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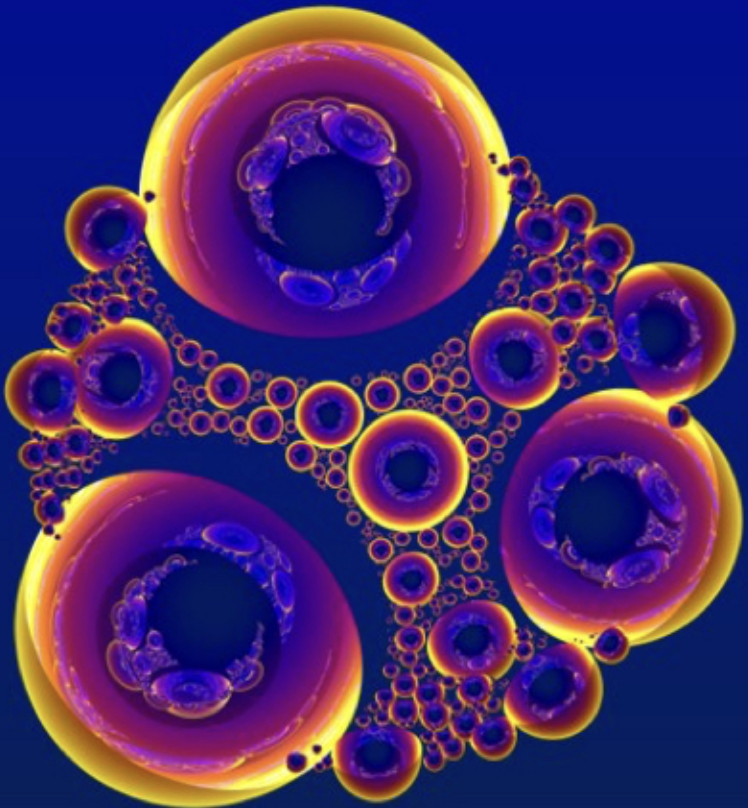

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On boundary points of minimal continuously Hutchinson invariant sets

Per Alexandersson Nils Hemmingsson Dmitry Novikov
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Abstract: A linear differential operator $T = Q(z)\frac{d}{dz} + P(z)$ with polynomial coefficients defines a continuous family of Hutchinson operators when acting on the space of positive powers of linear forms. In this context, T has a unique minimal Hutchinson-invariant set M_{CH}^{T} in the complex plane. Using a geometric interpretation of its boundary in terms of envelopes of certain families of rays, we subdivide this boundary into local and global arcs (the former being portions of integral curves of the rational vector field $\frac{Q(z)}{P(z)}\partial_z$), and singular points of different types which we classify below.

The latter decomposition of the boundary of M_{CH}^{T} is largely determined by its intersection with the plane algebraic curve formed by the inflection

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points of trajectories of the field $\frac{Q(z)}{P(z)}\partial_z$. We provide an upper bound for the number of local arcs in terms of degrees of P and Q. As an application of our classification, we obtain a number of global geometric properties of minimal Hutchinson-invariant sets.

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1 Introduction

Given a linear differential operator

$$T = Q(z)\frac{d}{dz} + P(z) \tag{1.1}$$

where P, Q are polynomials that are not identically vanishing, we say that a closed subset $S \subset \mathbb{C}$ is *continuously Hutchinson invariant* for T (T_{CH}-invariant set for short) if for any $u \in S$ and any arbitrary non-negative number t, the image $T(f)$ of the function $f(z) = (z-u)^t$ either has all roots in S or vanishes identically. In [AHN+24], we have initiated the study of general topological properties of T_{CH}-invariant sets.

The main motivation for the present study is that it covers an interesting and manageable special case of a more general inverse Pólya-Schur problem introduced in [ABS]. For the convenience of our readers, let us briefly recall what the Pólya–Schur problem/theory and its inverse are, see [CsCr, ABS].

The main question of the classical Pólya–Schur theory can be formulated as follows.

Problem 1.1. Given a subset $S \subset \mathbb{C}$ of the complex plane, describe the semigroup of all linear operators $T : \mathbb{C}[z] \to \mathbb{C}[z]$ sending any polynomial with roots in S to a polynomial with roots in S (or to 0).

Definition 1.2. If an operator T has the latter property, then we say that *S is a T-invariant set*, or that *T preserves S*.

So far, Problem 1.1 has only been solved for the circular domains (i.e., images of the unit disk under Möbius transformations), their boundaries [BB], and more recently for strips [BCh]. Even a very similar case of the unit interval is still open at present. It seems that for a somewhat general class of subsets $S \subset \mathbb{C}$, Problem 1.1 is currently out of reach of all existing methods.

In [ABS], the following inverse problem in the Pólya–Schur theory which seems both natural and more accessible than Problem 1.1 has been proposed.

Problem 1.3. Given a linear operator $T : \mathbb{C}[x] \to \mathbb{C}[x]$, characterize all closed T-invariant subsets of the complex plane. Alternatively, find a sufficiently large class of T-invariant sets.

Paper [ABS] concentrates on the fundamental case when T is a linear finite order differential operator with polynomial coefficients and shows that under some weak assumptions on these coefficients, there exists a unique minimal T-invariant set (and its analogs when T acts on polynomials of degree greater than or equal to a given positive integer n). Many basic properties of T-invariant sets such as their convexity, compactness etc are discussed in [ABS] as well as the delicate connection of Problem 1.3 to the classical complex dynamics.

However effective criteria characterizing T-invariant sets and explicit description of the minimal T-invariant set in somewhat interesting cases are currently missing which motivated the consideration in [AHN+24] of the action of T on integer and positive powers of linear forms. This situation is still quite interesting and appears to be more tractable.

In particular, the following results have been obtained in [AHN+24]:

- provided that either P or Q is not a constant polynomial, there is a unique *minimal* continuously Hutchinson invariant set M_{CH}^{T} for a given operator T (in what follows we will always assume that this condition is satisfied);
- the only T_{CH}-invariant set is the whole $\mathbb{C}$ unless $|\deg Q - \deg P| \leq 1$;

- a complete characterization of operators T for which M_{CH}^{T} has an empty interior has been obtained (see Section 2.1 for details).

In this paper, we will focus on operators whose minimal set M_{CH}^{T} has a nonempty interior.

Definition 1.4. For an operator T given by (1.1) with P and Q not vanishing identically, at each point z such that $PQ(z) \neq 0$, we define the *associated ray* $r(z)$ as the half-line $\{z + t\frac{Q(z)}{P(z)} \mid t \in \mathbb{R}^{+}\}$.

Remarkably, T_{CH}-invariant sets (and, in particular, the minimal one) can be characterized in terms of associated rays.

Theorem 1.5 (Theorem 3.18 in [AHN+24]). *A closed subset $S \subseteq \mathbb{C}$ is T_{CH}-invariant if and only if it satisfies the following two conditions:*

1. *S contains the roots of the polynomials P and Q;*
2. *for any point $z \notin S$, the associated ray $r(z)$ is disjoint from S.*

1.1 Main results

In the present paper, using Theorem 1.5, we provide a qualitative description of the boundary of minimal continuously Hutchinson invariant sets, including an exhaustive typology of its singular points. Our classification mainly depends on the intersection of the boundary $\partial\mathrm{M}_{CH}^{T}$ with the curve of inflections $\mathfrak{I}_R$ of the field $R(z)\partial_z = \frac{Q(z)}{P(z)}\partial_z$.

Definition 1.6. The *curve of inflections* $\mathfrak{I}_R$ of the vector field $R(z)\partial_z$ is defined as the closure of the set of points satisfying $\mathrm{Im}(R') = 0$, see [AHN+24]. It is a real plane algebraic curve of degree at most $d = 3\deg P + \deg Q - 1$ (in this paper, we will always have $d \geq 3$).

The curve of inflections splits the complex plane into *inflection domains* where the sign of $\mathrm{Im}(R')$ remains the same.

Points of $\partial\mathrm{M}^T_{CH}$ outside its intersection with $\mathfrak{I}_R$ are classified with the help of two correspondences Γ and Δ sending the boundary $\partial\mathrm{M}^T_{CH}$ to itself and defined as follows:

For a given point z of the boundary $\partial\mathrm{M}^T_{CH}$, $\Gamma(z)$ is essentially the intersection of M^T_{CH} with the integral curve of the rational field $R(z)\partial_z$ starting at z, where $R(z) = Q(z)/P(z)$. In contrast, Δ is the intersection of the associated ray $r(z)$ with the closure of $\partial\mathrm{M}^T_{CH}$ in the compactification $\mathbb{C} \cup \mathbb{S}^1$ of the complex plane (see Section 2.2). Formal definitions of Γ and Δ are given in Definition 4.1. Qualitatively, the boundary $\partial\mathrm{M}^T_{CH}$ is made of two kinds of arcs:

- *local arcs* which are integral curves of the field $R(z)\partial_z$ (i.e. $\Delta(z) = \emptyset$ and $\Gamma(z) \neq \emptyset$);
- *global arcs* at each point z of which the associated ray $r(z)$ is tangent to $\partial\mathrm{M}^T_{CH}$ elsewhere (i.e. $\Gamma(z) = \emptyset$ and $\Delta(z) \neq \emptyset$).

Local arcs are locally strictly convex real-analytic arcs (see Proposition 4.11). In contrast, global arcs (formed by points of global type) can fail to be C^1.

Local arcs inherit an obvious orientation from the vector field $R(z)\partial_z$. Global arcs also have canonical orientation, but its definition requires some work (see Section 4.5.2).

Local and global arcs connect special singular points of $\partial\mathrm{M}^T_{CH}$ which in most of the cases belong to the curve of inflections. The latter decomposes into three loci (singular, tangent and transverse), each determining its own variety of singular points.

Definition 1.7. The *curve of inflections* $\mathfrak{I}_R$ of the field $R(z)\partial_z$ decomposes into:

- the *singular locus* $\mathfrak{S}_R$ formed by the points where several branches of $\mathfrak{I}_R$ intersect;
- the *tangency locus* $\mathfrak{T}_R$ formed by the non-singular points where the field $R(z)\partial_z$ is tangent to $\mathfrak{I}_R$;
- *transverse locus* $\mathfrak{I}^*_R$ formed by the non-singular points of $\mathfrak{I}_R$ where the field $R(z)\partial_z$ is transverse to $\mathfrak{I}_R$.

The singular and the tangency loci are given by algebraic conditions. Therefore their intersection with $\partial\mathrm{M}^T_{CH}$ is controlled in terms of $\deg P$ and $\deg Q$. On the contrary, many points of the boundary can belong to the transverse locus $\mathfrak{I}_R^*$. We refine the definition of the correspondence Δ according to the value of $\frac{R(z)}{u-z}$ (which, by definition, is a positive number).

Definition 1.8. We define $\Delta(z) = (\overline{r(z)}\setminus\{z\})\cap\overline{\mathrm{M}_{CH}^T}$, where $\overline{r(z)}$, $\overline{\mathrm{M}_{CH}^T}$ are closures of $r(z), \mathrm{M}_{CH}^T$ in the compactification $\mathbb{C}\cup\mathbb{S}^1$ of $\mathbb{C}$, respectively.

For any $z \in \mathfrak{I}_R \setminus \mathcal{Z}(PQ)$, we have $\Delta(z) = \Delta^-(z)\cup\Delta^0(z)\cup\Delta^+(z)$ where $u \in \Delta(z)\cap\mathbb{C}$ belongs to:

- $\Delta^-(z)$ if $R'(z) \leq -\frac{R(z)}{u-z}$;
- $\Delta^0(z)$ if $R'(z) = -\frac{R(z)}{u-z}$;
- $\Delta^+(z)$ if $R'(z) \geq -\frac{R(z)}{u-z}$,

and $u \in \Delta(z)\cap\mathbb{S}^1$ belongs to

- $\Delta^-(z)$ if $R'(z) \leq 0$;
- $\Delta^0(z)$ if $R'(z) = 0$;
- $\Delta^+(z)$ if $R'(z) \geq 0$.

In particular, if $R'(z) > 0$, then $\Delta^-(z) = \emptyset$.

The main result of the present paper is a classification of boundary points of minimal continuously Hutchinson sets.

Theorem 1.9. *For any linear differential operator T given by (1.1), any point z of the boundary $\partial\mathrm{M}_{CH}^T$ of its minimal T_{CH}-invariant set belongs to one of the following types:*

- roots of polynomials P *and* Q *(at most* $\deg P + \deg Q$ *of them);*
- singular points of the curve of inflections *(at most* $2d$ *of them);*
- tangency points between the curve of inflections and the field $R(z)\partial_z$*;*

  - *straight segments, half-lines and lines (contained in at most* $\deg P + \deg Q + 1$ *lines);*
  - *at most* $2d^2$ *isolated points;*

- points of the transverse locus $\mathfrak{I}_R^*$ *belonging to one of the four subclasses:*
  - bouncing type*:* $\Delta^+(z) \neq \emptyset$ *and* $\Gamma \cup \Delta^-(z) \neq \emptyset$*;*
  - switch type*:* $\Delta^+(z) \neq \emptyset$ *and* $\Gamma(z) \cup \Delta^-(z) = \emptyset$*;*
  - C^1-inflection type*:* $\Delta^+(z) = \emptyset$*,* $\Delta^-(z) \neq \emptyset$ *and* $\Gamma(z) = \emptyset$*;*
  - C^2-inflection type*:* $\Delta^+(z) = \emptyset$ *and either* $\Delta^-(z) = \emptyset$ *or* $\Gamma(z) \neq \emptyset$*.*
- points not on the curve of inflections *belonging to one of the three subclasses:*
  - local type*:* $\Gamma(z) \neq \emptyset$ *and* $\Delta(z) = \emptyset$*;*
  - global type*:* $\Gamma(z) = \emptyset$ *and* $\Delta(z) \neq \emptyset$*;*
  - extruding type*:* $\Gamma(z) \neq \emptyset$ *and* $\Delta(z) \neq \emptyset$*.*

Here, $d = 3\deg P + \deg Q - 1$*.*

There can be many singular points of bouncing, extruding, C^1-inflection, C^2-inflection and switch types (we do not have a polynomial bound of their number in terms of $\deg P$ and $\deg Q$). An extensive description of their geometric features is given below:

- at *points of extruding type*, the boundary of $\partial \mathrm{M}_{CH}^T$ is not convex and it switches from a global to a local arc (see Section 4.6 and Figure 1);
- at *points of bouncing type*, $\partial \mathrm{M}_{CH}^T$ hits the curve of inflections, but does not cross it. In a neighborhood of such a point, the boundary $\partial \mathrm{M}_{CH}^T$ remains in the closure of the same inflection domain (see Section 5.2 and Figure 1);
- at *points of switch type*, $\partial \mathrm{M}_{CH}^T$ is strictly convex, crosses the curve of inflections and the boundary switches from a local to a global arc (see Section 5.5 and Figure 1);

- at points of C^1-*inflection type*, $\partial \mathrm{M}^T_{CH}$ crosses the curve of inflections and it switches from a global to another global arc having the opposite orientation. At such a point the curvature of $\partial \mathrm{M}^T_{CH}$ is discontinuous (see Section 5.4 and Figure 1);

- at points of C^2-*inflection type*, $\partial \mathrm{M}^T_{CH}$ crosses the curve of inflections and the boundary switches from a global arc to a local arc. Besides, the curvature of $\partial \mathrm{M}^T_{CH}$ is continuous at such a point (see Section 5.3 and Figure 1).

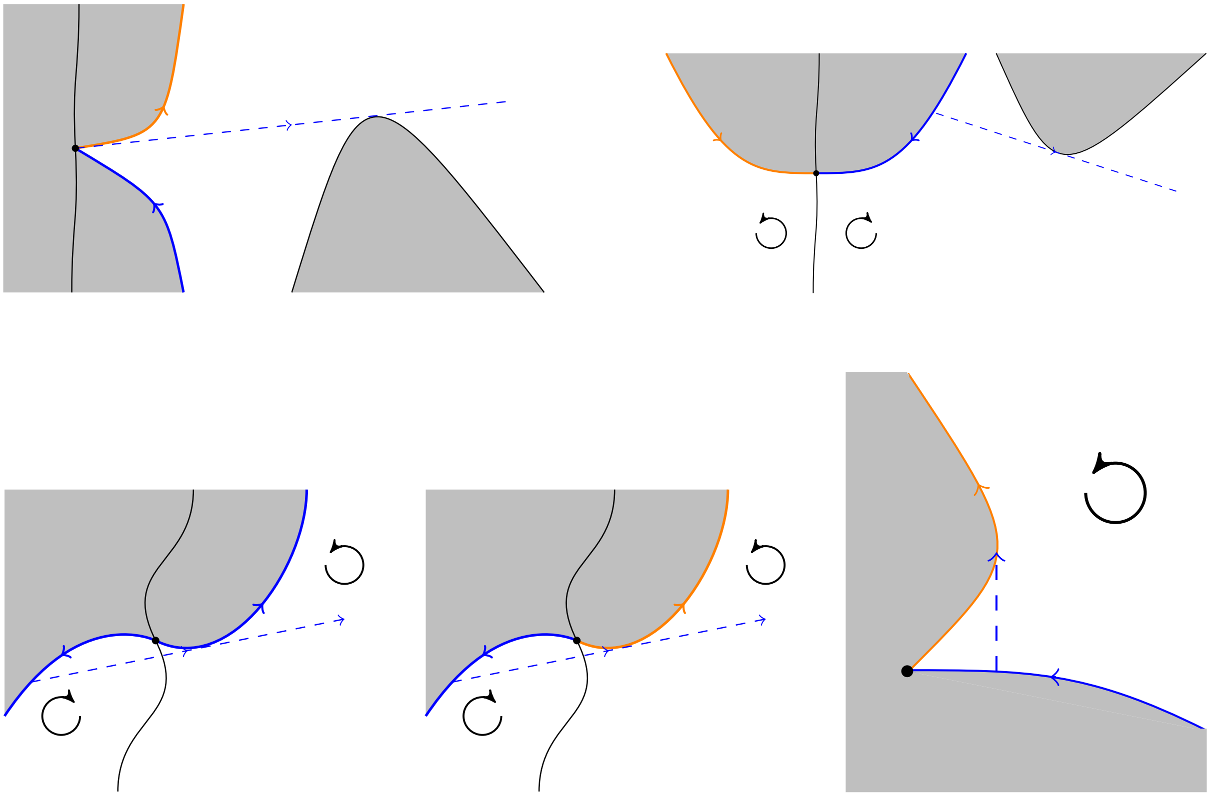

Figure 1: Top row from the left: Extruding type, Switch type. Bottom row from the left: C^1-inflection type, C^2-inflection type, Bouncing type. In the pictures, blue arcs are global arcs, red arcs are local arcs and the black arc is a germ of the curve of inflections. The pointed arrow is the associated ray indicating the support points, when applicable.

Our second main result is an upper bound on the number of points of C^1-inflection, C^2-inflection and switch type in terms of $d = 3\deg P + \deg Q - 1$.

Theorem 1.10. *For any operator T given by* (1.1)*, the numbers of points of switch, C^1-inflection and C^2-inflection type (respectively $|\mathcal{S}|$, $|\mathcal{I}_1|$ and $|\mathcal{I}_2|$) in $\partial \mathrm{M}_{CH}^T$ satisfy the following bounds:*

$$|\mathcal{S}| \leq e^{16d\ln(d)} + 46d^3;$$

$$2|\mathcal{I}_1| + |\mathcal{I}_2| \leq 2e^{16d\ln(d)} + 92d^3.$$

Corollary 1.11. *For any operator T given by* (1.1)*, the boundary $\partial \mathrm{M}_{CH}^T$ of the minimal set contains at most $d^{16d} + 46d^3 + d(2d+1)$ local arcs.*

In the last section of the paper, we deduce many results about the global geometry of minimal sets from the classification of boundary points. In several cases, an exact description can be given in terms of local and global arcs. In particular, we can prove that in generic case, the minimal T_{CH}-invariant set is connected in $\mathbb{C}$.

Theorem 1.12. *For any linear differential operator T given by* (1.1)*, the minimal continuously Hutchinson invariant set M_{CH}^T is a connected subset of $\mathbb{C}$ with the possible exception of the case when $R(z)$ is of the form $\lambda + \frac{\mu}{z} + o(z^{-1})$ with $\lambda \in \mathbb{C}^*$ and $\mu/\lambda \in \mathbb{R}$.*

In this latter case, (unless both P and Q are constants and then there is no reasonable notion of a minimal set), M_{CH}^T is formed by at most $\frac{1}{2}\deg P + \frac{1}{2}\deg Q$ connected components.

1.2 Organization of the paper

- In Section 2, we provide the basic background information on Hutchinson invariant sets developed in [AHN+24], including the results about their asymptotic geometry.

- In Section 3, we describe the local geometry around singular points of the vector field $R(z)\partial_z$ in terms of their local degree and principal value. We also describe the main properties of the curve of inflections defined by the equation $Im(R') = 0$ and we also introduce the notion of horns.

- In Section 4, we describe boundary points in the complement to the curve of inflections, introducing Γ and Δ correspondences.
- In Section 5, we classify boundary points in the generic locus of the curve of inflections, proving Theorems 1.9 and 1.10 (in Sections 5.6 and 5.8.3 respectively). Corollary 1.11 is also proved in Section 5.8.3.
- In Section 6, we apply the latter results to get precise descriptions of minimal sets in several cases. Theorem 1.12 is proven in Section 6.4.

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2 Preliminary results and basic properties of M_{CH}^{T}

The following notation will be important throughout this text.

Notation 2.1. Given an operator T as in (1.1), we define $p_\infty, q_\infty \in \mathbb{C}^*$, and $p, q \in \mathbb{N}$ so that

$$P(z) = p_\infty z^p + o(z^p);$$

$$Q(z) = q_\infty z^q + o(z^q).$$

Furthermore, we set $\lambda = \frac{q_\infty}{p_\infty} \in \mathbb{C}^*$ and $\phi_\infty = \arg(\lambda)$.

Similarly, for any point $\alpha \in \mathbb{C}$, we have $R(z) = r_\alpha (z-\alpha)^{m_\alpha} + o(|z-\alpha|^{m_\alpha})$ with $r_\alpha \neq 0$ and $m_\alpha \in \mathbb{Z}$. We denote by ϕ_α the argument of r_α.

Remark 2.2. Observe that frequently used affine changes of the variable z are applied to the vector field $R(z)\partial_z$ and not to the rational function $R(z)$ itself.

2.1 Regularity of the minimal set

For an operator T as in (1.1), its minimal set M_{CH}^{T} can be of three possible types:

- *regular* if M_{CH}^{T} coincides with the closure of its interior;
- *fully irregular* if M_{CH}^{T} has empty interior;
- *partially irregular* if M_{CH}^{T} has nonempty interior but is not regular.

Actually, irregularity is related to specific reality conditions. The characterization of operators for which M_{CH}^{T} is fully irregular is contained in Theorem 1.15 of [AHN+24].

Theorem 2.3. *For an operator T as in (1.1), the minimal set M_{CH}^{T} is fully irregular in the following cases:*

- $R(z) = \lambda$ *for some* $\lambda \in \mathbb{C}^*$*;*
- $R(z) = \lambda(z-\alpha)$ *for some* $\lambda \notin \mathbb{R}_{<0}$, $\alpha \in \mathbb{C}$ *and* $\deg Q = 1$*;*
- $R(z) = \lambda(z-\alpha)$ *for some* $\lambda \in \mathbb{R}_{>0}$, $\alpha \in \mathbb{C}$ *and* $\deg Q \geq 2$*;*
- *operators satisfying the following conditions (up to an affine change of variable):*
  - $R(z)$ *is real on* $\mathbb{R}$*;*
  - *roots of* P *and* Q *are real, simple and interlacing (i.e. the roots of* P *and* Q *alternate along the real axis);*
  - $|\deg Q - \deg P| \leq 1$*;*

- *if* $\deg Q - \deg P = 1$, *then* $\lambda \in \mathbb{R}_{>0}$.

In any other case, M^T_{CH} *has a nonempty interior.*

In this paper, we will always assume that M^T_{CH} has a nonempty interior.

Remark 2.4. If $\deg P + \deg Q \leq 1$, then M^T_{CH} is either fully irregular or coincides with $\mathbb{C}$ (see Theorem 1.15 of [AHN+24]). Therefore, our operators will always satisfy $\deg P + \deg Q \geq 2$. This implies in particular that $d = 3\deg P + \deg Q - 1$ satisfies $d \geq 3$.

Referring to the closure of the interior of M^T_{CH} as the *regular locus* and its complement in M^T_{CH} as the *irregular locus*, we observe that the latter is contained in very specific lines of the plane.

Definition 2.5. For a given rational function $R(z)$, a line Λ is called R-invariant if for any $z \in \Lambda$ such that $R(z)$ is defined, we have $z + R(z) \in \Lambda$.

In particular, up to an affine change of variable, we can assume $\Lambda = \mathbb{R}$ and thus $R(z)$ is a real rational function. Besides, a R-invariant line is automatically an irreducible component of the curve of inflections $\mathfrak{I}_R$.

Definition 2.6. For an operator T whose minimal set M^T_{CH} is not fully irregular, a *tail* is a semi-open straight segment $[\alpha, \beta[$ in M^T_{CH} satisfying the following conditions:

- the segment $]\alpha, \beta]$ belongs to an R-invariant line;
- for any $z \in]\alpha, \beta]$, $\frac{\beta - \alpha}{R(z)} \in \mathbb{R}_{>0}$;
- for any $z \in]\alpha, \beta]$, z is disjoint from the regular locus of M^T_{CH};
- α belongs to the regular locus of M^T_{CH};
- $\beta \in \mathcal{Z}(PQ)$;
- β is a root of the same multiplicity for both P and Q.

In particular, every tail belongs to a R-invariant line.

The following fact has been proven in Corollary 7.8 of [AHN+24].

Theorem 2.7. *For an operator T whose minimal set M_{CH}^{T} is not fully irregular, the irregular locus of M_{CH}^{T} is a (possibly empty) finite union of tails.*

In particular, if P and Q have no common roots, then the minimal set of the corresponding operator is either regular or fully irregular.

2.2 Extended complex plane

Following Theorem 1.5, T_{CH}-invariant sets are characterized by the position of the associated rays starting in their complements. Let us introduce a certain compactification[1] of $\mathbb{C}$ which comes very handy in our considerations. We baptise it the *extended complex plane* $\mathbb{C} \cup \mathbb{S}^1 \supset \mathbb{C}$.

The extended complex plane $\mathbb{C} \cup \mathbb{S}^1$ is set-theoretically the disjoint union of $\mathbb{C}$ and $\mathbb{S}^1$ endowed with the topology defined by the following basis of neighborhoods:

- for a point $x \in \mathbb{C}$, we choose the usual open neighborhoods of x in $\mathbb{C}$;
- for a direction $\theta \in \mathbb{S}^1$, we choose open neighborhoods of the form $I \cup C(z, I)$ where I is an open interval of $\mathbb{S}^1$ containing θ and $C(z, I)$ is an open cone with apex $z \in \mathbb{C}$ whose opening (i.e. the interval of directions) is I.

Definition 2.8. Given $R(z)$ as above, let $p \in \mathbb{C}$ be a non-singular point of $R(z)$. We define $\sigma(p)$ as the argument of $R(p)$. We think of $\sigma(p)$ as a point of the circle at infinity.

One can easily see that $\mathbb{S}^1$ of the extended plane $\mathbb{C} \cup \mathbb{S}^1$ can be identified with the above circle at infinity. The extended plane is compact and homeomorphic to a closed disk. In particular, usual straight lines in $\mathbb{C}$ have compact closures in $\mathbb{C} \cup \mathbb{S}^1$. (Below we will make no distinction between a real line in $\mathbb{C}$ and its closure in $\mathbb{C} \cup \mathbb{S}^1$). *Open*

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[1] Notice that the most frequently used compactification of $\mathbb{C}$ is $\bar{\mathbb{C}} = \mathbb{C}P^1$.

half-planes in $\mathbb{C}\cup\mathbb{S}^1$ are, by definition, connected components of the complement to a line.

Given a T_{CH}-invariant set $S \subset \mathbb{C}$, we denote by $\overline{S}$ its closure in the extended plane $\mathbb{C}\cup\mathbb{S}^1$.

The following result, but with a slightly different formulation, has been proved in Lemma 4.4 of [AHN+24].

Lemma 2.9. *Given an T_{CH}-invariant set $S \subset \mathbb{C}$, let $\alpha : [0,1] \to \mathbb{C}$ be such that:*

- $\forall t \in (0,1)$, $\alpha_t \in S^c$*;*
- $\sigma(\alpha_0) \neq \sigma(\alpha_1)$*;*
- $\sigma(\alpha)$ *is homotopic to the positive arc from* $\sigma(\alpha_0)$ *to* $\sigma(\alpha_1)$ *in the circle at infinity via a homotopy* $H(t,x) : [0,1]\times[0,1]$ *such that* $H(0,x_0) = \sigma(\alpha(0))$, $H(1,x_0) = \sigma(\alpha(1))$ *for all* $x_0 \in [0,1]$.

If X denotes the connected component of in $\mathbb{C}\cup\mathbb{S}^1$ containing the interval $]\sigma(\alpha_0),\sigma(\alpha_1)[$ in the complement of $r(\alpha_0)\cup\alpha\cup r(\alpha_1)$, then $X \subset S^c$.

2.3 Integral curves

Another result has been proved in Proposition A.2 of [AHN+24].

Proposition 2.10. *Given a T_{CH}-invariant set $S \subset \mathbb{C}$ and some point $z_0 \in S$, if there is a positively oriented integral curve $\gamma : [0,\epsilon[\to \mathbb{C}$ of the vector field $R(z)\partial_z$ such that $\lim_{t\to\epsilon}\gamma(t) = z_0$, then for any $t \in [0,\epsilon]$, $\gamma(t) \in S$.*

When referring to the proposition above, we say that the bounded *backward trajectories* of $R(z)\partial_z$ of points in any invariant set S belongs to S.

2.4 Root trails

For any point $u \in \mathbb{C}$, the *root trail* $\mathfrak{tr}_u$ of u is the closure of the set of points z such that the associated ray $r(z)$ contains u. Except for the trivial cases described in Section 3 of

[AHN+24], root trails are plane real-analytic curves. By definition, the root trail of any point of M_{CH}^{T} is also contained in M_{CH}^{T}. Furthermore, for any fixed $u \in \mathbb{C}$, we defined a *t-trace* (corresponding to u) as any continuous function $\gamma(t)$ such that

$$tQ(\gamma(t)) + (\gamma(t) - u)P(\gamma(t)) = 0$$

for all $t \geq 0$. That is, any t-trace $\gamma(t)$ is a concatenation of parts of $\mathfrak{tr}_u$ such that the resulting curve is continuous for any $t \geq 0$.

Lemma 2.11. *Consider a linear differential operator T given by (1.1) and some point $u \in \mathbb{C}$. Assuming that $R(z)$ is not of the form $\lambda(z-u)$, then*

(i) for any point $u \in \mathbb{C}$ and any point $z_0 \notin \mathcal{Z}(PQ)$ such that $z_0 \in \mathfrak{tr}_u$ and $R(z_0) + (u - z_0)R'(z_0) \neq 0$, the root trail $\mathfrak{tr}_u$ has a unique branch passing through z_0 and its tangent slope is the argument of $\frac{R^2(z_0)}{R(z_0)+(u-z_0)R'(z_0)}$ (mod π).

(ii) If $R(z_0) + (u - z_0)R'(z_0) = 0$ and $m \geq 2$ is the smallest integer such that $R^{(m)}(z_0) \neq 0$, then $\mathfrak{tr}_u$ has m intersecting branches at z_0. Their tangent slopes are:

$$\frac{\theta_0}{m} + \frac{k\pi}{m},$$

where θ_0 is the argument of $\frac{R(z_0)}{R^{(m)}(z_0)}$ and $k \in \mathbb{Z}/m\mathbb{Z}$.

Before proving Lemma 2.11 we prove the next two Lemmas.

Lemma 2.12. *If $\gamma(t)$ is smooth planar curve, $\gamma(0) = z_0$, and $\dot{\gamma}(t) = G(\gamma(t))$ for some function G holomorphic and non-vanishing at z_0 then the sign of the curvature of $\gamma(t)$ at z_0 coincides with the sign of $\operatorname{Im} G'(z_0)$.*

Indeed, then $\ddot{\gamma}(t) = G'(\gamma(t)) \cdot \dot{\gamma}(t)$. By definition, the sign of the curvature of $\gamma(t)$ at z_0 coincides with the sign of $\operatorname{Im} \frac{\ddot{\gamma}(t)}{\dot{\gamma}(t)}|_{t=0} = \operatorname{Im} G'(z_0)$.

Lemma 2.13. *Let F be a function holomorphic at z_0 with $F(z_0) \in \mathbb{R}$ and let $m = \operatorname{ord}_{z_0}(F - F(z_0))$. Then the germ of $I_F = \{\operatorname{Im} F = 0\}$ at z_0 consists of m smooth branches with tangent slopes $\frac{\theta_0}{m} + \frac{k\pi}{m}$, $k \in \mathbb{Z}/m\mathbb{Z}$, where $\theta_0 = -\arg F^{(m)}(z_0)$.*

If $m = 1$ and $\gamma(t)$ is a parameterization of I_F such that $F(\gamma(t)) \equiv F(z_0) + t$ then the sign of the curvature of $\gamma(t)$ coincides with the sign of $-\operatorname{Im}\left[\frac{F''(z_0)}{(F')^2(z_0)}\right]$.

Proof. Indeed, we have $F(z) = a_0 + a_m(z - z_0)^m + \ldots$, $a_0 \in \mathbb{R}$, so the branches of I_F are tangent to the m lines satisfying equation $\operatorname{Im} a_m(z - z_0)^m = 0$, which have slopes as stated.

For the second claim, note that $\dot{\gamma}(t) = \frac{1}{F'(\gamma(t))}$, so the claim follows from Lemma 2.12. □

Remark 2.14. Similar results hold for F having a pole at z_0 by considering $\frac{1}{F}$.

Proof of Lemma 2.11. Note that by definition

$$\mathfrak{tr}_u = \left\{z \in \mathbb{C} \text{ s.t. } \frac{R(z)}{u - z} \in \mathbb{R}_+\right\} \subset \left\{\operatorname{Im} \frac{R(z)}{u - z} = 0\right\},$$

and Lemma 2.11 follows from the Lemma 2.13 with $F(z) = \frac{R(z)}{u-z}$ and the fact that $\arg R(z_0) = \arg(u - z_0)$. □

Remark 2.15. The condition $R(z_0) + (u - z_0)R'(z_0) = 0$ means that the point $u = z_0 - \frac{R(z_0)}{R'(z_0)}$ is obtained as the first iteration of Newton's method of approximating roots of $R(z)$ with the starting point z_0.

When u is a point at infinity in the extended plane $\mathbb{C} \cup \mathbb{S}^1$, the root trail $\mathfrak{tr}_u$ of u is the closure of the points z where the argument of $R(z)$ coincides with u.

Lemma 2.16. *Consider a linear differential operator T given by (1.1) such that $R(z)$ is not constant. For any point u at infinity and any point $z_0 \notin \mathcal{Z}(PQ)$ such that $z_0 \in \mathfrak{tr}_u$, provided $R'(z_0) \neq 0$, the root trail $\mathfrak{tr}_u$ has a unique branch passing through z_0 and its tangent slope is the argument of $\frac{R(z_0)}{R'(z_0)}$ (mod π).*

If $R'(z_0) = 0$ and $m \geq 2$ is the smallest integer such that $R^{(m)}(z_0) \neq 0$, then $\mathfrak{tr}_u$ has m intersecting branches at z_0. Their tangent slopes are:

$$\frac{\theta_0}{m} + \frac{k\pi}{m},$$

where θ_0 is the argument of $\frac{R(z_0)}{R^{(m)}(z_0)}$ and $k \in \mathbb{Z}/m\mathbb{Z}$.

Proof. In this case $\mathfrak{tr}_u \subset \{\operatorname{Im}(R(z)/R(z_0)) = 0\}$ and the claim follows again from Lemma 2.13 □

Remark 2.17. From Lemmas 2.11 and 2.16 it immediately follows that if a root trail $\mathfrak{tr}_u$ can have $m \geq 2$ branches at some point z_0, then z_0 belongs to the curve of inflections $\mathfrak{I}_R$ (because $R(z_0)$ and $u - z_0$ are real colinear).

Besides, if $m \geq 3$, then $R^{(k)}(z_0) = 0$ for $2 \leq k \leq m-1$ and z_0 is a singular point of $\mathfrak{I}_R$.

