

A gallery of open problems in geometry that keep me puzzled and amused

Anton Petrunin 

Received 02 Sep 2025; Accepted 12 Oct 2025

Abstract: This is a collection of open questions in geometry that I think of as *puzzles*: they stick to my brain — I see grips, but no spare hands. Puzzle-charm is the only criterion for inclusion here; importance is ignored.

These puzzles grew out of my collection of solved puzzles [44]. I kept it aside since things around open questions change rapidly.

Please send me an email if you solve one of the puzzles, or see a good approach.

AMS Classification: 00A27; 51-XX

Key words and phrases: open problems in geometry

Homemade

1.1 Two-convexity

Note that an open connected set $\Omega \subset \mathbb{R}^n$ is convex (in the usual sense) if the following holds: for any triangle $\triangle$, if two sides of $\triangle$ lie in Ω , then $\triangle \subset \Omega$. This observation motivates the definition of two-convexity used in the following problem.

☞ *Let Ω be a simply connected open set in $\mathbb{R}^n$. Assume that Ω is two-convex; that is, if 3 faces of a 3-simplex $\triangle$ lie in Ω , then $\triangle \subset \Omega$.*

Is it true that any component of the intersection of Ω with any 2-plane is simply connected?

If the boundary of Ω is a smooth hypersurface, then the answer is “yes”. Also, the answer is “yes” if $n = 3$.

To prove the first statement, note that two-convexity implies that at most one of the principal curvatures of the boundary is negative. Then follow the Morse-type argument in Lefschetz theorem [19, see Section $\frac{1}{2}$]. In the case $n = 3$, one can mimic the Morse-type argument; it also follows from [3, 7.17].

The first argument also proves that *if Ω is a simply connected open two-convex set in $\mathbb{R}^n$ with a smooth boundary, then any component of the intersection of Ω with any affine subspace (not necessarily 2-dimensional) is simply connected*. The example below shows that a general 2-convex open set does not have this property.

One may hope to prove the statement by constructing an approximation of Ω by two-convex domains with smooth boundaries. Again, the example below shows that such an approximation does not exist for $n \geq 4$.

Example. We will construct a two-convex simply connected open set Ω in $\mathbb{R}^4$ such that the intersection of Ω with some hyperplane is not simply connected.

Set

$$\Pi = \{(x, y, z, t) \in \mathbb{R}^4 \mid y < x^2\}.$$

Let Π' be the image of Π under a generic rotation of $\mathbb{R}^4$, say $(x, y, z, t) \mapsto (z, t, x, y)$.

Since Π is open and two-convex, so is the intersection $\Omega = \Pi \cap \Pi'$. Note that we can choose (x, y, z, t) -coordinates so that Ω is an epigraph for a function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ like

$$f = \max\{\alpha_1 - \beta_1^2, \alpha_2 - \beta_2^2\},$$

where α_i and β_i are linear functions on the (x, y, z) -coordinate subspace. Thus, in particular, Ω is contractible.

Let L_{t_0} be the intersection of Ω with hyperplane $t = t_0$; it is a complement of two convex parabolic cylinders in general position. If these cylinders have a point of intersection, then $\pi_1 L_{t_0} = \mathbb{Z}$.

The discussion above implies that f cannot be approximated by smooth functions such that their Hessians have at most one negative eigenvalue at all points.

Remark. Two-convexity shows up in comparison geometry — the maximal open flat sets in the manifolds of nonnegative or nonpositive curvature are two-convex [39].

1.2 Braid space

Let us define *braid space* $\mathbb{B}^n$ as the universal covering of $\mathbb{C}^n$ infinitely branching along the complex hyperplanes $z_i = z_j$ for $i \neq j$. We suppose that $\mathbb{C}^n$ is equipped with the canonical Euclidean metric, and $\mathbb{B}^n$ is equipped with the lifted length metric.

In other words, if $\mathbb{W}^n$ denotes the complement of complex hyperplanes $z_i = z_j$ in $\mathbb{C}^n$, then the braid space $\mathbb{B}^n$ is the completion of the universal metric covering of $\mathbb{W}^n$.

The following problem is a special case of a conjecture formulated by Daniel Allcock [4, Conjecture 1.4], which is related to an earlier conjecture of Ruth Charney and Michael Davis [13, Conjecture 3], which in turn is motivated by the $K(\pi, 1)$ -conjecture [49, 41]. Dmitri Panov and the author are also responsible for a more general version of Allcock's conjecture [39, Conjecture 1.1].

