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Self-duality of multidimensional continued fractions

Hiroaki Ito

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Abstract: F. Schweiger introduced the fibred system in [12], to unify and generalize many known continued fraction algorithms. An advantage of a fibred system is that it often provides a systematic construction of an absolutely continuous invariant density. In this paper, we define and study the self-duality of fibred systems, a strong symmetry of a given system. We show that explicit algebraic self-duality holds in many systems and present a curious system with "partial" self-duality.

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1 Introduction

The classical continued fraction is a self map T on $[0,1]$ defined by

$$T : x \longmapsto \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

Its absolutely continuous invariant probabilistic density is

$$d\nu = \frac{1}{\log 2} \cdot \frac{1}{1+x} dx.$$

The cylinder set $\Delta[a_1, a_2, \dots, a_n]$ is the interval whose elements share an initial fraction:

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

Then we have

$$\nu(\Delta[a_1, \dots, a_n]) = \nu(\Delta[a_n, \dots, a_1]).$$

We say that continued fraction algorithm is **symmetric in measure** if this equality holds for all cylinder sets. To see this symmetry, a standard way is to consider its natural extension:

$$\hat{T} : [0,1]^2 \ni (x,y) \longmapsto \left(\frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \frac{1}{y + \left\lfloor \frac{1}{x} \right\rfloor} \right) \in [0,1]^2$$

with the invariant density

$$\frac{1}{(1+xy)^2} dxdy.$$

The map $\hat{T}$ is invertible:

$$\hat{T}^{-1} : (x,y) \longmapsto \left(\frac{1}{x + \left\lfloor \frac{1}{y} \right\rfloor}, \frac{1}{y} - \left\lfloor \frac{1}{y} \right\rfloor \right)$$

and the restriction of $\hat{T}^{-1}$ to the second coordinate is equal to T. The self-duality immediately follows from this fact.

To make concrete the tractable a natural extension for higher dimensional continued fractions, F. Schweiger constructed the dual algorithm $(B^\#, T^\#)$ of the fibred system (B, T). The pair $(B \times B^\#, T \times V^\#(k(x)))$ gives the natural extension of (B, T) where $V^\#(k)$ is a local inverse branch of $T^\#$. In this framework, if there exists an isomorphism ϕ which satisfies:

$$\begin{array}{ccc} B^\# & \xrightarrow{T^\#} & B^\# \\ \phi\big\downarrow & & \phi\big\downarrow \\ B & \xrightarrow{T} & B \end{array}$$

then the system is self-dual. We say that self-duality is realized by an intertwining map ϕ. We will define an algebraic self-duality. If such a map ϕ is found we simply say that the system (B, T) is **algebraic self-dual**. See section 2 for details. In this paper we start with an easy observation:

Theorem 1. If the fibred system (B, T) is full and algebraic self-dual, then it is symmetric in measure.

However, we do not know when the self-duality holds in general, nor how to construct the intertwining map ϕ for a given full-branched fibred system. In the later section, we shall construct ϕ for several fibred systems in [12] and also give examples of fibred systems which is not self-dual.

2 Invariant measure and self-duality

In this chapter, we briefly review the concept of higher dimensional continued fractions by F. Schweiger and show Theorem 1.

We say that the dynamical system (B, T) is a fibred system if $\{B(k) : k \in I\}$ is a partition of the set B where I is countable and $T|_{B(k)}$ is injective.

Definition 1. The fibred system (B,T) is multidimensional continued fraction (**m.c.f.**) if
1. $B \subset \mathbb{R}^n$,
2. For every digits $k \in I$, there is a matrix $A_T(k) = ((A_{ij})) \in GL(n+1,\mathbb{Z})$ such that $y = T(x)$, $x \in B(k)$ is given as

$$y_i = \frac{A_{i0} + \sum_{j=1}^n A_{ij}x_j}{A_{00} + \sum_{j=1}^n A_{0j}x_j}.$$

Remark 1. For all invertible $(n+1)\times(n+1)$-matrix (a_{ij}), we define a transformation $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies

$$f(x)_i = \frac{a_{i0} + \sum_{j=1}^n a_{ij}x_j}{a_{00} + \sum_{j=1}^n a_{0j}x_j},$$

and we denote by A_f the matrix $((a_{ij}))$. Then, we can verify $A_f A_g = A_{f\circ g}$.

Since $T|_{B(k)}$ is injective, there exists a local inverse branch of T

$$V(k) : T(B(k)) \to B(k), \quad x = V(k)y$$

We denote the inverse matrix of $A_T(k)$ by $((B_{ij}))$. Then $y = Tx$ is equivalent to

$$x_i = \frac{B_{i0} + \sum_{j=1}^n B_{ij}y_j}{B_{00} + \sum_{j=1}^n B_{0j}y_j}.$$

where B_{ij} satisfies $B_{00} + \sum_{j=1}^n B_{0j}y_j > 0$.

Definition 2. Let (B,T) be a m.c.f. with matrices $\{A_T(k) : k \in I\}$. The m.c.f. $(B^\#,T^\#)$ is dual algorithm if the following conditions holds:
1. $B(k_1,k_2,\cdots,k_n) \neq \emptyset$ if and only if $B^\#(k_n,k_{n-1},\cdots,k_1) \neq \emptyset$,
2. There is a partition $\{B^\#(k), k \in I\}$ of $B^\#$ such that the associated matrices $A_{T^\#}(k) = ((A^\#_{ij}))$ of $T^\#$ restricted $B^\#(k)$ are the **transposed matrices** of $A_T(k)$ such that $y = T^\#(x)$, $x \in B^\#(k)$ is given as

$$y_i = \frac{A^\#_{i0} + \sum_{j=1}^n A^\#_{ij}x_j}{A^\#_{00} + \sum_{j=1}^n A^\#_{0j}x_j}.$$

Given a multidimensional continued fraction algorithm (B, T), its dual map is formally defined by the transpose of A_T. We then try to find an appropriate dual space $B^\#$ and its decomposition $\{B^\#(k) : k \in I\}$ which satisfies condition 1.

After this construction, given an n-dimensional continued fraction we set

$$K(x, y) := \frac{1}{(1 + x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^{n+1}},$$

and we denote by $\omega(k_1, k_2, \cdots, k_s; y)$ the Jacobian of $V(k_1, k_2, \cdots, k_s) = V(k_1) \circ V(k_2) \circ \cdots \circ V(k_s)$. Then, we can see

$$K(V(k_1, \dots, k_s)x, y)\omega(k_1, \dots, k_s; x) = K(x, V^\#(k_s, \dots, k_1)y)\omega^\#(k_s, \dots, k_1; y) \tag{1}$$

by a straightforward calculation. For any $x \in B$, we define

$$D(x) := \{y \in B^\# : x \in \bigcap_{s=1}^{\infty} T^s B(k_s^\#(y), \cdots, k_1^\#(y))\}.$$

Then, it is known that the following assertion holds (see Chapter 3 in [12]):

Proposition 1.

$$h(x) = \int_{D(x)} K(x, y) dy$$

is invariant density for T.