2.4.1 Concavity of root trails

Proposition 2.18. *Let u be a point of the extended plane $\mathbb{C} \cup \mathbb{S}^1$ and z_0 be a point of $\mathfrak{tr}_u$ such that $z_0 \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$ and $z_0 \neq u$. We denote by L the tangent line to $\mathfrak{tr}_u$ at z_0. We define $f(z,u)$ to be:*

- $\frac{\left(R''(z_0)(u-z_0)^2+2R'(z_0)(u-z_0)+2R(z_0)\right](u-z_0)}{\left(R'(z_0)(u-z_0)+R(z_0)\right)^2}$ *if $u \in \mathbb{C}$;*
- $\frac{R''(z_0)R(z_0)}{R'(z_0)^2}$ *if u is a point at infinity.*

Then the germ of $\mathfrak{tr}_u$ at z_0 belongs to

(i) the same half-plane bounded by L as the associated ray $r(z_0)$ if $\operatorname{Im}(f)$ and $\operatorname{Im}(R'(z_0))$ have opposite signs.

(ii) They belong to distinct half-planes bounded by L if $\operatorname{Im}(f)$ and $\operatorname{Im}(R'(z_0))$ have the same sign.

Finally, $\mathfrak{tr}_u$ has an inflection point at z_0 if $\operatorname{Im}(f) = 0$.

Proof. Let $F_u(z) = \frac{R(z)}{u-z}$ for $u \in \mathbb{C}$ and $F_u(z) = u^{-1}R(z)$ for $u \in \mathbb{S}^1$ so that $\mathfrak{tr}_u = \{\operatorname{Im} F_u(z) = 0\}$. Let $c = F_u'(z_0)$. We have

$$L = \left\{z_0 + c^{-1}\mathbb{R}\right\} = \{z \,|\, \operatorname{Im}\left(c(z - z_0)\right) = 0\}.$$

If $\gamma(t)$ is a local parameterization of $(tr)_u$ at z_0 such that $F_u(\gamma(t)) = F_u(z_0) + t$ then $\dot{\gamma}(0) = c^{-1}$. Moreover, $(tr)_u \subset L_+ = \{\operatorname{Im} c(z - z_0) > 0\}$ if the curvature of $\gamma(t)$ is positive and $(tr)_u \subset L_- = \{\operatorname{Im} c(z - z_0) < 0\}$ otherwise.

The tangent ray $r(z_0) = \{z_0 + R(z_0)\mathbb{R}_+\}$ lies in L_+ if $\operatorname{Im} R(z_0)c > 0$ and in L_- otherwise.

By Lemma 2.13 the sign of curvature of $\gamma(t)$ is opposite to the sign of $\operatorname{Im}\left[\frac{F_u''(z_0)}{\left(F_u'\right)^2(z_0)}\right]$.

For $u \in \mathbb{C}$ we have $c = F_u'(z_0) = \frac{R'(z_0)(u-z_0)+R(z_0)}{(u-z_0)^2}$ and

$$F_u''(z_0) = \frac{R''(z_0)(u-z_0)^2 + 2R'(z_0)(u-z_0) + 2R(z_0)}{(u-z_0)^3},$$

so we are interested in signs of

$$\operatorname{Im} R(z_0)c = \operatorname{Im} R(z_0)\frac{R'(z_0)(u-z_0)+R(z_0)}{(u-z_0)^2} = \operatorname{Im} R'(z_0)$$

(recall that $\frac{R(z_0)}{u-z_0} > 0$) and

$$\operatorname{Im} \frac{F_u''(z_0)}{\left(F_u'\right)^2(z_0)} = \operatorname{Im} \frac{\left(R''(z_0)(u-z_0)^2 + 2R'(z_0)(u-z_0) + 2R(z_0)\right](u-z_0)}{\left(R'(z_0)(u-z_0)+R(z_0)\right)^2}. \tag{2.1}$$

For $u \in \mathbb{S}^1$ we have $c = u^{-1}R'(z_0)$ and we are interested in the signs of $\operatorname{Im} R'(z_0)$ and $\operatorname{Im} \frac{R''(z_0)R(z_0)}{\left(R'\right)^2(z_0)}$.

□

In the transverse locus $\mathfrak{I}_R^*$ of the curve of inflections, the concavity of root trails with respect to the line containing the associated ray depends on the sign of some geometrically meaningful real function.

Proposition 2.19. *Consider a point $z_0 \in \mathfrak{I}_R^* \setminus \mathcal{Z}(PQ)$ and some point $u \in \mathbb{C} \cup \mathbb{S}^1$. Assume that $R(z_0) + R'(z_0)(u-z_0) \neq 0$ (or $R'(z_0) \neq 0$ if u is a point at infinity). Let L be the line containing the associated ray $r(z_0)$.*

The germ of $\mathfrak{tr}_u$ at z_0 and the positive germ $\gamma_{z_0}^+$ of the integral curve of the field $R(z)\partial_z$ starting at z_0 belong to the same open half-plane bounded by L if $R'(z_0) + R(z_0)/(u-z_0)$ is negative ($R'(z_0) < 0$ if u is a point at infinity).

The germ of $\mathfrak{tr}_u$ at z_0 and $\gamma_{z_0}^+$ belong to opposite open half-planes bounded by L if $R'(z_0) + R(z_0)/(u-z_0)$ is positive ($R'(z_0) > 0$ if u is a point at infinity).

Proof. Without loss of generality, we assume that $z_0 = 0$ and $R(z) = 1 + R'(0)z + (a+bi)z^2 + o(z^2)$ with $R'(0) \in \mathbb{R}$, $a \in \mathbb{R}$ and $b \in \mathbb{R}_{>0}$ ($b \neq 0$ because $z_0 = 0$ belongs to the transverse locus of the curve of inflections). Necessarily $u > 0$. Since $b > 0$, γ_0^+ belongs to the upper half-plane.

By Lemma 2.11 (if $u \in \mathbb{C}$) and Lemma 2.16 (if $u \in \mathbb{S}^1$), $\mathfrak{tr}_u$ has a unique branch at 0 tangent to $\mathbb{R}$. Let $F_u(z) = \frac{R(z)}{u-z}$ for $u \in r(z_0)$ and $F_u(z) = R(z)$ for $u \in \mathbb{S}^1$, so $\mathfrak{tr}_u = \{\operatorname{Im} F_u(z) = 0\}$. Choose a parameterization $\gamma(t)$ of this branch in such a way that $F_u(\gamma(t)) = F_u(z_0) + t$. Then

$$\dot{\gamma}(0) = \frac{1}{F_u'(0)} = \frac{u - z_0}{R'(0) + \frac{R(0)}{u-z_0}} \qquad \text{or} \qquad \dot{\gamma}(0) = \frac{1}{R'(0)}$$

for $u \in \mathbb{C}$ or $u \in \mathbb{S}^1$ being a point at infinity, respectively. Therefore $\dot{\gamma}(0) > 0$ if $R'(0) + R(0)/(u - z_0) > 0$ (resp. $R'(0) > 0$) and $\dot{\gamma}(0) < 0$ otherwise.

By (2.1) the sign of the curvature of $\gamma(t)$ at 0 is opposite to the sign of $\operatorname{Im} R''(0) = \operatorname{Im} b > 0$, i.e. is negative. Thus $\gamma(t)$ lies in the lower half-plane (i.e. not in the same half-plane as γ_0^+) if $R'(0)+R(0)/(u - z_0) > 0$ is positive and in the same half-plane as γ_0^+ if $R'(0)+R(0)/(u - z_0) < 0$ ($R'(0) > 0$ and $R'(0) < 0$ resp. for $u \in \mathbb{S}^1$). Since $R'(z_0) \in \mathbb{R}$, the number $R'(z_0)+R(z_0)/(u - z_0)$ is invariant under the maps $z \longmapsto az + b$ and $z \longmapsto \bar{z}$ used for normalization, and the claim follows. □

2.4.2 Root trails and connected components of the minimal set

When $\deg Q - \deg P = 0$, root trails provide a bound on the number of connected components of the minimal set (in all other cases, it is known that M_{CH}^T is connected).

Proposition 2.20. *Consider a linear differential operator T given by* (1.1) *and satisfying* $\deg Q - \deg P = 0$. *Any connected component C of M_{CH}^T satisfies the following conditions:*

- *C contains at least one root of P;*
- *C contains at least one root of Q;*
- *the sum of orders of zeros and poles of $R(z)$ in C vanishes.*

Proof. We assume that a connected component C of M_{CH}^T is disjoint from $\mathcal{Z}(P)$. Note that $\deg Q - \deg P = 0$ implies that the union of the zeros of $tQ(z) + P(z)(z - u)$ for any $u \in \mathbb{C}$, $T > 0$ and $t \in [0, T]$ is bounded.

Hence, for any $u \in \mathrm{M}_{CH}^T \setminus C$, the root trail of u is disjoint from C, as otherwise there would be points in the complement of M_{CH}^T belonging to the root trail of u. Since M_{CH}^T coincides with the T_{CH}-extension of any point in M_{CH}^T (see Lemma 2.2 of [AHN+24]), it follows that C cannot belong to the minimal invariant set.

Suppose now that there is a component for which the sums of orders of the zeros of Q does not equal the sums of orders of the zeros of P. Then there is a component C such that the sums of the orders of the zeros of P, say d_0 is strictly greater than the sums of the orders of the zeros of Q, say d_1. Taking $u \in C$ we have that for all t, the zeros of $tQ(z) + P(z)(z-u)$ belonging to C have total degree $d_0 + 1$. However, when sending $t \to \infty$, d_1 of these zeros tend to the zeros of Q belonging to C and at most one tend to ∞. This implies that at least $d_0 - d_1 > 0$ of the end points of the root trail of u does not belong to C, a contradiction. □

We prove now that the interior $(\mathrm{M}_{CH}^T)^\circ$ of the minimal set satisfies (outside zeroes and poles of $R(z)$) a weak property of local connectedness.

Lemma 2.21. *For any linear differential operator T given by* (1.1) *we consider a point α of the boundary $\partial\mathrm{M}_{CH}^T$ that is neither a zero nor a pole of $R(z)$. For any sufficiently small neighborhood V of α, the connected component M_V of $V \cap \mathrm{M}_{CH}^T$ containing α has connected interior.*

Proof. If α does not belong to the regular locus of M_{CH}^T, then M_V is a line segment and hence has empty interior.

We have $R(z) = r_\alpha + o(z-\alpha)$ for some $r_\alpha \in \mathbb{C}^*$. Then, a continuity argument immediately shows that M_V has connected interior, as otherwise points in the complement of M_{CH}^T would have associated rays intersecting M_{CH}^T.

□

In the following, we prove that the closure of a connected component of the interior of M_{CH}^T cannot be disjoint from $\mathcal{Z}(P)$.

Lemma 2.22. *For any linear differential operator T given by* (1.1)*, one of the following statements holds:*

1. M_{CH}^{T} *is fully irregular;*
2. $\mathrm{M}_{CH}^{T} = \mathbb{C}$*;*
3. *the closure of any connected component of the interior* $(\mathrm{M}_{CH}^{T})^{\circ}$ *of the minimal set contains a root of* $P(z)$*;*
4. *the closure of any connected component of the interior* $(\mathrm{M}_{CH}^{T})^{\circ}$ *of the minimal set contains an endpoint of a tail.*

Proof. We suppose that we are not in the case (1), (2). Besides, we assume the existence of a connected component C of the interior $(\mathrm{M}_{CH}^{T})^{\circ}$ of the minimal set whose closure is disjoint from $\mathcal{Z}(P)$, contradicting statement (3).

We first prove that C cannot be the only connected component of $(\mathrm{M}_{CH}^{T})^{\circ}$. Indeed, roots of $P(z)$ that do not belong to the regular locus of M_{CH}^{T} (the closure of the interior) belong to tails (see Theorem 2.7) and they are not zeros or poles of $R(z)$. Besides, M_{CH}^{T} is assumed to be distinct from $\mathbb{C}$. Consequently, we have $|\deg Q - \deg P| \leq 1$. The only case where the regular locus of M_{CH}^{T} can be disjoint from $\mathcal{Z}(P)$ is when $R(z)$ is of the form λ or $\lambda(z-\alpha)$. In the first case, M_{CH}^{T} is known to be fully irregular. In the second case, either $\lambda \in \mathbb{R}_{>0}$ (and M_{CH}^{T} is fully irregular, see Theorem 2.3) or $\lambda \notin \mathbb{R}_{>0}$ and M_{CH}^{T} has no tails (and $P(z)$ has no root at all). We assume therefore that the interior M_{CH}^{T} has several connected components.

We denote by A the set of points of C that belong to the closure of another connected component of $(\mathrm{M}_{CH}^{T})^{\circ}$. By assumption, these are not these points are not roots of $P(z)$ and Lemma 2.21 shows that each of them is a zero of $R(z)$.

Since M_{CH}^{T} is minimal, there is a point $u \in \overline{\mathrm{M}_{CH}^{T} \setminus C}$ and a point $z_0 \in \mathfrak{tr}_u \cap C$. As the root trail $\mathfrak{tr}_u$ changes continuously in u, u may be chosen outside A. Since $|\deg Q - \deg P| \leq 1$, the zeros of $tQ(z) + P(z)(z-u)$ as $t \to 0$ tends to $\mathcal{Z}(P) \cup \{u\}$. Further, we can assume that $\gamma(t)$ does not equal ∞ for some finite t, as this would imply $\deg Q - \deg P = 1$ and $\lambda < 0$, in which case M_{CH}^{T} is equal to $\mathbb{C}$. Hence, the minimal set M_{CH}^{T} therefore contains a continuous path $\gamma(t)$ from an element of $\mathcal{Z}(P) \cup \{u\}$ to z_0 such that $\gamma(t)$ solves $tQ(\gamma(t)) + P(\gamma(t))(\gamma(t) - u) = 0$.

The path γ has to enter the component C and can do so either through a tail or an element of A. The path γ cannot contain any element $\alpha \in A$ because the equations $Q(\alpha) = 0$ and $tQ(\alpha) + P(\alpha)(\alpha - u) = 0$ (for some $t > 0$) imply $P(\alpha) = 0$, contradicting our assumption. Our assumption that neither (1), (2) nor (3) was satisfied thus implies (4). □

2.5 Asymptotic geometry of Hutchinson invariant sets

Let us recall the results of [AHN+24] concerning minimal Hutchinson invariant sets (see Theorems 1.11 and 1.12 of [AHN+24]).

Theorem 2.23. *For any operator T as in (1.1) with a minimal set M_{CH}^{T} having a nonempty interior, M_{CH}^{T} is:*

- *a compact contractible subset of $\mathbb{C}$ if $\deg Q - \deg P = 1$, and $Re(\lambda) \geq 0$;*
- *a noncompact non-trivial subset of $\mathbb{C}$ if $\deg Q - \deg P = 0$ or -1;*
- *trivial, i.e. equal to $\mathbb{C}$ otherwise.*

Besides, the closure $\overline{\mathrm{M}_{CH}^{T}}$ in the extended plane $\mathbb{C} \cup \mathbb{S}^1$ is contractible, connected and compact.

Thus, the only interesting cases for the description of $\partial \mathrm{M}_{CH}^{T}$ are those for which the values of $\deg Q - \deg P$ are 1, 0 or -1. In the latter two cases, we have more precise results given below.

2.5.1 $\deg Q - \deg P = -1$

The following statement has been proved in Corollary 6.2 of [AHN+24].

Proposition 2.24. *For an operator T as in (1.1) such that $\deg Q - \deg P = -1$. Then the complement of its minimal Hutchinson invariant set M_{CH}^{T} in $\mathbb{C}$ has exactly two connected components X_1, X_2. Each X_i contains infinite cones whose intervals of directions are arbitrarily close to $\left(\frac{\phi_\infty - \pi}{2}, \frac{\phi_\infty + \pi}{2}\right)$ and $\left(\frac{\phi_\infty + \pi}{2}, \frac{\phi_\infty + 3\pi}{2}\right)$ respectively.*

2.5.2 $\deg Q - \deg P = 0$

The following statement has been proven in Corollary 6.4 of [AHN+24].

Proposition 2.25. *Take any operator T as in (1.1) such that $\deg Q - \deg P = 0$. Then for any $\epsilon > 0$, there exists an open cone $\mathcal{C}$ whose interval of directions is $(\phi_\infty + \pi - \epsilon, \phi_\infty + \pi + \epsilon)$ and such that M_{CH}^T is contained in $\mathcal{C}$.*

3 Local analysis of the boundary of M_{CH}^T

We consider an operator T as in (1.1) whose minimal set M_{CH}^T has a nonempty interior.

Notation 3.1. For any point $\alpha \in \partial\mathrm{M}_{CH}^T$, we define $r_\alpha \in \mathbb{C}^*$, $m_\alpha \in \mathbb{Z}$ so that

$$R(z) = \frac{Q(z)}{P(z)} = r_\alpha (z-\alpha)^{m_\alpha} + o(|z-\alpha|^{m_\alpha}). \tag{3.1}$$

We also define $\phi_\alpha = arg(r_\alpha)$ and $d_\alpha : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ where $d_\alpha(\theta) = \phi_\alpha + m_\alpha \theta$.

3.1 Description of a tangent cone

Definition 3.2. For any $\alpha \in \partial\mathrm{M}_{CH}^T$, we define $\mathcal{K}_\alpha$ as the subset of $\mathbb{S}^1$ formed by directions θ such that there is a sequence $(z_n)_{n\in\mathbb{N}}$ satisfying the following conditions:

- for any $n \in \mathbb{N}$, $z_n \in (\mathrm{M}_{CH}^T)^c$;
- $z_n \longrightarrow \alpha$;
- $arg(z_n - \alpha) \longrightarrow \theta$.

We also define $\mathcal{L}_\alpha$ as the subset of $\mathbb{S}^1$ formed by directions θ such that the half-line $\alpha + e^{i\theta}\mathbb{R}^+$ does not intersect the interior of M_{CH}^T.

Lemma 3.3. *For any $\alpha \in \partial\mathrm{M}_{CH}^T$, the following statements hold:*

1. *$\mathcal{K}_\alpha$ and $\mathcal{L}_\alpha$ are nonempty closed subsets of $\mathbb{S}^1$;*

2. *$\mathcal{L}_\alpha \subset \mathcal{K}_\alpha$;*

3. *for any $\theta \in \mathcal{K}_\alpha$, $d(\theta) \in \mathcal{L}_\alpha$. In particular, K_α is invariant under d_α;*

4. *for any $\theta \in \mathcal{K}_\alpha$, there exists a closed interval $J \subset \mathcal{K}_\alpha$ of length at most π containing both θ and $d_\alpha(\theta)$;*

5. *$\mathcal{L}_\alpha \neq \mathbb{S}^1$.*

Proof. From Definition 3.2 it immediately follows that $\mathcal{K}_\alpha$ and $\mathcal{L}_\alpha$ are closed subsets of $\mathbb{S}^1$.

If $\alpha \in \partial \mathrm{M}^T_{CH}$, then we can find a sequence of points in the complement of M^T_{CH} approaching α. By compactness of $\mathbb{S}^1$, we can choose a subsequence for which the arguments converge to some limit. Thus $\mathcal{K}_\alpha$ is nonempty.

Then, for any $\theta \in \mathcal{K}_\alpha$, we have a sequence $(z_n)_{n\in\mathbb{N}}$ in the complement of M^T_{CH} accumulating to α with the limit slope θ. The associated rays $r(z_n)$ accumulate to $\alpha + e^{id_\alpha(\theta)}\mathbb{R}^+$. Since none of them intersects the interior of M^T_{CH}, the half-line $\alpha + e^{id_\alpha(\theta)}\mathbb{R}^+$ does not intersect it either and $d_\alpha(\theta) \in \mathcal{L}_\alpha$.

Besides, in the case where $\theta \neq d_\alpha(\theta)$, (up to taking a subsequence of $(z_n)_{n\in\mathbb{N}}$, there is a closed interval $J \subset \mathbb{S}^1$ such that:

- the endpoints of J are θ and $d_\alpha(\theta)$;
- the length of J is at most π;
- for any η in the interior of J, there is a bound $N(\eta)$ such that for any $n \geq N(\eta)$, the associated ray $r(z_n)$ intersects the half-line $\alpha + e^{i\eta}\mathbb{R}^+$ at some point $P_{\eta,n}$.

Existence of sequences $(P_{\eta,n})_{n\geq N(\eta)}$ proves that for any $\eta \in J$, one has $\eta \in \mathcal{K}_\alpha$.

Finally, $\mathcal{L}_\alpha \neq \mathbb{S}^1$ because in this case, M^T_{CH} would have empty interior. □

Remark 3.4. Note that in the case $\theta = d_\alpha(\theta)$, the interval J is a singleton $\{\theta\}$.

Let us deduce local description of $\mathcal{K}_\alpha$ and $\mathcal{L}_\alpha$ depending on the local invariants of α.

Corollary 3.5. *For any $\alpha \in \partial \mathrm{M}^T_{CH}$, the following statements hold:*

- *if* $|m_\alpha| \geq 2$, *then* $\mathcal{K}_\alpha = \mathcal{L}_\alpha$ *and they are contained in the finite set of arguments satisfying* $\theta \equiv \frac{\phi_\alpha}{1-m_\alpha}\ [\frac{2\pi}{1-m_\alpha}]$*;*

- *if* $m_\alpha = 1$, *then* $\phi_\alpha = 0$ *and* $\mathcal{K}_\alpha = \mathcal{L}_\alpha$*;*

- *if* $m_\alpha = 0$, *then* $\phi_\alpha \in \mathcal{L}_\alpha$*;*

- *if* $m_\alpha = -1$, *then* $\mathcal{K}_\alpha = \mathcal{L}_\alpha$ *and these sets are formed by at most two intervals, each of length at most* π *and having their midpoints at* $\frac{\phi_\alpha}{2}$ *and* $\frac{\phi_\alpha}{2} + \pi$*.*

Proof. We consider maximal interval J in $\mathcal{K}_\alpha$ (which is non-empty by Lemma 3.3). The images of J under the iterated action of d_α belong to $\mathcal{L}_\alpha$.

If $|m_\alpha| \geq 2$, then J is a singleton since otherwise the union of its iterates would coincide with $\mathbb{S}^1$ (contradicting Lemma 3.3). Thus J has to be a fixed point of the map d_α.

If $m_\alpha = 1$ and $\phi_\alpha \neq 0$, then J coincides with $\mathbb{S}^1$ because no other connected subset of the circle is preserved under the action of nontrivial rotation. Therefore d_α is the identity map.

If $m_\alpha = 0$, then for any $\theta \in \mathcal{K}_\alpha$, $d_\alpha(\theta) = \phi_\alpha$. Therefore $\phi_\alpha \in \mathcal{L}_\alpha$.

If $m_\alpha = -1$, then J is invariant under the action of $\theta \longmapsto \phi_\alpha - \theta$. Thus, either $\frac{\phi_\alpha}{2}$ or $\frac{\phi_\alpha}{2} + \pi$ is the bisector of J. If J is of length strictly bigger than π, then Lemma 3.3 shows that its complement (of length strictly smaller than π) is also contained in $\mathcal{L}_\alpha$. Therefore $L_\alpha = \mathbb{S}^1$ which is a contradiction. □

We obtain a bound on the number of petals of M^T_{CH} that can be attached to a boundary point.

Corollary 3.6. *For any linear differential operator* T *given by* (1.1) *we consider a point* α *of the boundary* $\partial \mathrm{M}^T_{CH}$*. Then for any sufficiently small open subset* $V \subset \mathbb{C}$ *the interior of the connected component* M_V *of* $V \cap \mathrm{M}^T_{CH}$ *containing* α *has at most:*

- $|1 - m_\alpha|$ *connected components if* $m_\alpha \neq 1$*;*

- $\deg P$ *connected components if* $m_\alpha = 1$

where $R(z) = \lambda(z-\alpha)^{m_\alpha} + o((z-\alpha)^{m_\alpha})$ *with* $\lambda \in \mathbb{C}^*$ *and* $m_\alpha \in \mathbb{Z}$.

Proof. If α is not a zero or a pole of $R(z)$, then Lemma 2.21 proves the statement. Besides, if $m_\alpha \notin \{0,1\}$, Corollary 3.5 proves that α is in the closure of at most $1-m_\alpha$ components.

In the remaining cases, α is a simple zero of $R(z)$. If α is also a root of degree d of P, then it is a root of degree $d+1$ of Q.

We can divide P and Q by $(z-\alpha)^d$ while keeping the same minimal set M^T_{CH} (because in this case $\mathcal{Z}(PQ)$ remains unchanged). Consequently, we can assume that α is not a root of P. Lemma 2.22 proves that for any connected component C of $(\mathrm{M}^T_{CH})^\circ$ such that α is in the closure of C, either some root of $P(z)$ belongs to the closure of C or some tail is attached to C. If α is in the closure of several connected components of $(\mathrm{M}^T_{CH})^\circ$, then a same root of P cannot be in the closure of two of them because $\overline{\mathrm{M}^T_{CH}}$ would fail to be contractible. Similarly a given tail is attached to only one connected component of $(\mathrm{M}^T_{CH})^\circ$ (and contains at least one root of P). Therefore, α is in the closure of at most $\deg P$ components. $\square$

3.2 Curve of inflections

In §A.3 of [AHN+24] we introduced the curve of inflections $\mathfrak{I}_R$ of an analytic vector field $R(z)\partial_z$. By definition, it is the closure in $\mathbb{C}$ of the subset of $\mathbb{C} \setminus \mathcal{Z}(PQ)$ at each point of which the integral curve of the vector field $R(z)\partial_z$ passing through this point has zero curvature. Here and throughout, $\mathcal{Z}(F)$ denotes the set of zeros of the function F. Below we provide some additional information about $\mathfrak{I}_R$.

For an operator T for which $R(z)$ is not of the form λ or $\lambda(z-\alpha)$ for some $\lambda \in \mathbb{C}^*$ and $\alpha \in \mathbb{C}$, the function $R'(z)$ is a non-constant rational function. Therefore the curve of inflections $\mathfrak{I}_R$ of $R(z)\partial_z$ (which is defined as the closure of the set of points for which $Im(R'(z)) = 0$) is a real plane algebraic curve.

We first characterize the points at which several local branches of the curve of inflections intersect.

Lemma 3.7. *A point $z_0 \in \mathfrak{I}_R$ belongs to exactly $m \geq 2$ local branches of $\mathfrak{I}_R$ in the following cases:*

1. *z_0 is a critical point of $R'(z)$ of order $m-1$ (including zeroes of order m of $R(z)$);*
2. *z_0 is a pole of $R(z)$ of order $m-1$.*

The $2m$ limit slopes of the local branches at z_0 form a regular $2m$-gon in $\mathbb{S}^1$.

Proof. This follows immediately from Lemma 2.13. □

Corollary 3.8. *The curve of inflections $\mathfrak{I}_R$ has at most $4\deg P + \deg Q - 2$ singular points.*

Proof. There are at most $\deg P$ poles of $R(z)$ and the critical points of $R'(z)$ are the zeroes of $R''(z)$. □

Lemma 3.9. *Let $F(z) : \mathbb{C} \to \mathbb{C}P^1$ be a non-constant rational function of degree d. Then the real algebraic curve $\Gamma = \overline{\{z \in \mathbb{C} | \operatorname{Im} F(z) = 0\}}$ is non-empty, has at most d connected components and has exactly d connected components for generic F.*

Proof. Clearly, as $F^{-1}(x) \neq \emptyset$ for any $x \in \mathbb{R} \setminus \{F(\infty)\}$, $\Gamma \neq \emptyset$ as well.

By the open mapping theorem, the map $F : \bar{\Gamma} \to \mathbb{R}P^1$, where $\bar{\Gamma}$ is the closure of Γ in $\mathbb{C}P^1$, is onto on each connected component of $\bar{\Gamma}$. Since F has degree d this means that Γ has at most d components.

Note that the ramification points of $F : \bar{\Gamma} \to \mathbb{R}P^1$ coincide with the ramification points of $F : \mathbb{C}P^1 \to \mathbb{C}P^1$ lying on $\bar{\Gamma}$. Thus if the ramification values of F are not in $\mathbb{R}P^1$ then the former map is an unramified cover of degree d, so has exactly d connected components. This means that the bound is sharp. □

3.2.1 Inflection domains

Definition 3.10. The curve of inflections $\mathfrak{I}_R$ subdivides $\mathbb{C}$ into two open (not necessarily connected) domains: $\mathfrak{I}^+$ given by $Im(R'(z)) > 0$ and $\mathfrak{I}^-$ given by $Im(R'(z)) < 0$.

Observe that in $\mathfrak{I}^+$ (resp. $\mathfrak{I}^-$), the integral curves of the vector field $R(z)\partial_z$ are turning counterclockwise (resp. clockwise).

3.2.2 Circle at infinity

Consider the closure of the curve of inflections $\mathfrak{I}_R$ in the extended complex plane $\mathbb{C} \cup \mathbb{S}^1$.

Lemma 3.11. *The intersection* $\mathfrak{I}_R \cap \mathbb{S}^1$ *is:*

- *is empty if* $\deg Q - \deg P = 1$ *and* $\lambda \notin \mathbb{R}$*;*
- *coincides with the set* $\{\frac{\phi_\infty}{2}, \frac{\phi_\infty}{2} + \frac{\pi}{2}, \frac{\phi_\infty}{2} + \pi, \frac{\phi_\infty}{2} + \frac{3\pi}{2}\}$ *if* $\deg Q - \deg P = -1$.

In the remaining cases:

- $\deg Q - \deg P = 1$ *and* $\lambda \in \mathbb{R}$*; or*
- $\deg Q - \deg P = 0$*; or*
- $\deg Q - \deg P \notin \{-1, 0, 1\}$

the set $\mathfrak{I}_R \cap \mathbb{S}^1$ *consists of* $2k$ *points forming a regular* $2k$*-gon for some* k *satisfying* $k \leq \max\{\deg P, \deg Q\} + 1$.

Proof. If $k = \deg Q - \deg P \in \mathbb{Z} \setminus \{0, 1\}$, then $R'(z)$ has an expansion of the form $k\lambda_k z^{k-1} + o(z^{k-1})$ near ∞ from which the characterization of the infinite branches of the real locus of $R'(z)$ follows by Lemma 2.13 applied to either $R'(z)$ or to $\frac{1}{R'(z)}$ depending on whether $k > 0$ or $k < 0$ (clearly both have the same real locus outside their poles).

If $\deg Q - \deg P = 0$, then $R(z)$ has an expansion $\lambda + \frac{A}{z^k} + o(z^{-k})$ for some $A \in \mathbb{C}^*$ and $k \in \mathbb{N}^*$ near ∞. (The case when $R(z)$ is constant is ruled out by the genericity assumptions). Therefore $R'(z)$ has an expansion $-\frac{Ak}{z^{k+1}} + o(z^{-k-1})$. We conclude that $\mathfrak{I}_R$ has $2k$ infinite branches whose limit directions form a regular $2k$-gon.

If $\deg Q - \deg P = 1$, then $R(z)$ has an expansion $\lambda z + A + Bz^{-k} + o(z^{-k})$ for some $A \in \mathbb{C}$, $B \in \mathbb{C}^*$, and $k \in \mathbb{N}^*$. (The case when $R(z)$ is a linear function is ruled out by the genericity assumptions). We obtain that $R'(z)$ is of the form $\lambda - \frac{Bk}{z^{k+1}} + o(z^{-k-1})$. Consequently, unless λ is real, the curve of inflections $\mathfrak{I}_R$ is compact in $\mathbb{C}$. If λ is real, the infinite branches of $\mathfrak{I}_R$ are asymptotically the same as that of the real locus of $-\frac{kB}{z^{k+1}}$. Therefore $\mathfrak{I}_R$ has $2k$ infinite branches whose limit directions form a regular $2k$-gon.

In these last two cases, we have $R'(z) = \frac{M}{z^{k+1}} + o(z^{-k-1})$ for some $M \in \mathbb{C}^*$ and $k \geq 1$. The number k is the ramification index of either $R - \lambda z$ (for $\deg Q - \deg P = 1$) or R (for $\deg Q - \deg P = 0$) at infinity, thus k cannot be bigger than the degree $\max\{\deg P, \deg Q\}$ of R. Therefore $k \leq \max\{\deg P, \deg Q\} + 1$.

$\square$

3.2.3 Singularities of the vector field

Next we deduce from Corollary 3.5 a proof of the statement that any root of $P(z)$ or $Q(z)$ belonging to $\partial \mathrm{M}_{CH}^{T}$ automatically belongs to the curve of inflections.

Corollary 3.12. *Consider an operator T as in (1.1) such that M_{CH}^{T} does not coincide with $\mathbb{C}$ and has a nonempty interior. Let α be a zero or a pole of $R(z)$ such that $\alpha \in \partial \mathrm{M}_{CH}^{T}$. Then α also belongs to the curve of inflections $\mathfrak{I}_R$. Additionally, the number of local branches of $\mathfrak{I}_R$ at α equals:*

- *$a + 1$ if α is a pole of order $a \geq 1$;*
- *$a - 1$ if α is a zero of order $a \geq 2$;*
- *some integer $b \geq 1$ if α is a simple zero.*

Proof. The statement is proved by direct computation of $Im(R')$ in case of a pole or a zero of order $a \geq 2$. If α is a simple zero of $R(z)$, then we have $R(\alpha + \epsilon) = R'(\alpha)\epsilon + o(\epsilon)$. If $\alpha \in \partial \mathrm{M}_{CH}^{T}$, then $\phi_\alpha = arg(R'(\alpha)) = 0$ (see Corollary 3.5). Thus $\alpha \in \mathfrak{I}_R$.

Unless $R(z)$ is linear, $R(z)$ is of the form $R'(\alpha)(z - \alpha) + M(z - \alpha)^d + o(|z - \alpha|^d)$ for some $d \geq 2$ and $M \in \mathbb{C}^*$. Thus $R'(z) = R'(\alpha) + Md(z - \alpha)^{d-1} + o(|z - \alpha|^{d-1})$. Consequently, the number of local branches of the equation $Im(R') = 0$ equals $d - 1$.

If $R(z) = \lambda(z - \alpha)$, then $Re(\lambda) \geq 0$ (otherwise $\mathrm{M}_{CH}^{T} = \mathbb{C}$) and $Im(\lambda) \neq 0$ (otherwise M_{CH}^{T} is fully irregular). It follows that $Im(R'(z))$ is a non-vanishing constant and the curve of inflections is empty. In this case, $\mathfrak{I}_R$ does not contain any zero or pole of $R(z)$ on the boundary of M_{CH}^{T}. $\square$

3.2.4 Tangency locus

Definition 3.13. For the rational vector field $R(z)\partial_z$, the *tangency locus* $\mathfrak{T}_R$ is the subset of the curve of inflections $\mathfrak{I}_R$ where $R(z)\partial_z$ is tangent to some branch of $\mathfrak{I}_R$.

Proposition 3.14. *For an operator T as in (1.1), the tangency locus $\mathfrak{T}_R$ is the union of:*

- *at most $\max\{\deg Q, \deg P\} + 1$ lines; and*
- *at most $2(3\deg P + \deg Q - 1)^2$ points.*

Proof. For any point $z \in \mathcal{T}_R$, an immediate computation involving the Taylor expansion of $R'(z)$ proves that z belongs to the intersection of the curve of inflections (given by $\operatorname{Im}(R') = 0$) with a real plane algebraic curve given by the equation $\operatorname{Im}(R''R) = 0$. Indeed, the tangent line to $\mathfrak{I}_R$ at some $z_0 \in \mathfrak{I}_R$ is given by the equation $R''(z_0) \cdot (z - z_0) \in \mathbb{R}$, and the associated ray direction is $R(z_0)$. The degrees of these two curves are respectively $\deg Q + 3\deg P - 1$ and $2\deg Q + 6\deg P - 2$. Therefore, Bézout's theorem implies that $\mathfrak{T}_R$ contains at most $2(\deg Q + 3\deg P - 1)^2$ such points and some irreducible components corresponding to the common factors of the two equations.

By definition of the tangency locus these irreducible components are the integral curves of $R(z)\partial_z$ contained in the curve of inflections. Such integral curves have identically vanishing curvature and therefore they are segments of straight lines. Therefore the relevant irreducible components are straight lines. But $\mathfrak{I}_R$ intersects $\mathbb{S}^1$ at most $2\max\{\deg Q, \deg P\} + 2$ points by Lemma 3.11. Thus the number of the lines is at most $\max\{\deg Q, \deg P\} + 1$. □

We deduce an estimate on the number of connected components of the transverse locus $\mathfrak{I}_R^*$ of the curve of inflections. Denote $d = 3\deg P + \deg Q - 1 = \deg \mathfrak{I}_R$.

Corollary 3.15. *For an operator T as in (1.1), the transverse locus $\mathfrak{I}_R^*$ of the curve of inflections is formed by at most $2d^2 + 6d + 2$ connected components.*

Proof. A connected component of $\mathfrak{I}_R^*$ is either a smooth closed loop (so a connected component of $\mathfrak{I}_R$) or an arc joining points at infinity, singular points of $\mathfrak{I}_R$ or isolated points of the tangency locus.