📖 *Is it true that $\mathbb{B}^n$ is CAT(0) for any n ?*

The answer is “yes” for $n \leq 3$. The case $n = 2$ is trivial, and the case $n = 3$ follows from the theorem of Ruth Charney and Michael Davis about branched coverings of the 3-sphere [12]; another proof is given by Dmitri Panov and the author [40]. The case of the real braid space with 4 strings was settled by Ruth Charney thru a detailed analysis of short galleries [11, Theorem 5.4]. The *real braid space* is defined as the inverse image of the real subspace $\mathbb{R}^n \subset \mathbb{C}^n$ under the projection $\mathbb{B}^n \rightarrow \mathbb{C}^n$. This is a convex subset of $\mathbb{B}^n$; therefore, the question about real braid space is a special case of our problem.

The space $\mathbb{B}^n$ has plenty of convex functions, which might help to understand its geometry. The following construction is inspired by the argument given by Ian Agol [1]: (1) choose n distinct points in $\mathbb{C}$ (that is, an element of $\mathbb{W}^n$), (2) choose a closed curve z in its complement, (3) consider the function $\mathbb{B}^n \rightarrow \mathbb{R}$ that returns the greatest lower bound of lengths of closed curves that are homotopic to z .

Existence of convex functions forbids geodesic loops in $\mathbb{B}^n$; this would also follow if $\mathbb{B}^n$ is CAT(0). To show that $\mathbb{B}^n$ is CAT(0) one has to forbid geodesic digons [3, 6.6].

1.3 Quotients of Hilbert space

📖 *Suppose R is a compact simply connected Riemannian manifold that is isometric to a quotient of the Hilbert space by a group of isometries (or, more generally, R is the target of a Riemannian submersion from the Hilbert space). Is it true that R is isometric to a biquotient? That is, is it true that R is a quotient of a compact Lie group G with a bi-invariant metric by a group of isometries?*

Any biquotient can appear as a quotient of the Hilbert space by a group of isometries. Let us sketch the argument suggested by Alexander Lytchak; see [33]. Another proof follows from the construction of Chuu-Lian Terng and Gudlaugur Thorbergsson

given in [46, Section 4].

Denote by G^n the direct product of n copies of G . Consider the map $\varphi_n : G^n \rightarrow G//H$ defined by

$$\varphi_n : (\alpha_1, \dots, \alpha_n) \mapsto [\alpha_1 \cdots \alpha_n]_H,$$

where $[x]_H$ denotes the H -orbit of x in G .

Note that φ_n is a quotient map for the action of $H \times G^{n-1}$ on G^n defined by

$$(\beta_0, \dots, \beta_n) \cdot (\alpha_1, \dots, \alpha_n) = (\beta_0 \cdot \alpha_1 \cdot \beta_1^{-1}, \beta_1 \cdot \alpha_2 \cdot \beta_2^{-1}, \dots, \beta_{n-1} \cdot \alpha_n \cdot \beta_n^{-1}),$$

where $\beta_i \in G$ and $(\beta_0, \beta_n) \in H < G \times G$.

Denote by ρ_n the product metric on G^n rescaled by the factor $\sqrt{n}$. Note that the quotient $(G^n, \rho_n)/(H \times G^{n-1})$ is isometric to $G//H = (G, \rho_1)//H$. Let $\varphi_n : (G^n, \rho_n) \rightarrow G//H$ be the corresponding quotient map; clearly, φ_n is a submetry.

As $n \rightarrow \infty$, the curvature of (G^n, ρ_n) converges to zero and its injectivity radius goes to infinity. Therefore, the ultralimit of (G^n, ρ_n) , with the identity element as the basepoint, is a Hilbert space $\mathbb{H}$, and the submetries φ_n ultraconverge to a submetry $\varphi : \mathbb{H} \rightarrow G//H$.

1.4 Nested convex surfaces

 *Describe the Riemannian metrics on $\mathbb{S}^n$ that are isometric to smooth nested convex surfaces; that is, a complete smooth convex hypersurface in a complete smooth convex hypersurface in ... in a Euclidean space.*

If $n = 2$, then by Alexandrov's embedding theorem these are all Riemannian metrics with nonnegative curvature.

