Anti-classification for flows on two-tori

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We prove that the classification of real-analytic vector fields on the two-torus up to orbital topological equivalence does not admit a complete numerical invariant that is a Borel function. Moreover, smooth vector fields that are difficult to classify appear in generic smooth 7-parameter families. In dimension 2, this improves the recent result of A. Gorodetski and M. Foreman [6] for non-classifiability of smooth diffeomorphisms up to continuous conjugacy.

1 Introduction

Classification results constitute one of the central parts of the modern theory of dynamical systems. For example, due to Denjoy theorem, rotation number is a complete invariant that classifies C^2-smooth circle diffeomorphisms $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ without periodic orbits up to continuous conjugacy. Separatrix skeletons, graphs, or schemes are used to classify planar vector fields up to orbital topological equivalence. Ornstein's theorem [14] states that entropy is a complete invariant that classifies, up to measure-preserving transformation, Bernoulli shifts on closed subsets of the space of bi-infinite sequences.

On the other hand, Yoccoz's example shows that rotation number cannot be used to classify circle diffeomorphisms up to *smooth* conjugacy in the case when the rotation number is Liouville. P. Kunde showed [13] that smooth conjugacy on the space of circle diffeomorphisms admits no complete numerical invariant that is a Borel function. This is an *anti-classification* result that captures the complicated nature of the equivalence relation.

Strong anti-classification results were obtained in ergodic theory. Consider the space X of C^∞-smooth diffeomorphisms of a torus. Let the equivalence relation be a measure-preserving conjugacy. In [8], M. Foreman and B. Weiss proved that this equivalence relation is not *Borel*: the set $\{(S,T) \in X \times X \mid S \sim T\}$ is not Borel with respect to the C^∞-topology in $X \times X$. Earlier in [7], M. Foreman, D. Rudolph, and B. Weiss proved that measure-isomorphism for measure-preserving ergodic maps on the interval is not a Borel equivalence relation. In [9], M. Gerber and P. Kunde proved that Kakutani equivalence relation for ergodic measure-preserving transformations is also not Borel.

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In Sec. 2, we will introduce Borel reducibility, the partial order on equivalence relations that produces the hierarchy of equivalence relations (see also [5]). For many natural equivalence relations, their place in this hierarchy is not known. An important breakthrough was a paper by M. Sabok [15] who showed that the isomorphism of separable C^* algebras is the maximal equivalence relation among all orbit equivalence relations. J. Zielinski [16] showed that the homeomorphism of compact metric spaces is also maximal among all orbit equivalence relations. It is an open question whether measure-isomorphism for measure-preserving ergodic maps has the same property.

One of the natural equivalence relations in dynamical systems theory is continuous conjugacy. In the space of diffeomorphisms, A. Gorodetski and M. Foreman [6] showed that this equivalence relation for smooth diffeomorphisms of $\mathbb{R}^2$ has no complete Borel numerical invariants. Moreover, for diffeomorphisms on $\mathbb{R}^5$, this equivalence relation is not Borel[1]. However, proofs involve classification of diffeomorphisms that are highly degenerate. Related results were obtained in the space of continuous interval maps and circle maps, see [1] and references therein.

Planar vector fields can be classified up to orbital topological equivalence using a combinatorial invariant (in the form of separatrix skeletons, schemes, or Leontovich-Mayer-Fedorov graphs). Classification of vector fields on the torus is more complicated: since circle maps can appear as Poincare maps, classification invariant should incorporate both the information about the behavior of separatrices and the rotation number of the Poincare map. We will see that this is sufficient to obtain non-classifiability results similar to [6, Theorem 2].

The proofs are not directly related to, but largely inspired by, new examples in the modern bifurcation theory for planar vector fields that arise from sparkling separatrix connections, see [12].

Let $\mathcal{V}^2(T^2)$ be the space of C^2-smooth vector fields on the two-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $\mathcal{V}^\omega(T^2)$ be the space of real-analytic vector fields on the two-torus.

Definition 1. Two vector fields $v, w \in \mathcal{V}^2(T^2)$ are orbitally topologically equivalent, $v \sim w$, if there exists a homeomorphism $H\colon T^2 \to T^2$ that is homotopic to identity, such that H takes orbits of v to orbits of w, preserving time orientation.

The main results of the paper are the following.

Theorem 2. *Orbital topological equivalence in $\mathcal{V}^2(T^2)$ has no complete Borel numerical invariant: there is no Borel function $g\colon \mathcal{V}^2(T^2) \to Y$ with Y a Polish space such that for all $v, w \in X$, we have $v \sim w$ if and only if $g(v) = g(w)$.*

Theorem 3. *Orbital topological equivalence in $\mathcal{V}^\omega(T^2)$ has no complete Borel numerical invariant: there is no Borel function $g\colon \mathcal{V}^\omega(T^2) \to Y$ with Y a Polish space such that for all $v, w \in X$, we have $v \sim w$ if and only if $g(v) = g(w)$.*

By a classical Kuratowski's theorem, all uncountable Polish spaces are Borel isomorphic. In particular, any Polish space Y is Borel isomorphic to $\mathbb{R}$, thus we refer to these statements as the absence of *numerical* invariants for orbital topological equivalence. Results also imply that there are no complete functional invariants in any Polish functional space.

Below we will formulate and prove a stronger version of Theorem 2: vector fields that are difficult to classify appear in a generic 7-parameter family, see Theorem 14.

[1] In [6], authors announced stronger results, but they were not published as of 07/2025.

Remark 4. *While circle diffeomorphisms appear as first-return maps for vector fields on the torus, result of [13] does not imply Theorem 2, since we consider a different equivalence relation.*

Remark 5. *While time-1 flows of vector fields $v \in \mathcal{V}(T^2)$ are diffeomorphisms of T^2, continuous conjugacy for resulting diffeomorphisms does not coincide as equivalence relation to orbital topological equivalence of corresponding vector fields. So Theorem 2 does not imply [6, Theorem 2]. However, equivalence of vector fields is considered to be much simpler than equivalence of planar diffeomorphisms (e.g. Newhouse phenomenon does not happen for flows of vector fields). So in a sense, our result is stronger than [6, Theorem 2]. Also, methods of [6] do not allow analytic diffeomorphisms, in contrast with Theorem 3.*

Recall that an equivalence relation $\sim$ on a set X is not Borel if the set $\{(S,T) \in X \times X \mid S \sim T\}$ is not Borel. Even though equivalence relations in Theorems 2, 3 do not admit Borel numerical invariants, it is likely that they are Borel. We already cited results [8],[7], [9] on non-Borel equivalence relations that naturally appear in dynamics. In particular, [6, Theorem 1] states that continuous conjugacy defines a non-Borel equivalence relation on diffeomorphisms of $\mathbb{R}^5$. The following questions are open.

Can we find a two-dimensional manifold M such that the orbital topological equivalence of C^ω vector fields on M is not Borel?

Can we find a generic finite-parameter family of vector fields v_ρ, $\rho \in \mathbb{R}^k$ on a two-dimensional manifold such that the orbital topological equivalence is not Borel: the graph $\{(\rho_1, \rho_2) \in \mathbb{R}^{2k} \mid v_{\rho_1} \sim v_{\rho_2}\}$ of the orbital topological equivalence relation on this family is not a Borel set?