Following Proposition 3.14, the tangent locus contains at most $2d^2$ isolated points. Each of them is the endpoint of two arcs of the transverse locus.

Lemma 3.11 proves that at most $2\max\{\deg P, \deg Q\} + 2$ arcs of the transverse locus go to infinity.

Lemma 3.7 provides the analog result for the multiple points of the curve of inflections. In the "worst" case, poles of $R(z)$ and critical points of $R'(z)$ are simple. At most four arcs of the transverse locus are incident to such points. There are at most $4\deg P + \deg Q - 2 \leq 2d$ such points (see Corollary 3.8) so they are incident to at most $4d$ arcs.

Adding these bounds, we obtain an upper bound $4d^2 + 10d + 4$ on the number of ends of non-compact connected components of the transverse locus, i.e. there are at most $2d^2 + 5d + 2$ non-compact connected components. By Lemma 3.9 the number of the compact connected components (loops) of $\mathfrak{I}_R$ is at most d, which gives the required upper bound. □

Corollary 3.16. *On each connected component of the transverse locus* $\mathfrak{I}_R^*$*, the sign of* $\mathrm{Im}(R''R)$ *remains constant. If* $\mathrm{Im}(R''R)$ *is positive (resp. negative), then for any point* z *of the component, the associated ray* $r(z)$ *points towards* $\mathfrak{I}^+$ *(resp.* $\mathfrak{I}^-$*).*

Proof. Any regular point z of the curve of inflections satisfying $\mathrm{Im}(R''(z)R(z)) = 0$ belongs to the tangency locus (see the proof of Proposition 3.14). A direct computation proves the rest of the claim. □

3.3 Horns

In this section, we introduce some curvilinear triangles called *horns* and find conditions under which we can conclude that they do not belong to the minimal set M_{CH}^T. Our aim is to prove that some parts of the boundary of the minimal sets are portions of integral curves of the vector field $R(z)\partial_z$.

3.3.1 Definitions

Recall that $\sigma(q)$ is the argument of $R(q)$, i.e. $\sigma(q) = \operatorname{Im}\log R(q)$ and $r(q) = q + R(q)\mathbb{R}_+$ is the associated ray.

Definition 3.17. Assume that a segment $\gamma_p^{p'}$ of the positive trajectory of $R(z)\partial_z$ starting at $p \notin \mathcal{Z}(PQ)$ and ending at p' doesn't intersect the curve of inflections except possibly at p. Assume that the total variation of σ along $\gamma_p^{p'}$ is less than $\pi/2$.

We define the *horn* ${}_p\triangle_{p''}^{p'}$ at p as an open curvilinear triangle formed by $\gamma_p^{p'}$ and tangents to this trajectory at p and p' intersecting at a point p''.

Definition 3.18. A horn ${}_p\triangle_{p''}^{p'}$ is called *small positive* (resp. *small negative*) if

1. for any point $u \in {}_p\triangle_{p''}^{p'}$, the argument $\sigma(u + tR(u))$ is monotone increasing (resp. decreasing) in the variable t as long as $t \geq 0$ and $u + tR(u) \in {}_p\triangle_{p''}^{p'}$
2. for any two points $u, v \in {}_p\triangle_{p''}^{p'}$, the scalar product $(R(u), R(v))$ is positive.

A horn ${}_p\triangle_{p''}^{p'}$ is called *small* if it is either small positive or small negative.

Remark 3.19. A small positive horn becomes a small negative one after conjugation, i.e. after replacing $R(z)$ with $\overline{R(\bar{z})}$. Indeed,

$$(R(u), R(v)) = \operatorname{Re} R(u)\overline{R(v)}$$

remains the same after the conjugation, and

$$\frac{d\sigma(u + tR(u))}{dt}(t) = \operatorname{Im}\frac{R'(u + tR(u))}{R(u + tR(u))}R(u)$$

changes sign.

Lemma 3.20. *The curve of inflections (given by* $\operatorname{Im} R' = 0$*) does not intersect small horns.*

Proof. We have that $\frac{d\sigma(u+tR(u))}{dt}|_{t=0} = \operatorname{Im} R'(u) \geq 0$. Assume that we have the equality at some $u \in {}_{p}\triangle^{p'}_{p''}$. Since ${}_{p}\triangle^{p'}_{p''}$ is open and R' is an open map, this assumption will imply that $\frac{d\sigma(u+tR(u))}{dt}|_{t=0}$ changes sign in ${}_{p}\triangle^{p'}_{p''}$, which contradicts the smallness assumptions. □

We define the cone complementary to ${}_{p}\triangle^{p'}_{p''}$ (in short, the *complementary cone*) to be the open cone ${}_{p''}\angle$ with the apex p'' bounded by part of the ray $r(p)$ starting at p'' and by the ray extending the segment $p'p''$.

Lemma 3.21. *Consider a point* p *which neither belongs to* $\mathcal{Z}(PQ)$ *nor to the interior of* M^T_{CH}*. Assume that the integral curve* γ *of the vector field* $R(z)\partial_z$ *containing* p *is not a straight line. Then there exists a horn* ${}_{p}\triangle^{p'}_{p''}$ *such that both* ${}_{p}\triangle^{p'}_{p''}$ *and its complementary cone* ${}_{p''}\angle$ *do not intersect* M^T_{CH}*.*

Proof. Let $D(p) = \{|z-p| < \delta_{\mathcal{Z}}(p) = \frac{1}{2}\operatorname{dist}(p, \mathcal{Z}(PQ))\}$.

First, assume that $p \notin \mathrm{M}^T_{CH}$. Then by definition, $r(p) \subset {\mathrm{M}^T_{CH}}^{c}$.

Choose some $\delta > 0$ and define $p_0 = p$ and

$$p_i = p_{i-1} + \delta R(p_{i-1}) \in r(p_i) \subset (\mathrm{M}^T_{CH})^c \cap D(p), \quad i = 1, \ldots, N = N(\delta) = O\left(\frac{\delta_{\mathcal{Z}}(p)}{\delta}\right),$$

(we stop when $p_{N+1} \notin D(p)$).

The broken line $\hat{\gamma}^{p_N}_p = \cup_{i=1}^N [p_{i-1}, p_i] \subset (\mathrm{M}^T_{CH})^c \cap D(p)$ is the Euler approximation to the positive trajectory γ^+_p of $R(z)\partial_z$ starting from p and converges to it (more exact, to the connected component $\gamma^{p'}_p \subset D(p)$ of $\gamma^+_p \cap D(p)$ containing p) as $\delta \to 0$. Thus $\gamma^{p'}_p \subset \overline{(\mathrm{M}^T_{CH})^c}$. Repeating this argument for all $\tilde{p} \notin \mathrm{M}^T_{CH}$ sufficiently close to p we see that

$$\gamma^{p'}_p \subset \left(\overline{(\mathrm{M}^T_{CH})^c}\right)^o = (\mathrm{M}^T_{CH})^c. \tag{3.2}$$

If $\gamma^{p'}_p$ is a subset of the curve of inflections then it is a part of a straight line, which is excluded by our assumption. Thus we can assume that for p' sufficiently close to p the curve $\gamma^{p'}_p$ intersects the curve of inflections only at p. Therefore $\gamma^{p'}_p$ is convex and, choosing p' closer to p if needed, we can assume that $\gamma^{p'}_p$ is of angle smaller than π. Therefore

$$\left(\bigcup\nolimits_{s\in\gamma^{p'}_p} r(s)\right)^{\circ} = {}_{p''}\angle \bigcup {}_{p}\triangle^{p'}_{p''} \subset (\mathrm{M}^T_{CH})^c. \tag{3.3}$$

Second, assume that $p \in \partial \mathrm{M}_{CH}^T$ and let $\gamma_p^{p'}$ be a part of the connected piece of $\gamma_p^+ \cap D(p)$ containing p such that $\gamma_p^{p'}$ is convex and of angle smaller than $\pi/2$. Let $p_i \notin \mathrm{M}_{CH}^T$ be a sequence of points tending to p and take $p_i' \in \gamma_{p_i}^+$ such that $\gamma_{p_i}^{p_i'}$ converges to $\gamma_p^{p'}$. By analyticity this convergence is uniform in C^1 sense as well. Therefore

$$\Big(\bigcup\nolimits_{s\in\gamma_p^{p'}} r(s)\Big)^{\circ} \subset \bigcup_i \Big(\bigcup\nolimits_{s\in\gamma_{p_i}^{p_i'}} r(s)\Big)^{\circ} \subset (\mathrm{M}_{CH}^T)^c, \tag{3.4}$$

which finishes the proof.

□

3.3.2 Small horns exist

Proposition 3.22. *For any point $p \notin \mathcal{Z}(PQ)$ such that the trajectory $\gamma(p)$ of R starting at p is not a straight line, there exists a small horn ${}_p\triangle_{p''}^{p'}$.*

Proof. Using an affine change of variables we can assume that $p = 0$ and $R(0) = 1$. By assumption $R(z)$ is not a real rational function. Let

$$R(u) = 1 + \rho(u) + ibu^m + O(u^{m+1}), \quad b > 0,\ \rho \in \mathbb{R}[u],\ m \geq 1 \tag{3.5}$$

be the Taylor expansion of $R(z)$ at 0 (the case $m = 1$ is covered by Lemma 3.23). Here we can assume that $b > 0$ by replacing $R(z)$ by $\overline{R(\bar{z})}$, if necessary.

First, we consider the case $m = 1$, i.e. $p \notin \mathfrak{I}_R$.

Lemma 3.23. *For every compact set K not intersecting the curve of inflections $\mathfrak{I}_R$, there is a $\delta = \delta(K) > 0$ such that for every $p \in K$, there is a small horn ${}_p\triangle_{p''}^{p'}$ of diameter greater than δ.*

Proof. Indeed, for any $p \in K$ the function $\operatorname{Re} R(u)\overline{R(v)}$ is positive and $\operatorname{Im} \frac{R'(u)}{R(u)} R(v)$ is non-zero at $(p, p) \in \mathbb{C}^2$, so this remains true for all $(u, v) \in \mathbb{C}^2$ such that $\operatorname{dist}((p, p), (u, v)) < \delta = \delta(p)$ by continuity. This means that any ${}_p\triangle_{p''}^{p'} \subset U_{\delta(p)}(p)$ is a small horn. The uniform lower bound follows from the continuity of $\delta(p)$. □

From now on we assume that $m \geq 2$. Our next goal is to find the asymptotics of γ_0 near 0 and the ${}_0\triangle_{p''}^{p'}$. We abuse notation by writing the germ of γ_0 as $\gamma_0 = \{x + i\gamma_0(x), x > 0\}$.

Lemma 3.24.

$$\gamma_0(x) = \tfrac{b}{m+1} x^{m+1} + O(x^{m+2}) \tag{3.6}$$

and

$${}_0\triangle_{p''}^{p'} \subset \{0 < x < \epsilon, 0 < y < \gamma_0(x)\}. \tag{3.7}$$

Proof. Note that

$$\gamma_0 \subset \{\operatorname{Im} F = 0\}, \text{ where } F' = \frac{1}{R}, F(0) = 0. \tag{3.8}$$

Indeed,

$$\tfrac{d}{dt} \operatorname{Im} F(\gamma_0(t)) = \operatorname{Im} \tfrac{d}{dt} F(\gamma_0(t)) = \operatorname{Im}\left(F' \cdot \dot{\gamma}_0(t)\right) = 0.$$

Now,

$$\frac{1}{R} = \frac{1}{1+\rho(u)} - \frac{ibu^m}{(1+\rho(u))^2} + O(u^{m+1}), \tag{3.9}$$

so

$$F(u) = u + \tilde{\rho}(u) - i\tfrac{b}{m+1} u^{m+1} + O(u^{m+2}), \quad \tilde{\rho} \in \mathbb{R}[u].$$

For $u = x + iy$ we get

$$\operatorname{Im} F(u) = y(1 + o(1)) - \tfrac{b}{m+1} x^{m+1} + O(u^{m+2}).$$

Recalling that γ_0 is tangent to the real axis, we have $y = o(x)$. Therefore

$$\{\operatorname{Im} F = 0\} = \left\{x + iy \,:\, y = \tfrac{b}{m+1} x^{m+1} + O(x^{m+2})\right\} \subset \{y \geq 0\}$$

near the origin, and the claim of the Lemma follows since $r(0) = \mathbb{R}_+$. □

Next, we have to check the two conditions in Definition 3.18 for ${}_0\triangle_{p''}^{p'}$ with p' sufficiently close to 0. The second condition is easy: since $R(0) = 1$ then the scalar product $(R(u), R(v))$ is positive for all $u, v \in {}_0\triangle_{p''}^{p'}$ by continuity.

To check the first condition set $u = x_1 + iy_1, v = x_2 + iy_2 = u + tR(u) \in {}_0\triangle_{p''}^{p'}$ with $t > 0$. By the second property of the small horns, we have $x_2 > x_1$. By (3.7) we have $y_i = O\left(x_i^{m+1}\right)$. Combining (3.9) and

$$R'(v) = \rho'(v) + imbv^{m-1} + O(v^m), \tag{3.10}$$

we get

$$\begin{aligned}\frac{R'(v)}{R(v)} &= \left(\rho'(x_2) + imbx_2^{m-1} + O(x_2^m)\right)\frac{1 + \rho(x_2) - ibx_2^m + O(x_2^{m+1})}{\left(1 + \rho(x_1)\right)^2} \\ &= \frac{\rho'(x_2)\left(1 + \rho(x_2)\right) + imbx_2^{m-1} + O(x_2^m)}{\left(1 + \rho(x_2)\right)^2}.\end{aligned}$$

Thus, using (3.5), we get for $\Phi(u, v) = \left(1 + \rho(x_2)\right)^2 \operatorname{Im} R(u)\frac{R'(v)}{R(v)}$ the equation

$$\begin{aligned}\Phi &= \operatorname{Im}\left(\left[1 + \rho(x_1) + ibx_1^m + O(x_1^{m+1})\right] \cdot \left[\rho'(x_2)\left(1 + \rho(x_2)\right) + imbx_2^{m-1} + O(x_2^m)\right]\right) \\ &= mbx_2^{m-1} + O(x_2^m) > 0,\end{aligned} \tag{3.11}$$

where we use $x_1 \le x_2$. This proves the first requirement of Definition 3.18. ☐

Corollary 3.25. *The germ of* $\mathfrak{I}_R$ *at* p *cannot lie between* γ_p^+ *and* $r(p)$*.*

Proof. This would mean that this germ lies inside ${}_p\triangle_{p''}^{p'}$ which is impossible by Proposition 3.22 and Lemma 3.20. ☐

3.3.3 Removing small horns

We will use the following general Lemma

Lemma 3.26. *Assume that for some open set* $U \subset \mathbb{C} \setminus \mathcal{Z}(PQ)$ *and every point* $u \in U$*, the associated ray* $r(u)$ *lies in the union* $U \cup \left(\mathrm{M}_{CH}^T\right)^c$*. Then* $\mathrm{M}_{CH}^T \cap U = \emptyset$*.*

Proof. Indeed, if not then $\mathrm{M}_{CH}^T \setminus U \subsetneq \mathrm{M}_{CH}^T$ will be again invariant, which contradicts minimality of M_{CH}^T. ☐

The crucial property of small horns is the following Lemma.

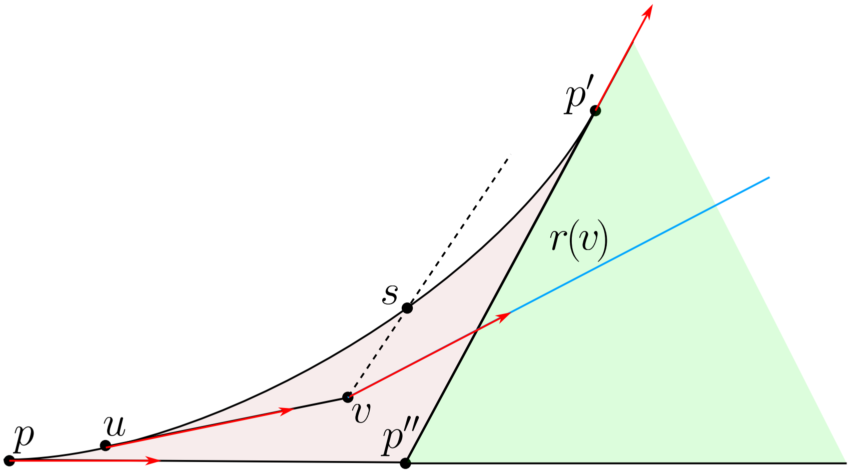

Figure 2: Removing small horns.

Lemma 3.27. *For any* $v \in {}_p\triangle^{p'}_{p''}$, *one has* $r(v) \subset {}_p\triangle^{p'}_{p''} \cup {}_{p''}\angle$.

Proof. We prove the statement assuming that the small horn ${}_p\triangle^{p'}_{p''}$ is positive, the negative case will follow by conjugation.

Let $u \in \gamma_p^{p'}$ be a point such that $v \in r(u)$. By definition of small horns, we have $\sigma(p) < \sigma(u) < \sigma(v)$, see Fig. 2.

The ray r_v does not intersect $\gamma_p^{p'}$. Indeed, assume that the ray $r(v)$ intersects $\gamma_p^{p'}$ at a point s. Then at the intersection point the slope of $\gamma_p^{p'}$ should be smaller than the slope of $r(v)$, i.e. $\sigma(s) < \sigma(v)$ which contradicts the requirement that the slope is monotone increasing along the segment joining v and s.

Also $r(v)$ cannot intersect pp'' since $\sigma(v) > \sigma(u)$ and $\sigma(u) > \sigma(p)$, where $u \in \gamma_p^{p'}$ such that $v \in r(u)$.

Thus $r(v)$ leaves ${}_p\triangle^{p'}_{p''}$ and enters ${}_{p''}\angle$ at some point of $p''p'$ with the slope $\sigma(p) < \sigma(v) < \sigma(p')$. Thus $r(v)$ never leaves ${}_{p''}\angle$. □

Proposition 3.28. *Assume that* ${}_p\triangle^{p'}_{p''}$ *is a small horn and* ${}_{p''}\angle \subset (\mathrm{M}^T_{CH})^c$. *Then* p *is not in the interior of* M^T_{CH}.

Proof. Follows from Lemmas 3.26 and 3.27. □

4 Boundary arcs

Recall that we consider an operator T whose minimal set M_{CH}^{T} is different from $\mathbb{C}$ and has a nonempty interior. We want to describe its boundary in combinatorial and dynamical terms. To do this, we introduce two set-valued functions.

Recall that in our terminology, $\overline{\mathrm{M}_{CH}^{T}}$ is the closure of M_{CH}^{T} in the extended plane $\mathbb{C} \cup \mathbb{S}^1$.

4.1 The correspondences Γ and Δ

Definition 4.1. For any $x \in \partial \mathrm{M}_{CH}^{T} \setminus \mathcal{Z}(PQ)$, we define:

- $\Gamma(x) = \{y \in \gamma_x^+ \mid y \neq x\} \cap \overline{\mathrm{M}_{CH}^{T}}$ where γ_x^+ is the positive trajectory of the vector field $R(z)\partial_z$ starting at x;
- $\Delta(x) = \{y \in r(x) \mid y \neq x\} \cap \overline{\mathrm{M}_{CH}^{T}}$.

Note that if $y \in \Gamma(x)$ or $y \in \Delta(x)$, and $x \in \partial \mathrm{M}_{CH}^{T}$, then $y \in \partial \mathrm{M}_{CH}^{T}$ as well.

Using correspondences Γ and Δ, we split the set of boundary points of M_{CH}^{T} disjoint from the curve of inflections into the following three types.

Definition 4.2. A point of $\partial \mathrm{M}_{CH}^{T} \setminus (\mathcal{Z}(PQ) \cup \mathfrak{I}_R)$ is a point of:

- *local type* if $\Gamma(z) \neq \emptyset$ and $\Delta(z) = \emptyset$;
- *global type* if $\Gamma(z) = \emptyset$ and $\Delta(z) \neq \emptyset$;
- *extruding type* if $\Gamma(z) \neq \emptyset$ and $\Delta(z) \neq \emptyset$.

By Proposition 4.7 these are the only possibilities for points in $\partial \mathrm{M}_{CH}^{T} \setminus \mathfrak{I}_R$.

4.2 Support lines

In this section, we prove that for a given point z, the condition $\Delta(z) \neq \emptyset$ means that the associated ray $r(z)$ is a *support line* of $\overline{\mathrm{M}_{CH}^{T}}$.

For any oriented support line of $\overline{\mathrm{M}_{CH}^{T}}$, we define the *co-orientation* of its support in the following way. The support point x is:

- a *direct support point* if the standard orientation of $\partial \mathrm{M}_{CH}^{T}$ and the orientation of the support line agree at x;
- an *indirect support point* otherwise.

In particular, if the support line is the positively oriented real axis, a support point x is called *direct* if the intersection of M_{CH}^{T} with a neighborhood of x is contained in the upper half-plane (see Figure 3 for examples of indirect support points).

Definition 4.3. Consider $z \in \mathbb{C}$ such that:

- z does not belong to the tangency locus $\mathcal{T}_R$ of the curve of inflections $\mathfrak{I}_R$;
- z is not a root of P or Q.

Then we say that $z \in \mathfrak{E}^+$ (resp. $\mathfrak{E}^-$) if the associated ray $r(z)$ is pointing inside the inflection domain $\mathfrak{I}^+$ (resp. $\mathfrak{I}^-$). This includes $z \in \mathfrak{I}^+$ (resp. $\mathfrak{I}^-$).

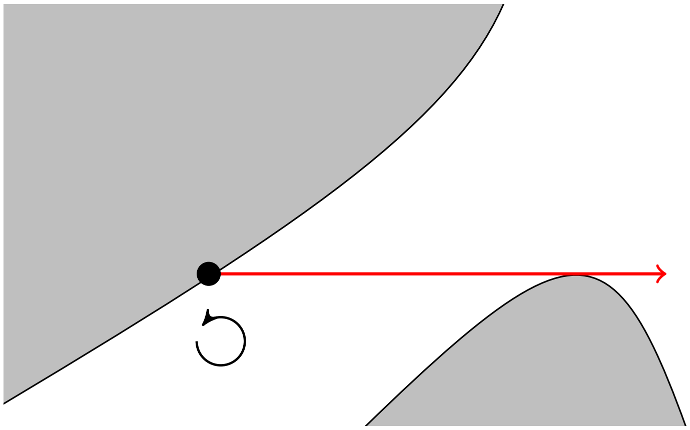

Figure 3: The point where the red arrow is tangent to M_{CH}^{T} is an indirect support point. The circular arrow indicates that the black point belongs to $\mathfrak{E}^+$.

Lemma 4.4. *Consider $z \in \partial \mathrm{M}_{CH}^{T} \setminus \mathcal{Z}(PQ)$ such that $z \in \mathfrak{E}^+$ (resp. $\mathfrak{E}^-$). If $y \in \Delta(z)$, then y is an indirect support point (resp. a direct support point).*

Proof. Without loss of generality, we can assume that $z \in \mathfrak{E}^+$, $z = 0$, $r(z) = \mathbb{R}_{>0}$ and $y = 1$. This implies that γ_0 lies in the upper half-plane. By Lemma 3.21 there is a neighborhood V of y such that $V \cap \left(\overline{\mathrm{M}^T_{CH}}\right)^\circ$ is contained in the lower half-plane. Therefore y is an indirect support point. □

Lemma 4.5. *Take $x, y \in \partial \mathrm{M}^T_{CH}$ such that:*

- $x, y \in \mathfrak{E}^- \cup \mathfrak{E}^+$
- *the associated rays $r(x)$ and $r(y)$ intersect at some point $m \in \mathbb{C}$;*
- $\sigma(y) \in]\sigma(x) - \pi, \sigma(x)[$.

Then the open cone Γ with apex m and the interval of directions $]\sigma(y), \sigma(x)[$ is disjoint from $\left(\mathrm{M}^T_{CH}\right)^\circ$ and there are the following subcases:

- *either $y \in \mathfrak{E}^+$ or $\Delta(y) \subset [y, m]$;*
- *either $x \in \mathfrak{E}^-$ or $\Delta(x) \subset [x, m]$.*

Proof. The path formed by the concatenation of segments $[x, m]$ and $[m, y]$ can be approached by a family paths joining x and y or a family of paths joining y and x whose interior points are disjoint from M^T_{CH}. Lemma 2.9 applies to one of these families of paths so Γ is disjoint from $\left(\mathrm{M}^T_{CH}\right)^\circ$.

Then, we assume by contradiction that $y \in \mathfrak{E}^-$ and some point $z \in \Delta(y)$ does not belong to $[y, m]$. Since Γ is disjoint from $\left(\mathrm{M}^T_{CH}\right)^\circ$, it follows that z is an indirect support point of the line containing $r(y)$ which contradicts Lemma 4.4. Consequently either $\Delta(y) \subset [y, m]$ or $y \notin \mathfrak{E}^-$.

An analogous argument proves that either $x \in \mathfrak{E}^-$ or $\Delta(x) \subset [x, m]$. □

4.3 Local arcs

In this section, we prove that local points (see Proposition 4.7) form *local arcs*.

Definition 4.6. A *local arc* of $\partial \mathrm{M}^T_{CH}$ is a maximal open arc of an integral curve of vector field $R(z)\partial_z$ that contains only local points. In particular, it is disjoint from $\mathcal{Z}(PQ)$ and $\mathfrak{I}_R$.

Local arcs are oriented by the vector field $R(z)\partial_z$.

Using the geometry of horns (see Section 3.3), we can show that every local point actually belongs to a local arc of $\partial \mathrm{M}^T_{CH}$.

Proposition 4.7. *Consider a point $p \in \partial \mathrm{M}^T_{CH}$ and such that $\Delta(p) = \emptyset$ and $p \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$. Then, the germ of the integral curve γ_p of $R(z)\partial_z$ passing through p belongs to $\partial \mathrm{M}^T_{CH}$.*

Without loss of generality, we assume that $p = 0$, $r(p) = \mathbb{R}_+$ and $p \in \mathfrak{G}^+$, so $\gamma_p^{p'}$ lies in the upper half-plane. The proof consists of two steps illustrated by Figure 4 and Figure 5 respectively.

Lemma 4.8. *M^T_{CH} lies above the integral curve γ_p of $R(z)\partial_z$ passing through p.*

Proof: see Fig. 4. By Lemma 3.21 there exists $p' \in \gamma_p^+$ such that the union ${}_p\triangle^{p'}_{p''} \cup {}_{p''}\angle$ is outside of M^T_{CH}. Let $q' \in \gamma_p^{p'}$, $q' \neq p, p'$, and let $q'' \in \mathbb{R}_+$ be the intersection point of $\mathbb{R}_+$ with the line tangent to $\gamma_p^{p'}$ at q'. By Proposition 3.22 we can assume that ${}_p\triangle^{q'}_{q''}$ is a *small* horn at p, ${}_p\triangle^{q'}_{q''} \subset {}_p\triangle^{p'}_{p''}$. Clearly, $\sigma(q') < \sigma(p')$.

The condition $\Delta(p) = \emptyset$ implies that $q'' \in (\mathrm{M}^T_{CH})^c$. Moreover, as $+\infty \notin \Delta(0)$, there is an open sector S with vertex on $\mathbb{R}$, containing $[q'', +\infty)$ and disjoint from M^T_{CH}.

For a point $\tilde{p}$ sufficiently close to p and lying below γ_p consider a horn ${}_{\tilde{p}}\triangle^{\tilde{q}'}_{\tilde{q}''}$ with vertices $\tilde{q}'$ and $\tilde{q}''$ close to q' and q'', respectively. By continuity, the part of $r(\tilde{p})$ starting from $\tilde{q}''$ lies in S. Also, $\tilde{q}'$ lies in the horn ${}_p\triangle^{p'}_{p''}$, so $r(\tilde{q}') \cap \mathrm{M}^T_{CH} = \emptyset$ by Lemma 3.21.

Thus the complementary cone ${}_{\tilde{q}''}\angle$ of $\tilde{p}$ with vertex $\tilde{q}''$ lies outside of M^T_{CH}. Therefore by Proposition 3.28 $\tilde{p} \notin \mathrm{M}^{T\,\circ}_{CH}$, so $\tilde{p} \in \overline{(\mathrm{M}^T_{CH})^c}$. As this remains true for any point in a sufficiently small neighborhood of $\tilde{p}$, we conclude $\tilde{p} \in \left(\overline{(\mathrm{M}^T_{CH})^c}\right)^\circ = (\mathrm{M}^T_{CH})^c$. Thus near p the set M^T_{CH} lies above γ_p.

□

Lemma 4.9. *The boundary $\partial \mathrm{M}^T_{CH}$ coincides with the integral curve γ_p in a neighborhood of p.*

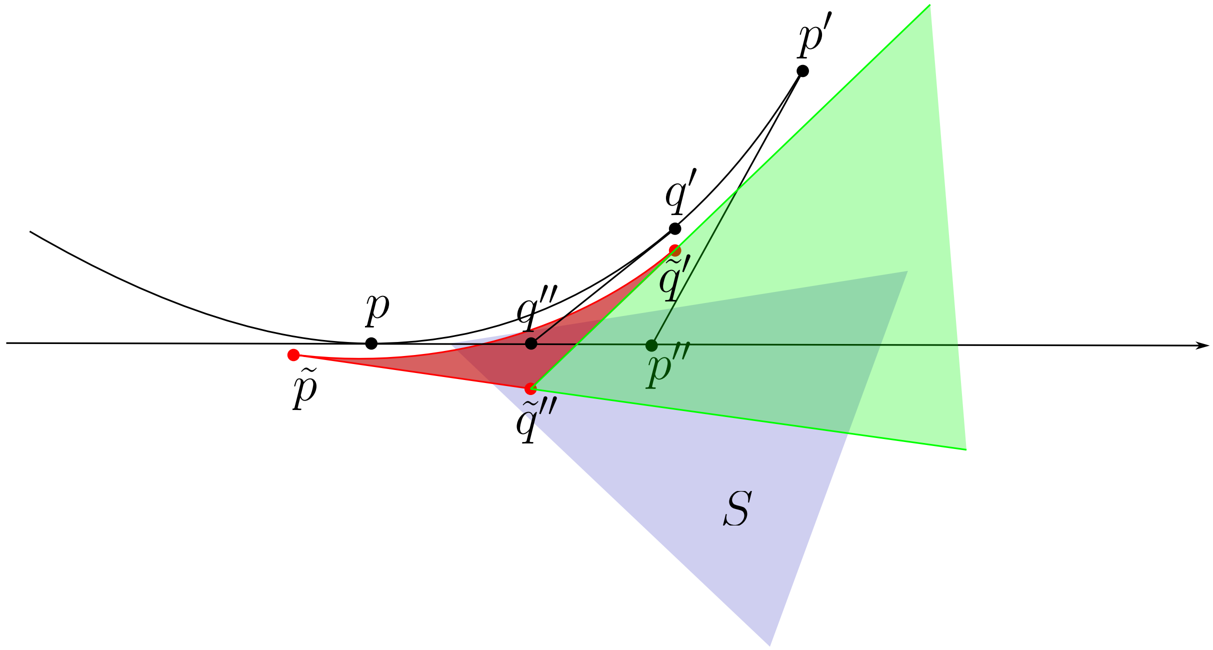

Figure 4: M_{CH}^T lies above the trajectory $\gamma_p^{p'}$.

Proof: See Figure 5. Lemma 4.8 and its proof implies that p lies on the boundary of a sector S with a vertex $s \in \mathbb{R}$, $s \neq p$, and disjoint from M_{CH}^T.

Recall that by Lemma 3.23 there is a lower bound δ on the size of small horns for all points close to p.

Assume that a point $q \notin \mathrm{M}_{CH}^T$ close to p lies above γ_p on a distance much smaller than δ and let ${}_q\triangle_{q''}^{q'}$ be its horn (necessarily small) of size $\delta/2$. Both ${}_q\triangle_{q''}^{q'}$ and ${}_{q''}\angle$ lie outside of M_{CH}^T.

Let $\tilde{p}$ be a point on γ_q close to q and in the negative direction from q, let $\tilde{p}''$ be the intersection of $r(\tilde{p})$ and the line $q'q''$. The horn ${}_{\tilde{p}}\triangle_{\tilde{p}''}^{q'}$ is smaller than δ, so is small.

The ray $\tilde{p}''q'$ lies outside of ${\mathrm{M}_{CH}^T}^{\circ}$. Moreover, as long as the ray $p'' + R(\tilde{p})\mathbb{R}_+ \subset r(\tilde{p})$ lies outside ${\mathrm{M}_{CH}^T}^{\circ}$ we have ${}_{\tilde{p}''}\angle \subset (\mathrm{M}_{CH}^T)^c$, so $\tilde{p} \notin {\mathrm{M}_{CH}^T}^{\circ}$ by Proposition 3.28.

These arguments work for all points $\tilde{p}$ sufficiently close to p and with slope $\sigma(\tilde{p})$ exceeding some negative number (namely the slope of the second side of S), in particular, for points slightly above γ_p^-, the negative trajectory of γ_p. But p lies in the horn of size δ

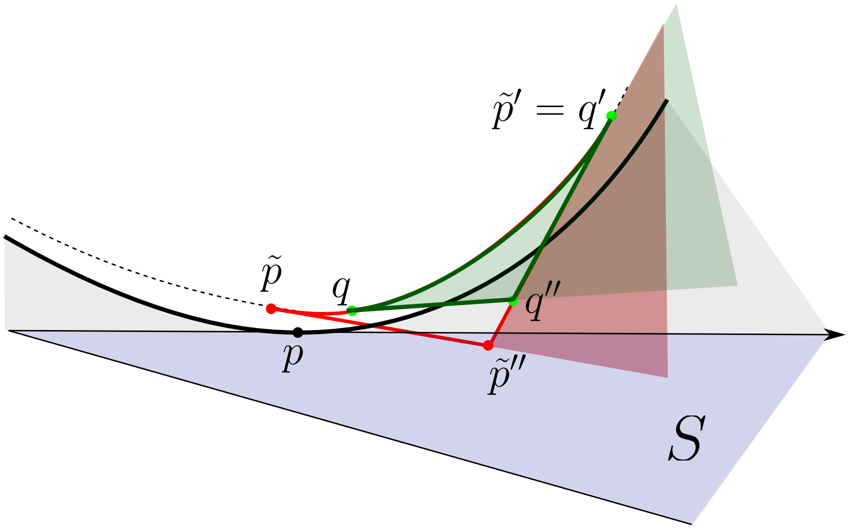

Figure 5: $\gamma_p^{p'}$ is boundary of M_{CH}^T

of such a point, which means that $p \notin \mathrm{M}_{CH}^T$ by Lemma 3.21, a contradiction.

□

Local analysis of horns (see Section 3.3) leads to the following results about the correspondence Γ.

Corollary 4.10. *Consider $z \in \partial\mathrm{M}_{CH}^T$ such that $z \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$. If $\Gamma(z) \neq \emptyset$ then z is either the starting point or a point of a local arc.*

Proof. We just have to prove that for some $y \in \Gamma(z)$ such that y is close enough to z, the arc α of integral curve between z and y belongs to $\partial\mathrm{M}_{CH}^T$. This follows from Lemma 3.21. □

Proposition 4.11. *Any local arc is a locally strictly convex real-analytic curve. Its orientation coincides with the standard topological orientation of $\partial\mathrm{M}_{CH}^T$ if it is contained in $\mathfrak{I}^+$ (and with the opposite orientation otherwise).*

Proof. As any integral curve of a real-analytic vector field, a local arc is a real-analytic curve in $\mathbb{R}^2$. The arc has to be locally convex because otherwise, the associated ray (which is contained in the tangent line) at some point would cross the interior of M_{CH}^T. Besides, direct computation shows that the curvature of an integral curve becomes zero only at points belonging to $\mathfrak{I}_R$. □

Let us check that a local arc of $\partial \mathrm{M}_{CH}^{T}$ cannot end inside an inflection domain. It cannot be periodic either.

Proposition 4.12. *Every local arc has an endpoint that belongs to* $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$.

Besides, if such an endpoint belongs to $\mathcal{Z}(PQ)$*, it is either a regular point or a simple pole of* $R(z)$*.*

Proof. Assume that the local arc γ is periodic and doesn't intersect $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$. Then following Proposition 4.11, γ is a strictly convex closed loop disjoint from $\mathcal{Z}(PQ)$ and M_{CH}^{T} is a strictly convex compact domain bounded by γ (in particular γ encompasses every point of $\mathcal{Z}(PQ)$). A neighborhood of γ is foliated by periodic integral curves γ_t of the vector field $R(z)\partial_z$ that are also disjoint from $\mathcal{Z}(PQ)$ and $\mathfrak{I}_R$, so strictly convex as well. Each of them cuts out a strictly convex compact domain $\mathcal{D}_t$. For each point z in the complement of some $\mathcal{D}_t$, $r(z)$ remains disjoint from $\mathcal{D}_t$, which by Lemma 3.26 contradicts the minimality of M_{CH}^{T}.