Direct calculations show that the metric has *nonnegative cosectional curvature*; the latter means that, at each point, the curvature tensor can be expressed as a linear combination with nonnegative coefficients of the curvature tensors of $\mathbb{S}^2 \times \mathbb{R}^{n-2}$.

(In dimension three, nonnegative cosectional curvature is equivalent to nonnegative sectional curvature; in dimension 4, it is equivalent to nonnegative curvature operator.)

It might happen that any metric with nonnegative cosectional curvature on $\mathbb{S}^n$ is isometric to a nested convex surface. (It's embarrassing, but I don't understand the situation with the Berger spheres.) If one drops completeness from the definition of nested convex surfaces and assumes that the cosectional curvature is strictly positive, then the answer is "yes" [43].

1.5 Maximal finite subgroups

☞ Suppose that a group Γ acts effectively, isometrically, and properly discontinuously on Euclidean space $\mathbb{E}^n$. Show that Γ contains at most 2^n maximal finite subgroups up to conjugation.

Let F be a maximal finite subgroup in Γ . Its fixed-point set $\text{Fix}(F)$ is an affine subset of $\mathbb{E}^n$. Furthermore, $\text{Fix}(F)$ is mapped to a singular set S_F in the quotient space $X = \mathbb{E}^n/\Gamma$. Moreover, (1) a conjugation of F does not change the set S_F ; and, since F is maximal, (2) S_F is a *simple* singular set; that is, S_F does not have a proper singular subset. So instead of counting subgroups up to conjugation, we may count simple singular sets in X . Crude bounds on this number were obtained by Tadashi Fujioka [16].

It is expected that if the number of simple singular sets is maximal, then each of them has dimension 0; that is, each set S_F is an isolated point in X . If this is indeed true, then the problem essentially boils down to the following well-known problem in discrete geometry: *show that one can choose at most 2^n points in $\mathbb{E}^n$ such that any triangle with vertices among the chosen points is acute or right*. This problem was posed by Paul Erdős and solved by Ludwig Danzer and Branko Grünbaum [14]. More on the subject can be found in the paper by Nina Lebedeva [30].

1.6 Shortcut in a connected set

☞ Let p and q be a pair of points in a compact connected subset $K \subset \mathbb{R}^n$. Is it possible to connect p to q by a curve γ such that the length of $\gamma \setminus K$ is arbitrarily small?

Be aware that there are compact connected sets which contain no nontrivial paths; for example, the *pseudoarc*.

The problem remains open for $n \geq 3$. In the case $n = 2$, an affirmative answer was obtained by Taras Banakh [7]. His argument is based on the following claim:

Suppose K is a connected compact set in the plane and $p, q \in K$. Then, given $\varepsilon > 0$ there is a sequence of compact connected subsets $K_0, \dots, K_n$ with pairs of points $p_i, q_i \in K_i$ for each i such that (1) $\text{diam } K_i < \varepsilon$ for each i , (2) $p_0 = p, q_n = q$, and (3) if ℓ_i denotes the line segment $[q_{i-1}p_i]$, then

$$\text{length } \ell_1 + \dots + \text{length } \ell_n < \varepsilon.$$

Assume that the claim is proved. Then we can prove the statement by applying it recursively to each K_i , choosing a much smaller value of ε at each step. Indeed, consider the closure L of the set of all points p_i and q_i for all iterations; it is a closed subset in K . By adding to L the line segments ℓ_i for all the iterations we obtain a curve connecting p to q that spends an arbitrarily small length outside of L . Hence the 2-dimensional case follows.

Now let us prove the claim. Consider a coordinate grid Γ on the plane with step $\frac{\varepsilon}{2}$. Since the intersection $\Gamma \cap K$ is compact, it can be covered by a finite collection of closed line segments $\{I_1, \dots, I_n\}$ in the grid such that the

$$\sum_i \text{length}(I_i \setminus K) < \varepsilon.$$

Take a minimal collection of sets $\{K_i\}$ such that each set K_i is a connected component of the intersection of K with a closed square of the grid and the union

$$\left(\bigcup_i I_i\right) \cup \left(\bigcup_j K_j\right)$$

is a connected set containing p and q . Observe that the collection $\{K_i\}$ is finite and it satisfies the conditions in the claim, assuming that K_i are ordered correctly. (The choice of points p_i and q_i is straightforward; the line segments ℓ_j can be chosen to lie in the union of $\{I_i\}$.)