We refer the reader to [2] for the list of open questions in descriptive set theory related to dynamical systems.

2 Preliminaries: Borel reduction

We refer the reader to [5] for an expository introduction to Borel reduction and the hierarchy of equivalence relations with respect to the Borel reduction.

Recall that a topological space X is called *Polish* if it is separable and completely metrizable (i.e. admits a complete metric that is compatible with the topology). The σ-algebra of *Borel sets* of X is the smallest σ-algebra containing all open sets. A map $f\colon X \to Y$ between two topological spaces is called *Borel* if for any open set A, $f^{-1}(A)$ is a Borel set.

Consider an equivalence relation on the set X. We write $x \sim_E y$ if x, y are equivalent with respect to E.

Definition 6. An equivalence relation E on X is *smooth* if there exists a Polish space Y and a Borel function $f\colon X \to Y$ such that $x \sim_E y$ holds if and only if $f(x) = f(y)$ for all $x, y \in X$.

Theorem 2 means that orbital topological equivalence of vector fields is non-smooth.

We will use the following non-smooth equivalence relation.

Definition 7. Let E_α be the equivalence relation on the circle $\mathbb{R}/\mathbb{Z}$ given by $x \sim_{E_\alpha} y$ if $x = y + n\alpha \mod 1$.

The following proposition is the particular case of the general result (see [11, Theorem 1.1]): *an equivalence relation that admits a non-atomic ergodic measure is not smooth.* This statement is a part of the Glimm-Effros dichotomy, first discovered in [10], [4] for group actions. (Here, a finite Borel measure on X is called ergodic with respect to an equivalence relation E if any measurable E-invariant set has zero or full measure; a measure is called non-atomic if the measure of each E-equivalence class is zero.) For completeness, we will give an elementary proof of the proposition below.

Proposition 8. *For any $\alpha \notin \mathbb{Q}$, for any nonempty open interval $I \subset \mathbb{R}/\mathbb{Z}$, E_α is a non-smooth equivalence relation both on I and on $I \setminus \{n\alpha\}_{n\in\mathbb{Z}}$.*

Proof. Let $R_\alpha(x) = x+\alpha$ be a rotation on a circle $\mathbb{R}/\mathbb{Z}$. If there is a Borel numerical invariant $f\colon I \to \mathbb{R}$ for E_α, then the sets $A_y = f^{-1}((-\infty, y)) \subset I$ must be Borel, and thus Lebesgue measurable. Since every set A_y is an intersection of R_α-invariant measurable set with I, and R_α is ergodic, the measure of each set A_y is equal to 0 or to $\mu(I)$. Let x be the supremum of the set $\{y \in \mathbb{R}, \mu(A_y) = \mu(I)\}$. Since $\mu(\bigcap A_n) = \mu(\varnothing) = 0$ and $\mu(\bigcup A_n) = \mu(I)$, x is a finite real number. Then $\mu(f^{-1}((-\infty, x + 1/n)) = \mu(I)$ for any n and thus $\mu(f^{-1}((-\infty, x]) = \mu(I)$. On the other hand, $\mu(f^{-1}((-\infty, x - 1/n)) = 0$ for any n and thus $\mu(f^{-1}((-\infty, x)) = 0$. We conclude that $f^{-1}(x)$ has measure $\mu(I)$. This is impossible since $f^{-1}(x)$ is the intersection of a single orbit of an irrational rotation with I and has measure 0. The proof for $I \setminus \{n\alpha\}_{n\in\mathbb{Z}}$ is analogous. □

Our main tool is a Borel reduction for equivalence relations.

Definition 9. Let $E \subset X \times X$ and $F \subset Y \times Y$ be equivalence relations on Polish spaces X and Y, respectively. A Borel function $f\colon X \to Y$ is called a *Borel reduction* of E to F if for all $x_1, x_2 \in X$, we have that $x_1 \sim_E x_2$ if and only if $f(x_1) \sim_F f(x_2)$.

We say that E is Borel reducible to F, and write $E \leq_B F$.

Informally, if E is Borel reducible to F, then F is "not less complicated" than E. In particular, an equivalence relation E is smooth if and only if $E \leq_B =_\mathbb{R}$, where $=_\mathbb{R}$ is the equality relation on $\mathbb{R}$.

Borel reducibility is a partial order; hence if $E_1 \leq_B E_2$ and E_1 is non-smooth, then E_2 is also non-smooth (otherwise $E_1 \leq_B E_2 \leq_B =_\mathbb{R}$). We conclude that Theorem 2 is implied by the following.

Theorem 10. *For each irrational number ϕ, there exists a smooth family of vector fields $v_{\rho,\phi} \subset \mathcal{V}^2(T^2)$, $\rho \in \mathbb{R}/\mathbb{Z}$, such that the equivalence relation E_ϕ on $(\mathbb{R}/\mathbb{Z}) \setminus \{n\phi\}_{n\in\mathbb{Z}}$ is Borel reducible to orbital topological equivalence: for $\rho_1, \rho_2 \in (\mathbb{R}/\mathbb{Z}) \setminus \{n\phi\}_{n\in\mathbb{Z}}$, we have $v_{\rho_1,\phi} \sim v_{\rho_2,\phi}$ if and only if $\rho_1 = \rho_2 \bmod n\phi$.*

This theorem is proved in the next three sections.

3 Construction of the family $v_{\rho,\phi}$.

Recall that we consider vector fields on the two-torus $T^2 = (\mathbb{R}/\mathbb{Z})^2$. Let $M_s = \mathbb{R}/\mathbb{Z} \times \{s\}$ be the meridians of the torus. Fix $\phi \in [0, 1] \setminus \mathbb{Q}$ and $\rho \in \mathbb{R}/\mathbb{Z}$.

We define $v_{\rho,\phi}$ in the following way.

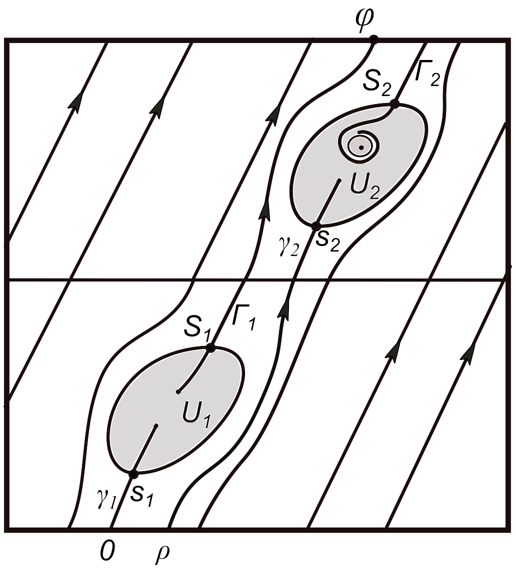

Figure 1: Phase curves of the vector field $v_{\rho,\phi}$.

- On neighborhoods of M_0 and $M_{0.5}$, define $v_{\rho,\phi} = (\phi, 1)$.
- Let the vector field $v_{\rho,\phi}$ in $\{0 < y < 0.5\}$ have two saddles s_1, S_1.