Now, we show that a local arc cannot go to infinity. When $|\deg Q - \deg P| \leq 1$, integral curves going to infinity enter the cones disjoint from M_{CH}^{T} and never leave them (see Section 2.5) and otherwise M_{CH}^{T} is trivial.

In the remaining cases, Poincaré-Bendixson theorem proves that a local arc γ has an ending point $y \in \partial \mathrm{M}_{CH}^{T}$. We assume by contradiction that $y \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$. We consider an arc β formed by a portion of the integral curve ending at y and a portion of the associated ray $r(y)$. Provided that β remains in the same inflection domain as y, the family of associated rays starting at the points of the arc β sweeps out a domain containing a cone (see Lemmas 2.9 and 3.21). Therefore, we have $\Delta(y) = \emptyset$. Proposition 4.7 then proves that the local arc can be continued in a neighborhood of y.

If $y \in \mathcal{Z}(PQ)$ and is a zero or a pole of $R(z)$, then $\mathcal{L}_y$ contains an interval of length at least π (see Definition 3.2). Corollary 3.5 proves that y is either a simple pole or a simple zero satisfying $\phi_y = 0$. In the latter case, y is a repelling singular point of $R(z)\partial_z$ and therefore cannot be the endpoint of a local arc. $\square$

As we will see in Section 4.6, in contrast with the case of ending points, a local arc can start inside an inflection domain at a point of extruding type.

4.4 Local connectedness of M_{CH}^T

Here we show that M_{CH}^T is locally connected, away from the part of the tangency locus that is formed by straight lines.

Lemma 4.13. M_{CH}^T *is locally connected outside of* $\mathfrak{I}_R \cup \mathcal{Z}(PQ)$*.*

Proof. If $z \in \partial\mathrm{M}_{CH}^T$ is a point of local type, then the boundary locally coincides with the integral curve passing through z by Lemma 4.9. Next, let $z \in \partial\mathrm{M}_{CH}^T \setminus (\mathcal{Z}(PQ) \cup \mathfrak{I}_R)$ with $\Delta(z) \neq \emptyset$. Let $y \in \Delta(z)$. As $z \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$, by Lemma 2.11 or Lemma 2.16 there is a unique germ of $\mathfrak{tr}_y$ passing through z, and it does so transversely to the integral curve of $R(z)\partial_z$ passing through z. Then taking the backward trajectories of $R(z)\partial_z$ of points in $\mathfrak{tr}_y$, all points on one side of $\mathfrak{tr}_y$ near z belong to M_{CH}^T. Taking now as a neighborhood basis a family of decreasing curvilinear quadrilaterals with two of the sides being trajectories of $R(z)\partial_z$ and two sides being smooth curves on either side of $\mathfrak{tr}_y$, it follows that all points on the other side of $\mathfrak{tr}_y$ have backward trajectories of $R(z)\partial_z$ intersecting $\mathfrak{tr}_y$ inside these neighborhoods, provided they are sufficiently small. As for any $x \in \mathrm{M}_{CH}^T$, its backward trajectory belongs to M_{CH}^T, it follows that M_{CH}^T is locally connected at z. $\square$

Lemma 4.14. M_{CH}^T *is locally connected at zeros and poles of* R*.*

Proof. If z_0 is a pole of R one can show using Proposition 3.12 in [AHN+24] and Corollary 3.5 that M_{CH}^T is locally connected at z_0. Next, for z_0 a zero of R it follows from the same corollary and by using the local portrait of $R(z)\partial_z$ near z_0. $\square$

Lemma 4.15. M_{CH}^T *is locally connected at all* z*, such that* γ_z *is not a straight line.*

Proof. Since the integral curve γ_z of vector field $R(z)\partial_z$ containing z is not a straight line, some germ of the negative part γ_z^- lies outside of $\mathfrak{I}_R$. Recall that if $y \in \mathrm{M}_{CH}^T$ then the negative part γ_y^- of γ_y necessarily lies in M_{CH}^T.

Let $\gamma_z^-(\epsilon) \subset \mathfrak{I}_R^c$ be the part of the negative trajectory γ_z^- lying strictly between z and $y = g_R^{-\epsilon}(z)$ where g_R^t is the flow of $R(z)\partial_z$. By Lemma 4.13 there is a neighborhood V_y of y of size smaller than ϵ such that $V_y \cap \mathrm{M}_{CH}^T$ is connected. Now, let $U_z = \cup_{0\le t\le\epsilon} g_R^t(V_y)$. Clearly U_z is a neighborhood of z. We claim that $U_z \cap \mathrm{M}_{CH}^T$ is connected. Indeed, if $w \in U_z \cap \mathrm{M}_{CH}^T$ then $w = g_R^t(w')$ for some $w' \in V_y \cap \mathrm{M}_{CH}^T$, $0 \le t \le \epsilon$. Since w' lies in the same connected component of $U_z \cap \mathrm{M}_{CH}^T$ as y and w and w' are jointed by $\gamma_w^- \subset U_z \cap \mathrm{M}_{CH}^T$ this means that $U_z \cap \mathrm{M}_{CH}^T$ is connected. As ϵ can be chosen arbitrarily small, the statement follows. □

We denote by $\mathcal{L}$ the union of all R-invariant lines.

Corollary 4.16. *$\partial\mathrm{M}_{CH}^T \setminus \mathcal{L}_R$ is parametrizable.*

By Carathéodory's theorem, the boundary of an open set is parametrizable if its boundary is locally connected. For each $z \in \partial\mathrm{M}_{CH}^T \setminus \mathcal{L}_R$ we have by Lemma 4.14 and Proposition 4.15 a neighborhood basis $\mathcal{N}(z)$ consisting of sets such that $U \cap \mathrm{M}_{CH}^T$ is connected. Define $U(z) \in \mathcal{N}(z)$ to be a set of the form $U(z) \subset B(z, \frac{1}{2}d(z, \mathcal{L}_R))$. The union

$$\bigcup_{z\in\partial\mathrm{M}_{CH}^T\setminus\mathcal{L}_R} U(z)$$

is an open cover of $\partial\mathrm{M}_{CH}^T \setminus \mathcal{L}_R$ and being a subset of $\mathbb{C}$, it has a countable subcover

$$\bigcup_{n\in\mathbb{N}} U(z_n).$$

For each z_n, $\overline{U(z_n) \cap \mathrm{M}_{CH}^T}$ is locally connected and by Lemma 2.21 and the fact that all irregular points are contained in $\mathcal{L}_R$, its boundary is a Jordan curve away from the poles and zeros of $R(z)$. Hence $\overline{\partial U(z_n) \cap \partial\mathrm{M}_{CH}^T}$ is parametrizable by Carathéodory's theorem, injectively away from the zeros and poles of $R(z)$. We start with a z_0 and use this parametrization of $\overline{\partial U(z_n) \cap \partial\mathrm{M}_{CH}^T}$. Then for the smallest $n = n_1$ such that $U(z_n) \cap U(z_0) \neq \emptyset, \partial\mathrm{M}_{CH}^T \cap U(z_n) \not\subset \partial\mathrm{M}_{CH}^T \cap U(z_0)$, we glue together the parametrizations of $\overline{\partial(U(z_{n_1}) \setminus U(z_0)) \cap \partial\mathrm{M}_{CH}^T}$ with that of $\overline{\partial U(z_0) \cap \partial \mathrm{M}_{CH}^T}$ along the end points of the parametrizations. We then have a parametrization of $(U(z_0) \cup U(z_{n_1})) \cap \partial \mathrm{M}_{CH}^T := \mathcal{B}_1$. We then take the smallest $n = n_2$ such that $U(z_{n_2}) \cap (U(z_0) \cup U(z_{n_1})) \neq \emptyset, \partial \mathrm{M}_{CH}^T \cap U(z_n) \not\subset \mathcal{B}_1$ and in the same way find a parametrization of $\mathcal{B}_2 := (U(z_0) \cup U(z_{n_1}) \cup U(z_{n_2})) \cap \partial \mathrm{M}_{CH}^T$. We proceed in this way inductively to get a parametrization of $\partial \mathrm{M}_{CH}^T \setminus \mathcal{L}_R$, potentially pinched at the zeros and poles of $R(z)$ (but not anywhere else).

4.5 Global arcs

4.5.1 Additional results about correspondence Δ

Lemma 4.17. *Consider $z \in \partial \mathrm{M}_{CH}^T \setminus \mathcal{Z}(PQ)$ such that $z \in \mathfrak{S}^+$ (resp. $\mathfrak{S}^-$). If $y \in \partial \mathrm{M}_{CH}^T$ and $y \in \Delta(z)$, then one of the following statements holds:*

- *$y \in \mathcal{Z}(PQ) \cup \mathfrak{I}_R$;*
- *$y \in \mathfrak{I}^-$ (resp. $\mathfrak{I}^+$).*

Proof. Without loss of generality, we assume that $z = 0$, $r(z) = \mathbb{R}^+$ and $z \in \mathfrak{S}^+$.

We consider $y \in \Delta(z)$ such that $y \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$. If $\Delta(y) = \emptyset$, then Proposition 4.7 shows that y belongs to a local arc. Besides, y is an indirect support point of the associated ray $r(z)$ (see Lemma 4.4). If $y \in \mathfrak{I}^+$, then the associated rays starting from a germ of the local arc at y sweep out a neighborhood of z and we get a contradiction. Therefore $y \in \mathfrak{I}^-$.

Now we consider the case where $\Delta(y) \neq \emptyset$ and assume by contradiction that $y \in \mathfrak{I}^+$. If $Im(R(y)) > 0$, then Lemma 4.5 provides an immediate contradiction. If $Im(R(y)) < 0$, then $\mathcal{L}_y$ (see Definition 3.2) contains an interval of length strictly larger than π such that $\sigma(y)$ is one of the ends. It follows from Corollary 3.5 that y is a simple zero of $R(z)$ (and therefore $y \in \mathcal{Z}(PQ)$).

If $r(y) = y + \mathbb{R}^-$, then for some small $\epsilon > 0$, points of the interval $]-\epsilon, \epsilon[$ are disjoint from the interior of M_{CH}^T. Associated rays starting from the points of $]-\epsilon, \epsilon[$ sweep out an open cone containing a neighborhood of y. This contradicts the assumption $y \in \partial \mathrm{M}_{CH}^T$.

Therefore, $r(y) = y + \mathbb{R}^+$ and $r(y) \subset r(z)$. In this case, for some small $\epsilon' > 0$, points of $]y - \epsilon', y[$ are disjoint from the interior of M_{CH}^T and their associated rays will sweep out a neighborhood of y if $y \in \mathfrak{I}^+$. Therefore, in that case we get that $y \in \mathfrak{I}^-$. Similar result holds for $z \in \mathfrak{E}^-$. □

Definition 4.18. For any point $z \in \partial \mathrm{M}_{CH}^T$ such that $z \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$ and $\Delta(z) \neq \emptyset$, we define $\Delta^{min}(z)$ (resp. $\Delta^{max}(z)$) as the infimum (resp. the supremum) in $\Delta(z)$ of the order induced by the orientation of the associated ray $r(z)$.

Besides, we define $L_z = |\Delta^{min}(z) - z|$.

Lemma 4.19. *For any $z \in \partial \mathrm{M}_{CH}^T$ such that $z \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$ and $\Delta(z) \neq \emptyset$, we have $\Delta^{min}(z) \neq z$ and $L_z \neq 0$.*

Proof. Without loss of generality, we assume that $z = 0$, $z \in \mathfrak{I}^+$ and $r(x) = \mathbb{R}_{>0}$. For any small enough real positive ϵ, we have $Re(R(\epsilon)), Im(R(\epsilon)) > 0$ and $\epsilon \in \mathfrak{I}^+$. If such an ϵ belongs to $\Delta(z)$, then it contradicts Lemma 4.17. □

Since $\overline{\mathrm{M}_{CH}^T}$ is compact in $\mathbb{C} \cup \mathbb{S}^1$, it follows immediately that for any z, $\Delta^{min}(z)$ is actually a point of $\partial \overline{\mathrm{M}_{CH}^T}$.

Definition 4.20. For any point $z \in \partial \mathrm{M}_{CH}^T$ such that $z \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$ and $\Delta(z) \neq \emptyset$, we define $\mathcal{U}(z)$ as the connected component of $(\mathrm{M}_{CH}^T)^c \setminus [z, \Delta^{min}(z)]$ incident to:

- the right side of $[z, \Delta^{min}(z)]$ if $z \in \mathfrak{I}^+$;
- the left side of $[z, \Delta^{min}(z)]$ if $z \in \mathfrak{I}^-$.

i.e. in the half-plane bounded by $r(z)$ different from that containing the germ of the trajectory of $R(z)\partial_z$ starting at z.

Lemma 4.21. *Consider $z \in \partial \mathrm{M}_{CH}^T$ such that $z \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$, $\Delta(z) \neq \emptyset$ and $z \in \mathfrak{I}^+$ (resp. $\mathfrak{I}^-$). For any $y \in \partial \mathrm{M}_{CH}^T \cap \partial \mathcal{U}(z)$ such that $y \in \mathfrak{I}^+$ (resp. $\mathfrak{I}^-$) and $\Delta(y) \neq \emptyset$, we have $\mathcal{U}(y) \subset \mathcal{U}(z)$.*

Besides, if $y \neq z$, we have $\mathcal{U}(y) \subsetneq \mathcal{U}(z)$.

Proof. By connectedness of M_{CH}^{T} in the case $\deg Q - \deg P = \pm 1$ and the asymptotic geometry of M_{CH}^{T} in the case $\deg Q - \deg P = 0$, it follows that the associated ray $r(y)$ intersects the associated ray $r(z)$. Applying Lemma 4.5 to $r(z)$ and $r(y)$, we see that $\Delta(y) \subset \partial\mathcal{U}(z)$. Thus $\mathcal{U}(y) \subset \mathcal{U}(z)$.

When $\mathcal{U}(y) = \mathcal{U}(z)$, the associated ray $r(y)$ has to coincide with $r(z)$ (with the same orientation since y and z belong to the same inflection domain). It follows that $y = z$. □

4.5.2 Orientation of global arcs

By Corollary 4.16, the following notion is well-defined.

Definition 4.22. *A global arc* in $\partial\mathrm{M}_{CH}^{T}$ is a maximal open connected arc formed by points of global type.

Furthermore, for a global arc α defined on (t_{min}, t_{max}), its end point is defined as long as

$$\omega_+(\alpha) := \bigcap_{t_0 \in (t_{min}, t_{max})} \overline{\{\alpha(t) : t > t_0\}}$$

is a singleton (and equals this element). The starting point is analogously defined if

$$\omega_-(\alpha) := \bigcap_{t_0 \in (t_{min}, t_{max})} \overline{\{\alpha(t) : t < t_0\}}$$

is a singleton. If $\omega_+(\alpha)$ is not a singleton, then it can only contain points contained in R-invariant lines, again by Corollary 4.16 and similarly for $\omega_-(\alpha)$. Regardless if they are singletons or not, we call the sets $\omega_\pm(\alpha)$ *end accumulation* and the *start accumulation*. In case they are in fact singletons, we will also call them end and starting points respectively. We have a geometrically meaningful way to define orientation on global arcs.

Lemma 4.23. *Any global arc $(\alpha_t)_{t\in I}$ can be oriented in such a way that for $t' > t$, we have:*

- $\alpha_t \in \partial\mathrm{M}_{CH}^{T} \cap \partial\mathcal{U}(\alpha_{t'})$;
- $\mathcal{U}(\alpha_{t'}) \supset \mathcal{U}(\alpha_t)$.

In particular, in $\mathfrak{I}^+$, the orientation of global arcs coincides with the standard topological orientation of $\partial \mathrm{M}_{CH}^T$ (it coincides with the opposite orientation in $\mathfrak{I}^-$).

In particular, a global arc is an interval, i.e. it cannot be a closed loop.

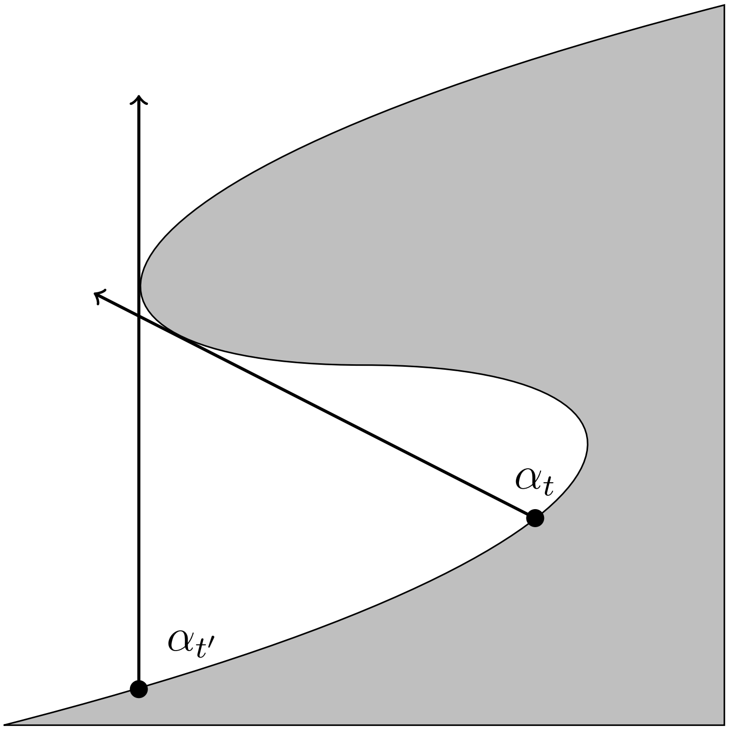

Figure 6: Two associated rays from the same global arc.

Proof. Removal of α_t from α cuts the arc into two pieces, one of which is contained in $\partial \mathcal{U}(\alpha_t)$ (see Figure 6). Lemma 4.21 then proves the inclusion of the sets of the form $\mathcal{U}(\alpha_t)$ as t sweeps out the interval I which provides a meaningful orientation on the global arc. □

Lemma 4.24. *Along a global arc α, the function $\sigma(z) = \arg(R(z))$ is a monotone mapping of α to an interval in $\mathbb{S}^1$ with length at most π.*

Besides, if $\sigma(\alpha_t) = \sigma(\alpha_{t'})$ for some $t > t'$, then $\Delta(\alpha_t)$ coincides with the point at infinity $\sigma(\alpha_t) = \sigma(\alpha_{t'})$ that also belongs to $\Delta(\alpha_{t'})$.

Proof. Consider two points α_t and $\alpha_{t'}$ of a global arc satisfying $t > t'$ for the canonical orientation. By Lemma 4.23, $\alpha_{t'} \in \partial\mathcal{U}(\alpha_t)$.

Without loss of generality, we assume that α is contained in $\mathfrak{I}^+$, $\alpha_t = 0$ and $r(\alpha_t) = \mathbb{R}^+$. If $\sigma(\alpha_{t'}) \in [-\pi, 0[$, any associated ray starting in a small enough neighborhood of $\alpha_{t'}$ will cross M^T_{CH}. If $\sigma(\alpha_{t'}) = 0$, then the interior of the strip bounded by $r(\alpha_t), r(\alpha_{t'})$ and the portion of global arc between α_t and $\alpha_{t'}$ is disjoint from M^T_{CH}. It follows that $\Delta(\alpha_t)$ contains only the point at infinity. In the remaining case, we have $\sigma(\alpha_{t'}) \in]0, \pi[$. □

Proposition 4.25. *Consider $z \in \partial\mathrm{M}^T_{CH}$ such that $z \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$ and $\Delta(z) \neq \emptyset$. Then, z is either the endpoint or a point of a global arc.*

Proof. We consider an arbitrarily small open arc α of $\partial\mathrm{M}^T_{CH} \cap \mathcal{U}(z)$ ending at z. By assumptions, α is disjoint from $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$. If some point $y \in \alpha$ satisfies $\Gamma(y) \neq \emptyset$, then α partially coincides with a local arc. Since the ending point of any local arc belongs to $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$ (see Proposition 4.12), comparison of the orientation of local arcs and the orientation of $\partial\mathrm{M}^T_{CH}$ in a given inflection domain (see Lemma 4.11) proves that z also belongs to this local arc. This is a contradiction. Therefore, any point y in the arc α satisfies $\Gamma(y) = \emptyset$. Proposition 4.7 then implies that each point of the arc α satisfies $\Delta(y) \neq \emptyset$ and is thus a point of global type. Therefore, z is either the endpoint or a point of a global arc containing α. □

Proposition 4.26. *If a point $z \in \partial\mathrm{M}^T_{CH}$ satisfies:*

- *$z \notin \mathcal{Z}(PQ) \cup \mathfrak{I}_R$;*
- *$\Delta(z) \neq \emptyset$;*
- *$\Gamma(z) = \emptyset$;*

then z belongs to a global arc.

Proof. Following Proposition 4.25, z is either the ending point or a point of a global arc. We consider a connected neighborhood V of z in $\partial\mathrm{M}^T_{CH}$ that is disjoint from $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$. Without loss of generality, we assume that V belongs to $\mathfrak{I}^+$.

We consider a point $y \in V$ such that the oriented arc from y to z in $\partial \mathrm{M}_{CH}^{T}$ has the same orientation as the standard topological orientation of the boundary. If $\Gamma(y) \neq \emptyset$, then y is either a point or the starting point of a local arc (Corollary 4.10) that can be continued til z (see Proposition 4.12) since V is disjoint from $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$. Therefore, $\Gamma(y) = \emptyset$ and it follows then from Proposition 4.7 that $\Delta(y) \neq \emptyset$. Thus, any such point y is a global point belonging to global arc α.

Then, we consider points $y \in V$ such that the oriented arc from y to z in $\partial \mathrm{M}_{CH}^{T}$ has the opposite orientation as the standard topological orientation of the boundary. If such point y satisfies $\Gamma(y) \neq \emptyset$, then it is a point or the starting point of a local arc. Since V is connected and disjoint from $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$, it contains at most one local arc starting at a point of extruding type. The complement of the closure of this local arc in V coincides with global arc β. By hypothesis, z is not a point of extruding type so it belongs to a global arc β. □

Proposition 4.27. *If $z_0 \in \mathbb{C}$ is the endpoint of a global arc α and is neither a zero nor a pole of $R(z)$, then $\Delta(z_0) \neq \emptyset$.*

Proof. We assume that α is parameterized by the interval $]0, 1[$ (with the correct orientation) and $\alpha(t) \to z_0$ as $t \to 1$. For any $n \geq 2$, we pick a point $\beta_n \in \Delta(\alpha(1 - 1/n))$. Since $\mathbb{C} \cup \mathbb{S}^1$ is compact, the sequence $(\beta_n)_{n \geq 2}$ has an accumulation point β. Since z_0 is the endpoint of α, the point β cannot coincide with z_0 (see Lemma 4.23). It follows that a family of associated rays accumulates on a half-line starting at z_0 and containing β. Since $\arg(R(z))$ is continuous in a neighborhood of z_0, we get that this half-line coincides with the associated ray $r(z_0)$. □

Definition 4.28. A point $z \in \partial \mathrm{M}_{CH}^{T}$ is a *non-convexity point* if there is a cone $\mathcal{C}$ at z of angle strictly bigger than π and a neighborhood V of z such that $\mathcal{C} \cap V \subset \mathrm{M}_{CH}^{T}$.

For a point z_0 for which $\Delta(z_0)$ consists of a single point u satisfying the condition $R(z_0) + (u - z_0)R'(z_0) \neq 0$, Lemma 2.11 proves that: (i) the root trail $\mathfrak{tr}_u$ has a unique branch at z_0, (ii) it is contained in M_{CH}^{T}, and (iii) its tangent slope is the argument of $\frac{R^2(z_0)}{R(z_0)+(u-z_0)R'(z_0)}$

(mod π). These lemma yields that if at some point z_0, $\Delta(z_0)$ contains more than one point, then $\partial \mathrm{M}_{CH}^{T}$ cannot be smooth at z_0:

Proposition 4.29. *At a point $z \notin \mathfrak{I}_R \cup \mathcal{Z}(PQ)$ such that $\Delta(z)$ contains at least two points, the boundary $\partial \mathrm{M}_{CH}^{T}$ has a non-convexity point.*

Proof. First assume that $\Delta(z)$ contains two points u, v both of which are not points at infinity. Lemma 2.11 proves that z belongs to two distinct root trails. Assuming that $R(z) + (u - z)R'(z)$ and $R(z) + (v - z)R'(z)$ are nonzero, the tangent slopes of these root trails at z are determined by the argument of $\frac{R^2(z)}{R(z)+(u-z)R'(z)}$ and $\frac{R^2(z)}{R(z)+(v-z)R'(z)}$. By hypothesis, we have $Im(R'(z)) \neq 0$ and these two branches intersect transversely at z and the claim follows, taking the backward trajectories of the two root-trails. If $R(z) + (u - z)R'(z) = 0$, then two branches of the root trail intersect transversely.

In the remaining case, $\Delta(z)$ contains exactly one point u satisfying the condition $R(z) + (u - z)R'(z) \neq 0$ and a point $\sigma(z)$ at infinity. Then the root trail of u at z has a slope given by the argument of $\frac{R^2(z)}{R(z)+(u-z)R'(z)}$ (or $\frac{R(z)}{R'(z)}$ if u is at infinity, see Lemma 2.16). Similarly, $R'(z) \notin \mathbb{R}$ so these curves intersect transversely at z. Summarizing we see that in all possible cases, the boundary $\partial \mathrm{M}_{CH}^{T}$ has a non-convexity point. □

4.6 Points of extruding type

Outside the local and the global arcs, the only singular boundary points in the complement of $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$ which can occur are points of extruding type.

Proposition 4.30. *Let z be a point of extruding type in $\partial \mathrm{M}_{CH}^{T}$. Then z is both the endpoint of a global arc and the starting point of a local arc.*

The boundary $\partial \mathrm{M}_{CH}^{T}$ is not C^1 at z and z is a non-convexity point.

Proof. See Figure 7 below. By definition of the correspondence Γ, and local considerations of R, z is the starting point of a local arc. Propositions 4.25 and 4.26 show that z is the ending point of a global arc.

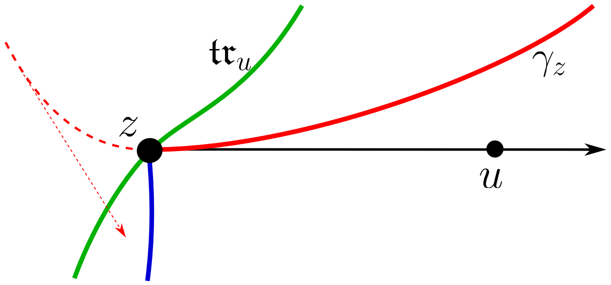

Figure 7: Negative part of γ_z cannot be on the boundary as $\mathfrak{tr}_u \subset \mathrm{M}^T_{CH}$.

For any point $u \in \Delta(z)$, the root trail $\mathfrak{tr}_u$ has a unique local branch at z and its tangent direction is the argument of $\frac{R^2(z)}{R(z)+(u-z)R'(z)}$ (mod π), see Lemma 2.11. Indeed, $R(z) + (u - z)R'(z) \neq 0$ because $u - z$ is real collinear to $R(z)$ while $\mathrm{Im}(R'(z)) \neq 0$. Since $z \notin \mathfrak{I}_R$, this branch transversely intersects the integral curve of $R(z)\partial_z$ containing z. Both of these branches are (semi)analytic curves contained in M^T_{CH} and the associated rays of the points lying on the negative part of γ_z intersect $\mathfrak{tr}_u \subset \mathrm{M}^T_{CH}$. Thus the negative part of γ_z is disjoint from $\partial\mathrm{M}^T_{CH}$. □

4.7 Boundary arcs in inflection domains

Proposition 4.31. *For any connected component $\mathcal{D}$ of the complement of the curve of inflections $\mathfrak{I}_R$, $\partial\mathrm{M}^T_{CH} \cap \mathcal{D}$ is a union of disjoint topological arcs. In each of them, local and global arcs have the same orientation. If* $\mathrm{Im}(R')$ *is positive (resp. negative) in $\mathcal{D}$ then the latter orientation coincides with (is opposite to) the topological orientation of $\partial\mathrm{M}^T_{CH}$.*

Proof. The statement about orientation follows from Proposition 4.11 and Lemma 4.23. Proposition 4.30 shows that a point of extruding type is incident to a local and a global arcs. It remains to prove that any point of $\mathcal{Z}(PQ) \cap \mathcal{D}$ is incident to at most two arcs.

Such a point z_0 is neither a zero nor a pole of $R(z)$, see Corollary 3.12. Therefore, Corollary 3.6 together with the fact that all irregular points are contained in $\mathcal{J}_R$ proves the statement. □

5 Singular boundary points on the curve of inflections

At points belonging to the curve of inflections the boundary $\partial \mathrm{M}_{CH}^T$ can display more complicated behaviours. In this section, we classify boundary points that belong to the transverse locus $\mathfrak{I}_R^*$ of the curve of inflections (see Definition 1.7). For the following definition, recall the Definition 1.8.

Definition 5.1. A point of $\partial \mathrm{M}_{CH}^T \setminus \mathcal{Z}(PQ)$ belonging to the transverse locus $\mathfrak{I}_R^*$ is a point of:

- *bouncing type* if $\Delta^+ \neq \emptyset$ and $\Gamma \cup \Delta^- \neq \emptyset$;
- *switch type* if $\Delta^+ \neq \emptyset$ and $\Gamma \cup \Delta^- = \emptyset$;
- C^1*-inflection type* if $\Delta^+ = \emptyset$, $\Delta^- \neq \emptyset$ and $\Gamma = \emptyset$;
- C^2*-inflection type* if $\Delta^+ = \emptyset$ and either $\Delta^- = \emptyset$ or $\Gamma \neq \emptyset$.

5.1 Horns at points of the transverse locus

At a point $p \in \mathfrak{I}_R^*$, the curve of inflections is smooth and the vector field $R(z)\partial_z$ is transversal to it. This means that by (3.5) we can assume that

$$R(u) = 1 + \rho u + (a + ib)u^2 + \dots \tag{5.1}$$

where we assumed that $p = 0$. The condition $\operatorname{Im} R'(0) = 0$ means that $\rho \in \mathbb{R}$, and the transversality condition is equivalent to $b \neq 0$. Without loss of generality we can assume that $b > 0$. In other words, $m = 2$ in (3.5) which implies that the integral curves locally look like cubic curves with inflections at these points.

We define the *diameter* of a horn ${}_p\triangle_{p''}^{p'}$ to be the least upper bound $t_0 > 0$ of all $t' > 0$ such that there is $u \in {}_p\triangle_{p''}^{p'}$ such that $u + tR(u) \in {}_p\triangle_{p''}^{p'}$ for all $t \in (0, t')$.

Lemma 5.2. *For $p \in \mathfrak{I}_R^*$, there exists a neighborhood Ω of p and $\epsilon > 0$ such that for all points $u \in \overline{\Omega_+}$, where $\Omega_+ = \mathfrak{I}^+ \cap \Omega$, there exists a small horn of diameter greater than ϵ.*

Proof. Without loss of generality we assume $p = 0$. Consider the function

$$T(u,t) = \frac{d\sigma(u + tR(u))}{dt}(t) = \operatorname{Im}\left[\frac{R'(u + tR(u))}{R(u + tR(u))}R(u)\right]$$

defined in $\mathbb{C}_u \times \mathbb{R}_t$. Note that by definition $\mathfrak{I}^+ = \{T(u,0) > 0\}$. Since $T(0,t) = 2bt + O(t^2)$, we have $\frac{\partial T}{\partial t}(0,0) = 2b > 0$. Therefore $\frac{\partial}{\partial t}T(u,t) > b > 0$ in some sufficiently small neighborhood $\widetilde{U} \times (-\tilde{\varepsilon}, \tilde{\varepsilon})$ of $(0,0)$. Let $\widetilde{U}_+ = \mathfrak{I}^+ \cap \widetilde{U}$. By definition of $\mathfrak{I}^+$, we have $\widetilde{U}_+ \times \{0\} \subset \{T > 0\}$. Taken together, this implies that $\widetilde{U}_+ \times [0, \tilde{\varepsilon}] \subset \{T > 0\}$ for some $\tilde{\varepsilon} > 0$. This means that the argument of $R(u + tR(u))$ is monotone increasing for $0 < t < \tilde{\varepsilon}$ and for every $u \in \widetilde{U}_+$.

By transversality of $\mathfrak{I}_R$ and R at 0 we have

$$\frac{\partial}{\partial t}T\big(g_R^t(0) + sR(g_R^t(0)), 0\big)|_{s=t=0} = \frac{\partial}{\partial s}T\big(g_R^t(0) + sR(g_R^t(0)), 0\big)|_{s=t=0} > 0,$$

where g_R^t is the flow of $R(z)\partial_z$.

Thus there is a neighborhood $\Omega \subset \widetilde{U}$ of 0 and $\epsilon < \tilde{\varepsilon}$ such that $T(g_R^t(u) + sR(g_R^t(u)), 0) > 0$ as soon as $u \in \Omega_+ = \mathfrak{I}^+ \cap \Omega$ and $t, s \in [0, \epsilon]$. Decreasing Ω, ϵ if needed, we can assume that

$$g_R^t(u) + sR(g_R^t(u)) \subset \widetilde{U}_+ \quad \text{for all } u \in \Omega_+ \text{ and } t, s \in [0, \epsilon].$$

Thus for every $u \in \Omega_+$, any horn of diameter at most ϵ lies in $\widetilde{U}_+$ and is therefore a small horn.

If $u \in \overline{\Omega_+}$ then the horn ${}_u\triangle^{u'}_{u''}$ of diameter ϵ lies in a union of small horns ${}_{\tilde{u}}\triangle^{\tilde{u}'}_{\tilde{u}''}$ of diameter ϵ, where $\tilde{u} \in {}_u\triangle^{u'}_{u''}$. Thus ${}_u\triangle^{u'}_{u''}$ is a small horn as well.

□

5.2 Points of bouncing type

Proposition 5.3. *Let z be a point of bouncing type in $\partial \mathrm{M}^T_{CH}$:*

$$\Delta^+(z) \neq \emptyset \quad \textit{and} \quad \Gamma(z) \cup \Delta^-(z) \neq \emptyset.$$

Then z is the ending point of a global arc and also the starting point of another arc which can either be local or global.

Further, z is a point of nonconvexity and for small enough neighborhoods V of z, one has $V \cap \partial \mathrm{M}^T_{CH} \cap \mathfrak{I}_R = \{z\}$.

Proof. Without loss of generality, we can assume that $z = 0$, $R(0) = 1$, and $R'(0) \in \mathbb{R} \setminus \{0\}$. Since z belongs to $\mathfrak{I}_R^*$, we get $R''(0) = a + bi$ with $a \in \mathbb{R}$ and $b \in \mathbb{R}^*$. Without loss of generality, we can assume that $b > 0$ and therefore γ_z lies in the upper half-plane.

Let $u_+ \in \Delta^+(0) \neq \emptyset$. If $u_+ \notin \Delta^0(0)$ then the germ of $\mathfrak{tr}_{u_+}$ at 0 is smooth and tangent to $\mathbb{R}$ at 0 and contained in the lower half-plane, see Lemma 2.11 and Proposition 2.19). Otherwise it consists of two branches transversal to $\mathbb{R}$ and orthogonal one to another. Denote by $\alpha_+ \subset \mathfrak{tr}_{u_+} \subset \mathrm{M}_{CH}^T$ the arc starting at 0 and lying in the lower right quadrant.

Similarly, since $\Gamma(0) \cup \Delta^-(0)$ is nonempty there exists an arc α_- (either portion of an integral curve γ_0 or a root trail $\mathfrak{tr}_{u_-}$ of a $u_- \in \Delta^-(0)$) starting at 0, contained in the upper right quadrant, and belonging to M_{CH}^T.

Denote by $\alpha = \alpha_- \cup \alpha_+$, and let V be a small neighborhood of 0 such that α cuts V into two parts. The part V_- of V to the left of α is entirely contained in M_{CH}^T (as $r(u)$ intersects $\alpha \subset \mathrm{M}_{CH}^T$ for any $u \in V_-$). This domain contains the intersection of V with a cone with vertex at 0 and of angle strictly larger than π. It also contains all the intersection of V with $\mathfrak{I}_R$ excepted for the point 0, see Lemmas 2.11, 2.16 and Remark 2.17.

It remains to prove that in a neighborhood of z, $\partial \mathrm{M}_{CH}^T$ is formed by exactly two arcs.