Definition 3. A fibred system (B, T) is called full if $T(B(\mathring{d})) = \mathring{B}$ for all d $\in I$.

Note that $D(x) = B^\#$ if the system (B, T) is full. By Proposition 1, we can obtain an invariant measure $\mu^\#$ for the dual algorithm $(B^\#, T^\#)$.

Lemma 1. The multidimensional c.f. (B, T) is full, then one has

$$\mu(B(k_1, k_2, \cdots, k_s)) = \mu^\#(B^\#(k_s, k_{s-1}, \cdots, k_1)).$$

Proof. For all $k_1, \dots .k_s \in I$, by Proposition 1,

$$\begin{aligned}\mu(B(k_1, k_2, \cdots, k_s)) &= \int_{B(k_1, k_2, \cdots, k_s)} \int_{B^\#} K(x, y) dy dx \\ &= \int_{B} \int_{B^\#} K(V(k_1, k_2, \cdots, k_s)x, y)\omega(k_1, k_2, \cdots, k_s; x) dy dx.\end{aligned}$$

By (1), we have

$$\begin{aligned}\mu(B(k_1,k_2,\cdots,k_s)) &= \int_B\int_{B^{\#}} K(x,V^{\#}(k_s,k_{s-1},\cdots,k_1)y)\omega^{\#}(k_s,k_{s-1},\cdots,k_1;y)dydx\\ &= \int_{B^{\#}}\int_B K(V^{\#}(k_s,k_{s-1},\cdots,k_1)y,x)\omega^{\#}(k_s,k_{s-1},\cdots,k_1;y)dxdy\\ &= \mu^{\#}(B^{\#}(k_s,k_{s-1},\cdots,k_1)).\end{aligned}$$

□

Definition 4. A n-dimensional continued fraction (B,T) is "algebraic self-dual" on $\mathcal{D}\subset I$ if the diagram

$$\begin{array}{ccc} B^{\#} & \xrightarrow{T^{\#}} & B^{\#}\\ \downarrow{\scriptstyle\phi} & & \downarrow\\ B & \xrightarrow{T} & B\end{array}$$

is commutative and ϕ is a bijective, differentiable, and measurable function map such that $\phi(\mathring{B^{\#}}(k)) = \mathring{B}(k)$ for all $k\in\mathcal{D}$.

For the regular continued fraction algorithm $([0,1),T)$, the matrix $A_T(k)$ is symmetric. Thus, it is clearly self-dual since $T=T^{\#}$.

Proof of Theorem 1. Note that the map ϕ is bijective, since (B,T) is algebraic self-dual (on $\mathcal{D}=I$). By substitution, for all $k_1,k_2,\cdots,k_s\in I$

$$\begin{aligned}\mu^{\#}(\phi^{-1}B(k_1,k_2,\cdots,k_s)) &= \int_{\phi^{-1}B(k_1,k_2,\cdots,k_s)}\int_B K(x,y)dydx\\ &= \int_{B(k_1,k_2,\cdots,k_s)}\int_{B^{\#}} K(X,Y)dYdX\\ &= \mu(B(k_1,k_2,\cdots,k_s)).\end{aligned}$$

Therefore, by Lemma 1, we have

$$\begin{aligned}\mu(B(k_1,k_2,\cdots,k_s)) &= \mu^{\#}(B^{\#}(k_1,k_2,\cdots,k_s))\\ &= \mu(B(k_s,k_{s-1},\cdots,k_1)).\end{aligned}$$

□

Note that the m.c.f. (B, T) is algebraic self-dual on $\mathcal{D} \subset I$, then for all $k_1, k_2, \cdots, k_s \in \mathcal{D}$

$$\mu(B(k_1, k_2, \cdots, k_s)) = \mu(B(k_s, k_{s-1}, \cdots, k_1)).$$

Definition 5. The set function

$$\tau(B(k_1, k_2, \cdots, k_s)) := \mu(B(k_s, k_{s-1}, \cdots, k_1)).$$

is called the polar measure for (B, T).

In fact, τ is also an invariant measure for T. And $\mu = \tau$ means symmetric in measure. F. Schweiger gave an equivalent condition for $\mu = \tau$ under the conditions in [10].

3 Selmer Algorithm

Let $E^{n+1} := \{x \in \mathbb{R}_{>}^{n+1} : x_0 \geq x_1 \geq \cdots \geq x_n \geq 0\}$. Then define

$$x \in E^{n+1} \longmapsto x' = (x_0 - x_n, x_2, \ldots, x_n).$$

There is an index $i = i(x)$, $0 \leq i \leq n$ such that

$$S(x) = (x_1, \ldots, x_i, x_0 - x_n, \ldots, x_n) \in E^{n+1}.$$

We obtain the bottom map $T_S : \Delta \longrightarrow \Delta$ which makes the diagram

$$\begin{array}{ccc} E^{n+1} & \xrightarrow{S} & E^{n+1} \\ {\scriptstyle p}\downarrow & & \downarrow \\ \Delta & \xrightarrow{T=T_S} & \Delta \end{array}$$

commutative. Since

$$A_T(i) = \begin{array}{c} \\ 0 \\ \\ \\ i \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} & & i & & & & n \end{array} \\ \left(\begin{array}{ccccccc} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ 1 & & & & & & -1 \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & & 1 \end{array}\right) \end{array} \quad \textbf{on } \Delta(i),$$

the 1-time partition is $\Delta(i) = \{x \in \Delta : x_i > 1 - x_n \geq x_{i+1}\}$ where $x_0 = 1$, $x_{n+1} = 0$. We can see $T(\Delta(i)) = \{x \in \Delta : x_i + x_n \geq 1\}$. Therefore, for $i = 0, 1, \dots, n-1$

$$T(\Delta(i)) = \bigcup_{i \leq j} \Delta(j), \quad T(\Delta(n)) = \Delta(n-1) \cup \Delta(n).$$

The system (Δ, T) is not full, but $(X = \Delta(n-1) \cup \Delta(n), T)$ is full-branched system. The dual map of Selmer's algorithm (X, T) is defined on $X^{\#} = \mathbb{R}^n_{\geq}$. It is known that Selmer algorithm (X, T) is ergodic and admits an absolutely continuous invariant measure (see [12]).

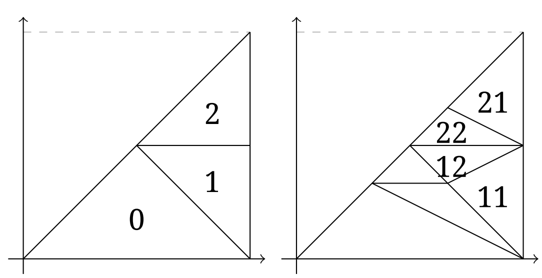

Fig. 1: The 1-time and 2-time partition of (Δ, T_S).