  One stable separatrix γ_1 of s_1 intersects M_0 at $(0,0)$, two unstable separatrices of s_1 form separatrix connections with two stable separatrices of S_1, an unstable separatrix Γ_1 of S_1 intersects $M_{0.5}$ at $(\phi/2, 1/2)$. Let U_1 be bounded by the separatrix connections; in U_1, the vector field $v_{r,\phi}$ has one stable and one unstable node.

  The correspondence map along $v_{r,\phi}$ between the sides of R_1 is given by $(x, 0) \to (x + \phi/2, 1/2)$, except it is undefined at $(0,0)$.
- For $\rho = 0$, the vector field $v_{0,\phi}$ in the domain $\{0.5 < y < 1\}$ has a similar structure, with saddles s_2, S_2 and separatrices γ_2, Γ_2 that intersect $M_{0.5}$ and M_1 at $(\phi/2, 1/2)$ and $(\phi, 1)$ respectively. The only difference is, that in the domain U_2 bounded by its separatrix connections, the vector field $v_{\rho,\phi}$ has one stable node and one unstable limit cycle with a stable node in it.
- For other values of ρ, in the domain $\{0.5 < y < 1\}$, we put $v_{\rho,\phi}(x, y) = v_{0,\phi}(x - \rho, y)$.

With this definition, $\rho \to v_{\rho,\phi}$ is a smooth family of vector fields for any fixed ϕ. The Poincare map under $v_{\rho,\phi}$ from the meridian M_0 of the torus to itself coincides with $x \to x + \phi$ everywhere, except it is undefined at the intersections with γ_1, γ_2. If $\rho \notin \{n\phi\}_{n\in\mathbb{Z}}$, then Γ_2 does not coincide with γ_1 and Γ_1 does not coincide with γ_2. In this case, we have $M_0 \cap \gamma_1 = \{-n\phi\}_{n=0}^{\infty}$ and $M_0 \cap \gamma_2 = \{\rho - n\phi\}_{n=0}^{\infty}$.

Recall that if a point a is non-singular for a vector field, $v(a) \neq 0$, then there exists a neighborhood U with a smooth chart $H\colon U \to \mathbb{R}^2$ such that $H_* v = (1, 0)$. A continuous curve γ is *topologically transverse* to the vector field if it does not pass through singular points, and for any point $a \in \gamma$ there exists its neighborhood U such that the image $H(\gamma \cap U)$ is a graph of a continuous function $x = x(y)$. The next lemma follows from elementary properties of correspondence maps.

Lemma 11. *For any simple closed loop $\alpha \subset T^2$ homotopic to the meridian M_0 that does not intersect $\overline{U}_1 \cup \overline{U}_2$ and is topologically transverse to $v_{\rho,\phi}$, there exists a homeomorphism $\xi \colon \alpha \to M_0$ with the following property: if $\xi(p_1) = p_2$, then either p_1, p_2 belong to the same trajectory of $v_{\rho,\phi}$, or $p_1 \in \Gamma_1 \cup \Gamma_2$ and $p_2 \in \gamma_1 \cup \gamma_2$.*

Proof. Lift $v_{\rho,\phi}$ to the vector field $\hat{v}$ on the cylinder $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$. Lift α and M_0 to the cylinder $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$ so that the lifts $\hat{\alpha}$, $\hat{M}_0 = \{y = 0\}$ do not intersect and $\hat{\alpha}$ is above $\hat{M}_0$. Then the correspondence map $\hat{\xi} \colon \hat{\alpha} \to \hat{M}_0$ along trajectories of $-\hat{v}$ is well-defined. Indeed, since both curves are topologically transverse to $\hat{v}$, the only obstructions for extending the correspondence map are intersections of $\hat{M}_0$ with stable separatrices of $\hat{v}$ and intersections of $\hat{\alpha}$ with unstable separatrices of $\hat{v}$. Since $\hat{M}_0$ and $\hat{\alpha}$ do not intersect $\overline{U}_1, \overline{U}_2$, these are the intersections of the lifts of $\gamma_{1,2}$ with $\hat{M}_0$, and of the lifts of $\Gamma_{1,2}$ with $\hat{\alpha}$. Correspondence map $\hat{\xi}$ extends continuously to these intersections. It descends to the map $\xi \colon \alpha \to M_0$ that satisfies assumptions of the lemma. □

4 Equivalent vector fields have E_ϕ-equivalent parameters

Lemma 12. *For irrational ϕ, let $v_{\rho,\phi}$ be vector fields constructed above. Suppose that $\rho_1, \rho_2 \notin \{n\phi\}_{n\in\mathbb{Z}}$.*

If vector fields $v_{\rho_1,\phi_1}, v_{\rho_2,\phi_2}$ are orbitally topologically equivalent, then $\phi_1 = \phi_2$ and $\rho_1 = \rho_2 + n\phi \bmod 1$.

Proof. Let $s_{1,2}, S_{1,2}, \gamma_{1,2}, \Gamma_{1,2}, U_{1,2}$ be as defined above for v_{ρ_1,ϕ_1}, and let $\tilde{s}_{1,2}, \tilde{S}_{1,2}, \tilde{\gamma}_{1,2}, \tilde{\Gamma}_{1,2}, \tilde{U}_{1,2}$ be analogous objects for v_{ρ_2,ϕ_2}.

Suppose that H is an orbital topological equivalence between v_{ρ_1,ϕ_1} and v_{ρ_2,ϕ_2}. Since H takes attractors and repellors of v_{ρ_1} to attractors and repellors of v_{ρ_2} respectively, and limit cycles to limit cycles, we have $H(U_1) = \tilde{U}_1$ and $H(U_2) = \tilde{U}_2$; $H(s_{1,2}) = \tilde{s}_{1,2}$ and $H(S_{1,2}) = \tilde{S}_{1,2}$; therefore $H(\gamma_1) = \tilde{\gamma}_1$ and $H(\gamma_2) = \tilde{\gamma}_2$. (Here we used that phase portraits of $v_{\rho,\phi}|_{U_1}$ and $v_{\rho,\phi}|_{U_2}$ are different, otherwise H could map γ_1 to $\tilde{\gamma}_2$ and γ_2 to $\tilde{\gamma}_1$.)

The curve $H(M_0)$ is topologically transverse to v_{ρ_2}, does not intersect $\tilde{U}_1 \cup \tilde{U}_2$ and is homotopic to M_0, since H is homotopic to identity. Using Lemma 11, define a homeomorphism $\xi \colon H(M_0) \to M_0$ along trajectories of v_{ρ_2,ϕ_2}. We get an orientation-preserving circle homeomorphism $\xi \circ H \colon M_0 \to M_0$.

Recall that the Poincare maps on M_0 under the action of v_{ρ_1,ϕ_1} and v_{ρ_2,ϕ_2} equal $x \to x + \phi_1$, $x \to x + \phi_2$ respectively. Since $\xi \circ H$ conjugates these Poincare maps, we have $\phi_1 = \phi_2$. From now on, we will omit subscripts 1, 2 in the notation ϕ_1, ϕ_2.