Lemma 5.4. *Assume that* $\Gamma(0) = \emptyset$ *and* $\Delta^-(0) \neq \emptyset$. *Then* 0 *is a starting point of a global arc.*

Proof. By Lemma 3.21 the points lying below the forward trajectory γ_0^+ of 0 are not in M_{CH}^T. Let q be a point lying slightly above γ_0 and near 0 such that $q \notin \mathrm{M}_{CH}^T$ (it exists since otherwise $\gamma_0^+ \subset \partial \mathrm{M}_{CH}^T$). Denote by $\tilde{p} = \tilde{p}(q)$ the first point on the backward trajectory γ_q^- starting from q such that $r(\tilde{p}) \cap \mathrm{M}_{CH}^T \neq \emptyset$, in particular $\tilde{p} \in \mathrm{M}_{CH}^T$ and therefore $\gamma_{\tilde{p}}^- \subset \mathrm{M}_{CH}^T$. This point exists since the point of intersection of γ_q^- with α has this property by definition of α.

Let $\gamma_{\tilde{p}}^q$ be the closed piece of trajectory of R joining $\tilde{p}$ and q. By definition of $\tilde{p}$, all point of the (evidently closed) set $\gamma_{\tilde{p}}^q \cap \mathrm{M}_{CH}^T$ except $\tilde{p}$ are necessarily in $\partial \mathrm{M}_{CH}^T$ and of local type. Proposition 4.7 then implies $\gamma_{\tilde{p}}^q \cap \partial \mathrm{M}_{CH}^T = \{\tilde{p}\}$: indeed, otherwise the set $\gamma_{\tilde{p}}^q \cap \mathrm{M}_{CH}^T$ cannot be closed.

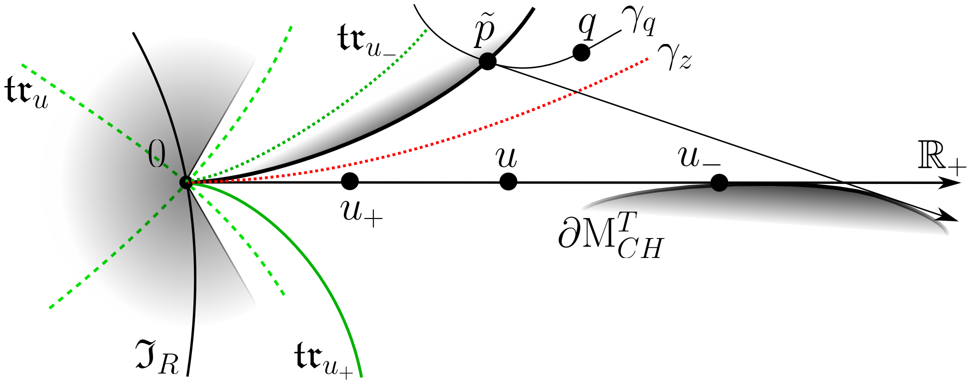

Figure 8: Bouncing type

The trajectory γ_0^+ is convex, so taking q sufficiently close to γ_0^+ we can assume that $r(q) \subset (\mathrm{M}^T_{CH})^c$ intersect γ_0^+ and therefore all trajectories of $R\partial_z$ close to and above γ_0^+. The points $\tilde p(q')$, where $q' \in r(q)$ lies above γ_0^+, form a global arc of $\partial \mathrm{M}^T_{CH}$ starting at 0 and lying between α_- and γ_0 in the upper right quadrant.

$\square$

If $\Gamma(0) \neq \emptyset$, then a local arc whose germ is contained in the upper half-plane starts at 0.

In both cases the portion of the boundary $\partial \mathrm{M}^T_{CH}$ in a neighborhood of 0 in the lower right quadrant is a global arc ending at z. Indeed, let $q \in {}_z\triangle_{z''}^{z'} \subset (\mathrm{M}^T_{CH})^c$ and denote again by $\tilde p = \tilde p(q)$ the first point on γ_q such that $r(\tilde p) \cap \mathrm{M}^T_{CH} \neq \emptyset$, in particular $\tilde p \in \mathrm{M}^T_{CH}$. As before, it lies on γ_q^- between q and $\gamma_q^- \cap \alpha$. Repeating the arguments of Lemma 5.4 we see that $\gamma_{\tilde p}^q \cap \mathrm{M}^T_{CH} = \{\tilde p\}$ and the points $\tilde p$ form a global arc of $\partial \mathrm{M}^T_{CH}$ lying in the lower half-plane and ending at z.

$\square$

5.3 Points of C^2-inflection type

Recall that a point $z \in \partial \mathrm{M}^T_{CH}$ is called a point of C^2-inflection type if $\Delta^+(z) = \emptyset$ and either $\Delta^-(z) = \emptyset$ or $\Gamma(z) \neq \emptyset$.

Proposition 5.5. *Consider a point p of $\partial \mathrm{M}^T_{CH} \setminus \mathcal{Z}(PQ)$ belonging to the transverse locus $\mathfrak{I}^*_R$. If $\Delta(p) = \emptyset$, then $\Gamma(p) \neq \emptyset$ and p is the starting point of a local arc.*

Proof. We keep the previous normalization $p = 0$, $R(0) = 1$, $\operatorname{Im} R''(0) = b > 0$. The positive trajectory γ_0^+ cuts $\mathfrak{I}^+$ into two parts, one containing the convex hull of γ_0 (denoted by Ω_{++}) and another one denoted by Ω_{+-}.

The arguments of Lemma 4.8 (using Lemma 5.2 instead of Lemma 3.23) can be repeated word by word as long as $\tilde{p} \in \mathfrak{I}^*_R \cap \partial\Omega_{++}$, see Figure 4. This proves that M^T_{CH} does not intersect Ω_{+-}. Together with $\Delta(0) = \emptyset$ this implies that some small sector $S_- = \{-\epsilon < \arg z < 0\}$ does not intersect M^T_{CH}.

Now, assume by contradiction, that there exists $q \in \Omega_{++} \setminus \mathrm{M}^T_{CH}$. Shrinking Ω_{++} if needed we can repeat the arguments of Lemma 4.9 as long as $\tilde{p} \in \mathfrak{I}^+$, see Figure 5, to conclude that $\Omega_{++} \cap \mathrm{M}^T_{CH} = \emptyset$.

Therefore there is a neighbourhood U_+ of 0 in $\mathfrak{I}^+$ which is disjoint from M^T_{CH}. We can assume that U_+ is the intersection of $\mathfrak{I}^+$ with a small disk centered at 0.

For sufficiently small $\epsilon > 0$, the set $U = U_+ \cup \{-\epsilon < \arg z < \epsilon\}$ is disjoint from M^T_{CH}; the part lying in the lower half-plane is in $U_+ \cup S_-$ and the part lying in the upper half-plane is in $U_+ \cup {}_0\triangle^{p'}_{p''} \cup {}_{p''}\angle$.

Now, take a small neighborhood U_- of 0 in Ω_- bounded by a convex curve transversal to R. For any $u \in U_-$, the ray $r(u)$, being close to R_+, lies inside $U \cup U_-$. By Lemma 3.26, this implies that $U_- \subset (\mathrm{M}^T_{CH})^c$, and therefore $0 \notin \mathrm{M}^T_{CH}$, a contradiction. Thus $\Omega_{++} \subset \mathrm{M}^T_{CH}$ and $\gamma_0 \subset \partial \mathrm{M}^T_{CH}$. □

Proposition 5.6. *Consider a point p of C^2-inflection type. Then there is a neighborhood V of p in which $\partial \mathrm{M}^T_{CH}$ is formed by:*

- *a portion of a local arc γ_p parameterized by an interval $[0, \epsilon[$, $\epsilon > 0$ with $\gamma_p(0) = p$;*
- *a portion of a global arc $\tilde{\gamma}_p$ parameterized by $[0, \epsilon[$ and such that $\tilde{\gamma}_p(0) = p$ and $\Delta(\tilde{\gamma}_p(t)) = \{\gamma_p(t)\}$.*

In particular, p *is simultaneously the starting point of a local arc and the starting point of a global arc.*

Proof. We again assume that $p = 0$ and $R(0) = 1$. Following the definition of a point of C^2-inflection type (see Theorem 1.9), we have that $\Delta^+(0) = \emptyset$ and either $\Delta^-(0) = \emptyset$ or $\Gamma(0) \neq \emptyset$. By Proposition 5.5, if $\Delta^-(0) = \emptyset$, then $\Gamma(0)$ is also nonempty. Therefore, in both cases 0 is the starting point of a local arc γ_0 and we deduce the shape of the boundary close to 0 from the assumption $\Delta^+(0) = \emptyset$.

The curve γ_0 divides the domain $\mathfrak{I}^+$ into two parts, and as before we denote the one containing a horn of 0 by Ω_{+-}.

Lemma 5.7. $\Omega_{+-} \cap \mathrm{M}_{CH}^T = \emptyset$.

Proof. At first, consider the case $R'(0) < 0$. Set $I_- := \partial\Omega_{+-} \cap \mathfrak{I}_R \setminus \{0\}$. We claim that $r(u) \cap \mathrm{M}_{CH}^T = \emptyset$ for all $u \in I_-$ sufficiently close to 0.

Set $\rho := -\left(R'(0)\right)^{-1}$. By assumption $\mathrm{M}_{CH}^T \cap \mathbb{R}_+ = \Delta^-(0)$ is a compact subset of $(\rho, +\infty]$, so $\mathrm{M}_{CH}^T \cap \mathbb{R}_+ \subset [\rho', +\infty]$, $\rho' > \rho$. Again, by closedness of M_{CH}^T and by Lemma 3.21 we can assume that for any $\epsilon > 0$ and all sufficiently small $0 < \delta' < \delta'(\epsilon)$

$$\mathrm{M}_{CH}^T \cap \{|\operatorname{Im} z| < \delta'\} \subset \{\operatorname{Re} z > \rho' - \epsilon, \operatorname{Im} z \leq 0\} \cup \{\operatorname{Re} z < \epsilon\}. \tag{5.2}$$

This means that $(\mathrm{M}_{CH}^T)^c$ contains not only the horn ${}_0\triangle_{p''}^{p'}$ but also the box

$$\Pi = \{-\delta' < \operatorname{Im} z \leq 0, \epsilon < \operatorname{Re} z < \rho + \epsilon\}$$

(we take $\epsilon < \frac{\rho' - \rho}{2}$).

For all $u \in I_-$, the slope $\sigma(u)$ is positive:

$$\operatorname{Im} R(u) = R'(0) \operatorname{Im} u + O(u^2) > 0, \tag{5.3}$$

as $\operatorname{Im} u < 0$ and $\operatorname{Re} u = O(\operatorname{Im} u)$ by transversality of $\mathfrak{I}_R$ and $r(0)$. Moreover, by (5.3) we have $\operatorname{Im}(u + tR(u)) = 0$ for $t = \rho + O(u)$, thus $r(u) \cap \mathbb{R}_+ = \rho + O(u)$. Therefore for any $\epsilon > 0$ and

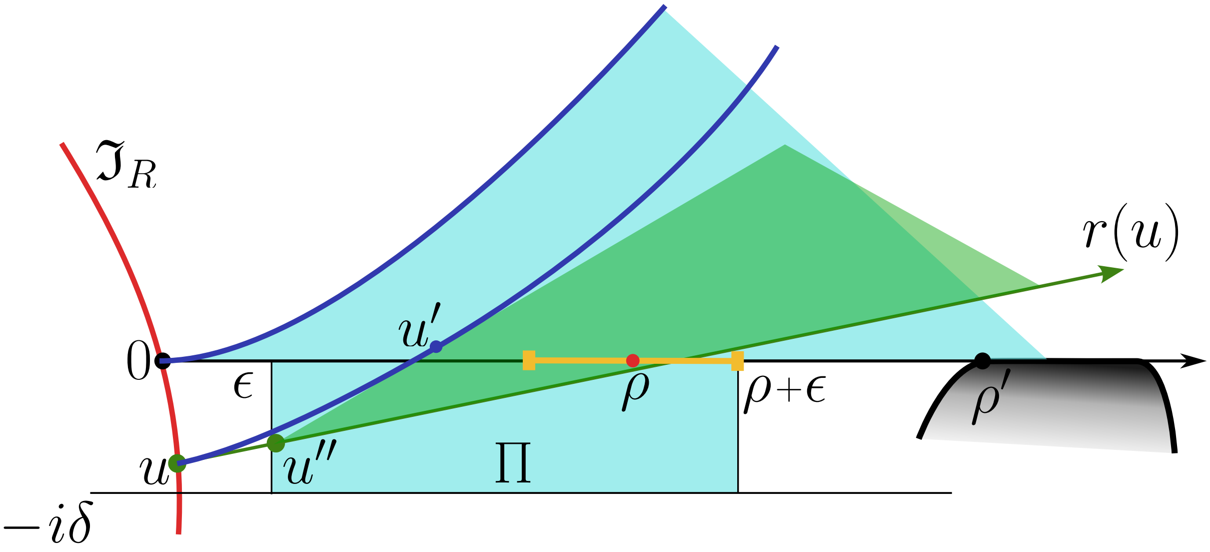

Figure 9: ${}_{u''}\angle$ lies in the complement to M^T_{CH}

for any point $u \in I_-$ sufficiently close to 0, the ray $r(u)$ has an arbitrarily small slope and $r(u) \cap \mathbb{R}_+ \in (\rho - \epsilon, \rho + \epsilon)$. Therefore

$$r(u) \cap \{\operatorname{Im} z > 0\} \subset {}_0\triangle^{p'}_{p''} \cup {}_{p''}\angle \subset (\mathrm{M}^T_{CH})^c \tag{5.4}$$

for all $u \in I_-$ sufficiently close to 0.

Let $u \in I_-$, $|\operatorname{Im} u| < \delta'$, and take $u'' \in r(u)$ with $\operatorname{Re} u'' = \epsilon$. If ϵ is sufficiently small then by Lemma 5.2 we can assume that the horn ${}_u\triangle^{u'}_{u''}$ is small.

The complementary cone ${}_{u''}\angle$ lies above $r(u)$ and to the right of $\{\operatorname{Re} z > \epsilon\}$. Thus

$${}_{u''}\angle \cap \{\operatorname{Im} z < 0\} \subset \Pi$$

and therefore ${}_{u''}\angle \cap \{\operatorname{Im} z \leq 0\} \subset (\mathrm{M}^T_{CH})^c$. Choosing ϵ sufficiently small we can assume that the angle of the sector ${}_{u''}\angle$ is small enough to conclude from (5.4) that ${}_{u''}\angle \cap \{\operatorname{Im} z > 0\} \subset (\mathrm{M}^T_{CH})^c$ as well. This implies that $u \in (\mathrm{M}^T_{CH})^c$ and therefore $\Omega_{+-} \subset (\mathrm{M}^T_{CH})^c$ as well.

If $R'(0) \geq 0$ then $\Delta^-(0) \subset \Delta^+(0) = \emptyset$, so $\Delta(0) = \emptyset$. Then the arguments of Lemma 4.8 are applicable for all $\tilde{p} \in \Omega_{+-}$, which proves the required claim in this case as well.

$\square$

Let $\tilde{\gamma}_0$ be the set of points in $\mathfrak{I}^-$ whose associated rays are tangent to the positive trajectory γ_0 of $R(z)\partial_z$ starting at 0.

Lemma 5.8. *For R as in (5.1) the local arc γ_0 is described by the relation $y(x) = \frac{b}{3}x^3 + o(x^3)$, $x \geq 0$ and the curve $\tilde{\gamma}_0$ is described by $y(x) = 5\frac{b}{3}x^3 + o(x^3)$, $x \leq 0$.*

Proof. Using (3.6) for (5.1) we see that

$$\gamma_0(t) = t + o(t) + i\left(\frac{b}{3}t^3 + O(t^4)\right)$$

with $a \in \mathbb{R}$ and therefore the slope $\sigma(\gamma_0(t)) = bt^2 + O(t^3)$.

The point $u \in \tilde{\gamma}_0$ whose associated ray $r(u)$ is tangent to γ_0 at $\gamma_0(t)$ has the form

$$u = \gamma_0(t) - sR(\gamma_0(t)) = t - s + o(t) + i\left(-sbt^2 + \frac{b}{3}t^3 + O(t^4)\right)$$

with the condition $\sigma(u) = \sigma(\gamma_0(t)) = bt^2 + O(t^3)$. The latter condition means that $s = 2t + o(t)$ and therefore

$$u = -t + o(t) - i\left(\frac{5}{3}bt^3 + O(t^4)\right).$$

☐

Lemma 5.9. *For any $u \in \tilde{\gamma}_0$, the ray $r(u)$ does not intersect $\tilde{\gamma}_0 \cup \gamma_0$ between u and the point of tangency $z = z(u)$ of $r(u)$ and γ_0.*

Proof. By Lemma 5.8, there is function $\gamma(x)$ such that $\gamma_0 \cup \tilde{\gamma}_0 = \{y = \gamma(x)\}$, with $\gamma(x) = \frac{b}{3}x^3 + o(x^3)$ for $x > 0$ and $\gamma(x) = \frac{5b}{3}x^3 + o(x^3)$ for $x < 0$. Both expressions, being power series, can be differentiated and produce asymptotic formulae for $\gamma'(x), \gamma''(x)$ as well. In particular, $\gamma''(x)$ is continuous and monotone on the interval $[\operatorname{Re}\alpha, \operatorname{Re} z]$.

Assume $r(u) \subset \{y = kx + b\}$. By construction, $\hat{\gamma} = \gamma(x) - kx - b$ vanishes at $\operatorname{Re} u$ and has a double zero at $\operatorname{Re} z$. Any other point of intersection of $r(u)$ and $\tilde{\gamma}_0 \cup \gamma_0$ will imply the existence of another zero of $\hat{\gamma}$ on $[\operatorname{Re} u, \operatorname{Re} z]$, thus $\hat{\gamma}$ will have four zeros on this interval counting multiplicities. By Rolle's Theorem this will imply the existence of two zeros of $\hat{\gamma}'' = \gamma''$ on $[\operatorname{Re} u, \operatorname{Re} z]$, which contradicts monotonicity of γ''. ☐

By Lemma 5.8 the curve $\tilde{\gamma}_0$ is tangent to $\mathbb{R}$, thus transversal to $\mathfrak{I}_R$ and divides Ω_- into two parts. Denote by Ω_{--} the closed part consisting of points whose associated rays do not intersect γ_0 and let Ω_{-+} denotes the second part. Clearly $\Omega_{-+} \subset \mathrm{M}^T_{CH}$.

Corollary 5.10. *Let* $u \in \tilde{\gamma}_0$ *and set* $u_+ := r(u) \cap \mathfrak{I}_R$ *and* $r_+(u) := r(u) \setminus \Omega_{--} = u_+ + R(u)\mathbb{R}_+$. *Then* $r_+(u) \cap \left(\mathrm{M}^T_{CH}\right)^\circ = \emptyset$.

Proof. Indeed, the piece of $r_+(u)$ between u_+ and the point of tangency $z = z(u)$ of $r(u)$ and γ_0 lies in Ω_{+-} by Lemma 5.9, and the remaining piece coincides with $r(z)$. Thus the claim follows from Lemmas 5.7 and 3.21. ☐

Lemma 5.11. *For any* $u \in \Omega_{--} \setminus \tilde{\gamma}_0$, *one has*

1. $r(u) \cap \tilde{\gamma}_0 = \emptyset$,
2. $r(u) \setminus \Omega_{--} \subset (\mathrm{M}^T_{CH})^c$.

Proof. Let γ_α^β be the piece of trajectory of $R(z)\partial_z$ containing u and with endpoints $\alpha = \alpha(u) \in \tilde{\gamma}_0$ and $\beta = \beta(u) \in \mathfrak{I}_R$: by Lemma 5.8 $\tilde{\gamma}_0$ is transversal to the trajectories of $R(z)\partial_z$ near 0 except γ_0. Let $D(u) = \cup_{z \in \gamma_\alpha^\beta} r(z)$ be the domain sweeped by the rays associated to the points of γ_α^β. Since γ_α^β is convex, $\partial D(u) = r(\alpha) \cup \gamma_\alpha^\beta \cup r(\beta)$. By Lemmas 5.9, 5.8, and 5.7, $\partial D(u) \cap \tilde{\gamma}_0 = \{\alpha\}$, so $r(u) \cap \tilde{\gamma}_0 = \emptyset$.

The boundary of $D_+(u) = D(u) \setminus \Omega_{--}$ consists of $r(\beta)$, the piece of $\mathfrak{I}_R$ lying between β and the point α_+ and $r_+(\alpha)$ which do not intersect $\mathrm{M}^T_{CH}{}^\circ$ by Lemma 5.7 and Corollary 5.10. This implies the second claim of the Lemma. ☐

Proof of Proposition 5.6. Take some $u \in \Omega_{--}^\circ$ and let $\Omega_{--}(u)$ be the curvilinear triangle bounded by γ_α^β, $\tilde{\gamma}_0$ and $\mathfrak{I}$. The ray $r(u)$ does not cross γ_α^β by convexity and does not cross $\tilde{\gamma}_0$ by Lemma 5.11. Therefore it leaves the domain $\Omega_{--}(u)$ through $\mathfrak{I}$, with $r(u) \setminus \Omega_{--}(u) \subset (\mathrm{M}^T_{CH})^c$, see Lemma 5.11. Thus $r(u) \subset \Omega_{--}(u) \cup (\mathrm{M}^T_{CH})^c$.

As $\Omega_{--}(u') \subset \Omega_{--}(u)$ for any $u' \in \Omega_{--}(u)$, this means that $r(u') \subset \Omega_{--}(u) \cup (\mathrm{M}^T_{CH})^c$ for all $u' \in \Omega_{--}(u)$, and therefore $\Omega_{--} \cap \mathrm{M}^T_{CH} = \emptyset$ by Lemma 3.26.

As $\Omega_{-+} \subset \mathrm{M}^T_{CH}$, we see that $\tilde{\gamma}_0 \subset \partial \mathrm{M}^T_{CH}$.

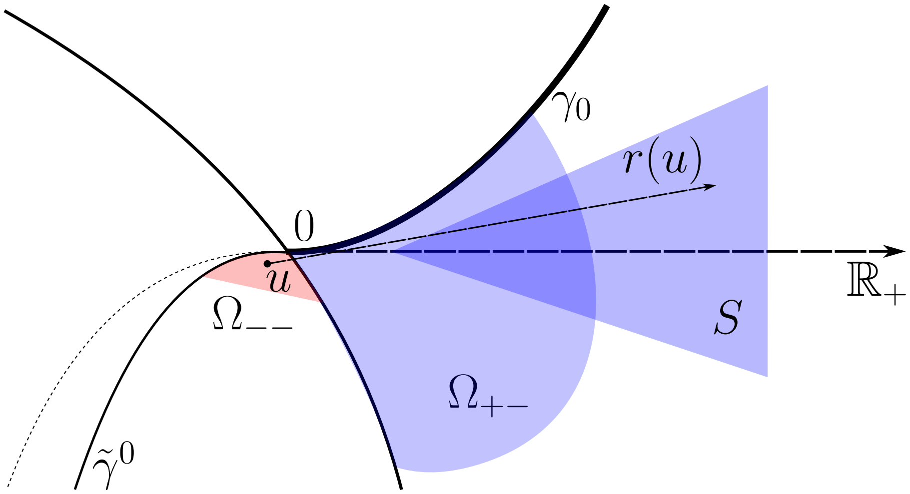

Figure 10: A point in the transverse locus of $\partial \mathrm{M}_{CH}^T \cap \mathfrak{I}_R$ with empty Δ correspondence is the starting point of a global arc.

□

□

5.4 Points of C^1-inflection type

Recall that a point $p \in \partial \mathrm{M}_{CH}^T \setminus \mathcal{Z}(PQ)$ belonging to the transverse locus $\mathfrak{I}_R^*$ is called a *point of C^1-inflection type* if $\Delta^+(p) = \Gamma(p) = \emptyset$ and $\Delta^-(p) \neq \emptyset$.

Proposition 5.12. *A point $p \in \partial \mathrm{M}_{CH}^T \cap \mathfrak{I}_R^*$ of C^1–inflection type is the starting point of two global arcs (one in each of the incident inflection domains).*

Proof. We assume that $p = 0$, $R(0) = 1$ and $\gamma_0 \subset \mathfrak{I}^+$ lies in the upper half-plane. We use $z = x + iy$ notations.

Exactly as in the case of bouncing type, the conditions $\Delta^- \neq \emptyset$ and $\Gamma = \emptyset$ imply that 0 is a starting point of a global arc, see Lemma 5.4. Denote this arc by $\eta = \{y = \xi(x)\}$. The arguments of Lemma 5.4 show that η lies between $\mathfrak{tr}_{+\infty}$ and $\mathfrak{tr}_{u_\infty}$, where $u_\infty = \sup \Delta^-$. Thus η has second order tangency with $\mathbb{R}_+$. Let $k(x)$ be the point of intersection of $r\left((x, \xi(x))\right)$

with $\mathbb{R}_+$. One can easily see that $k(x)$ is an increasing function, and $k(x) \to u_\infty$ as $x \to 0$. This means that $\lim_{x\to 0+} \xi(x)/x^2$ exists and is positive.

We repeat the arguments of the C^2-inflection case above one by one, with γ_0 replaced by η. The Lemma 5.7 can be repeated verbatim (necessarily $R'(0) < 0$), and we get $\Omega_{+-} \subset \mathrm{M}^T_{CH}$.

Let $\tilde{\eta} \subset \mathfrak{I}^-$ be the curve of points bounding (the germ at 0 of) the domain Ω_{--} consisting of points of $\mathfrak{I}^-$ whose associated rays do not intersect η. In particular, $r(u)$ is a supporting line to η for all $u \in \tilde{\eta}$. If γ_u^β is a trajectory of $R(z)\partial_z$ starting at $u \in \tilde{\eta}$ and ending at $\beta \in \mathfrak{I}_R$ then $\gamma_u^\beta \subset \Omega_{--}$ by convexity of γ_u^β, as in Lemma 5.11.

Lemma 5.13. *For any $u \in \tilde{\eta}$ the ray $r(u)$ does not intersect $\tilde{\eta}$ except at u itself.*

Proof. Note first that η lies above $r(u)$ for any $\alpha \in \tilde{\eta}$. Assume the opposite to the claim of the lemma, and let $u' \in r(u) \cap \tilde{\eta}$ be such a point of intersection. If $\sigma(u') < \sigma(u)$ then the ray $r(u')$ lies below $r(u)$ and therefore cannot intersect η. Similarly, if $\sigma(u') > \sigma(u)$ then $r(u)$ cannot intersect η. Therefore $\sigma(u') = \sigma(u)$, so u' is a point of tangency of $r(u)$ and $R(z)\partial_z$ just like u.

Now, recall that $0 \in \eta$ lies above $r(u)$. Moreover, as $u \in \mathfrak{I}^-$, the germ of $r(u)$ at u lies in Ω_{--}: for $v \in r(u)$ close to u we have $\sigma(v) < \sigma(u)$, so $r(v)$ doesn't intersect η. Thus the germ of $\tilde{\eta}$ at u lies above $r(u)$ as well. Thus both ends of the part $\tilde{\eta}_u^0$ of $\tilde{\eta}$ between u and 0 lie above $r(u)$, so $\tilde{\eta}_u^0$ intersects $r(u)$ in even number of points not including u, i.e. at least at three points including u, all of them the points of tangency of $r(\alpha)$ and $R(z)\partial_z$.

As $u \to 0$ the line $r(u)$ converges to $\mathbb{R}$ and the points of tangency necessarily converge to 0 as well. This means that $R\partial_z$ has point of tangency of order at least 3 with $\mathbb{R}$ at 0. However, $p \in \mathfrak{I}_R^*$ means that $\operatorname{Im} R''(0) \neq 0$, so the order of tangency of $R\partial_z$ with $\mathbb{R}$ at 0 is exactly 2 (as $\operatorname{Im} R(0) = \operatorname{Im} R'(0) = 0$). This contradiction proves the Lemma.

□

The same arguments as in Lemma 5.11 and in the proof of Proposition 5.6 now show that Ω_{--} satisfies the conditions of Lemma 3.26 and is therefore disjoint from M^T_{CH}. □

5.5 Points of switch type

Recall that a point $p \in \partial \mathrm{M}_{CH}^{T} \setminus \mathcal{Z}(PQ)$ belonging to the transverse locus $\mathfrak{I}_R^*$ is called a *point of switch type* if $\Delta^+(p) \neq \emptyset$ and $\Gamma(p) \cup \Delta^-(p) = \emptyset$.

Proposition 5.14. *The negative part $\gamma_0^-(t)$ of the integral curve $\gamma_0(t)$ is a part of the boundary $\partial \mathrm{M}_{CH}^{T}$.*

As before, we assume that $p = 0$, $R(0) = 1$ and $\operatorname{Im} R''(0) > 0$. The trajectory $\gamma_0(t)$ and the curve $\mathfrak{I}_R$ divide U into four domains $\Omega_{\pm,\pm}$, with Ω_{+-} intersecting the ${}_0\triangle_{p''}^{p'}$.

Lemma 5.15. *In the above notation, $\Omega_{++} \subset (\mathrm{M}_{CH}^{T})^c$.*

Proof. Recall that by Lemma 3.21 the union $H(0) = {}_0\triangle_{p''}^{p'} \cup {}_{p''}\angle$ does not intersect M_{CH}^{T}.

Assume first that $R'(0) \leq 0$ and set $\rho := -(R'(0))^{-1} \in \mathbb{R}_+ \cup \{+\infty\}$. The set $\Delta(0)$ is a relatively closed subset of $(0, \rho]$ not containing ρ (as $\Delta^-(0) = \emptyset$). Thus there exists a $\rho' < \rho$ such that the ray $[\rho', +\infty]$ is disjoint from M_{CH}^{T}. As both $[\rho', +\infty]$ and $\overline{\mathrm{M}_{CH}^{T}}$ are closed in $\mathbb{C} \cup \mathbb{S}^1$ there is some neighborhood $\widetilde{U}$ of $[\rho', +\infty]$ in $\mathbb{C} \cup \mathbb{S}^1$ disjoint from $\overline{\mathrm{M}_{CH}^{T}}$. Choose an open sector $S \subset \tilde{U}$ with the ray $(\rho', +\infty)$ as bisector, necessarily disjoint from M_{CH}^{T}.

By shrinking $\Omega_{\pm\pm}$ we can assume that $|\sigma(z)| < |\sigma(S)|$ for all $z \in \Omega_{\pm\pm}$, where $\pm\sigma(S)$ are the slopes of the sides of S. Moreover, we restrict ourselves to a neighborhood U of 0 so small that rays $r(z)$ do not intersect the interval $[0, \rho']$ for $z \in U \cap \{\operatorname{Re} z \geq 0\}$ and $z \neq 0$: this is possible since the trails of these points lie in the lower half-plane. In particular, for any $z \in \Omega_{++} \subset U \cap \{\operatorname{Re} z \geq 0\}$ the ray $r(z)$ intersects the boundary of $H(0) \cup S$ only once at $\gamma_0^+(t)$.

The same conclusion holds in the case $R'(0) > 0$. In this case $\sigma(z) > 0$ for all $z \in U$, $\operatorname{Im} z > 0$.

Since $\Gamma(0) = \emptyset$ there exists a point $q \in \Omega_{++} \setminus \mathrm{M}_{CH}^{T}$. We can now repeat the arguments of Lemma 4.9 for U: the crucial fact used in Lemma 4.9 was that $r(\tilde{p})$ doesn't intersect M_{CH}^{T}, and this holds for points of Ω_{++}. Therefore $\Omega_{++} \cap \mathrm{M}_{CH}^{T} = \emptyset$.

$\square$

Proposition 5.16. *Take $p = \gamma_0(t)$ for some $t < 0$ sufficiently close to 0. The open curvilinear triangle $H(p) \subset \Omega_{-+}$ bounded by $r(p)$, the curve $\gamma_0^-(t)$ and the inflection curve lies outside M_{CH}^T.*

Denote $x = \operatorname{Re} z$, $y = \operatorname{Im} z$.

Lemma 5.17. *One has $\frac{\partial}{\partial x}\sigma(x + iy) < 0$ as long as $z = x + iy$ lies in a sufficiently small sector $\{|z| < \delta, x < 0, |y| < -\epsilon x\}$ for some $\epsilon, \delta > 0$ depending on the rational function $R(z)$ only.*

Proof. Recall that $R(z) = 1 + az + bz^2 + ..$, with $\operatorname{Im} b > 0$. Then

$$\log R = az + \left(b - \frac{a^2}{2}\right) z^2 + \ldots$$

and

$$(\log R)' = a + (2b - a^2)z + \ldots$$

Therefore

$$\frac{\partial}{\partial x}\sigma(x + iy) = \frac{\partial}{\partial x} \operatorname{Im}(\log R) = \operatorname{Im}(\log R)' = \operatorname{Im}(2b - a^2)z + o(z) < 0$$

if $|\arg z - \pi| < \pi - \arg(2b - a^2)$ and $|z|$ is sufficiently small (note that $0 < \arg(2b - a^2) < \pi$ as $\operatorname{Im} b > 0$ and $a \in \mathbb{R}$). □

Lemma 5.18. *The associated ray $r(z)$ does not intersect $\gamma_0^-(t)$ for any $z \in H(p)$.*

Proof. First consider the case $\operatorname{Im} z < 0$. As $\gamma_0^-(t)$ is tangent to $\mathbb{R}$, we can assume that this part of $H(p)$ satisfies the conditions of Lemma 5.17, so $\sigma(z) > \sigma(z_+) > 0$, where $z_+ = z + t \in \gamma_0^-(t)$. Thus the ray $r(z)$ lies in the half-plane bounded by the line tangent to $\gamma_0^-(t)$ at z_+ and containing z. Therefore $r(z)$ does not intersect $\gamma_0^-(t)$: by convexity of $\gamma_0^-(t)$it lies in the other half-plane bounded by the line tangent to $\gamma_0^-(t)$ at z_+.

Now, assume that $\operatorname{Im} z \geq 0$. Recall that we have chosen U so small that for any $z \in U$, $\operatorname{Im} z > 0$, the intersection $r(z) \cap \mathbb{R} \subset (\rho', +\infty) \subset \mathbb{R}_+$. Thus $r(z) \cap \{\operatorname{Im} w < 0\} \subset \{\operatorname{Re} z > \rho' > 0\}$ which is disjoint from γ^-. □

Lemma 5.19. *The associated ray $r(z)$ does not intersect $r(p)$ for any $z \in H(p)$.*

Proof. Let γ_z^- be a part of the integral curve of $R\partial_z$ ending at z and let $z' \in \gamma_z^- \cap r(p)$ be the first (from z) point where γ_z^- enters $H(p)$. This point must necessarily lie on $r(p)$ because γ_z^- intersects neither the curve of inflection points $\mathfrak{I}_R$ nor γ^- by uniqueness of solutions of ODE. Thus $\sigma(z') \leq \sigma(p)$.

Assume now that the ray $r(z)$ intersects $r(p)$. Then $\sigma(z) > \sigma(p) \geq \sigma(z')$ as z lies below $r(p)$. Therefore the slope $\sigma(w)$, $w \in \gamma_z^-$, is not monotone, implying that γ_z^- has an inflection point. But this is impossible since γ_z^- does not intersect the curve of inflections. $\square$

Proof of Proposition 5.16. By minimality, it is enough to prove that $r(z) \subset H(p) \cup (\mathrm{M}_{CH}^T)^c$ for any $z \in H(p)$. By Lemmas 5.18, 5.19 the ray $r(z)$ does not intersect $r(p)$ and $\gamma_0^-(t)$. Thus $r(z)$ leaves $H(p)$ through the curve of inflections $\mathfrak{I}_R$ with a small slope.

If the slope is positive then $r(z) \setminus H(p) \subset U_{++} \cup H(0)$. In particular, this is the case for all $z \in H(p)$, $\operatorname{Im} z < 0$ by Lemma 5.17.

Assume now that $\sigma(z) < 0$ (and therefore $\operatorname{Im} z > 0$). Recall that we have chosen U so small that for any $z \in U$, $\operatorname{Im} z > 0$, the intersection $r(z) \cap \mathbb{R} \subset (s', +\infty) \subset \mathbb{R}_+$. Thus $r(z) \setminus H(p) \subset \Omega_{++} \cup H(0) \cup S$.

Taken together, $r(z) \subset H(p) \cup (\mathrm{M}_{CH}^T)^c$ for all $z \in H(p)$. Therefore $H(p) \subset (\mathrm{M}_{CH}^T)^c$ by Lemma 3.26.