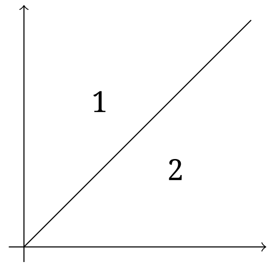

Fig. 2: The 1-time partition of $(X^{\#}, T_S^{\#})$.

Note that the 1-time partition of $X^{\#}$ is

$$X^{\#}(n-1) = \{x \in X^{\#} : x_{n-1} \leq x_n\},$$
$$X^{\#}(n) = \{x \in X^{\#} : x_{n-1} \geq x_n\}.$$

We construct the intertwining map ϕ for the Selmer algorithm.

For $n = 2$, the same method in the previous chapter gives

$$A_\phi = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

A simple analogy for the n-dimensional case works fine and we obtain

Proposition 2. Selmer's algorithm is algebraic self-dual. And

$$A_\phi = \begin{pmatrix} 2 & \cdots & 2 & 1 & 1 \\ \vdots & \iddots & & \vdots & \vdots \\ 2 & & & & \\ 1 & \cdots & & & 1 \\ 1 & \cdots & & 1 & 0 \end{pmatrix}.$$

Proof. We can see that for all $k \in \{n-1, n\}$

$$A_\phi A_{T^{\#}}(k) = A_T(k) A_\phi.$$

Let $\phi(B_1, \dots, B_n) = (b_1, \dots, b_n)$. Then, for all $(B_1, \dots, B_n) \in X^{\#}(n-1)$,

$$b_{n-1} + b_n - 1 = \frac{B_{n-1}}{2 + 2B_1 + \cdots + 2B_{n-2} + B_{n-1} + B_n} > 0$$

and

$$1 - 2b_n = \frac{B_n - B_{n-1}}{2 + 2B_1 + \cdots + 2B_{n-2} + B_{n-1} + B_n} > 0.$$

Thus, $\phi(X^{\#}(n-1)) \subset \Delta(n-1)$. We show $\phi(X^{\#}(n-1)) \supset \Delta(n-1)$. Since

$$A_{\phi}^{-1} = \begin{pmatrix} & & 1 & -1 & \\ & \iddots & -1 & & \\ 1 & \iddots & & & \\ -1 & & & 1 & 1 \\ & & & 1 & -1 \end{pmatrix},$$

for all $(b_1, \dots, b_n) \in \Delta(n-1)$, we can verify that $B_i \geq 0$, $i = 1, \dots, n$ and

$$B_n - B_{n-1} = \frac{1 - 2b_n}{b_{n-2} - b_{n-1}} > 0.$$

Thus, $\phi(X^{\#}(n-1)) = \Delta(n-1)$. Similarly, we can see $\phi(X^{\#}(n)) = \Delta(n)$. □

Corollary 1. Selmer's algorithm (X, T) is symmetric in measure. i.e., for all $a_1, a_2, \cdots, a_s \in \{n-1, n\}$

$$\mu(\Delta(a_1, a_2, \cdots, a_s)) = \mu(\Delta(a_s, a_{s-1}, \cdots, a_1)).$$

4 Brun Algorithm

Let $E^{n+1} := \{x \in \mathbb{R}_{>}^{n+1} : x_0 \geq x_1 \geq \cdots \geq x_n \geq 0\}$. Define

$$x \in E^{n+1} \longmapsto x' = (x_0 - x_1, x_2, \dots, x_n).$$

Then, there is an index $i = i(x)$, $0 \leq i \leq n$ such that

$$B(x) = (x_1, \dots, x_i, x_0 - x_1, \dots, x_n) \in E^{n+1}.$$

We obtain the bottom map $T : \Delta \longrightarrow \Delta$ which makes the diagram

$$\begin{array}{ccc} E^{n+1} & \xrightarrow{B} & E^{n+1} \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ \Delta & \xrightarrow{T=T_B} & \Delta \end{array}$$

commutative, where $p(x) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \dots, \frac{x_n}{x_0}\right)$. The 1-time partition of Δ is

$$\Delta(i) = \{x \in \Delta : x_i \geq 1 - x_1 > x_{i+1}\}$$

where $x_0 = 1$, $x_{n+1} = 0$. For digits $i \in \{0, 1, 2, \dots\}$, the matrix $A_T(i)$ is

$$A_T(i) = \begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ \end{array} \!\!\! \overset{\qquad\qquad\quad i}{\begin{pmatrix} & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ 1 & -1 & & & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}} \quad \textbf{on } \Delta(i).$$

The dual space of Brun's algorithm (Δ, T) is $\Delta^\# = \mathbb{R}_\geq \times [0, 1)^{n-1}$ and the 1-time partition of $\Delta^\#$ is

$$\Delta^\#(0) = \{x \in \Delta^\# : x_1 \geq 1\},$$
$$\Delta^\#(i) = \{x \in \Delta^\# : 0 \leq x_1 < 1, x_j < x_i, 1 \leq j \leq n\}.$$

We found the intertwining map ϕ of Brun algorithm for only $n = 2$:

$$A_\phi = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proposition 3. The Brun algorithm for 2-dimensions is self-dual.

Now, let (Δ, T_{BM}) be a multidimensional continued fraction with matrices

$$A_{T_{BM}}(i) = \begin{array}{c} \\ \\ \\ \\ i \\ \\ \\ \\ \end{array} \overset{\qquad\qquad\qquad i\qquad\qquad\qquad}{\begin{pmatrix} & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ 1 & -N & & & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}} \quad \textbf{on } \Delta[(i, N)].$$

This algorithm is called the Brun multiplicative algorithm. Since $\Delta^{\#} = [0, 1]^n$, we can get the invariant measure μ for T_{BM}. The invariant density when $n = 2$ is

$$\frac{1}{(1 + x)(1 + x + y)} + \frac{1}{(1 + y)(1 + x + y)}.$$

For the invariant measure in higher dimensions, see [1]. It is known that the multiplicative version is ergodic with respect to μ, see e.g. [12]. When $n = 2$, we found

$$\mu(\Delta[(1, 1), (2, 1)]) \neq \mu(\Delta[(2, 1), (1, 1)]),$$
$$\mu(\Delta[(1, 2), (2, 1)]) \neq \mu(\Delta[(2, 1), (1, 2)]),$$
$$\vdots$$

This implies that the polar measure τ doesn't coincide with μ.

5 Garrity-Schweiger Algorithm

Let $E^{n+1} := \{x \in \mathbb{R}^{n+1}_{>} : x_0 \geq x_1 \geq \cdots \geq x_n \geq 0\}$. We define

$$G(x) = (x_1, \dots, x_n, x_0 - x_1 - kx_n) \in E^{n+1}, \quad k = \left\lfloor \frac{x_0 - x_1}{x_n} \right\rfloor.$$

With the help of the projection

$$p : E^{n+1} \longrightarrow \Delta, \quad p(x_0, x_1, \dots, x_n) = \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right),$$

we obtain the bottom map $T : \Delta \longrightarrow \Delta$ which makes the diagram

$$\begin{array}{ccc} E^{n+1} & \xrightarrow{G} & E^{n+1} \\ p\downarrow & & p\downarrow \\ \Delta & \xrightarrow{T=T_G} & \Delta \end{array}$$

commutative. The map T is

$$T(x) = \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \cdots \frac{x_n}{x_1}, \frac{1 - x_1 - kx_n}{x_1}\right), \quad k = k(x) = \left\lfloor \frac{1 - x_1}{x_n} \right\rfloor.$$

The 1-time partition of Δ is

$$\Delta(k) = \{x \in B : 1 - x_1 - kx_n \geq 0 > 1 - x_1 - (k+1)x_n\}, \quad k \in \{0, 1, 2, \dots\}$$

and this fibred system is full.