Consider the set $A_1 = (M_0 \cap \gamma_1) = \{-n\phi\}_{n\in\mathbb{Z}} \times \{0\}$. The set $H(A_1)$ belongs to $H(M_0) \cap \tilde{\gamma}_1$. Since $\rho_2 \notin \{n\phi\}_{n\in\mathbb{Z}}$, the separatrix $\tilde{\gamma}_1$ does not form a separatrix connection with $\tilde{\Gamma}_{1,2}$; Lemma 11 implies that the set $\xi(H(A_1)) \subset M_0$ belongs to $\tilde{\gamma}_1$ as well.

Hence the circle homeomorphism $\xi \circ H$ takes the dense set $A_1 = \{-n\phi\}_{n\in\mathbb{Z}} \times \{0\}$ into a subset of $\{-n\phi\}_{n\in\mathbb{Z}} \times \{0\}$. We conclude that $\xi \circ H$ must be a rotation by $k\phi$, $k \in \mathbb{Z}$.

On the other hand, analogous arguments for γ_2 imply that $\xi \circ H$ takes $M_0 \cap \gamma_2 = \{\rho_1 - n\phi\}_{n\in\mathbb{Z}} \times \{0\}$ into a subset of $M_0 \cap \tilde{\gamma}_2 = \{\rho_2 - n\phi\}_{n\in\mathbb{Z}} \times \{0\}$, hence $\xi \circ H$ must be a rotation by $\rho_2 - \rho_1 + l\phi$, $l \in \mathbb{Z}$. We conclude that $\rho_2 - \rho_1 = m\phi \bmod 1$. □

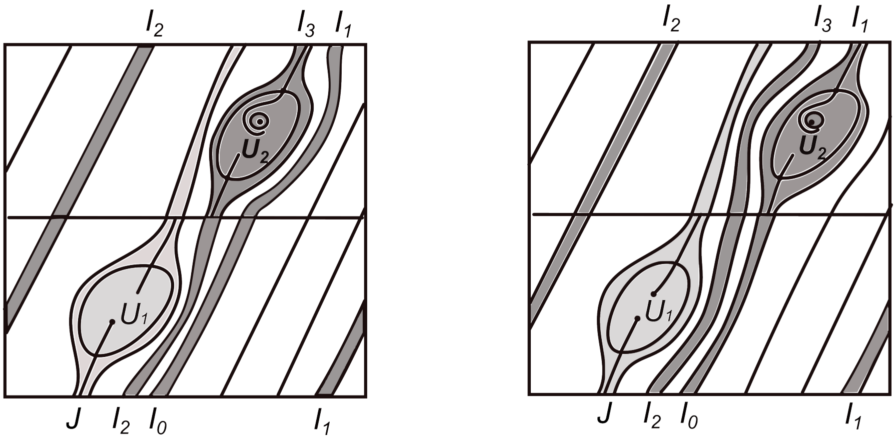

Figure 2: Domains V_1 (light-gray), V_2 (dark-gray) and V_3 (white) for vector fields v_{ρ_1} (left) and v_{ρ_2} (right) for $n = 3$.

5 Vector fields with E_ϕ-equivalent parameters are equivalent

Lemma 13. *For irrational ϕ, let $v_{\rho,\phi}$ be the vector field constructed above. Suppose that $\rho_1, \rho_2 \notin \{k\phi\}_{k\in\mathbb{Z}}$, and $\rho_1 = \rho_2 + n\phi \bmod 1$ for some integer n. Then $v_{\rho_1,\phi}$ and $v_{\rho_2,\phi}$ are orbitally topologically equivalent.*

Proof. We will write v_ρ instead of $v_{\rho,\phi}$ for brevity. Let ε be small so that the intervals

$$I_k = [\rho_2 + k\phi - \varepsilon, \rho_2 + k\phi + \varepsilon] = [\rho_1 - (n-k)\phi - \varepsilon, \rho_1 - (n-k)\phi + \varepsilon]$$

do not intersect for $k = 0, 1, \dots, n$ and do not intersect $J = [-\varepsilon, \varepsilon]$.

Let v_{ρ_1} be defined on the torus T, and v_{ρ_2} be defined on the torus $\tilde{T}$. We split T into a union of the following three connected sets:

- (1) V_1. Let V_1^1 be the union of arcs of trajectories of v_{ρ_1} that start at $(x, 0), x \in [-\varepsilon, \varepsilon] = J$, and end at $(x + \phi, 0)$. Let $V_1 = \overline{U}_1 \cup \overline{V_1^1}$.
- (2) V_2. Let V_2^1 be the union of arcs of trajectories of v_{ρ_1} that start at $(x, 0), x \in I_0$, and end at $(x + n\phi, 0) \in I_n$. Let $V_2 = \overline{U}_2 \cup \overline{V_2^1}$.
- (3) $V_3 = T \setminus V_1 \setminus V_2$.

The set V_1 contains the arc of γ_1 before its first intersection with M_0 that belongs to J. The set V_2 contains the arc of γ_2 and its 1st, 2nd, ..., nth intersections with M_0. Intersections happen inside I_{n-1}, ..., I_0 respectively.

Similarly, we define sets $\tilde{V}_1, \tilde{V}_2, \tilde{V}_3$ for v_{ρ_2}, using the same intervals $I_0, \dots I_n, J$ for the field v_{ρ_2}. Again, the set $\tilde{V}_1$ contains the arc of $\tilde{\gamma}_1$ before its first intersection with M_0 that belongs to J. The set $\tilde{V}_2$ contains the arc of γ_2 before its first intersection with M_0 that belongs to I_0, and the arc of Γ_2 that contains its 1st, 2nd, ..., $(n-1)$-th intersections with M_0. These intersections happen inside I_1, I_2, ..., I_n respectively. Fig. 2 shows domains V_1 – V_3 and $\tilde{V}_1$ – $\tilde{V}_3$. Now, construct H.

Clearly, $v_{\rho_1}|_{V_1}$ is orbitally topologically equivalent to $v_{\rho_2}|_{\tilde{V}_1}$. We will choose equivalence H that is identical on $J \times \{0\}$ and $(J + \phi) \times \{1\}$.

Vector fields $v_{\rho_1}|_{V_2}$ and $v_{\rho_2}|_{\tilde{V}_2}$ are also orbitally topologically equivalent. We will choose H that is identical on the bottom and top sides $I_0 \subset M_0$, $I_n \times \{1\}$ of V_2.

Finally, $V_3 \setminus M_0$ is a union of strips where v_{ρ_1} is orbitally topologically equivalent to the unit vector field; the same holds for v_{ρ_2} in $\tilde{V}_3 \setminus M_0$. Thus we can extend H to V_3 by setting H to be identity on $M_0 \cap \overline{V}_3 = M_0 \setminus (J \cup I_1 \cup I_2 \cup \dots I_{n-1})$ and extending it along trajectories of v_{ρ_1}, v_{ρ_2}. This completes the construction of the orbital topological equivalence. □

Lemmas 12 and 13 imply Theorem 10. Due to Sec. 2, this completes the proof of Theorem 2.