$\square$

Proposition 5.20. *Consider a point $p \in \partial \mathrm{M}_{CH}^T$ of switch type. Then there is a neighborhood V of p such that $\mathrm{M}_{CH}^T \cap V$ is contained in a half-disk centered at p. Besides, no neighborhood of p in M_{CH}^T can be contained in a cone centered at p with an angle strictly smaller than π. A point of switch type is the ending point of both a local and a global arcs.*

Proof. We essentially repeat the arguments of Lemma 4.8. Take $u \in \Delta(0) \neq \emptyset$ and let $\mathfrak{tr}_u^+$ be a germ of the branch of $\mathfrak{tr}_u \subset \mathrm{M}_{CH}^T$ lying in the lower-right quadrant. For any $q \in \mathfrak{tr}_u^+$ let $z = z(q) \in \gamma_q^+$ be the last point such that $r(z) \cap \mathrm{M}_{CH}^T \neq \emptyset$. This point exists since $\gamma_q^+ \neq \emptyset$ eventually enters ${}_0\triangle_{p''}^{p'} \subset (\mathrm{M}_{CH}^T)^c$. As in Lemma 5.4, $z \in \partial \mathrm{M}_{CH}^T$ and therefore $\operatorname{Im} z < 0$, as otherwise $z \in {}_0\triangle_{p''}^{p'}$. The points $z(q)$ form a global arc ending at 0. $\square$

5.6 Classification of boundary points

Here, we summarize several results obtained in the previous sections to finalize our classification of boundary points on $\partial \mathrm{M}_{CH}^{T}$.

Proof of Theorem 1.9. It follows from Corollary 3.8 that there are at most $4 \deg P + \deg Q - 2 \leq 2d$ singular points in the curve of inflections ($d = 3 \deg P + \deg Q - 1$). Proposition 3.14 proves that the tangency locus $\mathfrak{T}_R$ is formed by at most $2d^2$ isolated points and d lines.

The classification of points in $\mathfrak{T}_R^*$ is trivial. If Δ^+ is nonempty, then a point is of bouncing or switch type depending on whether $\Gamma \cup \Delta^-$ is empty or not. If Δ^+ is empty, then a point is of C^1-inflection or C^2-inflection type depending whether the conjunction of $\Delta^- \neq \emptyset$ and $\Gamma = \emptyset$ is satisfied or not.

Finally, for points outside $\mathcal{Z}(PQ) \cup \mathfrak{T}_R$, we just have to check that Γ and Δ cannot both be empty. This is proved in Proposition 4.7. □

5.7 Estimates concerning local and global arcs

We introduce the following notations:

- $|\mathcal{L}|$ is the number of local arcs;
- $|\mathcal{G}|$ is the number of global arcs;
- $|\mathcal{B}|$ is the number of points of bouncing type;
- $|\mathcal{E}|$ is the number of points of extruding type;
- $|\mathcal{I}_1|$ is the number of points of C^1–inflection type;
- $|\mathcal{I}_2|$ is the number of points of C^2–inflection type;
- $|\mathcal{S}|$ is the number of points of switch type.

We prove that the number of points of switch type provides an estimate for the number of local arcs (up to an error term depending only on $\deg P$ and $\deg Q$).

Lemma 5.21. *In the boundary $\partial \mathrm{M}^T_{CH}$, the ending point of every local arc, except at most $d(2d+1)$ of them, is a point of switch type where $d = 3\deg P + \deg Q - 1$. Conversely every point of switch type is the endpoint of some local arc.*
In other words, we have $|\mathcal{S}| \leq |\mathcal{L}| \leq |\mathcal{S}| + d(2d+1)$.

Proof. Proposition 5.20 proves that every point of switch type is the endpoint of some local arc. It remains to list all possible endpoints for local arcs.

Following Proposition 4.12, every local arc has an endpoint that either belongs to $\mathcal{Z}(PQ)$ or to $\mathfrak{I}_R$. For any point α which is the endpoint of a local arc, $\mathcal{L}_\alpha$ contains an interval of length at most π. It follows from Corollary 3.5 that such a point is either a simple pole of $R(z)$ or a point which is neither a zero or a pole of $R(z)$. Only two local arcs can have the same simple pole as their endpoint. Any other point is the endpoint of at most one local arc (because only one integral curve passes through such a point). Consequently, at most $3\deg P + \deg Q$ local arcs have an endpoint in $\mathcal{Z}(PQ)$.

It remains to count local arcs one endpoint of which belongs to $\mathfrak{I}_R \setminus \mathcal{Z}(PQ)$. Any such point is incident to a unique integral curve which implies that it can be the endpoint of only one local arc. If such a point belongs to the transverse locus of the curve of inflections, then it is a point of switch type (see Propositions 5.3, 5.6, 5.12 and 5.20). There are $|\mathcal{S}|$ of them. Following Proposition 3.14, the tangency locus of $\mathfrak{I}_R$ is formed by at most $2d^2$ points and d straight lines where $d = 3\deg P + \deg Q - 1$. As $\arg(R(z))$ is constant on each such line, they contain at most one endpoint of a local arc. Thus $|\mathcal{L}| \leq |\mathcal{S}| + d(2d+1)$. □

Similarly, we prove an estimate on the number of global arcs that do not start at a point of the transverse locus of the curve of inflections.

Lemma 5.22. *In the boundary $\partial \mathrm{M}^T_{CH}$, the starting point of every global arc, except at most $12d + 5d^2$ of them, is either a point of C^1-inflection, C^2-inflection or of bouncing type. Besides, we have $2|\mathcal{I}_1| + |\mathcal{I}_2| \leq |\mathcal{G}| \leq |\mathcal{B}| + 2|\mathcal{I}_1| + |\mathcal{I}_2| + 12d + 5d^2$.*

Proof. Proposition 5.6 shows that every point of C^2-inflection is the starting point of a global arc while Proposition 5.12 shows that every point of C^1-inflection is the starting point of two global arcs. It follows that $2|\mathcal{I}_1| + |\mathcal{I}_2| \leq |\mathcal{G}|$.

Then, we list every possible starting point for a global arc (Lemma 4.23 proves that global arcs cannot be closed loops).

Since points of extruding type are not starting points of global arcs (see Proposition 4.30), every global arc either starts at a point at infinity or at a point of $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$.

We first count the number of global arcs which can start at infinity. If $\deg Q - \deg P = 1$, then by Theorem 2.23 we know that M_{CH}^T is compact. If $\deg Q - \deg P = -1$, then Proposition 6.9 proves that M_{CH}^T has only one connected component while its complement has two connected components. Therefore, we have at most four infinite global arcs in this case. If $\deg Q - \deg P = 0$, the complement of M_{CH}^T is connected so each connected component has at most two infinite global arcs. Following Proposition 2.20, M_{CH}^T has at most $\deg P + \deg Q$ connected components. Therefore the number of global arcs starting at infinity is at most $2 \deg P + 2 \deg Q$. In the only case where $4 > 2 \deg P + 2 \deg Q$ while $\deg Q - \deg P = -1$, $Q(z)$ is constant while $\deg P = 1$. In this case, M_{CH}^T is a straight line (see Proposition 6.7).

Let us consider the points of the transverse locus of $\mathfrak{I}_R$. Each point of C^1−inflection type is the starting point of exactly two global arcs (see Proposition 5.12). Each point of bouncing or C^2−inflection type is the starting point of exactly one global arc (see Propositions 5.3 and 5.6). No global arc starts at a point of switch type (see Proposition 5.20).

Now let us consider the tangency locus of $\mathfrak{I}_R$. It is formed by at most $2d^2$ points and $\deg P + \deg Q + 1$ R-invariant lines (see Definition 2.5). For each such line, at most 4 global arcs can have start accumulation $\omega_-(\alpha)$ belonging to it, because each line has two sides and rays have two possible directions. Otherwise, the rays starting from these global arcs intersect the interior of M_{CH}^T near the other global arcs. Using Corollary 3.6, we conclude that each of the $2d^2$ remaining points of the tangent locus is the starting point of at most two global arcs. For the same reasons, each singular point of $\mathfrak{I}_R$ that does not belong to $\mathcal{Z}(PQ)$ is the starting point of at most two global arcs. There are $4 \deg P + \deg Q - 2 \leq 2d$

such points (see Corollary 3.8).

It remains to estimate the number of global arcs that can start at a root α of $P(z)$ or $Q(z)$ in terms of the local degree m_α of $R(z)$ in α. Corollary 3.6 proves that α is the starting point of at most:

- two arcs if $m_\alpha = 0$;
- $2(1 - m_\alpha)$ arcs if $m_\alpha \leq -1$;
- $2 \deg P$ arcs if $m_\alpha \geq 1$.

Consequently, in the worst case scenario, the roots of $P(z)$ and $Q(z)$ are simple and disjoint so at most $2 \deg P(\deg P + 2)$ global arcs can start at these points.

Therefore, the number of global arcs whose starting point does not belong to $\mathfrak{I}_R^*$ is at most $(2 \deg P + 2 \deg Q) + 4(\deg P + \deg Q + 1) + 4d^2 + 4d + 2 \deg P(\deg P + 2)$.

If $\deg P = 0$, then $\deg Q = 1$ (otherwise M_{CH}^T is trivial) and M_{CH}^T is fully irregular (and has therefore no global arcs) so we can replace the obtained bound by the slightly weaker (but more practical) upper bound $12d + 5d^2$. □

5.8 Long arcs

In order to prove Theorem 1.10, we introduce a new decomposition of the boundary $\partial \mathrm{M}_{CH}^T$.

Definition 5.23. For any linear differential operator T given by (1.1), we define the *long arcs* as the maximal arcs formed by gluing local and global arcs along points of extruding or bouncing type (see Sections 4.6 and 5.2).

In particular, a long arc belongs to the closure of a unique inflection domain. Consequently, local and global arcs of a same long arc share the same orientation (see Section 4.5.2). This defines the orientation a long arc.

5.8.1 Estimates concerning long arcs

Drawing on the estimates of Section 5.7, we prove that the number of long arcs corresponds to the number of intersections between $\partial \mathrm{M}_{CH}^{T}$ and the transverse locus of $\mathfrak{I}_R$ that are not of bouncing type (in other words, where the boundary of the minimal set crosses the curve of inflections).

Lemma 5.24. *Every long arc except at most $28d^2+52d$ of them goes from a point of switch type to a point of C^1-inflection or C^2-inflection type. The number $|\mathcal{A}|$ of long arcs satisfies the following inequalities:*

$$2|\mathcal{S}| \leq |\mathcal{A}| \leq 2|\mathcal{S}| + 26d + 14d^2;$$

$$2|\mathcal{I}_1| + 2|\mathcal{I}_2| \leq |\mathcal{A}| \leq 4|\mathcal{I}_1| + 2|\mathcal{I}_2| + 26d + 14d^2.$$

Proof. Inequality $2|\mathcal{S}| \leq |\mathcal{A}|$ follows from the fact that every point of switch type is the ending point of two long arcs. Similarly, points of C^1-inflection or C^2-inflection type are the starting points of two long arcs so we obtain $2|\mathcal{I}_1| + 2|\mathcal{I}_2| \leq |\mathcal{A}|$.

We denote by $|\mathcal{A}_L|$ the number of long arcs containing a local arc. We already know that the endpoint of a local arc cannot be a point of extruding or bouncing type. Therefore, the endpoint of a local arc contained in a long arc is also the endpoint of the long arc. We deduce then from Lemma 5.21 that the endpoints of all but at most $d(2d+1)$ these long arcs containing a local arc are points of switch type: $|\mathcal{A}_L| \leq |\mathcal{S}| + d(2d+1)$.

Then, denote by $|\mathcal{A}_G|$ the number of long arcs that do not contain a local arc. The start accumulations of these long arcs are in particular start accumulations of a global arc and cannot be points of bouncing type. We deduce from the proof of Lemma 5.22 that the start accumulations of these long arcs, except at most $12d+5d^2$ of them are in fact starting points and they are points of C^1-inflection or C^2-inflection type. In other words, we have $|\mathcal{A}_G| \leq 2|\mathcal{I}_1| + |\mathcal{I}_2| + 12d + 5d^2$.

Finally, we deduce that the total number of long arcs satisfies $|\mathcal{A}| \leq |\mathcal{S}| + 2|\mathcal{I}_1| + |\mathcal{I}_2| + 13d + 7d^2$. Combining this inequality with $2|\mathcal{S}| \leq |\mathcal{A}|$, we obtain that $|\mathcal{S}| \leq 2|\mathcal{I}_1| +$

$|\mathcal{I}_2| + 13d + 7d^2$ and therefore $|\mathcal{A}| \leq 4|\mathcal{I}_1| + 2|\mathcal{I}_2| + 26d + 14d^2$. We obtain similarly that $|\mathcal{A}| \leq 2|\mathcal{S}| + 26d + 14d^2$. This also provides a bound on the number of long arcs that do not go from a point of switch type to a point of C^1-inflection or C^2-inflection type. $\square$

5.8.2 Admissible long arcs

We will refer to a long arc going from a point of switch type to a point of C^1-inflection or C^2-inflection type (or the opposite) as an *admissible long arc*.

Definition 5.25. We associate to each admissible long arc α a combinatorial symbol s_α that contains the following information:

- the connected component of the transverse locus $\mathfrak{I}_R^*$ containing the starting point of α;
- the connected component of $\mathfrak{I}_R^*$ containing the endpoint of α;
- the sign of the inflection domain α belongs to.

Chains of consecutive long arcs define *patterns* formed by the concatenation of the combinatorial symbols of their long arcs.

Chains of consecutive long arcs are given the orientation induced by the topological orientation of M_{CH}^T. Therefore, the orientation of a chain coincides with the orientation of the long arc inside positively oriented domains of inflection (and does not coincide with the orientation of the long arc inside negatively oriented domains of inflections).

Remark 5.26. In particular, at a point z of switch, C^1-inflection or C^2-inflection type in a chain, the associated ray $r(z)$ and the orientation of the chain points towards the same domain of inflection.

5.8.3 Bounding the number of transverse intersection points between $\partial \mathrm{M}_{CH}^{T}$ and the transverse locus of $\mathfrak{I}_R$

Using the fact that an associated ray cannot cross the curve of inflections more than d times (where $d = 3\deg P + \deg Q - 1$), we prove a bound on the number of chains of admissible long arcs that can realize a given pattern.

Lemma 5.27. *In the boundary $\partial \mathrm{M}_{CH}^{T}$ of the minimal set, there cannot be $2d+2$ disjoint chains of $2d$ admissible long arcs that realize the same pattern.*

Proof. We assume by contradiction the existence of $2d+2$ disjoint chains $\gamma_1, \dots, \gamma_{2d+2}$ of $2d$ admissible long arcs realizing the same pattern. We refer to the admissible long arc of γ_i corresponding to the j^{th} symbol as $\alpha_{i,j}$.

By definition of the combinatorial symbol, for a given j, the arcs $\alpha_{i,j}$ for $1 \le i \le 2d+2$ belong to the same inflection domain $\mathcal{D}_j$.

We denote by $\beta_1, \dots, \beta_{2d+1}$ the connected components of the transverse locus $\mathfrak{I}_R^*$ at the endpoints of long arcs ordered according to the orientation of the chains.

We are going to prove the existence of a point z of β_{d+1} such that its associated ray $r(z)$ also intersects d arcs among $\beta_1, \dots, \beta_{2d+1}$, defining a straight line that intersects transversely a real algebraic curve of degree at most d in at least $d+1$ points, obtaining the desired contradiction.

A first observation is that the chains $\gamma_1, \dots, \gamma_{2d+2}$ cannot cross each other (because the interior of M_{CH}^{T} is connected near endpoints of admissible arcs). Since the complex plane is simply connected, this fact implies that for a given j, the intersection points of the chains $\gamma_1, \dots, \gamma_{2d+2}$ with β_j determine a cyclic order that does not depend on j. We assume therefore that the indices of $\gamma_1, \dots, \gamma_{2d+2}$ are elements of $\mathbb{Z}/(2d+2)\mathbb{Z}$ and correspond to the previously defined cyclic ordering.

In a given inflection domain $\mathcal{D}_j$, it may happen that the linear orders of these intersection points with β_j and β_{j+1} respectively are different. They may differ by a rotation if β_j and β_{j+1} do not belong to the same connected components of the boundary of $\partial\mathcal{D}_j$ (if

$\mathcal{D}_j$ is not simply connected).

It follows that for any $1 \leq j \leq 2d$, for every $k \in \mathbb{Z}/(2d+2)\mathbb{Z}$ except possibly one, there is a quadrilateral in $\mathbb{C}$ bounded by $\alpha_{j,k}$, $\alpha_{j,k+1}$, one portion of β_j and one portion of β_{j+1}. The exception correspond to the case where the linear orderings do not match. Since we have $2d+2$ chains, it follows that we can assemble $2d$ of these strips into a unique long strip $\mathcal{S}$ bounded by some γ_k, γ_{k+1} and portions of β_0 and β_{2d+1}.

The chains γ_k and γ_{k+1} are oriented in such a way that for any point $z \in \partial\mathcal{S} \cap \beta_0$, associated ray $r(z)$ points inside $\mathcal{S}$. Since the orientation of γ_k and γ_{k+1} coincides with the topological orientation of M_{CH}^T, we can find a point z of $\partial\mathcal{S} \cap \beta_{d+1}$ in the complement of M_{CH}^T. Thus, for any such point z, associated ray $r(z)$ cannot cross chains γ_k and γ_{k+1} and has to leave $\mathcal{S}$ through β_0 or β_{2d+1}. Therefore, $r(z)$ has to cross either $\beta_1, \dots, \beta_{d+1}$ or $\beta_{d+1}, \dots, \beta_{2d+1}$. This ends the proof. □

We deduce a bound on the number of long arcs.

Corollary 5.28. *For any linear differential operator T given by* (1.1), *the number $|\mathcal{A}|$ of long arcs in $\partial\mathrm{M}_{CH}^T$ satisfies*

$$|\mathcal{A}| \leq 2e^{16d\ln(d)} + 92d^3$$

where $d = 3\deg P + \deg Q - 1$.

Proof. Since the number of connected components of the transverse locus $\mathfrak{I}_R^*$ is at most $2d^2+6d+2$ (see Corollary 3.15), the number of possible combinatorial symbols for an admissible long arc is $2(2d^2+6d+2)^2$ (see Definition 5.25). Therefore, the number of possible patterns for a chain of $2d$ admissible long arcs is at most $2^{2d}(2d^2+6d+2)^{4d}$. Since $d \geq 3$ (see Remark 2.4), we have $2d^2+6d+2 \leq \frac{5}{2}d^2$ and we obtain the weaker (but simpler) upper bound $(25/2)^{2d}d^{8d}$.

Then, using Lemma 5.27, we deduce that the number of disjoint chains of $2d$ admissible long arcs is at most $(25/2)^{2d}(2d+1)d^{8d}$.

The number of non-admissible long arcs is bounded by $28d^2+52d$ (Lemma 5.24). It follows that in the worst case, each non-admissible arc is located between two chains of

$2d-1$ admissible long arcs. Consequently, the number of long arcs is bounded by

$$(25/2)^{2d}(2d)(2d+1)d^{8d} + (28d^2+52d) + (28d^2+52d+1)(2d-1).$$

Since $d \geq 3$, this upper bound can be weakened to $2e^{16d\ln(d)} + 92d^3$. □

Theorem 1.10 then follows from the fact that $|\mathcal{S}|$ and $2|\mathcal{I}_1| + |\mathcal{I}_2|$ are bounded by the number of long arcs (see Lemma 5.24). Corollary 1.11 then follows from the combination of Theorem 1.10 with Lemma 5.21.

6 Global geometry of minimal sets

At present we do not know a general recipe how to describe non-trivial M_{CH}^{T}. Nevertheless we can prove some general statements about their global geometry and provide some illuminating examples.

Recall that M_{CH}^{T} is nontrivial if and only if $\deg Q - \deg P \in \{-1,0,1\}$.

In some cases, description of the convex hull $Conv(\mathrm{M}_{CH}^{T})$ is easier to obtain. The following has been proved as Corollary 5.16 in [AHN+24].

Proposition 6.1. *Consider a linear differential operator T given by* (1.1)*.Then the intersection of all convex Hutchinson invariant set coincides with the convex hull $Conv(\mathrm{M}_{CH}^{T})$ of the minimal set M_{CH}^{T}.*

The local analysis of boundary points carried on in the previous sections provides interesting partial results towards a characterization of points where $\partial\mathrm{M}_{CH}^{T}$ is locally convex.

6.1 Local convexity of the boundary

Local analysis in terms of correspondences Γ and Δ shows that corner points of M_{CH}^{T} have to satisfy very specific conditions.

Corollary 6.2. *For a linear differential operator T given by (1.1), consider a point α which is a corner point of the boundary $\partial \mathrm{M}^T_{CH}$. In other words, there is a neighborhood V of α such that $V \cap \mathrm{M}^T_{CH}$ is contained in a cone with apex α and with the opening strictly smaller than π. Then one of the following statements hold:*

- *α is a simple zero of $R(z)$ satisfying $\phi_\alpha = 0$ (see 3.1);*
- *α is a common root of $P(z)$ and $Q(z)$ of the same multiplicity (i.e. α is neither a zero nor a pole of $R(z)$).*

Besides, if α is a cusp (neighborhoods of α in M^T_{CH} can be included in cones of arbitrarily small opening angle), then one of the following statements holds:

- *α is a common root of $P(z)$ and $Q(z)$ of the same multiplicity;*
- *M^T_{CH} is fully irregular and contained in a half-line.*

Proof. Corollary 3.5 immediately implies that α cannot be a pole or a multiple zero of $R(z)$. Besides, if α is a simple zero, it has to satisfy the condition $\phi_\alpha = 0$. Now we assume that α is neither a zero nor a pole of $R(z)$. It remains to prove that $\alpha \in \mathcal{Z}(PQ)$.

Assume that $\alpha \notin \mathcal{Z}(PQ)$. In this case if some point of the forward trajectory of $R(z)\partial_z$ starting at α belongs to M^T_{CH}, then a germ of the integral curve starting at α is contained in M^T_{CH} (see Proposition 2.10) and α cannot be a corner point. We conclude that $\Gamma(\alpha) = \emptyset$.

If $\Delta(\alpha)$ contains some point y, then a branch of the root trail $\mathfrak{tr}_y$ containing α belongs to M^T_{CH} (see Lemmas 2.11 and 2.16). Thus in this case α cannot be a corner point and we get $\Delta(\alpha) = \emptyset$.

Now we take a cone $\mathcal{C}$ with apex at α, of angle at least π which is locally disjoint from M^T_{CH}. If $r(\alpha)$ is contained in $\mathcal{C}$, but is not one of the two limit rays, we can freely remove a neighborhood of α from M^T_{CH} and still get an invariant set. In any other case, we can find an arc contained in a neighborhood of α and the complement of M^T_{CH} whose associated rays sweep out a domain containing α. Thus we get a contradiction in this case as well which implies that α has to be in $\mathcal{Z}(PQ)$.

Finally if α is a simple zero of $R(z)$ and a cusp, then we have $\mathcal{L}_\alpha = \mathbb{S}^1$ (see Definition 3.2). It follows that M_{CH}^T has empty interior. All such cases have been completely classified in Section 7 of [AHN+24]. □

Further local analysis provides necessary conditions under which boundary points belong to locally convex parts of $\partial\mathrm{M}_{CH}^T$.

Proposition 6.3. *For a linear differential operator T given by* (1.1)*, consider a point $\alpha \in \partial\mathrm{M}_{CH}^T$ such that there is a neighborhood V of α with the property that $V \cap \mathrm{M}_{CH}^T$ is contained in a closed half-plane whose boundary contains α.*

If $\alpha \in \mathcal{Z}(PQ)$, then one of the following statements holds:

- *α is a simple pole of $R(z)$;*
- *α is a simple zero of $R(z)$ satisfying $\phi_\alpha = 0$ (see 3.1);*
- *α is a common root of $P(z)$ and $Q(z)$ of the same multiplicity (i.e. α is neither a zero nor a pole of $R(z)$).*

If $\alpha \in \mathfrak{I}_R^* \setminus \mathcal{Z}(PQ)$, then α is a point of switch type.
If $\alpha \notin \mathfrak{I}_R \cup \mathcal{Z}(PQ)$, then one of the following statements holds:

- *α is a point of local type;*
- *α is a point of global type and for any $u \in \Delta(\alpha)$, either $\mathrm{Im}(f(u,\alpha)) = 0$ or $\mathrm{Im}(f(u,\alpha))$ and $\mathrm{Im}(R'(\alpha))$ have opposite signs (for f defined as in Proposition 2.18).*

Proof. The case $\alpha \in \mathcal{Z}(PQ)$ follows from Corollary 3.5. If $\alpha \in \mathfrak{I}_R^* \setminus \mathcal{Z}(PQ)$ and $\Delta^-(\alpha) \neq \emptyset$, then M_{CH}^T contains both the germ of an integral curve of the field $-R(z)\partial_z$ at α and the germ of the root trail $\mathfrak{tr}_u$ for some $u \in \Delta^-(\alpha)$. Proposition 2.19 implies that M_{CH}^T cannot be convex at α. Besides, if $\Gamma(\alpha) \neq \emptyset$, then M_{CH}^T cannot be convex in α either because a germ of an integral curve having an inflection point at α is contained in M_{CH}^T. In the remaining cases, we have $\Gamma(\alpha) \cup \Delta^-(\alpha) = \emptyset$. If $\Delta^+(\alpha) \neq \emptyset$, this characterizes points of switch type (see

Theorem 1.9). If $\Delta^+(\alpha) = \emptyset$, then we obtain a point of C^2-inflection type, α is the starting point of a local arc and $\Gamma(\alpha)$ is therefore nonempty (see Proposition 5.5).

Now we consider the case $\alpha \notin \mathfrak{I}_R \cup \mathcal{Z}(PQ)$. If $\Gamma(\alpha) \neq \emptyset$, then α is a point of local type (α cannot be a point of extruding type because of Proposition 4.30). If $\Gamma(\alpha) = \emptyset$, then it follows from Proposition 4.7 that $\Delta(\alpha) \neq \emptyset$. Proposition 2.18 then provides the necessary condition. □

6.2 Case $\deg Q - \deg P = -1$

We have a rational vector field $R(z)\partial_z$ satisfying $R(z) = \frac{\lambda}{z} + \frac{\mu}{z^2} + o(1/z^2)$ with $\lambda \in \mathbb{C}^*$ and $\mu \in \mathbb{C}$.

6.2.1 Horizontal locus and special line

We define the following loci.

Definition 6.4. The *horizontal locus* $\mathcal{H}_R$ is the closure in $\mathbb{C}$ of the set formed by points $z \notin \mathcal{Z}(PQ)$, for which $\sigma(z) = \frac{\arg(\lambda)\pm\pi}{2}$.

We also denote by $\mathcal{L}_R$ the *special line* formed by points z given by the equation $Im(z/\lambda) = Im(\mu/\lambda^2)$.

For the sake of simplicity, the vector field $R(z)\partial_z$ is normalized by an affine change of variable as $R(z) = -\frac{1}{z} + o(z^{-2})$ ($\lambda = -1$ and $\mu = 0$). The line $\mathcal{L}_R$ then coincides with the real axis $\mathbb{R}$.

Lemma 6.5. *$\mathcal{H}_R$ is a real plane algebraic curve of degree at most $\deg P + \deg Q$. It has two asymptotic infinite branches. The line $\mathcal{L}_R$ is the asymptotic line for both of them.*

Proof. Curve $\mathcal{H}_R$ can be seen as the pull-back of the real axis under the mapping $R(z) : \mathbb{CP}^1 \to \mathbb{CP}^1$. We have $R(\infty) = 0$ and ∞ is a simple root of $R(z)$. Therefore, $\mathcal{H}_R$ is smooth near ∞.

It remains to show that the tangent line to $\mathcal{H}_R$ at infinity coincides with the real axis. Actually the tangent line is the line at which the linearization of $R(z)$ at ∞ attains real zeroes. Since this linearization is exactly $-\frac{1}{z}$, the result follows. □

Corollary 6.6. *The closure* $\overline{\mathrm{M}_{CH}^{T}}$ *of the minimal set in the extended plane contains asymptotic directions* 0 *and* π*. Besides, the curve* $\mathcal{H}_R$ *is contained in the minimal set* M_{CH}^{T}*.*

Proof. Looking at separatrices of the vector field $R(z)\partial_z$ and using Proposition 2.10 we get that the closure $\overline{\mathrm{M}_{CH}^{T}}$ in the extended plane contains asymptotic directions 0 and π. The associated rays of points of $\mathcal{H}_R$ are thus asymptotically tangent to M_{CH}^{T} and $\mathcal{H}_R$ is contained in the minimal set. □

Proposition 6.7. *Consider a linear differential operator* T *given by* (1.1) *such that* $\deg Q - \deg P = -1$*. Then the minimal convex Hutchinson invariant set* $Conv(\mathrm{M}_{CH}^{T})$ *is a bi-infinite strip (domain bounded by two parallel lines).*

More precisely, $Conv(\mathrm{M}_{CH}^{T})$ *is the smallest strip containing* $\mathcal{H}_R \cup \mathcal{Z}(PQ)$*.*

Proof. The minimal convex Hutchinson invariant set $Conv(\mathrm{M}_{CH}^{T})$ is the complement of the union of every open half-plane disjoint from M_{CH}^{T}. Since $\mathcal{H}_R$ is contained in M_{CH}^{T} (Corollary 6.6), these open half-planes have to be disjoint from $\mathcal{H}_R$. Conversely, any open half-plane H disjoint from $\mathcal{H}_R$ is such that $Im(R(z))$ is either positive or negative for every $z \in H$. Therefore, provided H does not contain any zero or pole of $R(z)$, one can conclude that it can be removed from any Hutchinson invariant set. In other words, $Conv(\mathrm{M}_{CH}^{T})$ is the complement to the union of all half-planes disjoint from $\mathcal{H}_R \cup \mathcal{Z}(PQ)$. Since $\mathcal{H}_R$ has asymptotically horizontal infinite branches, the boundary line of every half-plane disjoint from $\mathcal{H}_R$ has to be horizontal.

It remains to prove that such half-planes exist. It follows from the asymptotic description of $\mathcal{H}_R$ in Lemma 6.5 that $|Im(z)|$ is bounded on $\mathcal{H}_R$. Therefore we can find two (disjoint) open half-planes that are also disjoint from $\mathcal{H}_R$. These half-planes contain half-planes which, in addition, are disjoint from $\mathcal{Z}(PQ)$. □

6.2.2 Asymptotic geometry of the minimal set

Following Proposition 6.7, $Conv(\mathrm{M}_{CH}^T)$ is the smallest horizontal strip containing the curve $\mathcal{H}_R \cup \mathcal{Z}(PQ)$. The closure of the projection of $Conv(\mathrm{M}_{CH}^T)$ on the vertical axis is an interval $[y^-, y^+]$ where $y^- \leq 0 \leq y^+$.

Lemma 6.8. *For $0 < y < y_0$, denote by M_t the intersection point between the associated ray $r(t+iy)$ and the horizontal line $Im(z) = y_0$. Then the following statements hold:*

- *for $t \longrightarrow +\infty$, $\mathrm{Re}(M_t) \longrightarrow -\infty$ if $y < \frac{y_0}{2}$;*
- *for $t \longrightarrow +\infty$, $\mathrm{Re}(M_t) \longrightarrow +\infty$ if $\frac{y_0}{2} < y < y_0$.*

Analogous statements hold for $t \longrightarrow -\infty$ or $y_0 < y < 0$.

Proof. For large values of t, we have $\mathrm{Re}(R(z)) = -\frac{1}{t} + o(t^{-1})$ and $\mathrm{Im}(R(z)) = \frac{y}{t^2} + o(t^{-2})$. Provided t is large enough, $\mathrm{Im}(R(z))$ is positive and the associated ray $r(z)$ intersects the line $Im(z) = y_0$. Then the real part of the intersection point equals $t - (y_0 - y)\frac{t}{y} + o(t)$. After simplification, we obtain $\frac{(2y-y_0)t}{y} + o(t)$. The sign of the main term is then determined by the sign of $2y - y_0$. □

Proposition 6.9. *If $\deg Q - \deg P = -1$, the minimal set M_{CH}^T is connected in $\mathbb{C}$.*

Proof. Following Proposition 2.24, the complement $(\mathrm{M}_{CH}^T)^c$ of M_{CH}^T in $\mathbb{C}$ has exactly two connected components and it has been proved in Proposition 6.7 that each of them contains a half-plane. We refer to the domain containing an upper half-plane as $\mathcal{D}^+$ and to the domain containing a lower half-plane as $\mathcal{D}^-$. Since M_{CH}^T contains $\mathcal{H}_R$, we deduce that $\mathrm{Im}(R(z))$ is positive on $\mathcal{D}^+$ and negative on $\mathcal{D}^-$.

Proving that M_{CH}^T is connected in $\mathbb{C}$ amounts to showing that $\mathcal{D}^+$ and $\mathcal{D}^-$ have only one topological end. We will prove this statement for $\mathcal{D}^+$ (the proof for $\mathcal{D}^-$ is identical). We assume by contradiction that $\mathcal{D}^+$ has a topological end $\varkappa$ distinct from the end of the upper half-plane contained in $\mathcal{D}^+$ (we will refer to this end as the *main end* of $\mathcal{D}^+$).

For any sequence $\{z_n\}$ of points in $(\mathrm{M}_{CH}^T)^c$ approaching $\varkappa$, we have (up to taking a subsequence) the sequence $\{\arg(z_n)\}$ converging either to 0 or to π (since otherwise, $\varkappa$

would not be distinct from the main end). Let's assume without loss of generality that it is 0. Again, we can assume that $\{Im(z_n)\}$ converges to some value $y_e \in [0, y^+]$.

If $y_e > 0$, then Lemma 6.8, shows that for any horizontal line L_f with $y_f \in]y_e, 2y_e[$, the associated rays of the points in ${\mathrm{M}_{CH}^T}^c$ converging to the end $\varkappa$ sweep out points of L_f whose real part is arbitrarily close to $+\infty$. Assuming that y_e is the maximal possible limit value, we deduce that no infinite component of M_{CH}^T can separate $\varkappa$ from the upper main end containing asymptotic directions of $]0, \pi[$.

Hence, for a sequence $\{z_n\}$ of points in $(\mathrm{M}_{CH}^T)^c$ approaching $\varkappa$, the only accumulation value of $\{Im(z_n)\}$ is 0. In this case, the associated rays $r(z_n)$ accumulate onto the $\mathbb{R}$-axis which is therefore contained in the closure of $\mathcal{D}^+$.

Now we prove that the open upper half-plane defined by $\mathrm{Im}(z) > 0$ is disjoint from M_{CH}^T. We assume by contradiction the existence of a point z_0 such that $y_0 = \mathrm{Im}(z_0)$ is positive and $z_0 \in \mathrm{M}_{CH}^T$. We denote by L_{y_0} the horizontal line formed by points satisfying $\mathrm{Im}(z) = y_0$. Since there is a family of associated rays accumulating onto the $\mathbb{R}$-axis, there exists a path $(t + if(t))_{t\in\mathbb{R}}$ such that for any t, $f(t) \in]0, \frac{y_0}{4}[$ and $t + if(t) \in \mathcal{D}^+$. Applying Lemma 6.8 to the intersection between L_{y_0} and the family of associated rays starting from $t + if(t)$, a continuity argument proves that z_0 belongs to some associated ray of the family (as t moves from $-\infty$ to $+\infty$, the intersection of the associated rays with L_{y_0} moves from the right end to the left end of this horizontal line). Therefore, z_0 cannot belong to M_{CH}^T and the open upper half-plane defined by $\mathrm{Im}(z) > 0$ is disjoint from M_{CH}^T.

Then, there are interior points of connected a component X of M_{CH}^T located above $\varkappa$ whose imaginary value is negative. It follows that the associated rays of points of $\mathcal{D}^+$ approaching $\varkappa$ intersect the interior of X (these associated rays accumulate on the $\mathbb{R}$-axis). Therefore, there is no such end $\varkappa$ and M_{CH}^T is connected. □

Proposition 6.10. *There is a compact set K and a positive constant $B > 0$ such that the intersection $\mathrm{M}_{CH}^T \cap K^c$ is contained in the closure of the domain bounded by the hyperbolas* *given by*

$$y = \frac{y^+}{2}\left(1 + \frac{B}{x}\right), \quad y = \frac{y^-}{2}\left(1 + \frac{B}{x}\right) : \quad x > 0 \tag{6.1}$$

$$y = \frac{y^-}{2}\left(1 - \frac{B}{x}\right), \quad y = \frac{y^+}{2}\left(1 - \frac{B}{x}\right) : \quad x < 0. \tag{6.2}$$

Proof. By Lemma 6.5 for any y in $J = [y^-, \frac{y^-}{2}[\cup]\frac{y^+}{2}, y^+]$, there is a positive constant $A > 0$ such that the union of the two semi-infinite horizontal strips characterized by $\mathrm{Im}(z) \in J$ and $|\mathrm{Re}(z)| > A$ is disjoint from $\mathcal{H}_R$.