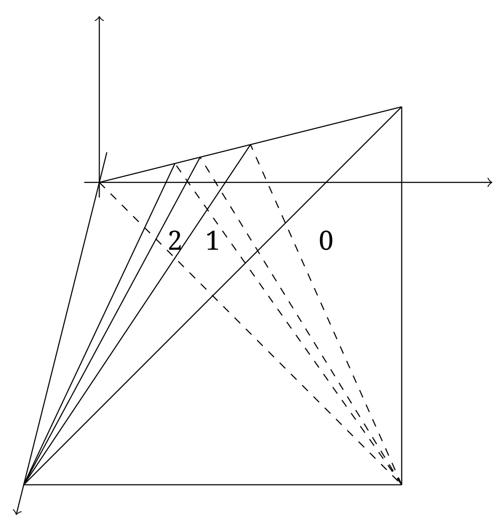

Fig. 3: The 1-time partition of (Δ, T_G) for $n = 3$.

This continued fraction algorithm for $n = 2$ was introduced by Garrity in [4], which is commonly known as the triangle algorithm or Garrity's triangle algorithm. An alternative extension of Garrity's triangle algorithm is studied in [2]. On the other hand, the following map was introduced by Schweiger in [9]:

$$F : x \longmapsto \left(\frac{x_2}{x_1} - \left\lfloor \frac{x_2}{x_1} \right\rfloor, \frac{x_3}{x_1}, \ldots, \frac{x_n}{x_1}, \frac{1}{x_1} - 1 \right)$$

on $[0,1) \times \mathbb{R}_{\geq}^{n-1}$. The dynamical system $([0,1) \times \mathbb{R}_{\geq}^{n-1}, F)$ is isomorphic to the dual algorithm of (Δ, T_G). It is known that the (Δ, T_G) is ergodic with respect to the Lebesgue measure for $n = 2$ (see [7]). It was shown that this system is also ergodic with respect to absolutely continuous invariant measures for all dimensions [5]. In this paper, we call the system (Δ, T_G) the Garrity-Schweiger algorithm.

We show that the m.c.f. (Δ, T_G) is algebraic self-dual. Since

$$A_T(k) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & & 1 \\ 1 & -1 & 0 & \cdots & -k \end{pmatrix} \quad \textbf{on } \Delta(k),$$

the dual map $T^{\#}$ is

$$T^{\#}(x) = \left(\frac{1 - x_n}{x_n}, \frac{x_1}{x_n}, \cdots, \frac{x_{n-2}}{x_n}, \frac{x_{n-1} - kx_n}{x_n} \right).$$

This dual map is defined on

$$\Delta^{\#} = \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n-1, 0 \leq x_n < 1\} = \mathbb{R}_{\geq}^{n-1} \times [0,1)$$

and the 1-time partition is given by

$$\Delta^{\#}(k) = \{x \in \Delta^{\#} : x_{n-1} - kx_n \geq 0 > x_{n-1} - (k+1)x_n\}, \quad k = k^{\#}(x) = \left\lfloor \frac{x_{n-1}}{x_n} \right\rfloor.$$

Since the Garrity-Schweiger algorithm is full system, the invariant density h is

$$\int_{\mathbb{R}_{\geq}^{n-1} \times [0,1)} K(x,y) dy \sim \frac{1}{x_1 \cdots x_{n-1}(1 + x_n)}.$$

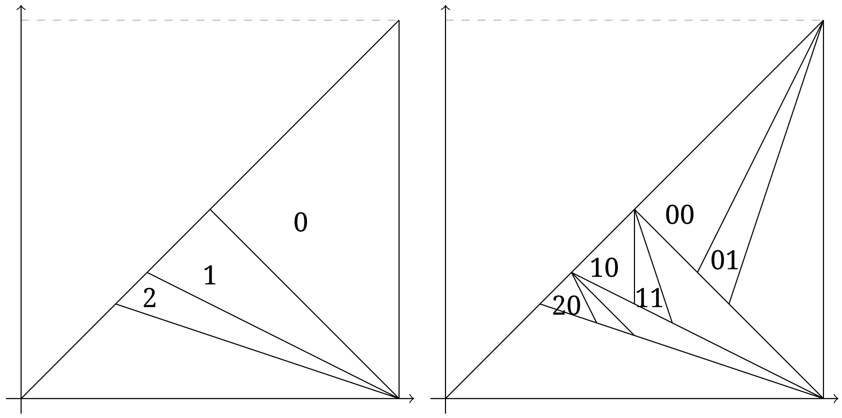

Fig. 4: The 1-time and 2-time partition of (Δ, T_G).

We found that this algorithm is self-dual for $n = 2$ and the matrix A_ϕ is given by

$$A_\phi = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Here, we describe our heuristic method to find such a matrix. First we assume that $A_\phi = ((a_{ij}))$ has integer entries. From $A_\phi A_{T^\#} = A_T A_\phi$, we see A_ϕ is symmetric. Assume that ϕ sends $\Delta^\#(k) \cap \Delta^\#(k+1)$ to $\Delta(k) \cap \Delta(k+1)$. In particular, if $\phi(0,0) = (1,0)$, then we see

$$a_{11} = a_{21}, \quad a_{31} = 0.$$

Put $a_{11} = x$, A_ϕ has to have a form

$$\begin{pmatrix} x & x & 0 \\ x & * & * \\ 0 & * & * \end{pmatrix}.$$

Further if $\phi(k,1) = (\frac{1}{k+1}, \frac{1}{k+1})$, then

$$\frac{x + ka_{22}}{x + kx} = \frac{ka_{32} + a_{33}}{x + kx} = \frac{1}{k+1}.$$

Therefore we have $x = 1$, $a_{22} = a_{32} = 0, a_{33} = 1$ and the condition $A_\phi A_{T^\#}(k) = A_T(k) A_\phi$, $\phi(\Delta^\#(k)) = \Delta(k)$ are guaranteed. Thus, we obtain the following.