6 Genericity of nonclassifiable vector fields

In this section, we prove a stronger version of Theorem 2: vector fields that are orbitally topologically equivalent to $v_{\rho,\phi}$ form (at least) a codimension-7 submanifold $\mathcal{M}$ in the space $\mathcal{V}^2(T^2)$ of smooth vector fields. This implies that they appear in generic smooth 7-parameter families of vector fields.

Theorem 14. *For any irrational ϕ, there exists a codimension-7 continuous submanifold $\mathcal{M} \subset \mathcal{V}^2(T^2)$ such that any vector field $v \in \mathcal{M}$ is orbitally topologically equivalent to some vector field of the form $v_{\rho,\phi}$ described in Theorem 2.*

To define the set $\mathcal{M}$, we will need the notion of the rotation number. Let $f \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a circle homeomorphism, and let $F \colon \mathbb{R} \to \mathbb{R}$ be its lift to the real line. The rotation number of the circle homeomorphism f is given by

$$\operatorname{rot} f = \lim_{n\to\infty} \frac{F^n(x)}{n}.$$

The rotation number is rational if and only if f has a periodic orbit and depends continuously on f. In a 1-parameter family f_t, if $\frac{d}{dt} f_t > 0$, then the rotation number is monotonic with respect to t, and strictly monotonic whenever $\operatorname{rot}(f_t)$ is irrational.

We will also need the notion of the characteristic number of a saddle. Recall that if $\lambda_1 < 0 < \lambda_2$ are the eigenvalues of the linearization matrix of a vector field at a saddle singular point, then $|\lambda_1| : |\lambda_2|$ is called the characteristic number of a saddle. It is invariant under smooth changes of space and time variables. If L_1, L_2 are transversals to the stable and unstable separatrices of a saddle of a vector field v, then the correspondence map along v from L_1 to L_2 is defined on semi-transversals; let these transversals be given by $\{x > 0\}$ and $\{y > 0\}$ in local coordinates on L_1, L_2. This correspondence map is called the Dulac map. The following lemma is known to specialists.

Lemma 15. *Dulac map for the saddle with characteristic number μ has the form $x \to cx^\mu(1 + o(1))$ on a neighborhood of zero where c is a nonzero constant.*

Proof. The proof repeats the computation in the proof of Lemma 5 in [12]. Change variables so that saddle separatrices become coordinate axes, and L_1, L_2 become $\{y = 1\}$ and $\{x = 1\}$ respectively. Differential equation takes the form $x' = xg_1(x, y), y' = yg_2(x, y)$. After time change, equation becomes $x' = x, y' = yg(x, y)$, with g smooth, $g(0, 0) = -\mu$. Rescaling $x \to x/\varepsilon$, $y \to y/\varepsilon$ in an ε-neighborhood of zero brings the equation to the form $x' = x, y' = y\tilde{g}(x, y)$ with $|\tilde{g}(x, y) - g(0, 0)| < O(\varepsilon)|x| + O(\varepsilon)|y|$. Trajectory of the new vector field that starts at $(x_0, 1)$ has the form $x(t) = x_0 e^t$, $\log y(t) = \int_0^t \tilde{g}(x, y)dt$. Hence it takes the time $T = -\log x_0$ for this trajectory to land on the transversal $(1, y)$. An estimate on $\tilde{g}(x, y)$ above implies that $y(T) = C(x_0)e^{-\mu T} = C(x_0) \cdot x_0^\mu$ with $1 - O(\varepsilon) < C(x) < 1 + O(\varepsilon)$. Thus the Dulac map along the initial vector field v between $\{y = \varepsilon\}$ and $\{x = \varepsilon\}$ has the form $y = c(x) \cdot x^\mu$ with $1 - O(\varepsilon) < c(x)/c_0 < 1 + O(\varepsilon)$ for certain c_0. Since the correspondence map between $L_1 = \{y = 1\}$ and $\{y = \varepsilon\}$ is smooth, as well as the correspondence map between $L_2 = \{x = 1\}$ and $\{x = \varepsilon\}$, we conclude that on a sufficiently small neighborhood of zero on L_1, the Dulac map from L_1 and L_2 has the form $y = d(x) \cdot x^\mu$ with $1 - O(\varepsilon) < d(x)/d_0 < 1 + O(\varepsilon)$ for certain d_0. Since ε was arbitrary, this implies the statement. □

Proof of Theorem 14. **Construction of $\mathcal{M}$.**

Take a vector field $v_{\rho,\phi}$ with small ρ. Modify it if needed to guarantee that on a neighborhood of $\{x = 0\}$, we have $v_{\rho,\phi} = (\phi, 1)$. Consider its small neighborhood $\mathcal{U}$ in the space $\mathcal{V}^2(T^2)$ of C^2-smooth vector fields in T^2. Let $s_1(v), s_2(v), S_1(v), S_2(v)$ be saddles of v, $v \in \mathcal{U}$, that are close to s_1, s_2, S_1, S_2. Let l_1 be an interval transverse to the left separatrix connection of s_1 and S_1 of $v_{\rho,\phi}$; let $\alpha(v)$ and $\beta(v)$ be first intersections of separatrices of $s_1(v), S_1(v)$ with l_1, in a local chart on l_1. Define $\delta_1(v) = \alpha(v) - \beta(v)$, which is a smooth function of v. In a similar way, define functions $\delta_2(v), \delta_3(v), \delta_4(v)$ for each of the separatrix connections of v. Note that we have $\delta_k(v_{\rho,\phi}) = 0$, $k = 1, 2, 3, 4$, since $v_{\rho,\phi}$ has four separatrix connections. Define

$$\mathcal{M}^0 = \{v \in \mathcal{V}^2(T^2) \mid \delta_k(v) = 0, k = 1, 2, 3, 4\}.$$

Denote the characteristic numbers of the saddles $s_1(v), s_2(v), S_1(v), S_2(v)$ by $\mu_1(v), \mu_2(v), \nu_1(v), \nu_2(v)$ respectively.

For any $v \in \mathcal{M}^0$, let P_v be the Poincare map along v from M_0 to itself. Formally, P_v is undefined at the first intersections of separatrices $\gamma_1(v)$, $\gamma_2(v)$ with M_0, but it extends continuously to these points. Denote these points $A(v), B(v)$. On the left semi-neighborhood of $A(v)$, the map P_v is a composition of two Dulac maps, from M_0 to l_1 and from l_1 to M_0. Similarly, P_v is a composition of two Dulac maps on the right semi-neighborhood of $A(v)$. Due to Lemma 15, P_v has the following form:

$$\begin{array}{l} x \to P_v(A(v)) + C_1(v) \cdot (x - A(v))^{\mu_1(v) \cdot \nu_1(v)}(1 + o(1)) \text{ for } x < A(v) \\ x \to P_v(A(v)) + C_2(v) \cdot (x - A(v))^{\mu_1(v) \cdot \nu_1(v)}(1 + o(1)) \text{ for } x > A(v) \end{array}.$$

Note that constants $C_1(v)$ and $C_2(v)$ do not necessarily coincide. Since the Poincare map from M_0 to itself along $v_{\rho,\phi}$ is identity, we get that $\mu_1(v_{\rho,\phi}) \cdot \nu_1(v_{\rho,\phi}) = 1$; similarly, $\mu_2(v_{\rho,\phi}) \cdot \nu_2(v_{\rho,\phi}) = 1$.