1.7 Involution of the 3-sphere

▮ Suppose that a closed geodesic γ is the fixed-point set of an isometric involution of $(\mathbb{S}^3, g)$. Assume that the sectional curvature of g is at least 1. Is it true that $\text{length } \gamma \leq 2\pi$?

A nonsharp universal upper bound on $\text{length } \gamma$ follows from a result of Tadashi Fujioka [16]. The same question can be asked about spherical polyhedral metrics with curvature at least 1 in the sense of Alexandrov.

If, instead of an isometric involution, we have an $\mathbb{S}^1$ -action, then the answer is “yes”. It follows since the quotient space $(\mathbb{S}^3, g)/\mathbb{S}^1$ is a convex disc with sectional curvature at least 1, and γ projects isometrically to its boundary.

1.8 Faceting Fubini–Study

A *polyhedral metric* on a compact manifold is a length metric such that, for some triangulation, each simplex is isometric to a simplex in Euclidean space. Such a metric is said to be nonnegatively curved if the total angle around each codimension-two simplex does not exceed 2π ; this is equivalent to nonnegative curvature in the sense of Alexandrov.

▮ Let d be the Fubini–Study metric on the complex projective plane $\mathbb{CP}^2$. Is it possible to find a sequence of nonnegatively curved polyhedral metrics d_n on $\mathbb{CP}^2$ that converges to d ?

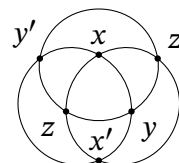
Nonnegatively curved polyhedral metrics on $\mathbb{CP}^2$ have a number of interesting properties. For example, any such metric is Kähler; that is, one can choose a complex

structure on each simplex of its triangulation that is respected by the gluings. Furthermore, these complex structures coincide with the standard complex structure on $\mathbb{CP}^2$ for a right parametrization, and in this parametrization, the singular set forms an algebraic submanifold. Such metrics on $\mathbb{CP}^2$ were constructed and studied by Dmitri Panov [38]; see also his paper with Martin de Borbon [15].

1.9 Octahedron comparison

Suppose Γ is a graph with vertices $v_1, \dots, v_n$. We say that a metric space X meets the Γ -comparison if, for any set of points in X labeled by the vertices of Γ , there is a model configuration $\tilde{v}_1, \dots, \tilde{v}_n$ in the Hilbert space $\mathbb{H}$ such that if v_i is adjacent to v_j , then $|\tilde{v}_i - \tilde{v}_j|_{\mathbb{H}} \leq |v_i - v_j|_X$, and otherwise, $|\tilde{v}_i - \tilde{v}_j|_{\mathbb{H}} \geq |v_i - v_j|_X$; here $|p - q|_M$ denotes the distance between points p and q in the metric space M .

Let us denote by O_3 the graph of the octahedron. For the labeling shown on the diagram, O_3 -comparison means that any six-point configuration $x, x', y, y', z, z' \in X$ can be mapped to $\mathbb{H}$ so that the diagonals xx' , yy' , and zz' do not get shorter and the edges do not get longer.



📦 Is it true that O_3 -comparison holds in any geodesic $\text{CAT}(0)$ space?

This problem is motivated by the question of Michael Gromov [21, 1.19₊(e)].

Note that the 4-cycle C_4 is an induced subgraph of O_3 . Therefore, O_3 -comparison implies C_4 -comparison; the latter describes the $\text{CAT}(0)$ -comparison [2]. In general, C_4 -comparison does not imply O_3 -comparison, but for intrinsic metrics it is unknown; a weaker inequality was proved by Tetsu Toyoda [48].

An affirmative answer might lead to a classification of six-point spaces that admit an isometric embedding in a geodesic $\text{CAT}(0)$ space. The five-point case was solved by Tetsu Toyoda [47] (see also [31]).

On the other hand, Nina Lebedeva and the author proved that O_3 -comparison holds in products of trees [32]. Therefore, a counterexample would mean that O_3 -comparison is something new and interesting; so both answers are good.

Postscript. The problem has been solved by Jihun Kim [27].

Overheard

2.10 Group generated by central symmetries

▮ Consider the Lobachevsky space of sufficiently large dimension. Does it admit a discrete cocompact isometric action generated by

(a) elements of finite order?

(b) central symmetries?

(c) rotations around subspaces of codimension k ?