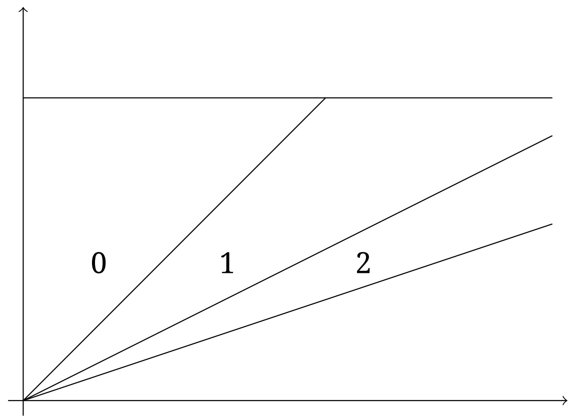

Fig. 5: The 1-time partition of $(\Delta^{\#}, T_G)$.

Proposition 4. The Garrity-Schweiger algorithm is algebraic self-dual. And

$$A_\phi = \begin{pmatrix} 1 & \cdots & 1 & \\ \vdots & \iddots & & \\ 1 & & & \\ & & & 1 \end{pmatrix}.$$

Proof. By a straightforward calculation, we can see that for all $x \in \Delta$

$$\phi \circ T^{\#}(x) = T \circ \phi(x).$$

Let $\phi(B_1, B_2, \ldots, B_n) = (b_1, b_2, \ldots, b_n)$. Then, since

$$A_\phi^{-1} = \begin{pmatrix} & & & 1 & \\ & & 1 & -1 & \\ & \iddots & \iddots & & \\ 1 & -1 & & & \\ & & & & 1 \end{pmatrix},$$

we have

$$B_{n-1} - \left\lfloor \frac{B_{n-1}}{B_n} \right\rfloor B_n = \frac{1 - b_1 - \left\lfloor \dfrac{1 - b_1}{b_n} \right\rfloor b_n}{b_{n-1}}.$$

Therefore, we can see $\phi(\Delta^{\#}\mathring{}(k)) = \mathring{\Delta}(k)$ for all $k \in \mathbb{Z}_{\geq 0}$. □

Corollary 2. The Garrity-Schweiger algorithm is symmetric in measure.

We introduce the slow version of the Garrity-Schweiger map. F. Schweiger defined the Flip-flop map in [13]. It is known that the jump transformation of the map is Garrity's triangle map (See also [3]). Similarly, we can see that the jump transformation of the n-dimensional Flip-flop map is the Garrity-Schweiger map and it is algebraic self-dual.

Let $\Delta = \{x \in \mathbb{R}^n_> : 1 \geq x_1 \geq \cdots \geq x_n > 0\}$. Let the cylinder set of the Selmer algorithm and Brun algorithm be $\Delta_S(i)$ and $\Delta_B(i)$ respectively. Then, since

$$\Delta_S(i) = \{x \in \Delta : x_i > 1 - x_n \geq x_{i+1}\}, \quad \Delta_B(i) = \{x \in \Delta : x_i > 1 - x_1 \geq x_{i+1}\},$$

and $x_0 = 1$ $x_{n+1} = 0$, we have

$$\Delta = \Delta_S(0) \cup \Delta_B(n).$$

Now, we define the map $T : \Delta \to \Delta$ as

$$A_T = \begin{cases} \begin{pmatrix} 1 & & & -1 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} & \textbf{on } \Delta_S(0), \\ \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & -1 & & \end{pmatrix} & \textbf{on } \Delta_B(n). \end{cases}$$

We consider the jump transformation over the cylinder $\Delta_S(0)$, then we obtain a map with matrices

$$\begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & -1 & & \end{pmatrix} \begin{pmatrix} 1 & & & -1 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}^k = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & -1 & & -k \end{pmatrix}$$

This map is Garrity-Schweiger map T_G. The dual space is $\mathbb{R}^n$, and the invariant density is

$$\frac{1}{x_1 x_2 \cdots x_n}.$$

Proposition 5. The n-dimensional Flip-Flop algorithm is algebraic self-dual. And

$$A_\phi = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \iddots & \\ 1 & & \end{pmatrix}.$$

6 Poincaré Algorithm

Finally, we give an example that is not self-dual. Note that the algorithm below is conjugate to the original Poincaré algorithm. There are maps F, G that the original map is $F \circ G$, but ours is $G \circ F$, where G is the sorting map into non-increasing order. We commonly refer to the projective map of the original map $F \circ G$ is the Parry-Daniel map. See e.g. [8], [12, Chapter 21] or [11, Chapter 14].

Let $E^{n+1} := \{x \in \mathbb{R}^{n+1}_{>} : x_1 \geq x_2 \geq \cdots \geq x_{n+1} \geq 0\}$. Then define

$$F : x \in E^{n+1} \longmapsto x' = (x_1 - x_2, x_2 - x_3, \dots, x_n - x_{n+1}, x_{n+1}).$$

There is an element σ of symmetric group $\mathcal{S}_{n+1}$ such that

$$P(x) = (x'_{\sigma(1)}, x'_{\sigma(2)}, \dots, x'_{\sigma(n+1)}) = G \circ F(x) \in E^{n+1}.$$

In this section, we consider the normalized map $T_P : \Delta \longrightarrow \Delta$ which makes the diagram

$$\begin{array}{ccc} E^{n+1} & \xrightarrow{P} & E^{n+1} \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ \Delta & \xrightarrow{T = T_P} & \Delta \end{array}$$

commutative, where $p(x) = \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \dots, \frac{x_{n+1}}{x_1}\right)$.

Then for all digit $\sigma \in \mathcal{S}_{n+1}$, $y = Tx$, $x \in \Delta(\sigma)$ is given by

$$y_i = \frac{\sum_{j=1}^{n+1} A_{\sigma^{-1}(i)j} x_{j-1}}{\sum_{j=1}^{n+1} A_{\sigma^{-1}(1)j} x_{j-1}}$$

where $x_0 = 1$ and

$$A_T(e) = ((A_{i,j})) := \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \\ & & & 1 \end{pmatrix}.$$

The dual map is defined on $\Delta^{\#} = \mathbb{R}^n_{>}$. Therefore the invariant density is

$$\frac{1}{x_1 x_2 \cdots x_n}.$$

Note that

$$A_T(\sigma) = ((A_{\sigma^{-1}(i)j})) = \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} & \sigma^{-1}(i) & \sigma^{-1}(i)+1 & n \end{array} \\ \begin{pmatrix} & \vdots & \vdots & \vdots \\ \cdots & 1 & -1 & \\ & & & \\ \cdots & & & 1 \\ & & & \end{pmatrix} \end{array},$$

$$A_{T^{\#}}(\sigma) = \begin{array}{c} \\ 1 \\ \\ i \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} & \sigma(1) & \sigma(i) & & \sigma(i-1) \end{array} \\ \begin{pmatrix} \cdots & 1 & \vdots & & \vdots \\ & & & & \\ \cdots & & 1 & \cdots & -1 \\ & & & & \end{pmatrix} \end{array}.$$