Let $\mathcal{M}^1 \subset \mathcal{U}$ be given by

$$\mathcal{M}^1 = \{v \in \mathcal{M}^0 \mid \mu_1(v) \cdot \nu_1(v) = 1, \mu_2(v) \cdot \nu_2(v) = 1\}.$$

Then $\mathcal{M}^1$ is a codimension-6 smooth submanifold in $\mathcal{V}^2(T^2)$. Condition on characteristic numbers implies that P_v has nonzero one-sided derivatives on both sides of $A(v), B(v)$ for all $v \in \mathcal{M}^1$. So P_v is a C^2-smooth circle homeomorphism with two break points $A(v), B(v)$.

Finally, the set $\mathcal{M}$ is given by

$$\mathcal{M} = \{v \in \mathcal{M}^1, \operatorname{rot}(P_v) = \phi\}.$$

Codimension of $\mathcal{M}$.

We will prove that $\mathcal{M}$ is a codimension-1 continuous submanifold (i.e. the graph of a continuous function) in $\mathcal{M}^1$. Indeed, let $R_t\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation by the angle t. Let $v = (v^1, v^2)$, and let

$$\mathcal{L} = \{v \in \mathcal{M}^1 \mid \frac{v^1(0)}{v^2(0)} = \frac{v^1_{\rho,\phi}(0)}{v^2_{\rho,\phi}(0)}\}.$$

Since on a neighborhood of a boundary of the unit square, we have $v_{\rho,\phi} = (1, \phi)$, the vector field $R_t^* v$ is well-defined on the torus for small t. Moreover, $v \in \mathcal{M}^1$ implies $R_t^* v \in \mathcal{M}^1$ for small t, therefore the set $\mathcal{M}^1$ can be locally represented as a Cartesian product $\mathbb{R} \times \mathcal{L}$ in the smooth chart $(t, v) \to R_t^* v(x, y)$.

We have $\frac{d}{dt} P_{R_t^* v_{\rho,\phi}} > 0$ due to the construction of $v_{\rho,\phi}$, thus $\frac{d}{dt} P_{R_t^* v} > 0$ for v close to $v_{\rho,\phi}$. Properties of the rotation number imply that for any v, the set $\mathcal{M}$ intersects each fiber $R_t^* v$ on a single point, i.e. $\mathcal{M}$ is a graph in $\mathcal{M}^1$. Since $\operatorname{rot}(\cdot)$ is continuous, $\mathcal{M}$ is the graph of the continuous function. Hence $\mathcal{M}$ is a continuous manifold of codimension 7.

Finding orbitally topologically equivalent $v_{R,\phi}$.

For any $v \in \mathcal{M}$, we will find R such that v is orbitally topologically conjugate to $v_{R,\phi}$. It is an easy generalization of a classical Denjoy theorem that a circle homeomorphism with breaks that has irrational rotation number is continuously conjugate to the rotation $x \to x + \operatorname{rot}(f)$ (see e.g. [3, Theorem 2.4]). Let $\xi\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ conjugate P_v to the rotation $x \to x + \phi$. Post-composing ξ with rotation, we may and will assume that $\xi(A(v)) = 0$. Let

$$R = \xi(B(v)).$$

Constructing conjugacy.

Now, we will prove that $v \in \mathcal{M}$ is orbitally topologically conjugate to $v_{R,\phi}$. Let γ_1^R, γ_2^R denote separatrices of the vector field $v_{R,\phi}$.

Lift $v, v_{R,\phi}$ to vector fields $\tilde{v}, \tilde{v}_{R,\phi}$ in closed cylinders $C = \mathbb{R}/\mathbb{Z} \times [0, 1]$. Comparing phase portraits, we can see that $\tilde{v}$ is orbitally topologically equivalent to $\tilde{v}_{R,\phi}$ on C. Choose topological equivalence H to coincide with ξ on the lower boundary of C, $H(x, 0) = (\xi(x), 0)$. This is possible since ξ takes $A(v)$ to 0 and $B(v)$ to R, i.e. matches intersection points of separatrices of $\tilde{v}$ with M_0 to the intersection points of separatrices of $\tilde{v}_{R,\phi}$ with M_0.

Since H maps trajectories of $\tilde{v}$ to trajectories of $\tilde{v}_{R,\phi}$, we get $H(P_v(x), 1) = (\xi(x) + \phi, 1)$ on the upper boundary of C. Hence $H(x, 1) = (\xi(P_v^{-1}(x)) + \phi, 1) = (\xi(x) - \phi + \phi, 1) = (\xi(x), 1)$. Since $H(x, 0) = (\xi(x), 0)$ and $H(x, 1) = (\xi(x), 1)$, the map H descends to a continuous map on T^2. This completes the proof.

□

Remark 16. *We could simplify the family $v_{\rho,\phi}$ by replacing two saddles in U_1 and/or U_2 with a single saddle that has a separatrix loop (cf. the next section), thus improving the codimension. However, in this case, the Poincare map will be necessarily critical with non-symmetric critical points. Such circle maps are not well-studied, and we could not find a reference to the analogue of the Denjoy theorem that applies to this case.*

7 Analytic vector fields

In this section, we prove Theorem 3. We provide an explicit analytic family with no complete numerical invariants; the proof is computer-assisted.

Consider the family of Hamiltonian vector fields

$$v_{\phi,b,c,d} = (\frac{d}{dy}u_{\phi,b,c,d}, -\frac{d}{dx}u_{\phi,b,c,d})$$

with the Hamiltonian

$$u_{\phi,b,c,d}(x,y) = x - \phi y + (\cos y - 1)(b\sin(x-y) + c\sin(x) + d\cos(y)). \quad (1)$$

We will prove the following theorem; it implies Theorem 3 due to Proposition 8.

Theorem 17. *For some open interval $I \subset \mathbb{R}/\mathbb{Z}$, the equivalence relation E_ϕ on $I \setminus \{n\phi\}_{n\in\mathbb{Z}}$ is Borel reducible to the orbital topological equivalence on a subset of the family $v_{\phi,b,c,d}$.*

Namely, there exist analytic functions D and ρ defined on an open set V in $\mathbb{R}^3$, such that the function ρ is non-constant on c for fixed b, ϕ, and two vector fields $v_1 = v_{\phi_1,b_1,c_1,D(\phi_1,b_1,c_1)}$ and $v_2 = v_{\phi_2,b_2,c_2,D(\phi_2,b_2,c_2)}$ for $(\phi_{1,2}, b_{1,2}, c_{1,2}) \in U$, irrational ϕ_1, ϕ_2, and $\rho(\phi_1, b_1, c_1) \notin \{n\phi_1\}_{n\in\mathbb{Z}}$, $\rho(\phi, b_2, c_2) \notin \{n\phi_2\}_{n\in\mathbb{Z}}$ are orbitally topologically equivalent if and only if $\phi_1 = \phi_2 = \phi$ and $\rho(\phi, b_1, c_1) = \rho(\phi, b_2, c_2) \bmod n\phi$ in $\mathbb{R}/\mathbb{Z}$.

