde{L}, \widetilde{S} \right] = y^{-1}z^{-1} \begin{pmatrix} y^2 & x \\ xy^2 & x^2 + z^2 \end{pmatrix}, \quad \left[\widetilde{R}, \widetilde{S} \right] = x^{-1}z^{-1} \begin{pmatrix} y^2 + z^2 & x \\ xy^2 & x^2 \end{pmatrix},$$

$$\left[\widetilde{L}, \widetilde{S} \right] \left[\widetilde{R}, \widetilde{S} \right] = x^{-1}y^{-1}z^{-2} \begin{pmatrix} x^2y^2 + y^4 + y^2z^2 & x^3 + xy^2 \\ x^3y^2 + xy^4 + 2xy^2z^2 & x^4 + x^2y^2 + x^2z^2 \end{pmatrix}.$$

Note that we can start the Cohn tree with the matrices

$$A = \begin{pmatrix} y^2 & x \\ xy^2 & x^2 + z^2 \end{pmatrix}, \quad B' = \begin{pmatrix} y^2 + z^2 & x \\ xy^2 & x^2 \end{pmatrix},$$

which are certain quantisations of Arnold's cat-matrices. They generate a subgroup in the commutator of Γ_L, which could be of special interest.

11. Concluding remarks

As we have seen, there are many questions about Markov polynomials, which are still to be answered. Already the question if they provide (up to change of sign of any two of them) all integer Laurent-polynomial solutions to the generalised Markov equation (2) is still open. This can be considered as a Laurent-polynomial analogue of the strong approximation conjecture by Bourgain, Gamburd and Sarnak [3] about connectivity of the Markov graph of the mod-p solutions of the Markov equation (see recent progress in this direction in [8]).

Note that from Theorem 3.1 it follows that all Markov polynomials are distinct, which implies that for a generic choice of the hyperbolic structure on a once-punctured torus, no two simple closed geodesics have the same length [27]. On the relation with hyperbolic geometry and the Teichmüller theory we refer to [12, 13, 26].

We have shown also that there are very plausible conjectures about the coefficients of Markov polynomials. They can be reformulated combinatorially using the weighted perfect matching interpretation of these coefficients proposed in [17, 27, 5]. It would be interesting to explore if this combinatorial interpretation can help to prove some of them (in particular, the Factor-4 conjecture).

Finally, the Laurent version Γ_L of the modular group seems to be interesting in its own right and deserves a more detailed study (generators, relations, commutator, etc.).

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