Consider some positive number $B > A$ and introduce the domain D_B characterized by the inequalities:

- $\mathrm{Im}(z) > y^+$ if $\mathrm{Re}(z) \in [-B, B]$;
- $\mathrm{Im}(z) > g(t)$ where $g(t) = \frac{y^+}{2}\frac{|t|+B}{|t|}$ if $t = \mathrm{Re}(z) \notin [-B, B]$.

For any point z such that $\mathrm{Im}(z) > y^+$, the associated ray $r(z)$ remains in D_B. Now we assume that $z = t + iy$ satisfies the conditions

$$|t| > B \quad \text{and} \quad \frac{y^+}{2}\frac{|t| + B}{|t|} < |y| \leq y^+.$$

Without loss of generality, we assume that $t < -B$.

In order to prove that the associated ray $r(z)$ remains in D_B, we have to show that for any $t < -B$ and any $s \in [t, -B]$, we have

$$\frac{\mathrm{Im}(R(z))}{\mathrm{Re}(R(z))} > \frac{g(s) - g(t)}{s - t}.$$

Since $\frac{g(s)-g(t)}{s-t} \leq \frac{By^+}{2st} \leq -\frac{y^+}{2t}$, we just have to prove that

$$\frac{\mathrm{Im}(R(z))}{\mathrm{Re}(R(z))} > -\frac{y^+}{2t}.$$

In our case $\mathrm{Re}(R(z)) = -\frac{1}{t} + o(t^{-2})$ and $\mathrm{Im}(R(z)) = \frac{y}{t^2} + o(t^{-3})$ imply that

$$\frac{\mathrm{Im}(R(z))}{\mathrm{Re}(R(z))} = -\frac{y}{t} + o(t^{-2}).$$

Since $y - \frac{y^+}{2} > \frac{y^+B}{2|t|} > 0$, the inequality holds provided B is large enough.

By replacing y^+ by y^-, we get an analogous result for the lower part of the complement to M^T_{CH}. □

6.2.3 Examples

Consider a family of operators of the form $T_\alpha = Q(z)\frac{d}{dz} + P(z)$ where $Q(z) = (z-\alpha)^k$ and $P(z) = z(\alpha - z)^k$ with the common root $\alpha \in \mathbb{C}$ of degree $k \in \mathbb{N}^*$.

The family T_α provides a rich assortment of examples. We have $R(z) = -\frac{1}{z}$. The special line is the real axis $\mathbb{R}$ which coincides with the horizontal locus $\mathcal{H}_R$. Besides, the integral curves of $R(z)\partial_z$ are hyperbolas (level sets of xy).

Proposition 6.11. *If $\alpha \in \mathbb{R}$, then the minimal set M_{CH}^T of operator T_α coincides with the real axis $\mathbb{R}$.*

Proof. This follows immediately from Proposition 6.7 and Proposition 6.9. □

If α does not belong to the real axis, we get different pictures depending on whether or not α belongs to the imaginary axis. Without loss of generality, we will assume that $Im(\alpha) > 0$.

Proposition 6.12. *If α is of the form $y_0 i$ with $y_0 > 0$, then the minimal set M_{CH}^T is the union of the segment $[\frac{y_0}{2}i, y_0 i]$ with the horizontal strip formed by points z satisfying $0 \leq Im(z) \leq \frac{y_0}{2}$.*

Proof. From Proposition 6.7 it follows immediately that the convex hull of M_{CH}^T is contained in the strip bounded by $\mathbb{R}$ and the horizontal line $Im(z) = Im(y_0)$. For any point of segment $[0, y_0 i]$, the associated ray contains α so $[0, y_0 i] \subset \mathrm{M}_{CH}^T$.

For any point of the horizontal strip given by the inequalities $0 \leq Im(z) \leq \frac{y_0}{2}$, a simple computation proves that its associated ray intersects the segment $[0, y_0 i]$.

Finally, for any point z such that $Im(z) > \frac{y_0}{2}$ and $Re(z) \neq 0$, the associated ray is disjoint from the segment $[0, y_0 i]$. This completely characterizes the minimal set. □

The latter case provides an example of a partially irregular minimal set whose irregularity locus is contained in a R-invariant line (the imaginary axis in this case).

In the general case, the boundary of M_{CH}^T is more complicated. Up to conjugation, we can restrict us to the case when $\mathrm{Re}(\alpha), \mathrm{Im}(\alpha) > 0$.

Proposition 6.13. *If α is of the form $x_0 + y_0 i$ with $x_0, y_0 > 0$, then the minimal set M_{CH}^T of T_α is bounded by the following arcs:*

- *the real $\mathbb{R}$-axis ;*
- *global arc $(t, f_1(t))$ where $f_1(t) = \frac{y_0 t}{2t - x_0}$ for $t \in [x_0, +\infty[$;*
- *local arc $(t, f_2(t))$ where $f_2(t) = \frac{x_0 y_0}{t}$ for $t \in [x_0, x_e]$;*
- *global arc $(t, f_3(t)$ where $f_3(t) = \frac{x_0 y_0 t}{(2\sqrt{x_0 t} + x_0)^2}$ for $t \in [0, x_e]$;*
- *global arc $(t, f_4(t))$ where $f_4(t) = \frac{y_0 t}{2t - x_0}$ for $t \in]-\infty, 0]$.*

Here, (x_e, y_e) is a point of extruding type. Its coordinates are $x_e = (3 + 2\sqrt{2})x_0$ and $y_e = \frac{y_0}{3+2\sqrt{2}}$.

Proof. The convex hull of M_{CH}^T is contained in the strip bounded by $\mathbb{R}$ and the horizontal line $Im(z) = Im(y_0)$, see Proposition 6.7. The arcs $(t, f_1(t))$ and $(t, f_4(t))$ are characterized by the fact that the associated rays starting from their points contain $x_0 + iy_0$ (this can be checked by a direct computation). In particular, they belong to two distinct branches of the same hyperbola. Besides, the domain $\mathcal{D}$ between $\mathbb{R}^-$ and arc $(t, f_4(t))$ is automatically contained in M_{CH}^T.

Following Proposition 2.10, the backward trajectory of the vector field $R(z)\partial_z$ starting at $x_0 + y_0 i$ is contained in M_{CH}^T. The domain between this portion of the integral curve and the arc $(t, f_1(t))$ is also contained in M_{CH}^T.

We denote by $\mathcal{D}'$ the domain in the open right upper quadrant where the associated ray intersects the domain $\mathcal{D}$. At each point $(t, \gamma(t))$ of the upper boundary of $\mathcal{D}'$, the associated ray is tangent to the branch of hyperbola $(s, f_4(s))$ for some $s \leq 0$. Since $R(z) = -\frac{1}{z}$, the argument of $t + i\gamma(t)$ equals the negative of the slope of $(s, f_4(s))$ at s. Since $\frac{df_4}{ds}(s) = -\frac{x_0 y_0}{(2s - x_0)^2}$, we get

$$\frac{\gamma(t)}{t} = \frac{x_0 y_0}{(2s - x_0)^2}.$$

Since the tangent line has to intersect the imaginary axis at $2\gamma(t)i$, we obtain the following equation:

$$\frac{f_4(s) - 2\gamma(t)}{s} = -\frac{x_0 y_0}{(2s - x_0)^2}.$$

Replacing $\gamma(t)$ by $\frac{x_0 y_0 t}{(2s-z_0)^2}$, we get $t = \frac{s^2}{x_0}$.

Since s is the negative square root of $x_0 t$, we deduce that $\gamma(t) = \frac{x_0 y_0 t}{(2\sqrt{x_0 t}+x_0)^2}$. In particular, for $s = -x_0$, we get $t = x_0$ and $\gamma(x_0) = \frac{y_0}{9}$.

The arc γ and the backward trajectory starting at $x_0 + iy_0$ (which is a branch of hyperbola) intersect each other at some point $x_e + iy_e$. From a computation, we obtain $x_e = (3 + 2\sqrt{2})x_0$ and therefore $y_e = \frac{y_0}{3+2\sqrt{2}}$.

It is then geometrically clear that for any point z above the curve formed by arcs defined by functions f_1, f_2, f_3, f_4, the associated ray cannot intersect any of these arcs. □

The latter example provides an illustration of a point of extruding type. Since the boundary arcs are explicit algebraic curves, we can obtain the exact picture shown in Figure 11.

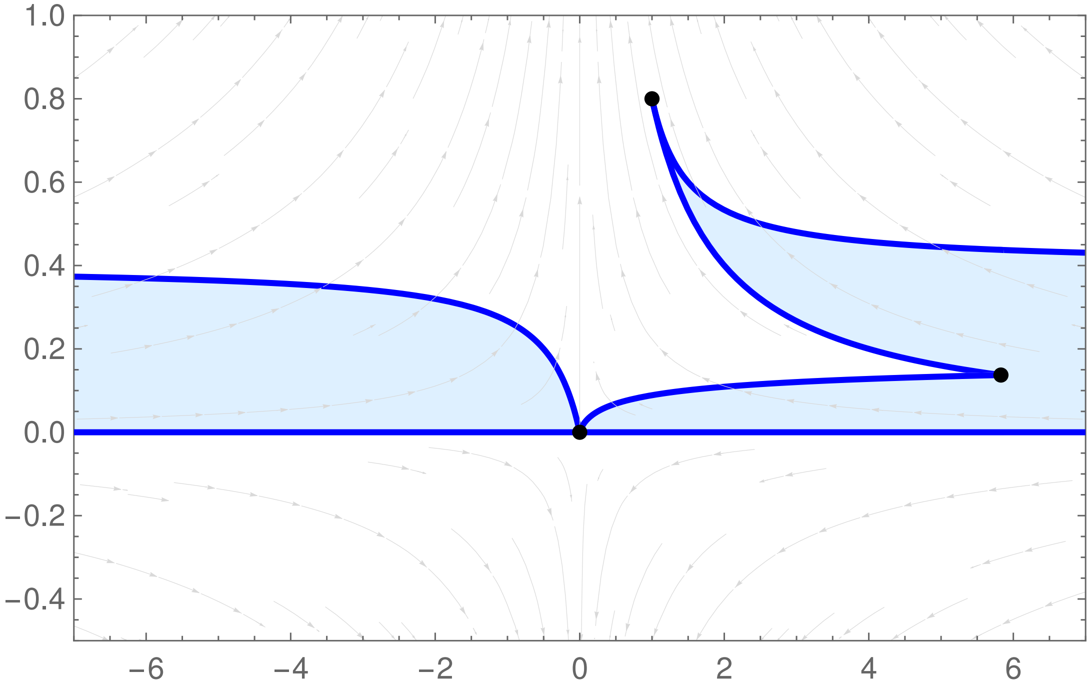

Figure 11: The case when $\alpha = 1 + 0.8i$.

6.3 Case $\deg Q - \deg P = 0$

For the sake of simplicity, we normalize the vector field $R(z)\partial_z$ by an affine change of variable so that $R(z) = 1 + \frac{\mu}{z^\kappa} + o(z^{-\kappa-1})$ for some $\mu \in \mathbb{C}^*$ and $\kappa \geq 1$. The case of a constant vector field is already treated in Section 2.3 of [AHN+24].

Under the assumptions $\mathrm{Im}(\mu) \neq 0$ and $\kappa = 1$, we are going to prove that the minimal set M^T_{CH} is connected. Firstly we show that M^T_{CH} is regular and disjoint from the curve of inflections $\mathfrak{I}_R$ outside a compact set.

Lemma 6.14. *Assuming that* $\mathrm{Im}(\mu)(-1)^\kappa > 0$, *there is a cone* $\mathcal{C}$ *and a compact set* K *such that:*

- *for any* $z \in \mathcal{C}$, $\mathrm{Im}(R(z)) > 0$ *and* $\mathrm{Im}(R'(z)) > 0$*;*
- $\mathrm{M}^T_{CH} \subset \mathcal{C} \cup K$*.*

Besides, M^T_{CH} *is a regular subset of* $\mathbb{C}$*.*

Proof. Computing $R(t)$ and $R'(t)$ for a negative real number t, we obtain that $R(t) = 1 + \frac{\mu}{t^\kappa} + o(t^{-\kappa-1})$ and thus $\mathrm{Im}(R(t)) \sim \mathrm{Im}(\mu)t^{-\kappa}$. The sign of the former is thus the sign of $\mathrm{Im}(\mu)(-1)^\kappa$. Similarly we obtain that it is also the sign of $\mathrm{Im}(R'(t))$ for t close enough to infinity and negative.

It follows from Proposition 2.25 that M^T_{CH} is contained in an infinite cone $\mathcal{C}_0$ whose asymptotic directions are $]\pi - \epsilon, \pi + \epsilon[$ for some $\epsilon \in]0, \frac{\pi}{2}[$. Besides, since the asymptotic directions of infinite branches of the algebraic curves defined by equations $\mathrm{Im}(R) = 0$ and $\mathrm{Im}(R') = 0$ are not horizontal, $\mathcal{C}_0$ and thus M^T_{CH} are covered by the union of a cone $\mathcal{C}$ and a compact set K such that for any $z \in \mathcal{C}$, $\mathrm{Im}(R(z)) > 0$ and $\mathrm{Im}(R'(z)) > 0$.

Any R-invariant line (see Definition 2.5) has to be horizontal and therefore it intersects the cone $\mathcal{C}$. Thus some points of any R-invariant line Λ have the associated rays that are not contained in Λ. Therefore, there are no R-invariant lines for such a vector field $R(z)\partial_z$. The minimal set M^T_{CH} has hence no tails and Theorem 2.23 guarantees that M^T_{CH} is regular. □

Corollary 6.15. *Assuming that* $\mathrm{Im}(\mu)(-1)^{\kappa} > 0$*, consider a sequence* $(\alpha_n)_{n\in\mathbb{N}}$ *of points of* $\partial \mathrm{M}^{T}_{CH}$ *such that* $|\alpha_n| \to +\infty$ *and* $\Delta(\alpha_n) \neq \emptyset$ *for any* $n \in \mathbb{N}$*. Then there exist a subsequence* $(\alpha_{f(n)})_{n\in\mathbb{N}}$ *and a line* $\mathcal{L}_{y_0}$ *given by* $\mathrm{Im}(z) = y_0$ *such that:*

- *the line* $\mathcal{L}_{y_0}$ *is disjoint from the interior of* M^{T}_{CH}*;*
- *the line* $\mathcal{L}_{y_0}$ *contains a point of* $\partial\mathrm{M}^{T}_{CH}$*;*
- $\mathrm{Re}(\alpha_{f(n)}) \to -\infty$*;*
- $\mathrm{Im}(\alpha_{f(n)}) \leq y_0$ *for any* $n \in \mathbb{N}$*.*

Proof. Up to taking a subsequence, we can also assume that every a_n belongs to the cone $\mathcal{C}$ defined in Lemma 6.14. Lemma 4.17 implies that for any n, points of $\Delta(\alpha_{f(n)})$ belong to $\mathfrak{I}^{-}$, $\mathfrak{I}_R$ or $\mathcal{Z}(PQ)$. Therefore, following Lemma 6.14, points of $\Delta(\alpha_n)$ accumulate in a compact set as $n \to \infty$. We denote by z_0 one of their accumulation points and by L_{y_0} the horizontal line containing z_0 (here $y_0 = \mathrm{Im}(z_0)$).

Thus, up to taking a subsequence of α, we get a sequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \to y_0$ and $y_n \in \Delta(\alpha_n)$ for any $n \in \mathbb{N}$. As α_n goes to infinity while $\Delta(\alpha_n)$ remains in a compact set, the associated rays $r(\alpha_n)$ accumulate on L_{y_0}. Thus the line L_{y_0} is disjoint from the interior of M^{T}_{CH}. Besides, since $\alpha_n \in \mathcal{C}$ for any $n \in \mathbb{N}$, we have $Im(R(\alpha_n)) > 0$ and therefore $\mathrm{Im}(\alpha_n) \leq y_0$. □

Lemma 6.16. *Provided that* $\mathrm{Im}(\mu) \neq 0$ *and* $\kappa = 1$*, no integral curve has a horizontal asymptotic line at infinity.*

Proof. Since $R(z) = 1 + \frac{\mu}{z} + o(z^{-2})$, the integral curve $\gamma(t)$ satisfies $\mathrm{Re}(\gamma(t)) \sim t$ as $t \to \pm\infty$. Then $Im(\gamma'(t)) = \frac{Im(\mu)}{t} + o(t^{-1})$. We obtain that $Im(\gamma(t))$ has logarithmic growth as $t \to \pm\infty$ and therefore the integral curve has no asymptotic lines at infinity. □

Corollary 6.17. *Provided that* $\mathrm{Im}(\mu) \neq 0$ *and* $\kappa = 1$*, the minimal set* M^{T}_{CH} *is connected in* $\mathbb{C}$*. Besides,* $\partial\mathrm{M}^{T}_{CH}$ *has exactly two infinite arcs: one is a local arc starting at infinity while the other is a global arc ending at infinity.*

Proof. Without loss of generality, we can assume that $\mathrm{Im}(\mu) > 0$. Proposition 2.20 shows that there are finitely many connected components of M^T_{CH}. Moreover they are attached to the point $\pi \in \mathbb{S}^1$ at infinity in some linear order. We refer to these components of M^T_{CH} as $X_1, \dots, X_k$ where X_1 is the lowest component while X_k is the highest component. Besides the boundary ∂X_i of any component X_i has exactly two topological ends. We call them the lower end ∂X_i^- and the upper end ∂X_i^+.

Since $\partial \mathrm{M}^T_{CH} \cap \mathfrak{I}_R$ is contained in a compact set (see Lemma 6.14), points of ∂X that are close enough to infinity are either of local, global or of extruding types. Proposition 4.12 proves that every local arc has an endpoint in $\mathfrak{I}_R \cup \mathcal{Z}(PQ)$. Thus the ends of ∂X are represented either by a local arc starting at infinity or by a global arc. Since $\mathrm{Im}(\mu) > 0$, these points belong to $\mathfrak{I}^-$ so the orientation constraint shows that only the upper end can be represented by a local arc. Otherwise, the local arc would have the point at infinity as its endpoint. Equivalently, any lower end ∂X_i^- has to be represented by an infinite global arc ending at infinity (see Lemma 4.23).

For any component X_i, the lower end ∂X_i^- of its boundary is approached by a sequence of points of global type. Applying Corollary 6.15 to such a sequence we prove the existence of a horizontal line L_i lying below the component X_i and disjoint from the interior of M^T_{CH}. Thus no component of M^T_{CH} lying below the line L_i can contain an infinite local arc because the latter has no asymptotic line at infinity (see Lemma 6.16). Consequently, among the ends of $\partial \mathrm{M}^T_{CH}$, only ∂X_k^+ can be represented by a local arc.

It remains to prove that M^T_{CH} has only one connected component. Assuming that $k > 1$, we consider the upper end ∂X_1^+. We already know that it can be approached by points $(\alpha_n)_{n\in\mathbb{N}}$ for which $\Delta(a_n) \neq \emptyset$. Applying Corollary 6.15, we prove the existence of a line L such that:

- L is disjoint from the interior of M^T_{CH};
- there is some point $z_0 \in L \cap \partial \mathrm{M}^T_{CH}$;
- points of $(\alpha_n)_{n\in\mathbb{N}}$ lie below the line L.

We deduce from the first and the third bullet points that the interior of component X_1 lies below line L.

Besides, since for each $n \in \mathbb{N}$, α_n belongs to $\mathfrak{I}^-$, $\Delta(\alpha_n)$ is a direct support point of M_{CH}^T for the associated ray $r(\alpha_n)$ (see Lemma 4.4). Then, z_0 is also a direct support point of M_{CH}^T for (oriented) line L. It follows that the interior of the component of M_{CH}^T containing z_0 lies above line L. In other words, z_0 belongs to some component X_i such that $i > 1$.

For any point z of ∂X_i close enough to ∂X_i^-, the associated ray $r(z)$ has to cross the portion of the line L formed by points whose real part is smaller than $\mathrm{Re}(z_0)$ (otherwise, the associated ray would have to cross the interior of X_i). This is impossible since z lies on or above L and $Im(R(z)) > 0$. This is a contradiction. There is no such component X_i and M_{CH}^T is connected. A neighborhood of its upper end is contained in a local arc while a neighborhood of its lower end is contained in a global arc. □

6.4 Connected components of minimal sets

Putting together partial results for the different values of $\deg Q - \deg P$, we are able to state a bound on the number of connected components of M_{CH}^T in $\mathbb{C}$. It is already known that the closure of M_{CH}^T in the extended plane $\mathbb{C} \cup \mathbb{S}^1$ is always connected.

Proof of Theorem 1.12. For any operator T satisfying $|\deg Q - \deg P| > 1$, it has been proved in Theorem 1.11 of [AHN+24] that $\mathrm{M}_{CH}^T = \mathbb{C}$. Besides, when $\deg Q - \deg P = 1$, Section 6.3 and Corollary 5.20 of the same paper proves that M_{CH}^T is connected and contractible. For $\deg Q - \deg P = -1$, it follows from Proposition 6.9.

The only case where there could be several connected components is $\deg Q - \deg P = 0$. If $R(z)$ is constant, then there are two situations. If P, Q are both constant, then there is no meaningful notion of minimal set (see Section 2.3.1 in [AHN+24]). Otherwise, M_{CH}^T is formed by parallel half-lines starting at points of $\mathcal{Z}(PQ)$. Since every point of $\mathcal{Z}(PQ)$ is a common root of P and Q (otherwise $R(z)$ would not be constant) we get that there are at most $\frac{1}{2}\deg P + \frac{1}{2}\deg Q$ such half-lines.

If $R(z)$ is not constant, then we have $R(z) = \lambda + \frac{\mu}{z} + o(z^{-1})$ for some $\lambda \in \mathbb{C}^*$ and $\mu \in \mathbb{C}$. If $\mathrm{Im}(\mu/\lambda) \neq 0$, then Corollary 6.17 proves that M_{CH}^T is connected. Otherwise, Proposition 2.20 provides an upper bound $\frac{1}{2} \deg P + \frac{1}{2} \deg Q$. □

6.5 Case $\deg Q - \deg P = 1$

In [AHN+24] we found that, outside a rather trivial case[2], a necessary and sufficient condition for the compactness of M_{CH}^T in case $\deg Q - \deg P = 1$ is $\mathrm{Re}(\lambda) \geq 0$. Moreover in case $\mathrm{Re}(\lambda) < 0$, we get $\mathrm{M}_{CH}^T = \mathbb{C}$.

We will describe M_{CH}^T for $\mathrm{Re}(\lambda) = 0$. Unfortunately, in the most interesting situation $\mathrm{Re}(\lambda) > 0$, we do not have a general description of M_{CH}^T, but we provide a number of partial results, observations and examples.

6.5.1 $\mathrm{Re}(\lambda) = 0$

In this case a complete characterization of $\partial \mathrm{M}_{CH}^T$ can be carried out.

Theorem 6.18. *Consider a linear differential operator T given by (1.1) such that $\deg Q - \deg P = 1$ and $\mathrm{Re}(\lambda) = 0$. In this case, the neighborhood of infinity is foliated by a family $\mathcal{C}$ of closed integral curves of the vector field $R(z)\partial_z$.*

The boundary $\partial \mathrm{M}_{CH}^T$ of the minimal set of T is described as the first closed leaf (according to the natural ordering starting at infinity) of the family $\mathcal{C}$ containing a point of $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$.

If the latter leaf γ contains a point of the curve of inflections $\mathfrak{I}_R$, then the latter point is a tangency point between γ and $\mathfrak{I}_R$. Moreover it is the first leaf that is non-strictly convex (the curvature at the tangency point vanishes).

In particular, $\partial \mathrm{M}_{CH}^T$ is formed by finitely many local arcs. It is real-analytic and convex (but can fail to be strictly convex). It contains neither zeros nor poles of $R(z)\partial_z$.

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[2] When $\deg P = 0$ and $\deg Q = 1$, M_{CH}^T coincides with the unique root of $Q(z)$ when $\lambda \notin \mathbb{R}_{<0}$ and coincides with $\mathbb{C}$ otherwise.

Proof. It follows from $\lambda \in \mathbb{C}^*$ and $\mathrm{Re}(\lambda) = 0$ that $\mathrm{Im}(\lambda) \neq 0$. The curve of inflections $\mathfrak{I}_R$ is therefore compact. The neighborhood of infinity is foliated by a family $\mathcal{C}$ of integral curves of vector field $R(z)\partial_z$. The orientation of these integral curves depends on the sign of $\mathrm{Im}(\lambda)$. By compactness of $\mathfrak{I}_R$, another neighborhood $\mathcal{C}'$ of infinity is foliated by strictly convex integral curves (the curvature of integral curves vanishes precisely on $\mathfrak{I}_R$).

We first consider the case when some point α of $\mathcal{Z}(PQ)$ belongs to $\mathcal{C}'$. Denoting by γ the periodic leaf α belongs to, we deduce from Proposition 2.10 that γ belongs to M_{CH}^T and bounds a strictly convex domain $\mathcal{D}$. Provided the complement of $\mathcal{D}$ does not contain any other point of $\mathcal{Z}(PQ)$, we obtain that $\mathcal{D}$ coincides with M_{CH}^T. Since α is disjoint from $\mathfrak{I}_R$, it follows from Corollary 3.12 that it cannot be a zero or a pole of $R(z)$ (α is a root of both P and Q of the same multiplicity).

In the remaining cases, we can assume that $\mathcal{Z}(PQ)$ is disjoint from $\mathcal{C}'$. The cylinder $\mathcal{C}$ is bounded by a singular curve formed by separatrices (integral curves connecting singularities of $R(z)\partial_z$). We denote by Σ the union of these separatrices and by $\mathcal{S}$ the smallest simply connected subset containing Σ. By Proposition 2.10, Σ and $\mathcal{S}$ are contained in M_{CH}^T. For the same reason, a point z of cylinder $\mathcal{C}$ is contained in M_{CH}^T if and only if the periodic integral curve containing z belongs entirely to M_{CH}^T. Therefore, the boundary of M_{CH}^T coincides with some periodic integral curve of the cylinder $\mathcal{C}$.

Since the associated rays cannot cross the interior of M_{CH}^T, its boundary $\partial\mathrm{M}_{CH}^T$ (which is a periodic integral curve) has to be convex. Therefore, it is contained in the domain of inflection of infinity (or in its boundary). Since the domain $\mathcal{C}'$ does not belong to the interior of M_{CH}^T (its complement is clearly a T_{CH}-invariant set), these conditions characterize the boundary γ of $\mathcal{C}'$ as the boundary of M_{CH}^T.

The curve γ cannot cross the curve of inflections because it is convex. If it did not intersect $\mathcal{I}_R$ there would be a strictly smaller invariant set whose boundary is an integral curve between $\mathcal{I}_R$ and γ. Thus γ has a tangency point with $\mathcal{I}_R$. At this point, the curvature of γ vanishes.

The boundary $\partial\mathrm{M}_{CH}^T$ is formed by local arcs joining points of $\mathcal{Z}(PQ)$ (with the same multiplicity of P and Q) and some points of the tangency locus. By Proposition 4.7, there arcs are strictly convex and real-analytic. □

6.5.2 $\mathrm{Re}(\lambda) > 0$

As we mentioned above, we do not have a general description of M^T_{CH}, but only a number of interesting examples. Observe that in this case ∞ is a sink of $R(z)\partial_z$).

A qualitative description of the convex hull $Conv(\mathrm{M}^T_{CH})$ is the best that we can obtain with our current knowledge.

Proposition 6.19. *Consider a linear differential operator T given by (1.1) with $\deg Q - \deg P = 1$. The boundary $\partial Conv(\mathrm{M}^T_{CH})$ of the convex hull $Conv(\mathrm{M}^T_{CH})$ of the minimal set is formed by:*

- *finitely many straight segments;*
- *finitely many portions of integral curves of vector field $R(z)\partial_z$.*

In particular, the latter are strictly convex and belong to local arcs of $\partial \mathrm{M}^T_{CH}$. In particular, $\partial Conv(\mathrm{M}^T_{CH})$ is piecewise-analytic.

Proof. We denote by $\mathcal{S}$ the set of points where the boundary $\partial Conv(\mathrm{M}^T_{CH})$ is strictly convex. They also belong to $\partial \mathrm{M}^T_{CH}$ (these points belong to the support of the hull). It follows from Theorem 1.9 that outside finitely many points, $\mathcal{S}$ is formed by either local or global arcs of $\partial \mathrm{M}^T_{CH}$. If such a point z belongs to a global arc, then the line containing the associated ray $r(z)$ is a support line of $Conv(\mathrm{M}^T_{CH})$ at z and every point of $\Delta(z)$. It follows that $[z, \Delta^{max}(z)]$ is a straight segment contained in $\partial Conv(\mathrm{M}^T_{CH})$. Consequently any arc of $\mathcal{S}$ has to be a portion of local arc.

We know that $\partial Conv(\mathrm{M}^T_{CH})$ is formed by straight segments and portions of local arcs. It remains to prove that there are finitely many of them. We consider an arc α of $\partial Conv(\mathrm{M}^T_{CH})$ contained in a local arc γ of $\partial \mathrm{M}^T_{CH}$. The endpoint of α (with the orientation defined by $R(z)\partial_z$) has at the same time to be the endpoint of γ (since otherwise the associated rays starting at points of α would intersect M^T_{CH}). Therefore, the endpoint of every such arc α

in $\partial Conv(\mathrm{M}^T_{CH})$ belongs to $\mathcal{Z}(PQ) \cup \mathfrak{I}_R$ (see Proposition 4.12). Since there are finitely many such points in $\mathcal{S}$, there are finitely many such arcs in $\partial Conv(\mathrm{M}^T_{CH})$.

If the boundary of the convex hull is not formed by finitely many straight segments and portions of integral curves, then there are infinitely many corner points of angle smaller than π between the pairs of consecutive straight segments of the boundary. It follows from Corollary 6.2 that these points belong to $\mathcal{Z}(PQ)$. Therefore, we have finitely many corner points and finitely many straight segments. $\square$

In the examples below (including a very interesting family of operators in which $Q(z)$ has simple roots and $P(z) = Q'(z)$), $Conv(\mathrm{M}^T_{CH})$ is a polygon.

Proposition 6.20. *Consider a linear differential operator T given by (1.1), such that every root α of $Q(z)$ is simple and satisfies $P(\alpha) \neq 0$ and $\phi_\alpha = 0$.*

Then, $Conv(\mathrm{M}^T_{CH})$ coincides with the convex hull of $\mathcal{Z}(Q)$.

Proof. The argument is similar to the one used in the proof of the classical Gauss–Lucas theorem (see [Mor]). If the differential form $\frac{P(z)dz}{Q(z)}$ has all positive residues, then the roots of $P(z)$ are contained in the convex hull of $\mathcal{Z}(Q)$.

The proof is based on consideration of the electrostatic force F created by the system of point charges placed at the poles of $\frac{P(z)dz}{Q(z)}$ where the value of each charge equals the residue at the corresponding pole. This electrostatic force F equals the conjugate of $\frac{P(z)dz}{Q(z)}$ and one can show that if we take any line L not intersecting the convex hull of $\mathcal{Z}(Q)$ then at any point $p \in L$, F points inside the half-plane of $\mathbb{C} \setminus L$ not containing $\mathcal{Z}(Q)$. Now recall that the associated ray has the same direction as the conjugate of P/Q. Thus, the associated ray $r(p)$ does not intersect the convex hull of $\mathcal{Z}(Q)$. $\square$

6.5.3 The first family of examples

Consider a family of operators of the form $T_\lambda = Q(z)\frac{d}{dz} + P(z)$ where $Q(z) = \lambda(z-1)^k z$ and $P(z) = (z-1)^k$ for some principal coefficient $\lambda \in \mathbb{C}^*$ and some degree $k \in \mathbb{N}^*$.

Integral curves of the vector field $R(z)\partial_z$ are logarithmic spirals parametrized by $\gamma(t) = \gamma(0)e^{\lambda t}$. In particular, they are concentric circles for $Re(\lambda) = 0$.

Depending on the value of λ, the shape of the minimal set M_{CH}^{T} can change drastically. Namely,

- if $Re(\lambda) < 0$, then $\mathrm{M}_{CH}^{T} = \mathbb{C}$ (see Theorem 1.11 of [AHN+24]);
- if $Re(\lambda) = 0$, then M_{CH}^{T} is the closed unit disk (see Theorem 6.18);
- if $Re(\lambda) > 0$ and $Im(\lambda) = 0$, then M_{CH}^{T} is the segment $[0, 1]$.

When $Re(\lambda) > 0$ and $Im(\lambda) \neq 0$, M_{CH}^{T} has a more complicated shape we describe below in terms of local and global arcs. Up to conjugation, we will assume that $\mathrm{Im}(\lambda) > 0$.

Proposition 6.21. *If λ satisfies $\mathrm{Re}(\lambda), \mathrm{Im}(\lambda) > 0$, then the minimal set M_{CH}^{T} of operator T_λ is bounded by the following arcs:*

- *local arc γ where $\gamma(t) = e^{-\lambda t}$ and $t \in]0, t_0[$;*
- *global arc α where $\alpha(t) = \frac{1}{1+\lambda t}$ and $t \in]0, t_1[$.*

These two arcs intersect at 1 and the point $\gamma(t_0) = \alpha(t_1)$ of extruding type characterized as the first intersection point between α and γ defined on $\mathbb{R}_{>0}$.

Proof. The backward trajectory of the vector field $R(z)\partial_z$ starting at 1 is parametrized by $\gamma(t) = e^{-\lambda t}$ and $t \in [0, \infty)$. Proposition 2.10 shows that this arc is entirely contained in M_{CH}^{T}.

Points z for which the associated ray contains 1 are characterized by the condition $\frac{1-z}{\lambda z} \in \mathbb{R}_{>0}$. They form an arc parametrized by $\alpha(t) = \frac{1}{1+\lambda t}$ for $t \in [0, +\infty[$. This arc is also contained in M_{CH}^{T}.

Since $R(z) = \lambda z$, it is geometrically clear that these two arcs bound M_{CH}^{T}. The boundary $\partial \mathrm{M}_{CH}^{T}$ is formed by a portion of each of them with two singular points at 1 (when $t = 0$) and the first intersection point in the parametrization. There are different ways to see that such an intersection occurs. One of them is to note that $\lim_{t\to\infty} \alpha(t) = 0$ and

$\lim_{t\to\infty}\arg(\alpha'(t)) = \lim_{t\to\infty}\arg(\frac{-\lambda}{(1+\lambda t)^2}) = \arg(\frac{-1}{\lambda})$ exists. Since we have $\mathrm{Re}(\lambda), \mathrm{Im}(\lambda) > 0$, it follows that the asymptotically straight arc $\alpha(t)$ and the logarithmic spiral $\gamma(t)$ intersect infinitely many times. The endpoint distinct from 1 common to α and γ is the first intersection point between the two parametrized arcs defined on $\mathbb{R}_{>0}$. $\square$

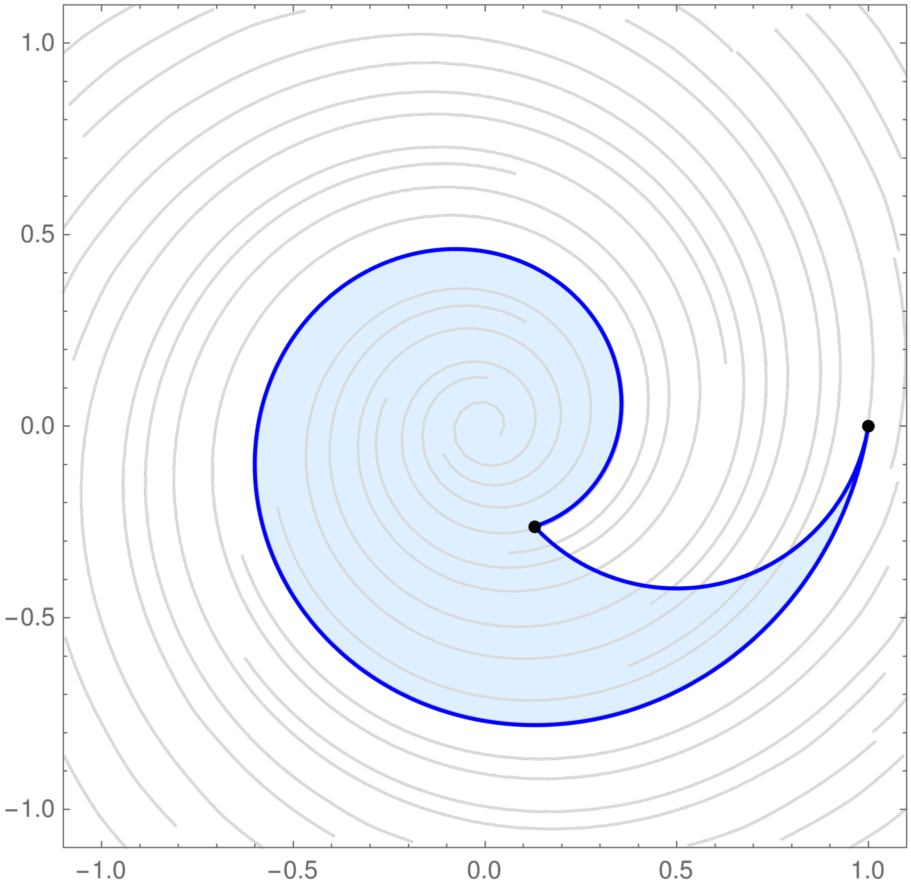

Figure 12: Illustration of the boundary of the minimal set when $\lambda = 1 + 6i$ in Prop. 6.21.