Proof. First, let us explain how the second part of the theorem implies its first part. Define an analytic function $C(\phi, r)$ implicitly on some open set by a condition $\rho(\phi, b_0, C(\phi, r)) = r$. Then for fixed irrational ϕ, the Borel reduction of E_ϕ on $I \setminus \{n\phi\}_{n\in\mathbb{Z}}$ to the orbital topological equivalence is given by $\rho \to v_{\phi,b_0,C(\phi,\rho),D(\phi,b_0,C(\phi,\rho))}$. This implies the first part of Theorem 3.

Clearly, the vector field $v_{\phi,b,c,d}$ is well-defined on the torus $T^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$. For simplicity of notation, we will lift it to the annulus $0 \le y \le 1$. It is easy to check that for $\phi \ne 0$, the vector field is transversal to $y = 0$, $y = 1$. Its phase curves are level curves of $u_{\phi,b,c,d}$. Since $u_{\phi,b,c,d}(x, 0) = x$ and $u_{\phi,b,c,d}(x, 2\pi) = x - 2\pi\phi$, the correspondence map from $\{y = 0\}$ to $\{y = 1\}$ is $x \to x + 2\pi\phi$ whenever defined.

Fig. 3 shows the level curves of $u_{\phi,b,c,d}$ for $\phi = 1/3, b = 2, c = 1, d \approx 1$. The following statements on vector fields $v_{\phi,b,c,d}$ were verified numerically for $\phi = 1/3, b = 2$, $c \in [0.7, 1.1]$, $d \in [0.9, 1.3]$.

- Vector fields $v_{\phi,b,c,d}$ have six singular points on the torus.

This was checked by applying Python's fsolve method (that uses the Powell's hybrid method) with the dense mesh of parameter values and initial guesses. Singular points are marked on Fig. 3.

- These six singular points are hyperbolic; there are three saddle points, two minima of $u_{\phi,b,c,d}$ and one maximum of $u_{\phi,b,c,d}$.

This was checked by computing the Jacobians of the linear part of $v_{\phi,b,c,d}$. Jacobians remain greater than 2 in modulus.

Let s_1, s_2, s_3 be the saddles of $v_{\phi,b,c,d}$, numbered left to right in x-coordinate, $x \in [0, 2\pi]$.

- The curves $\{y = 0\}, \{y = 1\}$, the level curve $\{u_{\phi,b,c,d}(x,y) = 1, 0 < y < 1\}$, and a connected component of the level curve $\{u_{\phi,b,c,d}(x, y) = 5, 0 < y < 1\}$ divide the torus into two domains V_1 and V_2; one of them (V_1) contains s_1 and a minimum of $u_{\phi,b,c,d}$, the other (V_2) contains s_2, s_3, and the remaining minimum and maximum of $u_{\phi,b,c,d}$.

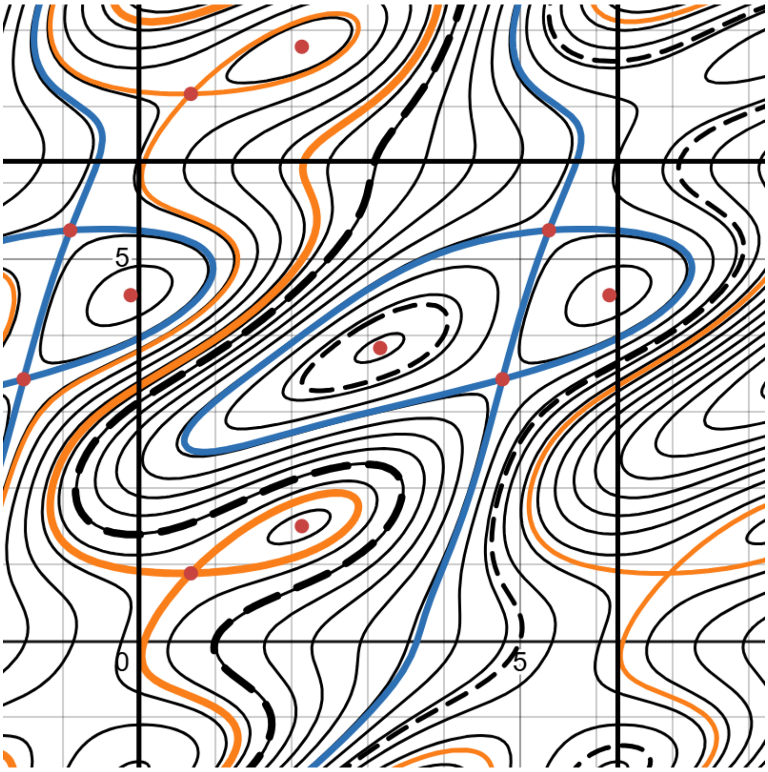

Figure 3: Level curves of $u_{\phi,b,c,d}$ (phase curves of the vector fields $v_{\phi,b,c,d}$) for $\phi = 1/3$, $c = 1, b = 2, d = 1.0016$. The black square has sides of length 2π. Separatrices of the saddles s_1, s_2, s_3 are shown in thick. Thick dots represent singular points of $v_{\phi,b,c,d}$. Dashed lines represent level curves $u_{\phi,b,c,d} = 1, u_{\phi,b,c,d} = 5$.

This was checked by (1) plotting these level curves for $u_{1/3,2,1,1}$ (see Fig. 3) and (2) verifying that for all $c \in [0.7, 1.1]$, $d \in [0.9, 1.3]$, values of $u_{1/3,2,c,d}$ at its critical points in $0 < y < 1$ are not equal to $1 \pm 2\pi k, 5 \pm 2\pi k$. This implies that critical points cannot move from one strip to another as parameters vary.

Since level curves of $u_{\phi,b,c,d}$ are phase curves for $v_{\phi,b,c,d}$ and $u_{\phi,b,c,d}(x, 0) = x$, we conclude that the correspondence map from $\{y = 0\}$ to $\{y = 1\}$ is defined near $x = 1, 5$. Consider the domain V_1. Since $u_{\phi,b,c,d}$ is monotonic on $y = 0$, out of four separatrices of s_1, only one can intersect $\{y = 0\}$ and only one can intersect $\{y = 1\}$. Since separatrices are the only obstruction from extending Poincare maps, exactly one separatrix must intersect $\{y = 0\}$ and one must intersect $\{y = 1\}$. Thus the remaining two separatrices of s_1 must form a separatrix loop in V_1 as shown in Fig. 3, and the correspondence map from $\{y = 0\}$ to $\{y = 1\}$ in V_1 is well-defined except a single point of intersection with a separatrix (to which it extends continuously).

Suppose that $u_{\phi,b,c,d}(s_2) = u_{\phi,b,c,d}(s_3)$. Since $u_{\phi,b,c,d}$ is monotonic on $y = 0$ and $y = 1$, in the strip V_2, only two of the eight separatrices of s_2, s_3 can intersect $\{y = 0\}$ and $\{y = 1\}$. Since separatrices are the only obstruction from extending Poincare maps, exactly two of these separatrices intersect $\{y = 0\}$ and $\{y = 1\}$. The remaining six separatrices must form three separatrix connections. There are two possibilities: we either have three separatrix connections between s_2 and s_3, or one connection and two separatrix loops. To check that we always have the first possibility as shown on Fig. 3, we verified the following.