Ernest Vinberg [52, 51] proved that there are no such actions generated by reflections. This solves part (c) for $k = 1$; the remaining questions are completely open.

2.11 Milnor's cartography problem

▮ Let Ω be a round disc of radius $\alpha < \frac{\pi}{2}$ on the unit sphere $\mathbb{S}^2$, and let Ω' be a convex domain in $\mathbb{S}^2$ with the same area as Ω . Show that there is a $(1, \frac{\alpha}{\sin \alpha})$ -bi-Lipschitz map from Ω' to the plane.

The question was asked by John Milnor [36]. Polar coordinates with the origin at the center of Ω provide an example of a $(1, \frac{\alpha}{\sin \alpha})$ -bi-Lipschitz map and this constant $\frac{\alpha}{\sin \alpha}$ is optimal.

If, in the formulation of the problem, one replaces “area” with “perimeter”, then the answer is “yes”. It can be proved using the idea of Gary Lawlor [29].

Postscript. The problem has since been solved by the author [45].

2.12 Convex hull in CAT(0)

Let X be a geodesic metric space. Recall that a set $C \subset X$ is called *convex* if, for any two points $x, y \in C$, every geodesic from x to y lies in C . The *closed convex hull* of a subset $K \subset X$ is defined as the intersection of all closed convex sets containing K .

▮ *Let K be a compact subset of a complete length CAT(0) space X . Is it true that the closed convex hull of K is compact?*

Note that the space is not assumed to be locally compact; otherwise, the statement is evident. I believe in a counterexample, perhaps even with negatively pinched curvature (in the sense of Alexandrov).

The closed convex hull of a compact set (in particular, of a three-point set) might look ugly even in a smooth 3-dimensional manifold. If in doubt, check the note of Alexander Lytchak and the author [35].

This problem was mentioned by Mikhael Gromov [20, 6.B₁(f)] and restated by Eva Kopecká and Simeon Reich [28], but it seems to have been known in mathematical folklore for a long time.

The problem remains open even when K consists of three points. A counterexample is known if the CAT(0) assumption is weakened to the existence of a conical geodesic bicombed; it was constructed by Giuliano Basso, Yannick Krifka, and Eleftherios Soultanis [8].

Postscript. The problem has been solved by Ariel Goodwin [18, 34].

2.13 Homeomorphism of the torus

Let $\mathbb{T}$ denote the 2-dimensional torus.

A homeomorphism $h : \mathbb{T} \rightarrow \mathbb{T}$ is called *simple* if it is supported on an embedded open disc; that is, there is an embedded open disc $\Delta \subset \mathbb{T}$ such that h is the identity outside of Δ .

The *disc norm* of a homeomorphism $h : \mathbb{T} \rightarrow \mathbb{T}$ is defined as the least number of simple homeomorphisms $h_1, \dots, h_n$ such that

$$h = h_1 \circ \dots \circ h_n.$$

▮ *Is there a homeomorphism of the 2-dimensional torus with arbitrarily large disc norm?*

Analogous questions for the cylinder and the sphere are solved, but for the Möbius band the problem is open. This problem is discussed by Dmitri Burago, Sergei Ivanov, and Leonid Polterovich in [10].

Postscript. Jonathan Bowden, Sebastian Hensel, and Richard Webb [9] constructed unbounded quasimorphisms on the group of homeomorphisms of the torus that are isotopic to the identity, which implies an affirmative answer to the problem.

2.14 Open projection

Let us denote by $\mathbb{D}^n$ a closed n -dimensional ball.

Recall that a map is called *open* if the image of any open set is open.

▮ *Is it possible to construct an embedding $\mathbb{D}^4 \hookrightarrow \mathbb{S}^2 \times \mathbb{R}^2$ such that the projection $\mathbb{D}^4 \rightarrow \mathbb{S}^2$ is an open map?*

It is easy to construct an embedding $\mathbb{D}^3 \hookrightarrow \mathbb{S}^3$ such that its composition with the Hopf fibration is open. Therefore, there is an open map $f_3 : \mathbb{D}^3 \rightarrow \mathbb{S}^2$.

Composing f_3 with any open map $\mathbb{D}^4 \rightarrow \mathbb{D}^3$, one gets an open map $f_4 : \mathbb{D}^4 \rightarrow \mathbb{S}^2$. However, this map is not a projection of an embedding $\mathbb{D}^4 \hookrightarrow \mathbb{S}^2 \times \mathbb{R}^2$.