Then, we have

$$\Delta(\sigma)=\begin{cases}\{x\in\Delta: & x_{\sigma^{-1}(i)-1}-x_{\sigma^{-1}(i)}>x_{\sigma^{-1}(i+1)-1}-x_{\sigma^{-1}(i+1)},\ \text{for } i=1,2,\dots,n\ (i\neq j-1,j),\\ & x_{\sigma^{-1}(j-1)-1}-x_{\sigma^{-1}(j-1)}>x_n,\ x_n>x_{\sigma^{-1}(j+1)-1}-x_{\sigma^{-1}(j+1)}\},\ \text{if } j\neq 1,n+1,\\ \{x\in\Delta: & x_n>x_{\sigma^{-1}(2)-1}-x_{\sigma^{-1}(2)},\\ & x_{\sigma^{-1}(i)-1}-x_{\sigma^{-1}(i)}>x_{\sigma^{-1}(i+1)-1}-x_{\sigma^{-1}(i+1)},\ \text{for } i=2,\dots,n\},\ \text{if } j=1,\\ \{x\in\Delta: & x_{\sigma^{-1}(i)-1}-x_{\sigma^{-1}(i)}>x_{\sigma^{-1}(i+1)-1}-x_{\sigma^{-1}(i+1)},\ \text{for } i=1,2,\dots,n-1,\\ & x_{\sigma^{-1}(n)-1}-x_{\sigma^{-1}(n)}>x_n\},\ \text{if } j=n+1\end{cases}$$

and

$$\Delta^{\#}(\sigma)=\{x\in\Delta^{\#}:x_{\sigma(i+1)-1}-x_{\sigma(i)-1}>0 \text{ for } i=1,\dots,n\}$$

where $x_0=1$ and j is a integer satisfies $\sigma(n+1)=j$.

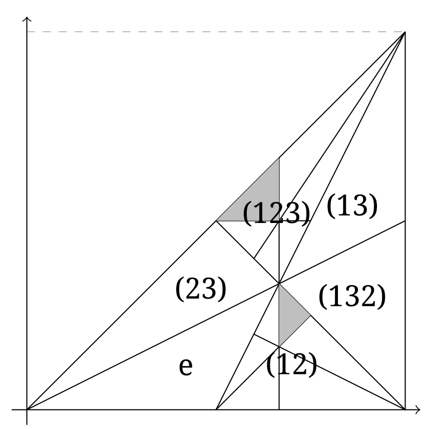

Fig. 6: $\Delta[(12),(123)]$ and $\Delta[(123),(12)]$ for $n=2$

In the case $n=2$, the domain Δ has six partitions $\Delta[e]$, $\Delta[(12)]$, $\Delta[(23)]$, $\Delta[(13)]$, $\Delta[(123)]$, and $\Delta[(132)]$ (see Fig. 6). By direct computation we have

$$\mu(\Delta[(12),(123)]) = \int_{[\frac{2}{3},\frac{3}{4})}\int_{[x-\frac{1}{2},-x+1)} \frac{1}{xy}dxdy = \frac{\log 2}{2}\log\frac{9}{8} = 0.0408\cdots,$$
$$\mu(\Delta[(123),(12)]) = \int_{[\frac{1}{2},\frac{2}{3})}\int_{[\frac{1}{2},x)} \frac{1}{xy}dxdy = \frac{1}{2}\left(\log\frac{4}{3}\right)^2 = 0.0413\cdots.$$

Thus,

$$\mu(\Delta[(12),(123)]) \neq \mu(\Delta[(123),(12)]).$$

From Theorem 1, this algorithm is not algebraic self-dual for $n = 2$.

All the same, we shall prove that this algorithm is algebraic self-dual on $\{e,(13),(123),(132)\}$ with the intertwining map ϕ defined as

$$A_\phi = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let us explain our empirical method to find this intertwining map. At first, we follow the heuristic way as before. Assume that $A_\phi = ((a_{ij}))$ has integer entries. From $A_\phi A_{T^\#}(\sigma) = A_T(\sigma)A_\phi$, we see it is symmetric. If $\phi(1,1) = (\frac{2}{3},\frac{1}{3})$, then we see

$$a_{11}+a_{12}+a_{13} : a_{21}+a_{22}+a_{23} : a_{31}+a_{32}+a_{33} = 3 : 2 : 1. \tag{2}$$

Assume for now that $\Delta^\#[e] \cap \Delta^\#[(12)]$ is mapped to $\Delta[e] \cap \Delta[(12)]$. Then from

$$\phi(1,y) = \left(\frac{a_{21}+a_{22}+a_{23}y}{a_{11}+a_{12}+a_{13}y}, \frac{a_{31}+a_{32}+a_{33}y}{a_{11}+a_{12}+a_{13}y}\right),$$

if $\lim_{y\to\infty}\phi(1,y) = (\frac{1}{2},0)$, then there exists an integer k that we have

$$A_\phi = \begin{pmatrix} * & * & 2k \\ * & * & k \\ 2k & k & 0 \end{pmatrix}.$$

However in this case, it is natural to assume $\lim_{x\to\infty}\phi(x,x)=(0,0)$, and then we have

$$a_{22}+a_{23}=a_{32}=0.$$

This implies $k=0$ and clearly we have $\phi(\Delta^{\#}(\sigma))\neq\Delta(\sigma)$ which does not fit our purpose. After this wrong trial, we reach the correct assumption that $\Delta^{\#}[e]\cap\Delta^{\#}[(12)]$ is mapped to $\Delta[e]\cap\Delta[(23)]$ (See Fig. 7). Indeed if $\lim_{y\to\infty}\phi(1,y)=(0,0)$, then A_ϕ has the form

$$A_\phi=\begin{pmatrix} * & * & * \\ * & * & 0 \\ * & 0 & 0 \end{pmatrix}.$$

From $\lim_{x\to\infty}\phi(x,x)=(\frac{1}{2},0)$, we obtain

$$\frac{a_{22}}{a_{12}+a_{13}}=\frac{1}{2}.$$

Considering (2), by several trials we found

$$A_\phi=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

which satisfies all the conditions on $\Delta[e]\cup\Delta[(13)]\cup\Delta[(123)]\cup\Delta[(132)]$. (See Appendix A.) We define an involution on $\mathcal{S}_n$.

Definition 6. We denote the set of involutions of the symmetric group by

$$Inv(\mathcal{S}_n)=\{\sigma\in\mathcal{S}_n : \sigma^2=e\}.$$

The cardinality of this set $\#Inv(\mathcal{S}_n)$:$1,2,4,10,26,76,\cdots$ are also known as telephone numbers and various studies have been made on these numbers (see Section 5.1.4 of [6]).