- The function $u_{\phi,b,c,d}$ is monotonic on the straight line segment joining the minimum and the maximum of $u_{\phi,b,c,d}$ in V_2.

This was checked by computing the directional derivative of $u_{\phi,b,c,d}$ at the points of this segment, with step size equal to 0.01 of its length. If separatrices of s_2, s_3 formed separatrix loops, minimum and maximum of u would be inside these loops and the segment $[s_2, s_3]$ would intersect the level set $u_{\phi,b,c,d}(x, y) = u_{\phi,b,c,d}(s_2)$ at least twice, which contradicts monotonicity of $u_{\phi,b,c,d}$ on $[s_2, s_3]$. Hence s_2, s_3 form three separatrix connections. The correspondence map from $\{y = 0\}$ to $\{y = 1\}$ is everywhere defined, except two intersection points with separatrices to which it extends continuously. Thus it coincides with $x \to x + 2\pi\phi$ as noted above.

We claim that the condition $u_{\phi,b,c,d}(s_2) = u_{\phi,b,c,d}(s_3)$ defines a graph of an analytic function $D = D(\phi, b, c)$ in the parameter space for $c \in [0.7, 1.1]$, $d \approx 2$, $\phi \approx 1/3$. This follows from the property below.

- Derivative of $u_{\phi,b,c,d}(s_3) - u_{\phi,b,c,d}(s_2)$ with respect to d is positive for all $c \in [0.7, 1.1]$, $d \in [0.9, 1.3]$.

Derivative was computed via implicit function theorem. It remains between 2.05 and 2.15.

Hence for each $c \in [0.7, 1.1]$, the interval $d \in [0.9, 1.3]$ contains at most one value d such that $u_{\phi,b,c,d}(s_2) = u_{\phi,b,c,d}(s_3)$. This value $d = D(\phi, b, c)$ was determined numerically for $\phi = 1/3, b = 2$, $c \in [0.7, 1.1]$ and remains in $[0.9, 1.3]$. Since the condition is open, the same holds for all $b \approx 2$, $\phi \approx 1/3$, and $D(\phi, b, c)$ is well-defined. It is analytic since singular points of $v_{\phi,b,c,d}$ depend analytically on parameters.

For any vector field $v_{\phi,b,c,D(\phi,b,c)}$, let U_1 be the domain bounded by the separatrix connection of s_1; let U_2 be the domain bounded by the separatrix connections of s_2, s_3. Let stable separatrices of s_1, s_2 be γ_1, γ_2, let unstable separatrices of s_1, s_3 be Γ_1, Γ_2. Let $\rho(\phi, b, c) = u_{\phi,b,c,d}(s_2) - u_{\phi,b,c,d}(s_1)$ be the distance between the intersection points of γ_1 and γ_2 with the meridian $M_0 = \{y = 0\}$. Numerically, we checked the following.

- The value $\rho(\phi, b, c)$ is not constant on the graph $(\phi, b, c, D(\phi, b, c))$: namely, for $c = 0.7$ we have $d(c) \approx 1.23$ and $\rho \approx 3.40$ while for $c = 1.1$ we get $d \approx 0.92$ and $\rho \approx 3.62$.

The function ρ remains non-constant on c for $\phi \approx 1/3, d \approx 2$ since it is analytic.

The remaining part of the proof is the same as for Theorem 10. While the vector field $v_{\phi,b,c,D(\phi,b,c)}$ with $\rho = \rho(\phi, b, c)$ is not orbitally topologically equivalent to $v_{\rho,\phi}$, the only difference is the explicit shape of the phase portrait inside U_1, U_2; the Poincare map and the behavior of separatrices $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$ is the same. Hence the proof of Lemma 13 applies to the family $v_{\phi,b,c,D(\phi,b,c)}$ with $\rho = \rho(\phi, b, c)$, for any fixed b close to 2 and any fixed irrational ϕ close to 1/3, without any modification. In the proof of Lemma 12, we also used the fact that an orbital topological equivalence H matches corresponding saddles, $H(s_k) = \tilde{s}_k$. For our family, the latter follows from the fact that s_2, s_3 form three separatrix connections while s_1 has a separatrix loop. Hence the proof of Lemma 12 applies for the family $v_{\phi,b,c,D(\phi,b,c)}$ with minor modification. These lemmas imply the second part of Theorem 17, which completes its proof. □

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References

[1] H. Bruin and B. Vejnar. Classification of one dimensional dynamical systems by countable structures. *Journal of Symbolic Logic*, 88(2):562–578, 2023.

[2] J. Buzzi, N. Chandgotia, M. Foreman, S. Gao, F. García-Ramos, A. Gorodetski, F. L. Maitre, F. Rodríguez-Hertz, and M. Sabok. Open questions in descriptive set theory and dynamical systems, 2023. https://arxiv.org/pdf/2305.00248.

[3] A. Dzhalilov, I. Liousse, and D. Mayer. Singular measures of piecewise smooth circle homeomorphisms with two break points. *Discrete and Continuous Dynamical Systems*, 24(2):381–403, 2009.

[4] E. G. Effros. Polish transformation groups and classification problems. *General Topology and Modern Analysis (Proc. Conf., Univ. California, Riverside, Calif., 1980)*, pages 217–227, 1981.

[5] M. Foreman. What is a Borel reduction? *Notices Amer. Math. Soc.*, 65(10):1263–1268, 2018.

[6] M. Foreman and A. Gorodetski. Anti-classification results for smooth dynamical systems. Preliminary report, 2022. arXiv:2206.09322.

[7] M. Foreman, D. J. Rudolph, and B. Weiss. The conjugacy problem in ergodic theory. *Annals of Mathematics*, 173(3):1529–1586, 2011.

[8] M. Foreman and B. Weiss. Measure preserving diffeomorphisms of the torus are unclassifiable. *Journal of the European Mathematical Society*, 24(8):2605–2690, 2022.

[9] M. Gerber and P. Kunde. Non-classifiability of ergodic flows up to time change. *Inventiones Mathematicae*, pages 1–93, 2025.

[10] J. Glimm. Locally compact transformation groups. *Trans. Amer. Math. Soc.*, 101(1):124–138, 1961.

[11] L. A. Harrington, A. S. Kechris, and A. Louveau. A Glimm-Effros dichotomy for Borel equivalence relations. *J. Amer. Math. Soc.*, 3(4):903–928, 1990.

[12] Y. Ilyashenko, Y. Kudryashov, and I. Schurov. Global bifurcations in the two-sphere: a new perspective. *Inventiones Mathematicae*, 213:461 – 506, 2018.

[13] P. Kunde. There is no complete numerical invariant for smooth conjugacy of circle diffeomorphisms, 2022. arXiv:2209.02137.

[14] D. Ornstein. Bernoulli shifts with the same entropy are isomorphic. *Advances in Mathematics*, 4(3):337–352, 1970.

[15] M. Sabok. Completeness of the isomorphism problem for separable C*-algebras. *Invent. math.*, 204:833–868, 2016.

[16] J. Zielinski. The complexity of the homeomorphism relation between compact metric spaces. *Advances in Mathematics*, 291:635–645, 2016.