The question was asked by ε - δ (an anonymous mathematician) [50].

2.15 Zero-curvature along geodesics

▮ *Let M be a surface of genus 2 with a Riemannian metric of nonpositive curvature. Show that, with probability 1, a random geodesic in M runs into the set of negative curvature.*

I learned this problem from Dmitri Burago; it originated from a discussion with Keith Burns, Michael Brin, and Yakov Pesin [25]. Weisheng Wu [53] proved the statement under the additional assumption that the set of points with negative curvature has finitely many connected components. It seems that *there are no known examples of a surface M with a nonperiodic geodesic that runs in the set of zero curvature.*

2.16 Ambrose problem

▮ *Let (M, g) be a simply connected complete Riemannian manifold, and let $p \in M$. Denote by h the pullback of the metric tensor g to the tangent space $T_p M$ along the exponential map $\exp_p : T_p M \rightarrow M$; in general, h might degenerate at some points. Does h define the manifold (M, g) and the point p up to isometry?*

This simple-looking question was asked a long time ago by Warren Ambrose [5]. Affirmative answers were obtained in three cases: (1) in dimension 2, (2) for analytic metrics, and (3) for generic metrics; see [23, 24, 26, 6] and the references therein.

2.17 Optical fibers

Suppose that V is a body in $\mathbb{R}^3$ bounded by two plane figures F_1 and F_2 (the ends) and a smooth surface Σ (a tube). Suppose that a light ray entering V thru one end and bouncing with perfect reflection from the interior walls of Σ will emerge from the other end with probability 1; moreover, a light ray that starts in the interior of V , after bouncing, emerges from one of the ends with probability 1. In this case, we say that V is an *optical fiber*.

A small tubular neighborhood of a smooth curve $\gamma : [a, b] \rightarrow \mathbb{R}^3$ with disk ends provides an example of an optical fiber. Let us describe a more general example, which can be constructed in the same way using any plane figure bounded by a smooth curve instead of a disk. Suppose a plane curve is given by a periodic parametrization $\theta \mapsto (x(\theta), y(\theta))$. Choose a parallel normal frame e_1, e_2 along γ (that is, $e'_i(t) \parallel \dot{\gamma}(t)$ for all t), and consider the tube $[a, b] \times \mathbb{S}^1 \hookrightarrow \mathbb{R}^3$ defined by

$$(t, \theta) \mapsto \gamma(t) + x(\theta) \cdot e_1(t) + y(\theta) \cdot e_2(t)$$

(The condition that the frame is parallel implies that any normal plane to γ cuts the tube at a right angle.) This way we get an *optical fiber* with congruent ends.

Are there any constructions of optical fibers different from the one described above? In other words:

▮ *Is it always possible to slice an optical fiber into plane figures that meet the walls at right angles?*

In particular,

▮ *Is there an optical fiber with noncongruent ends?*

By Liouville's theorem, it is clear that the ends must have the same area.

If the walls are only assumed to be *piecewise* smooth, then one can construct an optical fiber with a pair of equidecomposable figures at the ends. (The construction is

the same, but one splits the tube into a few tubes along the way and then reassembles them.)

The problem is based on two questions posted on MathOverflow [37, 42].

2.18 Guth’s sponge

📖 Show that there is $\varepsilon_n > 0$ such that any bounded open set in $\mathbb{R}^n$ with volume at most ε_n admits a length-increasing embedding into a unit ball.

The question was asked by Larry Guth in his thesis [22]. It is open in all dimensions $n \geq 2$. In dimension 2, Parker Glynn-Adey proved a weaker version using a strip between parallel lines instead of the ball [17].

Acknowledgment.

This work was partially supported by the following grants: NSF grants DMS 0103957, 0406482, 0905138, 1309340, 2005279, and Simons Foundation grants 245094 and 584781.