Theorem 2. The n-dimensional Poincaré algorithm is algebraic self-dual on $w_0 Inv(\mathcal{S}_{n+1})$ where

$$w_0=\begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}.$$

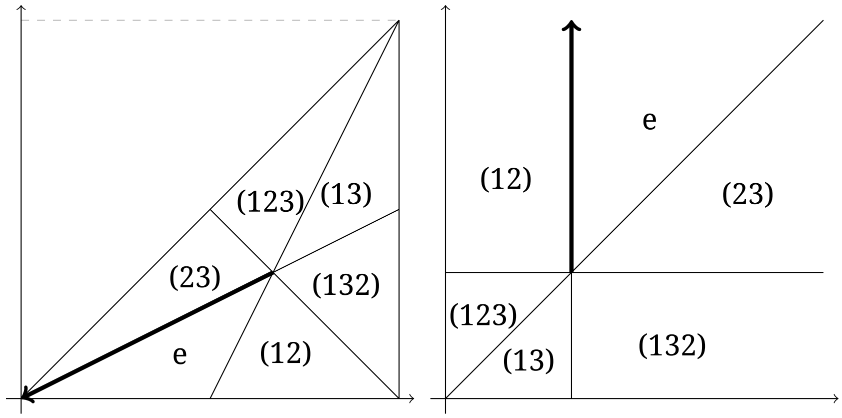

Fig. 7: The 1-time partition of (Δ, T_P) and $(\Delta^{\#}, T_P^{\#})$.

And

$$A_\phi = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \iddots & \\ 1 & & \end{pmatrix}.$$

Proof. Let $M = ((a_{i,j}))$ be a monomial $(0,1)$–matrix with $a_{i,j} = 1$. i.e., there is exactly one “1” in each row and each column. This has the one-to-one correspondence with permutation, we denote

$$M \longleftrightarrow \begin{pmatrix} \cdots & i & \cdots \\ \cdots & j & \cdots \end{pmatrix}.$$

Then, we have

$$w_0\sigma^{-1} = \begin{pmatrix} \sigma(1) & \sigma(2) & \cdots & \sigma(n+1) \\ n+1 & n & \cdots & 1 \end{pmatrix} \longleftrightarrow (A_{\sigma^{-1}(i)j})B,$$
$$\sigma w_0 = \begin{pmatrix} 1 & 2 & \cdots & n+1 \\ \sigma(n+1) & \sigma(n) & \cdots & \sigma(1) \end{pmatrix} \longleftrightarrow B(A_{\sigma^{-1}(i)j})^t.$$

Therefore, we have

$$\begin{aligned}\{\sigma \in \mathcal{S}_{n+1} : B(A_{\sigma^{-1}(i)j})^t = (A_{\sigma^{-1}(i)j})B\} &= \{\sigma \in \mathcal{S}_{n+1} : (w_0\sigma)^2 = e\} \\ &= w_0 Inv(\mathcal{S}_{n+1}).\end{aligned}$$

Let $\phi(B_1, B_2, \dots, B_n) = (b_1, b_2, \dots, b_n)$ and $\sigma \in w_0 Inv(\mathcal{S}_{n+1})$. By the definition of $Inv(\mathcal{S}_{n+1})$,

$$\begin{aligned}\sigma^{-1}(i) &= w_0\sigma w_0(i) \\ &= w_0\sigma(n+1-i+1) \\ &= n+1-\sigma(n+1-i+1)+1\end{aligned}$$

and we have

$$n - \sigma(n-i+2) + 1 = \sigma^{-1}(i) - 1. \tag{3}$$

We show $\phi(\Delta^{\#}(\sigma)) \supset \Delta(\sigma)$. Let $(b_1, b_2, \dots, b_n) \in \Delta(\sigma)$.

For $i \neq n-j+1, n-j+2$, by (3),

$$\begin{aligned}B_{\sigma(i+1)-1} - B_{\sigma(i)-1} &= \frac{b_{n-\sigma(i+1)+1} - b_{n-\sigma(i+1)+2}}{b_n} - \frac{b_{n-\sigma(i)+1} - b_{n-\sigma(i)+2}}{b_n} \\ &= \frac{b_{\sigma^{-1}(n-i+1)-1} - b_{\sigma^{-1}(n-i+1)} - b_{\sigma^{-1}(n-i+2)-1} + b_{\sigma^{-1}(n-i+2)}}{b_n} \\ &> 0.\end{aligned}$$

For $i \neq n-j+1, n-j+2$, since $\sigma(n+1) = j$ and $\sigma \in w_0 Inv(\mathcal{S}_{n+1})$, by (3), $\sigma(n-j+2) = 1$. Then we have

$$\begin{aligned}B_{\sigma(n-j+2)-1} - B_{\sigma(n-j+1)-1} &= 1 - \frac{b_{n-\sigma(n-j+1)+1} - b_{n-\sigma(n-j+1)+2}}{b_n} \\ &= \frac{b_n - b_{\sigma^{-1}(j+1)-1} + b_{\sigma^{-1}(j+1)}}{b_n} \\ &> 0\end{aligned}$$

and

$$\begin{aligned} B_{\sigma(n-j+3)-1} - B_{\sigma(n-j+2)-1} &= \frac{b_{n-\sigma(n-j+3)+1} - b_{n-\sigma(n-j+3)+2}}{b_n} - 1 \\ &= \frac{b_{\sigma^{-1}(j-1)-1} - b_{\sigma^{-1}(j-1)} - b_n}{b_n} \\ &> 0. \end{aligned}$$

Similarly, we have $\phi(\Delta^{\#}(\sigma)) \subset \Delta(\sigma)$. □

Corollary 3. The n-dimentional Poincaré algorithm is symmetric in measure on $w_0 Inv(\mathcal{S}_{n+1})$, i.e., for all $\sigma_1, \sigma_2, \dots, \sigma_s \in w_0 Inv(\mathcal{S}_{n+1})$,

$$\mu(\Delta[\sigma_1, \sigma_2, \dots, \sigma_s]) = \mu(\Delta[\sigma_s, \sigma_{s-1}, \dots, \sigma_1]).$$

A Self-duality of 2-dimensional Poincaré Algorithm

We see that the 2-dimensional Poincaré Algorithm (Δ, T_P) is algebraic self-dual on $\{e, (13), (123), (132)\}$ with the intertwining map ϕ defined as

$$A_\phi = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \\ 1 & & \end{pmatrix}.$$

Note that $\Delta = \{\mathbf{x} \in \mathbb{R}^2 : 1 \geq x_1 \geq x_2 > 0\}$, $\Delta^{\#} = \mathbb{R}^2_{\geq}$.