References

- [1] I. Agol. Braid group on 4 strands. MathOverflow. 105
- [2] S. Alexander, V. Kapovitch, and A. Petrunin. *Alexandrov geometry—foundations*, volume 236 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2024. 110
- [3] S. Alexander, V. Kapovitch, and A. Petrunin. *An invitation to Alexandrov geometry: CAT(0) spaces*, volume 2395 of *Lecture Notes in Mathematics*. Springer, Cham, 2 edition, 2026. 103, 105

- [4] D. Allcock. Completions, branched covers, Artin groups, and singularity theory. *Duke Math. J.*, 162(14):2645–2689, 2013. [104](#)
- [5] W. Ambrose. Parallel translation of Riemannian curvature. *Ann. of Math. (2)*, 64:337–363, 1956. [114](#)
- [6] P. Angulo Arday. *Cut and conjugate points of the exponential map, with applications*. 2014. Thesis (Ph.D.)–Universidad Autónoma de Madrid. [114](#)
- [7] T. Banakh. Running most of the time in a connected set. *MathOverflow*. [108](#)
- [8] G. Basso, Y. Krifka, and E. Soultanis. A non-compact convex hull in generalized non-positive curvature. *Math. Ann.*, 390(4):5863–5882, 2024. [112](#)
- [9] J. Bowden, S. Hensel, and R. Webb. Quasi-morphisms on surface diffeomorphism groups. *J. Amer. Math. Soc.*, 35(1):211–231, 2022. [113](#)
- [10] D. Burago, S. Ivanov, and L. Polterovich. Conjugation-invariant norms on groups of geometric origin. In *Groups of diffeomorphisms*, volume 52 of *Adv. Stud. Pure Math.*, pages 221–250. Math. Soc. Japan, Tokyo, 2008. [113](#)
- [11] R. Charney. The Deligne complex for the four-strand braid group. *Trans. Amer. Math. Soc.*, 356(10):3881–3897, 2004. [105](#)
- [12] R. Charney and M. Davis. Singular metrics of nonpositive curvature on branched covers of Riemannian manifolds. *Amer. J. Math.*, 115(5):929–1009, 1993. [105](#)
- [13] R. Charney and M. Davis. The $K(\pi, 1)$ -problem for hyperplane complements associated to infinite reflection groups. *J. Amer. Math. Soc.*, 8(3):597–627, 1995. [104](#)
- [14] L. Danzer and B. Grünbaum. Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V. L. Klee. *Math. Z.*, 79:95–99, 1962. [107](#)

- [15] M. de Borbon and D. Panov. A Miyaoka-Yau inequality for hyperplane arrangements in $\mathbb{CP}^n$. *J. Lond. Math. Soc. (2)*, 113(4):Paper No. e70525, 114, 2026. [110](#)
- [16] T. Fujioka. Uniform boundedness on extremal subsets in Alexandrov spaces. *J. Geom. Anal.*, 35(2):Paper No. 46, 28, 2025. [107](#), [109](#)
- [17] P. Glynn-Adey. *Width, Ricci Curvature, and Bisecting Surfaces*. ProQuest LLC, Ann Arbor, MI, 2016. Thesis (Ph.D.)—University of Toronto. [116](#)
- [18] A. Goodwin. A finite set with noncompact closed convex hull in a hadamard space, 2026. [112](#)
- [19] M. Gromov. Sign and geometric meaning of curvature. *Rend. Semin. Mat. Fis. Milano*, 61:9–123, 1991. [103](#)
- [20] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993. [112](#)
- [21] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces*. Modern Birkhäuser Classics. Basel: Birkhäuser, 2007. [110](#)
- [22] L. Guth. *Area-contracting maps between rectangles*. ProQuest LLC, Ann Arbor, MI, 2005. Thesis (Ph.D.)—Massachusetts Institute of Technology. [116](#)
- [23] J. Hebda. Metric structure of cut loci in surfaces and Ambrose’s problem. *J. Differential Geom.*, 40(3):621–642, 1994. [114](#)
- [24] J. Hebda. Heterogeneous Riemannian manifolds. *Int. J. Math. Math. Sci.*, pages Art. ID 187232, 7, 2010. [114](#)
- [25] F. Rodriguez Hertz. On the geodesic flow of surfaces of nonpositive curvature, 2003. [114](#)