$\underline{\Delta[e]}$; Since

$$A_T(e) = \begin{pmatrix} 1 & -1 & \\ & 1 & -1 \\ & & 1 \end{pmatrix}, \ A_{T^{\#}}(e) = \begin{pmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \end{pmatrix},$$

$A_T(e) A_\phi = A_\phi A_{T^{\#}}(e)$ holds. And

$$T(b_1, b_2) = \left(\frac{b_1 - b_2}{1 - b_1}, \frac{b_2}{1 - b_1} \right), \quad T^{\#}(B_1, B_2) = (-1 + B_1, -B_1 + B_2).$$

Thus, the cylinder set $\Delta[e]$ and $\Delta^{\#}[e]$ are

$$\begin{aligned}\Delta[e] &= \{(b_1,b_2)\in\Delta : 1-b_1>b_1-b_2,\ b_1-b_2>b_2\}\\ &= \{(b_1,b_2)\in\Delta : 1-2b_1+b_2>0,\ b_1>2b_2\},\\ \Delta^{\#}[e] &= \left\{(B_1,B_2)\in\Delta^{\#} : -1+B_1>0,\ -B_1+B_2>0\right\}.\end{aligned}$$

We show $\phi(\Delta^{\#}[e])=\Delta[e]$. Let $\phi(B_1,B_2)=(b_1,b_2)$. Then,

$$b_1=\frac{1+B_1}{1+B_1+B_2},\ b_2=\frac{1}{1+B_1+B_2}$$

and

$$B_1=\frac{b_1}{b_2}-1,\ B_2=\frac{1}{b_2}-\frac{b_1}{b_2}.$$

Thus, we can see $\phi(\Delta^{\#}[e])=\Delta[e]$ by the following calculation

$$1-2b_1+b_2=\frac{-B_1+B_2}{1+B_1+B_2},\quad b_1-2b_2=\frac{-1+B_1}{1+B_1+B_2}$$

and

$$B_1-1=\frac{b_1}{b_2}-2,\quad B_2-B_1=\frac{1-2b_1+b_2}{b_2}.$$

$\underline{\Delta[(12)]}$;

$$A_T((12))A_\phi=\begin{pmatrix} & 1 & -1\\ 1 & -1 & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & \\ 1 & & \end{pmatrix}=\begin{pmatrix} & 1 & \\ & & 1\\ 1 & & \end{pmatrix},$$

$$A_\phi A_T^{\#}((12))=\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & \\ 1 & & \end{pmatrix}\begin{pmatrix} & 1 & \\ 1 & -1 & \\ -1 & & 1\end{pmatrix}=\begin{pmatrix} & & 1\\ 1 & & \\ & 1 & \end{pmatrix}.$$

$\underline{\Delta[(23)]}$;

$$A_T((23))A_\phi = \begin{pmatrix} 1 & -1 & \\ & & 1 \\ & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \\ 1 & & \end{pmatrix} = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix},$$

$$A_\phi A_T^\#((23)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \\ 1 & & \end{pmatrix}\begin{pmatrix} 1 & & \\ -1 & & 1 \\ & 1 & -1 \end{pmatrix} = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}.$$

$\underline{\Delta[(13)]}$; Since

$$A_T((13)) = \begin{pmatrix} & & 1 \\ & 1 & -1 \\ 1 & -1 & \end{pmatrix},$$

$A_T((13))A_\phi = A_\phi A_{T^\#}((13))$ holds.

$$\begin{aligned}\Delta[(13)] &= \{(b_1, b_2) \in \Delta \,:\, b_2 > b_1 - b_2,\ b_1 - b_2 > 1 - b_1\} \\ &= \{(b_1, b_2) \in \Delta \,:\, 2b_2 - b_1 > 0,\ 2b_1 - b_2 - 1 > 0\}, \\ \Delta^\#[(13)] &= \left\{(B_1, B_2) \in \Delta^\# \,:\, B_1 - B_2 > 0,\ 1 - B_1 > 0\right\}.\end{aligned}$$

We show $\phi(\Delta^\#[e]) = \Delta[e]$. Let $\phi(B_1, B_2) = (b_1, b_2)$. Then, we can see $\phi(\Delta^\#[(13)]) = \Delta[(13)]$ by the following calculation

$$2b_2 - b_1 = \frac{1 - B_1}{1 + B_1 + B_2}, \quad 2b_1 - b_2 - 1 = \frac{B_1 - B_2}{1 + B_1 + B_2}$$

and

$$B_1 - B_2 = \frac{2b_1 - b_2 - 1}{b_2}, \quad 1 - B_1 = 2 - \frac{b_1}{b_2}.$$

$\underline{\Delta[(123)]}$; Since

$$A_T((123)) = \begin{pmatrix} & & 1 \\ 1 & -1 & \\ & 1 & -1 \end{pmatrix}, \quad A_{T^\#}((123)) = \begin{pmatrix} & 1 & \\ & -1 & 1 \\ 1 & & -1 \end{pmatrix},$$

$A_T(123)A_\phi = A_\phi A_{T^\#}(123)$ holds.

$$\begin{aligned} \Delta[(123)] &= \{(b_1,b_2)\in\Delta : b_2 > 1-b_1,\ 1-b_1 > b_1-b_2\} \\ &= \{(b_1,b_2)\in\Delta : b_1+b_2-1>0,\ 1-2b_1+b_2>0\}, \\ \Delta^\#[(123)] &= \left\{(B_1,B_2)\in\Delta^\# : -B_1+B_2>0,\ 1-B_2>0\right\}. \end{aligned}$$

We show $\phi(\Delta^\#[(123)]) = \Delta[(123)]$. Let $\phi(B_1,B_2) = (b_1,b_2)$. Then, we can see $\phi(\Delta^\#[(123)]) = \Delta[(123)]$ by the following calculation

$$b_1+b_2-1 = \frac{1-B_2}{1+B_1+B_2}, \quad 1-2b_1+b_2 = \frac{-B_1+B_2}{1+B_1+B_2}$$

and

$$-B_1+B_2 = \frac{1-2b_1+b_2}{b_2}, \quad 1-B_2 = \frac{b_1+b_2-1}{b_2}.$$

$\underline{\Delta[(132)]}$; Since

$$A_T((132)) = \begin{pmatrix} & 1 & -1 \\ & & 1 \\ 1 & -1 & \end{pmatrix}, \quad A_{T^\#}((132)) = \begin{pmatrix} & & 1 \\ 1 & & -1 \\ -1 & 1 & \end{pmatrix},$$

$A_T(132)A_\phi = A_\phi A_{T^\#}(132)$ holds.

$$\begin{aligned} \Delta[(132)] &= \{(b_1,b_2)\in\Delta : b_1-b_2 > b_2,\ b_2 > 1-b_1\} \\ &= \{(b_1,b_2)\in\Delta : b_1-2b_2>0,\ b_1+b_2-1>0\}, \\ \Delta^\#[(132)] &= \left\{(B_1,B_2)\in\Delta^\# : 1-B_2>0,\ -1+B_1>0\right\}. \end{aligned}$$

We show $\phi(\Delta^{\#}[(132)]) = \Delta[(132)]$. Let $\phi(B_1, B_2) = (b_1, b_2)$. Then, we can see $\phi(\Delta^{\#}[(132)]) = \Delta[(132)]$ by the following calculation

$$b_1 - 2b_2 = \frac{-1 + B_1}{1 + B_1 + B_2}, \quad b_1 + b_2 - 1 = \frac{1 - B_2}{1 + B_1 + B_2}$$

and

$$1 - B_2 = \frac{b_1 + b_2 - 1}{b_2}, \quad -1 + B_1 = \frac{b_1}{b_2} - 2.$$

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AUTHOR

Hiroaki Ito
Montanuniversität Leoben
Franz Josef-Straße, 18
8700, Leoben, Austria
email: hiroaki.i-2.68@outlook.jp