- [26] J. Itoh. The length of a cut locus on a surface and Ambrose’s problem. *J. Differential Geom.*, 43(3):642–651, 1996. [114](#)
- [27] J. Kim. Octahedron comparison in geodesic CAT(0) spaces: An exact computer-assisted proof candidate. Version 0.1.0, 2026. [111](#)
- [28] E. Kopecká and S. Reich. Nonexpansive retracts in Banach spaces. In *Fixed point theory and its applications*, volume 77 of *Banach Center Publ.*, pages 161–174. Polish Acad. Sci. Inst. Math., Warsaw, 2007. [112](#)
- [29] G. Lawlor. A new area-maximization proof for the circle. *The Mathematical Intelligencer*, 20(1):29–31, 1998. [112](#)
- [30] N. Lebedeva. Alexandrov spaces with maximal number of extremal points. *Geom. Topol.*, 19(3):1493–1521, 2015. [107](#)
- [31] N. Lebedeva and A. Petrunin. 5-point CAT(0) spaces after Tetsu Toyoda. *Anal. Geom. Metr. Spaces*, 9(1):160–166, 2021. [110](#)
- [32] N. Lebedeva and A. Petrunin. Trees meet octahedron comparison. *J. Topol. Anal.*, 17(5):1205–1209, 2025. [111](#)
- [33] N. Lebedeva, A. Petrunin, and V. Zolotov. Bipolar comparison. *Geom. Funct. Anal.*, 29(1):258–282, 2019. [105](#)
- [34] A. Lytchak. Non-compact convex hulls in cat(0) spaces revisited, 2026. [112](#)
- [35] A. Lytchak and A. Petrunin. About every convex set in any generic Riemannian manifold. *J. Reine Angew. Math.*, 782:235–245, 2022. [112](#)
- [36] J. Milnor. A problem in cartography. *Amer. Math. Monthly*, 76:1101–1112, 1969. [111](#)
- [37] J. O’Rourke. Light rays bouncing in twisted tubes. MathOverflow. [116](#)

- [38] D. Panov. Polyhedral Kähler manifolds. *Geom. Topol.*, 13(4):2205–2252, 2009. [110](#)
- [39] D. Panov and A. Petrunin. Sweeping out sectional curvature. *Geom. Topol.*, 18(2):617–631, 2014. [104](#)
- [40] D. Panov and A. Petrunin. Ramification conjecture and Hirzebruch’s property of line arrangements. *Compos. Math.*, 152(12):2443–2460, 2016. [105](#)
- [41] L. Paris. $K(\pi, 1)$ conjecture for Artin groups. *Ann. Fac. Sci. Toulouse Math. (6)*, 23(2):361–415, 2014. [104](#)
- [42] A. Petrunin. Optic fibers after joseph o’rourke. MathOverflow. [116](#)
- [43] A. Petrunin. Polyhedral approximations of Riemannian manifolds. *Turkish J. Math.*, 27(1):173–187, 2003. [107](#)
- [44] A. Petrunin. *PIGTIKAL (puzzles in geometry that I know and love)*, volume 2 of *Association for Mathematical Research Monographs*. Association for Mathematical Research, 2022. [102](#)
- [45] A. Petrunin. Milnor’s cartography problem, 2026. [112](#)
- [46] C.-L. Terng and G. Thorbergsson. Submanifold geometry in symmetric spaces. *J. Differential Geom.*, 42(3):665–718, 1995. [106](#)
- [47] T. Toyoda. An intrinsic characterization of five points in a CAT(0) space. *Anal. Geom. Metr. Spaces*, 8(1):114–165, 2020. [110](#)
- [48] T. Toyoda. Inequalities on six points in a CAT(0) space. *Int. Math. Res. Not. IMRN*, (6):rnaf060, 2025. [110](#)
- [49] H. van der Lek. *The Homotopy Type of Complex Hyperplane Complements*. PhD thesis, University of Nijmegen, 1983. [104](#)

- [50] ε - δ (user:10330). Open map $\mathbb{D}^4 \rightarrow \mathbb{S}^2$. MathOverflow. [114](#)
- [51] È. B. Vinberg. The non-existence of crystallographic groups of reflections in Lobachevsky spaces of large dimension. *Trans. Mosc. Math. Soc.*, 1985:75–112, 1984. [111](#)
- [52] Èh. B. Vinberg. Discrete reflection groups in Lobachevskij spaces of high dimension. *Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova*, 132:62–68, 1983. [111](#)
- [53] W. Wu. On the ergodicity of geodesic flows on surfaces of nonpositive curvature. *Ann. Fac. Sci. Toulouse Math. (6)*, 24(3):625–639, 2015. [114](#)

AUTHOR

Anton Petrunin 
 Department of Mathematics
 Pennsylvania State University
 109 McAllister Building
 University Park, PA 16802
 United States
email: petrunin@math.psu.edu