6.5.4 A second family of examples

Consider the family $T = z(z^k - 1)\frac{d}{dz} + (z^k + 1)$, where k is a positive integer. We are going to prove that for any k, the minimal set M^T_{CH} is the unit disk.

Lemma 6.22. *Set* $f(z) = z + t\frac{z(z^k-1)}{z^k+1}$*, with* $t > 0$*. Then* $|f(z)| > 1$ *whenever* $|z| > 1$*.*

Proof. We substitute $z = r^{\frac{1}{k}}e^{i\theta}$ with $r > 1$. After some algebraic manipulations, we find that

$$\frac{|f(z)|^2}{|z|^2} = \frac{|f(r^{\frac{1}{k}}e^{i\theta})|^2}{r^{2/k}} = 1 + t\frac{2r^2 - 2 + r^2t - 2\cos(\theta k)rt + t}{r^2 + 2\cos(\theta k)r + 1}. \tag{6.3}$$

Setting $c := \cos k\theta$ and rewriting further, we get

$$\frac{|f(r^{\frac{1}{k}}e^{i\theta})|^2}{r^{2/k}} = 1 + t\frac{2(r^2-1) + t\left((r-c)^2 + (1-c^2)\right)}{(r+c)^2 + (1-c^2)}. \tag{6.4}$$

Since $-1 \leq c \leq 1$, it follows that

$$\frac{|f(r^{\frac{1}{k}}e^{i\theta})|^2}{r^{2/k}} > 1 + t\frac{r^2-1}{(r+1)^2} > 1.$$

Consequently, $|f(z)| > |z|$ whenever $|z| > 1$ and the statement follows. □

Lemma 6.23. *The separatrices of the vector field* $R(z)\partial_z = \frac{z(z^k-1)}{z^k+1}\partial_z$ *are the arcs of the unit circle, connecting roots of* $P(z)$ *with roots of* $Q(z)$*.*

Proof. Assuming that z is not a root of Q, we have that

$$\int \frac{z^k+1}{z(z^k-1)}dz = k^{-1}\log\left(\frac{(1-z^k)^2}{z^k}\right).$$

Now for $z = e^{i\theta}$, we find that

$$\operatorname{Im}\log((1-z^k)^2/z^k) = \arg((1-z^k)^2/z^k) = \arg(-2 + e^{ik\theta} + e^{-ik\theta}) = \pi.$$

As the integral trajectories are level curves of $\operatorname{Im}\int\frac{dz}{R(z)}$ away from zeros or poles of $R(z)$, it follows that the unit circle consists of the integral trajectories of $R(z)\partial_z$. Since the roots of P lie on the unit circle and the zeros of Q on the unit circle have positive residues, it follows that these integral trajectories must be separatrices that are contained in M_{CH}^T. □

Corollary 6.24. *For* $T = z(z^k-1)\partial_z + (z^k+1)$*, the minimal set* M_{CH}^T *coincides with the unit disk.*

Proof. By Lemma 6.22, we have that all the associated rays for points lying outside the unit disk never intersect the unit disk. Therefore, M^T_{CH} is contained in the unit disk. Since the unit circle consists of separatrices of $-R(z)\partial_z$ (see Lemma 6.23), it follows that M^T_{CH} contains the unit circle. The associated ray of any point (distinct from 0) of the open unit disk intersects the unit circle so M^T_{CH} coincides with the unit disk. □

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On boundary points of minimal sets

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Anti-classification for flows on two-tori

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Abstract: We prove that the classification of real-analytic vector fields on the two-torus up to orbital topological equivalence does not admit a complete numerical invariant that is a Borel function. Moreover, smooth vector fields that are difficult to classify appear in generic smooth 7-parameter families. In dimension 2, this improves the recent result of A. Gorodetski and M. Foreman [6] for non-classifiability of smooth diffeomorphisms up to continuous conjugacy.

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Key words and phrases: Borel equivalence relations, Borel hierarchy, low-dimensional dynamics, rotation numbers

1 Introduction

Classification results constitute one of the central parts of the modern theory of dynamical systems. For example, due to Denjoy theorem, rotation number is a complete invariant that classifies C^2-smooth circle diffeomorphisms $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ without periodic

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orbits up to continuous conjugacy. Separatrix skeletons, graphs, or schemes are used to classify planar vector fields up to orbital topological equivalence. Ornstein's theorem [14] states that entropy is a complete invariant that classifies, up to measure-preserving transformation, Bernoulli shifts on closed subsets of the space of bi-infinite sequences.

On the other hand, Yoccoz's example shows that rotation number cannot be used to classify circle diffeomorphisms up to *smooth* conjugacy in the case when the rotation number is Liouville. P. Kunde showed [13] that smooth conjugacy on the space of circle diffeomorphisms admits no complete numerical invariant that is a Borel function. This is an *anti-classification* result that captures the complicated nature of the equivalence relation.

Strong anti-classification results were obtained in ergodic theory. Consider the space X of C^∞-smooth diffeomorphisms of a torus. Let the equivalence relation be a measure-preserving conjugacy. In [8], M. Foreman and B. Weiss proved that this equivalence relation is not *Borel*: the set $\{(S,T) \in X \times X \mid S \sim T\}$ is not Borel with respect to the C^∞-topology in $X \times X$. Earlier in [7], M. Foreman, D. Rudolph, and B. Weiss proved that measure-isomorphism for measure-preserving ergodic maps on the interval is not a Borel equivalence relation. In [9], M. Gerber and P. Kunde proved that Kakutani equivalence relation for ergodic measure-preserving transformations is also not Borel.

In Sec. 2, we will introduce Borel reducibility, the partial order on equivalence relations that produces the hierarchy of equivalence relations (see also [5]). For many natural equivalence relations, their place in this hierarchy is not known. An important breakthrough was a paper by M. Sabok [15] who showed that the isomorphism of separable C^* algebras is the maximal equivalence relation among all orbit equivalence relations. J. Zielinski [16] showed that the homeomorphism of compact metric spaces is also maximal among all orbit equivalence relations. It is an open question whether measure-isomorphism for measure-preserving ergodic maps has the same property.

One of the natural equivalence relations in dynamical systems theory is continuous conjugacy. In the space of diffeomorphisms, A. Gorodetski and M. Foreman [6] showed that this equivalence relation for smooth diffeomorphisms of $\mathbb{R}^2$ has no complete Borel numerical invariants. Moreover, for diffeomorphisms on $\mathbb{R}^5$, this equivalence relation is not Borel[1]. However, proofs involve classification of diffeomorphisms that are highly degenerate. Related results were obtained in the space of continuous interval maps and circle maps, see [1] and references therein.

Planar vector fields can be classified up to orbital topological equivalence using a combinatorial invariant (in the form of separatrix skeletons, schemes, or Leontovich-Mayer-Fedorov graphs). Classification of vector fields on the torus is more complicated: since circle maps can appear as Poincare maps, classification invariant should incorporate both the information about the behavior of separatrices and the rotation number of the Poincare map. We will see that this is sufficient to obtain non-classifiability results similar to [6, Theorem 2].

The proofs are not directly related to, but largely inspired by, new examples in the modern bifurcation theory for planar vector fields that arise from sparkling separatrix connections, see [12].

Let $\mathcal{V}^2(T^2)$ be the space of C^2-smooth vector fields on the two-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $\mathcal{V}^\omega(T^2)$ be the space of real-analytic vector fields on the two-torus.

Definition 1.1. Two vector fields $v, w \in \mathcal{V}^2(T^2)$ are orbitally topologically equivalent, $v \sim w$, if there exists a homeomorphism $H : T^2 \to T^2$ that is homotopic to identity, such that H takes orbits of v to orbits of w, preserving time orientation.

The main results of the paper are the following.

Theorem 1.2. *Orbital topological equivalence in $\mathcal{V}^2(T^2)$ has no complete Borel numerical invariant: there is no Borel function $g : \mathcal{V}^2(T^2) \to Y$ with Y a Polish space such that for all $v, w \in X$, we have $v \sim w$ if and only if $g(v) = g(w)$.*

Theorem 1.3. *Orbital topological equivalence in $\mathcal{V}^\omega(T^2)$ has no complete Borel numerical invariant: there is no Borel function $g : \mathcal{V}^\omega(T^2) \to Y$ with Y a Polish space such that for all $v, w \in X$, we have $v \sim w$ if and only if $g(v) = g(w)$.*

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[1] In [6], authors announced stronger results, but they were not published as of 07/2025.

By a classical Kuratowski's theorem, all uncountable Polish spaces are Borel isomorphic. In particular, any Polish space Y is Borel isomorphic to $\mathbb{R}$, thus we refer to these statements as the absence of *numerical* invariants for orbital topological equivalence. Results also imply that there are no complete functional invariants in any Polish functional space.

Below we will formulate and prove a stronger version of Theorem 1.2: vector fields that are difficult to classify appear in a generic 7-parameter family, see Theorem 6.1.

Remark 1.4. While circle diffeomorphisms appear as first-return maps for vector fields on the torus, result of [13] does not imply Theorem 1.2, since we consider a different equivalence relation.

Remark 1.5. While time-1 flows of vector fields $v \in \mathcal{V}(T^2)$ are diffeomorphisms of T^2, continuous conjugacy for resulting diffeomorphisms does not coincide as equivalence relation to orbital topological equivalence of corresponding vector fields. So Theorem 1.2 does not imply [6, Theorem 2]. However, equivalence of vector fields is considered to be much simpler than equivalence of planar diffeomorphisms (e.g. Newhouse phenomenon does not happen for flows of vector fields). So in a sense, our result is stronger than [6, Theorem 2]. Also, methods of [6] do not allow analytic diffeomorphisms, in contrast with Theorem 1.3.

Recall that an equivalence relation $\sim$ on a set X is not Borel if the set $\{(S,T) \in X \times X \mid S \sim T\}$ is not Borel. Even though equivalence relations in Theorems 1.2, 1.3 do not admit Borel numerical invariants, it is likely that they are Borel. We already cited results [8],[7], [9] on non-Borel equivalence relations that naturally appear in dynamics. In particular, [6, Theorem 1] states that continuous conjugacy defines a non-Borel equivalence relation on diffeomorphisms of $\mathbb{R}^5$. The following questions are open.

Can we find a two-dimensional manifold M such that the orbital topological equivalence of C^ω vector fields on M is not Borel?

Can we find a generic finite-parameter family of vector fields v_ρ, $\rho \in \mathbb{R}^k$ on a two-dimensional manifold such that the orbital topological equivalence is not Borel: the graph

$\{(\rho_1, \rho_2) \in \mathbb{R}^{2k} \mid v_{\rho_1} \sim v_{\rho_2}\}$ *of the orbital topological equivalence relation on this family is not a Borel set?*

We refer the reader to [2] for the list of open questions in descriptive set theory related to dynamical systems.

2 Preliminaries: Borel reduction

We refer the reader to [5] for an expository introduction to Borel reduction and the hierarchy of equivalence relations with respect to the Borel reduction.

Recall that a topological space X is called *Polish* if it is separable and completely metrizable (i.e. admits a complete metric that is compatible with the topology). The σ-algebra of *Borel sets* of X is the smallest σ-algebra containing all open sets. A map $f : X \to Y$ between two topological spaces is called *Borel* if for any open set A, $f^{-1}(A)$ is a Borel set.

Consider an equivalence relation on the set X. We write $x \sim_E y$ if x, y are equivalent with respect to E.

Definition 2.1. An equivalence relation E on X is *smooth* if there exists a Polish space Y and a Borel function $f : X \to Y$ such that $x \sim_E y$ holds if and only if $f(x) = f(y)$ for all $x, y \in X$.

Theorem 1.2 means that orbital topological equivalence of vector fields is non-smooth.

We will use the following non-smooth equivalence relation.

Definition 2.2. Let E_α be the equivalence relation on the circle $\mathbb{R}/\mathbb{Z}$ given by $x \sim_{E_\alpha} y$ if $x = y + n\alpha \bmod 1$.

The following proposition is the particular case of the general result (see [11, Theorem 1.1]): *an equivalence relation that admits a non-atomic ergodic measure is not smooth.* This statement is a part of the Glimm-Effros dichotomy, first discovered in [10], [4] for group actions. (Here, a finite Borel measure on X is called ergodic with respect to an equivalence relation E if any measurable E-invariant set has zero or full measure; a measure is called non-atomic if the measure of each E-equivalence class is zero.) For completeness, we will give an elementary proof of the proposition below.

Proposition 2.3. *For any* $\alpha \notin \mathbb{Q}$, *for any nonempty open interval* $I \subset \mathbb{R}/\mathbb{Z}$, E_α *is a non-smooth equivalence relation both on* I *and on* $I \setminus \{n\alpha\}_{n\in\mathbb{Z}}$.

Proof. Let $R_\alpha(x) = x+\alpha$ be a rotation on a circle $\mathbb{R}/\mathbb{Z}$. If there is a Borel numerical invariant $f : I \to \mathbb{R}$ for E_α, then the sets $A_y = f^{-1}((-\infty, y)) \subset I$ must be Borel, and thus Lebesgue measurable. Since every set A_y is an intersection of R_α-invariant measurable set with I, and R_α is ergodic, the measure of each set A_y is equal to 0 or to $\mu(I)$. Let x be the supremum of the set $\{y \in \mathbb{R}, \mu(A_y) = \mu(I)\}$. Since $\mu(\bigcap A_n) = \mu(\varnothing) = 0$ and $\mu(\bigcup A_n) = \mu(I)$, x is a finite real number. Then $\mu(f^{-1}((-\infty, x + 1/n)) = \mu(I)$ for any n and thus $\mu(f^{-1}((-\infty, x]) = \mu(I)$. On the other hand, $\mu(f^{-1}((-\infty, x - 1/n)) = 0$ for any n and thus $\mu(f^{-1}((-\infty, x)) = 0$. We conclude that $f^{-1}(x)$ has measure $\mu(I)$. This is impossible since $f^{-1}(x)$ is the intersection of a single orbit of an irrational rotation with I and has measure 0. The proof for $I \setminus \{n\alpha\}_{n\in\mathbb{Z}}$ is analogous. □

Our main tool is a Borel reduction for equivalence relations.

Definition 2.4. Let $E \subset X \times X$ and $F \subset Y \times Y$ be equivalence relations on Polish spaces X and Y, respectively. A Borel function $f : X \to Y$ is called a *Borel reduction* of E to F if for all $x_1, x_2 \in X$, we have that $x_1 \sim_E x_2$ if and only if $f(x_1) \sim_F f(x_2)$.

We say that E is Borel reducible to F, and write $E \leq_B F$.

Informally, if E is Borel reducible to F, then F is "not less complicated" than E. In particular, an equivalence relation E is smooth if and only if $E \leq_B =_\mathbb{R}$, where $=_\mathbb{R}$ is the equality relation on $\mathbb{R}$.

Borel reducibility is a partial order; hence if $E_1 \leq_B E_2$ and E_1 is non-smooth, then E_2 is also non-smooth (otherwise $E_1 \leq_B E_2 \leq_B =_\mathbb{R}$). We conclude that Theorem 1.2 is implied by the following.

Theorem 2.5. *For each irrational number ϕ, there exists a smooth family of vector fields $v_{\rho,\phi} \subset \mathcal{V}^2(T^2)$, $\rho \in \mathbb{R}/\mathbb{Z}$, such that the equivalence relation E_ϕ on $(\mathbb{R}/\mathbb{Z}) \setminus \{n\phi\}_{n\in\mathbb{Z}}$ is Borel reducible to orbital topological equivalence: for $\rho_1, \rho_2 \in (\mathbb{R}/\mathbb{Z})\setminus\{n\phi\}_{n\in\mathbb{Z}}$, we have $v_{\rho_1,\phi} \sim v_{\rho_2,\phi}$ if and only if $\rho_1 = \rho_2 \bmod n\phi$.*

This theorem is proved in the next three sections.

3 Construction of the family $v_{\rho,\phi}$.

Recall that we consider vector fields on the two-torus $T^2 = (\mathbb{R}/\mathbb{Z})^2$. Let $M_s = \mathbb{R}/\mathbb{Z} \times \{s\}$ be the meridians of the torus. Fix $\phi \in [0, 1] \setminus \mathbb{Q}$ and $\rho \in \mathbb{R}/\mathbb{Z}$.

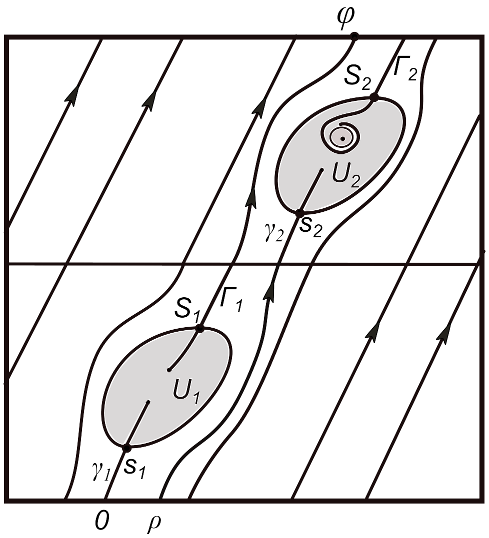

Figure 1: Phase curves of the vector field $v_{\rho,\phi}$.

We define $v_{\rho,\phi}$ in the following way.

- On neighborhoods of M_0 and $M_{0.5}$, define $v_{\rho,\phi} = (\phi, 1)$.
- Let the vector field $v_{\rho,\phi}$ in $\{0 < y < 0.5\}$ have two saddles s_1, S_1.

One stable separatrix γ_1 of s_1 intersects M_0 at $(0,0)$, two unstable separatrices of s_1 form separatrix connections with two stable separatrices of S_1, an unstable separatrix Γ_1 of S_1 intersects $M_{0.5}$ at $(\phi/2, 1/2)$. Let U_1 be bounded by the separatrix connections; in U_1, the vector field $v_{r,\phi}$ has one stable and one unstable node.

The correspondence map along $v_{r,\phi}$ between the sides of R_1 is given by $(x,0) \to (x+\phi/2, 1/2)$, except it is undefined at $(0,0)$.

- For $\rho = 0$, the vector field $v_{0,\phi}$ in the domain $\{0.5 < y < 1\}$ has a similar structure, with saddles s_2, S_2 and separatrices γ_2, Γ_2 that intersect $M_{0.5}$ and M_1 at $(\phi/2, 1/2)$ and $(\phi, 1)$ respectively. The only difference is, that in the domain U_2 bounded by its separatrix connections, the vector field $v_{\rho,\phi}$ has one stable node and one unstable limit cycle with a stable node in it.
- For other values of ρ, in the domain $\{0.5 < y < 1\}$, we put $v_{\rho,\phi}(x,y) = v_{0,\phi}(x-\rho, y)$.

With this definition, $\rho \to v_{\rho,\phi}$ is a smooth family of vector fields for any fixed ϕ. The Poincare map under $v_{\rho,\phi}$ from the meridian M_0 of the torus to itself coincides with $x \to x+\phi$ everywhere, except it is undefined at the intersections with γ_1, γ_2. If $\rho \notin \{n\phi\}_{n\in\mathbb{Z}}$, then Γ_2 does not coincide with γ_1 and Γ_1 does not coincide with γ_2. In this case, we have $M_0 \cap \gamma_1 = \{-n\phi\}_{n=0}^{\infty}$ and $M_0 \cap \gamma_2 = \{\rho - n\phi\}_{n=0}^{\infty}$.

Recall that if a point a is non-singular for a vector field, $v(a) \neq 0$, then there exists a neighborhood U with a smooth chart $H : U \to \mathbb{R}^2$ such that $H_* v = (1,0)$. A continuous curve γ is *topologically transverse* to the vector field if it does not pass through singular points, and for any point $a \in \gamma$ there exists its neighborhood U such that the image $H(\gamma \cap U)$ is a graph of a continuous function $x = x(y)$. The next lemma follows from elementary properties of correspondence maps.

Lemma 3.1. *For any simple closed loop $\alpha \subset T^2$ homotopic to the meridian M_0 that does not intersect $\overline{U}_1 \cup \overline{U}_2$ and is topologically transverse to $v_{\rho,\phi}$, there exists a homeomorphism $\xi : \alpha \to M_0$ with the following property: if $\xi(p_1) = p_2$, then either p_1, p_2 belong to the same trajectory of $v_{\rho,\phi}$, or $p_1 \in \Gamma_1 \cup \Gamma_2$ and $p_2 \in \gamma_1 \cup \gamma_2$.*

Proof. Lift $v_{\rho,\phi}$ to the vector field $\hat{v}$ on the cylinder $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$. Lift α and M_0 to the cylinder $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$ so that the lifts $\hat{\alpha}$, $\hat{M}_0 = \{y = 0\}$ do not intersect and $\hat{\alpha}$ is above $\hat{M}_0$. Then the correspondence map $\hat{\xi}\colon \hat{\alpha} \to \hat{M}_0$ along trajectories of $-\hat{v}$ is well-defined. Indeed, since both curves are topologically transverse to $\hat{v}$, the only obstructions for extending the correspondence map are intersections of $\hat{M}_0$ with stable separatrices of $\hat{v}$ and intersections of $\hat{\alpha}$ with unstable separatrices of $\hat{v}$. Since $\hat{M}_0$ and $\hat{\alpha}$ do not intersect $\overline{U}_1, \overline{U}_2$, these are the intersections of the lifts of $\gamma_{1,2}$ with $\hat{M}_0$, and of the lifts of $\Gamma_{1,2}$ with $\hat{\alpha}$. Correspondence map $\hat{\xi}$ extends continuously to these intersections. It descends to the map $\xi\colon \alpha \to M_0$ that satisfies assumptions of the lemma. □

4 Equivalent vector fields have E_ϕ-equivalent parameters

Lemma 4.1. *For irrational ϕ, let $v_{\rho,\phi}$ be vector fields constructed above. Suppose that $\rho_1, \rho_2 \notin \{n\phi\}_{n\in\mathbb{Z}}$.*

If vector fields $v_{\rho_1,\phi_1}, v_{\rho_2,\phi_2}$ are orbitally topologically equivalent, then $\phi_1 = \phi_2$ and $\rho_1 = \rho_2 + n\phi \bmod 1$.

Proof. Let $s_{1,2}, S_{1,2}, \gamma_{1,2}, \Gamma_{1,2}, U_{1,2}$ be as defined above for v_{ρ_1,ϕ_1}, and let $\tilde{s}_{1,2}, \tilde{S}_{1,2}, \tilde{\gamma}_{1,2}, \tilde{\Gamma}_{1,2}, \tilde{U}_{1,2}$ be analogous objects for v_{ρ_2,ϕ_2}.

Suppose that H is an orbital topological equivalence between v_{ρ_1,ϕ_1} and v_{ρ_2,ϕ_2}. Since H takes attractors and repellors of v_{ρ_1} to attractors and repellors of v_{ρ_2} respectively, and limit cycles to limit cycles, we have $H(U_1) = \tilde{U}_1$ and $H(U_2) = \tilde{U}_2$; $H(s_{1,2}) = \tilde{s}_{1,2}$ and $H(S_{1,2}) = \tilde{S}_{1,2}$; therefore $H(\gamma_1) = \tilde{\gamma}_1$ and $H(\gamma_2) = \tilde{\gamma}_2$. (Here we used that phase portraits of $v_{\rho,\phi}|_{U_1}$ and $v_{\rho,\phi}|_{U_2}$ are different, otherwise H could map γ_1 to $\tilde{\gamma}_2$ and γ_2 to $\tilde{\gamma}_1$.)

The curve $H(M_0)$ is topologically transverse to v_{ρ_2}, does not intersect $\tilde{U}_1 \cup \tilde{U}_2$ and is homotopic to M_0, since H is homotopic to identity. Using Lemma 3.1, define a homeomorphism $\xi\colon H(M_0) \to M_0$ along trajectories of v_{ρ_2,ϕ_2}. We get an orientation-preserving circle homeomorphism $\xi\circ H\colon M_0 \to M_0$.

Recall that the Poincare maps on M_0 under the action of v_{ρ_1,ϕ_1} and v_{ρ_2,ϕ_2} equal $x \to x+\phi_1$,

$x \to x + \phi_2$ respectively. Since $\xi \circ H$ conjugates these Poincare maps, we have $\phi_1 = \phi_2$. From now on, we will omit subscripts 1, 2 in the notation ϕ_1, ϕ_2.

Consider the set $A_1 = (M_0 \cap \gamma_1) = \{-n\phi\}_{n\in\mathbb{Z}} \times \{0\}$. The set $H(A_1)$ belongs to $H(M_0) \cap \tilde{\gamma}_1$. Since $\rho_2 \notin \{n\phi\}_{n\in\mathbb{Z}}$, the separatrix $\tilde{\gamma}_1$ does not form a separatrix connection with $\tilde{\Gamma}_{1,2}$; Lemma 3.1 implies that the set $\xi(H(A_1)) \subset M_0$ belongs to $\tilde{\gamma}_1$ as well.

Hence the circle homeomorphism $\xi \circ H$ takes the dense set $A_1 = \{-n\phi\}_{n\in\mathbb{Z}} \times \{0\}$ into a subset of $\{-n\phi\}_{n\in\mathbb{Z}} \times \{0\}$. We conclude that $\xi \circ H$ must be a rotation by $k\phi$, $k \in \mathbb{Z}$.

On the other hand, analogous arguments for γ_2 imply that $\xi \circ H$ takes $M_0 \cap \gamma_2 = \{\rho_1 - n\phi\}_{n\in\mathbb{Z}} \times \{0\}$ into a subset of $M_0 \cap \tilde{\gamma}_2 = \{\rho_2 - n\phi\}_{n\in\mathbb{Z}} \times \{0\}$, hence $\xi \circ H$ must be a rotation by $\rho_2 - \rho_1 + l\phi$, $l \in \mathbb{Z}$. We conclude that $\rho_2 - \rho_1 = m\phi \bmod 1$. □

5 Vector fields with E_ϕ-equivalent parameters are equivalent

Lemma 5.1. *For irrational ϕ, let $v_{\rho,\phi}$ be the vector field constructed above. Suppose that $\rho_1, \rho_2 \notin \{k\phi\}_{k\in\mathbb{Z}}$, and $\rho_1 = \rho_2 + n\phi \bmod 1$ for some integer n. Then $v_{\rho_1,\phi}$ and $v_{\rho_2,\phi}$ are orbitally topologically equivalent.*

Proof. We will write v_ρ instead of $v_{\rho,\phi}$ for brevity. Let ε be small so that the intervals

$$I_k = [\rho_2 + k\phi - \varepsilon, \rho_2 + k\phi + \varepsilon] = [\rho_1 - (n-k)\phi - \varepsilon, \rho_1 - (n-k)\phi + \varepsilon]$$

do not intersect for $k = 0, 1, \dots, n$ and do not intersect $J = [-\varepsilon, \varepsilon]$.

Let v_{ρ_1} be defined on the torus T, and v_{ρ_2} be defined on the torus $\tilde{T}$. We split T into a union of the following three connected sets:

- (1) V_1. Let V_1^1 be the union of arcs of trajectories of v_{ρ_1} that start at $(x, 0), x \in [-\varepsilon, \varepsilon] = J$, and end at $(x + \phi, 0)$. Let $V_1 = \overline{U}_1 \cup \overline{V_1^1}$.
- (2) V_2. Let V_2^1 be the union of arcs of trajectories of v_{ρ_1} that start at $(x, 0), x \in I_0$, and end at $(x + n\phi, 0) \in I_n$. Let $V_2 = \overline{U}_2 \cup \overline{V_2^1}$.
- (3) $V_3 = T \setminus V_1 \setminus V_2$.

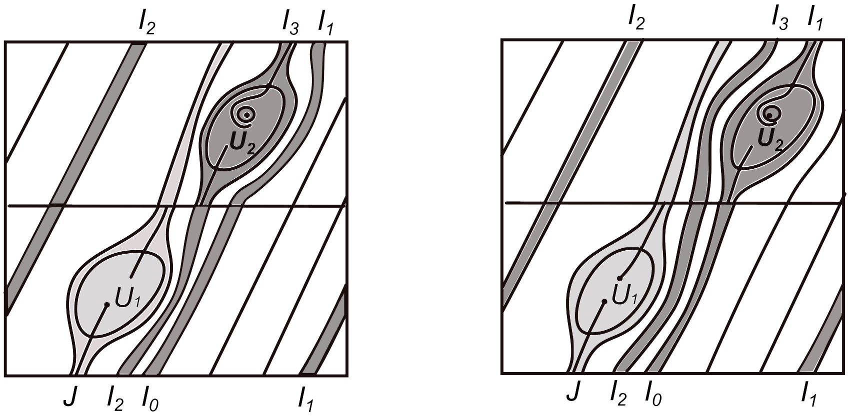

Figure 2: Domains V_1 (light-gray), V_2 (dark-gray) and V_3 (white) for vector fields v_{ρ_1} (left) and v_{ρ_2} (right) for $n = 3$.

The set V_1 contains the arc of γ_1 before its first intersection with M_0 that belongs to J. The set V_2 contains the arc of γ_2 and its 1st, 2nd, …, nth intersections with M_0. Intersections happen inside $I_{n-1}, \dots, I_0$ respectively.

Similarly, we define sets $\tilde{V}_1, \tilde{V}_2, \tilde{V}_3$ for v_{ρ_2}, using the same intervals $I_0, \dots I_n, J$ for the field v_{ρ_2}. Again, the set $\tilde{V}_1$ contains the arc of $\tilde{\gamma}_1$ before its first intersection with M_0 that belongs to J. The set $\tilde{V}_2$ contains the arc of γ_2 before its first intersection with M_0 that belongs to I_0, and the arc of Γ_2 that contains its 1st, 2nd, …, $(n-1)$-th intersections with M_0. These intersections happen inside $I_1, I_2, \dots, I_n$ respectively. Fig. 2 shows domains $V_1 - V_3$ and $\tilde{V}_1 - \tilde{V}_3$. Now, construct H.

Clearly, $v_{\rho_1}|_{V_1}$ is orbitally topologically equivalent to $v_{\rho_2}|_{\tilde{V}_1}$. We will choose equivalence H that is identical on $J \times \{0\}$ and $(J + \phi) \times \{1\}$.

Vector fields $v_{\rho_1}|_{V_2}$ and $v_{\rho_2}|_{\tilde{V}_2}$ are also orbitally topologically equivalent. We will choose H that is identical on the bottom and top sides $I_0 \subset M_0$, $I_n \times \{1\}$ of V_2.

Finally, $V_3 \setminus M_0$ is a union of strips where v_{ρ_1} is orbitally topologically equivalent to the unit vector field; the same holds for v_{ρ_2} in $\tilde{V}_3 \setminus M_0$. Thus we can extend H to V_3 by setting H to be identity on $M_0 \cap \overline{V}_3 = M_0 \setminus (J \cup I_1 \cup I_2 \cup \dots I_{n-1})$ and extending it along trajectories of v_{ρ_1}, v_{ρ_2}. This completes the construction of the orbital topological equivalence. $\square$

Lemmas 4.1 and 5.1 imply Theorem 2.5. Due to Sec. 2, this completes the proof of Theorem 1.2.

6 Genericity of nonclassifiable vector fields

In this section, we prove a stronger version of Theorem 1.2: vector fields that are orbitally topologically equivalent to $v_{\rho,\phi}$ form (at least) a codimension-7 submanifold $\mathcal{M}$ in the space $\mathcal{V}^2(T^2)$ of smooth vector fields. This implies that they appear in generic smooth 7-parameter families of vector fields.

Theorem 6.1. *For any irrational ϕ, there exists a codimension-7 continuous submanifold $\mathcal{M} \subset \mathcal{V}^2(T^2)$ such that any vector field $v \in \mathcal{M}$ is orbitally topologically equivalent to some vector field of the form $v_{\rho,\phi}$ described in Theorem 1.2.*

To define the set $\mathcal{M}$, we will need the notion of the rotation number. Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a circle homeomorphism, and let $F : \mathbb{R} \to \mathbb{R}$ be its lift to the real line. The rotation number of the circle homeomorphism f is given by

$$\operatorname{rot} f = \lim_{n\to\infty} \frac{F^n(x)}{n}.$$

The rotation number is rational if and only if f has a periodic orbit and depends continuously on f. In a 1-parameter family f_t, if $\frac{d}{dt} f_t > 0$, then the rotation number is monotonic with respect to t, and strictly monotonic whenever $\operatorname{rot}(f_t)$ is irrational.

We will also need the notion of the characteristic number of a saddle. Recall that if $\lambda_1 < 0 < \lambda_2$ are the eigenvalues of the linearization matrix of a vector field at a saddle singular point, then $|\lambda_1| : |\lambda_2|$ is called the characteristic number of a saddle. It is invariant under smooth changes of space and time variables. If L_1, L_2 are transversals to the stable and unstable separatrices of a saddle of a vector field v, then the correspondence map along v from L_1 to L_2 is defined on semi-transversals; let these transversals be given by $\{x > 0\}$ and $\{y > 0\}$ in local coordinates on L_1, L_2. This correspondence map is called the Dulac map. The following lemma is known to specialists.

Lemma 6.2. *Dulac map for the saddle with characteristic number* μ *has the form* $x \to cx^{\mu}(1 + o(1))$ *on a neighborhood of zero where* c *is a nonzero constant.*

Proof. The proof repeats the computation in the proof of Lemma 5 in [12]. Change variables so that saddle separatrices become coordinate axes, and L_1, L_2 become $\{y = 1\}$ and $\{x = 1\}$ respectively. Differential equation takes the form $x' = xg_1(x, y), y' = yg_2(x, y)$. After time change, equation becomes $x' = x, y' = yg(x, y)$, with g smooth, $g(0, 0) = -\mu$. Rescaling $x \to x/\varepsilon$, $y \to y/\varepsilon$ in an ε-neighborhood of zero brings the equation to the form $x' = x, y' = y\tilde{g}(x, y)$ with $|\tilde{g}(x, y) - g(0, 0)| < O(\varepsilon)|x| + O(\varepsilon)|y|$. Trajectory of the new vector field that starts at $(x_0, 1)$ has the form $x(t) = x_0e^t$, $\log y(t) = \int_0^t \tilde{g}(x, y)dt$. Hence it takes the time $T = -\log x_0$ for this trajectory to land on the transversal $(1, y)$. An estimate on $\tilde{g}(x, y)$ above implies that $y(T) = C(x_0)e^{-\mu T} = C(x_0) \cdot x_0^{\mu}$ with $1 - O(\varepsilon) < C(x) < 1 + O(\varepsilon)$. Thus the Dulac map along the initial vector field v between $\{y = \varepsilon\}$ and $\{x = \varepsilon\}$ has the form $y = c(x) \cdot x^{\mu}$ with $1 - O(\varepsilon) < c(x)/c_0 < 1 + O(\varepsilon)$ for certain c_0. Since the correspondence map between $L_1 = \{y = 1\}$ and $\{y = \varepsilon\}$ is smooth, as well as the correspondence map between $L_2 = \{x = 1\}$ and $\{x = \varepsilon\}$, we conclude that on a sufficiently small neighborhood of zero on L_1, the Dulac map from L_1 and L_2 has the form $y = d(x) \cdot x^{\mu}$ with $1 - O(\varepsilon) < d(x)/d_0 < 1 + O(\varepsilon)$ for certain d_0. Since ε was arbitrary, this implies the statement. □

Proof of Theorem 6.1. **Construction of $\mathcal{M}$.**

Take a vector field $v_{\rho,\phi}$ with small ρ. Modify it if needed to guarantee that on a neighborhood of $\{x = 0\}$, we have $v_{\rho,\phi} = (\phi, 1)$. Consider its small neighborhood $\mathcal{U}$ in the space $\mathcal{V}^2(T^2)$ of C^2-smooth vector fields in T^2. Let $s_1(v), s_2(v), S_1(v), S_2(v)$ be saddles of v, $v \in \mathcal{U}$, that are close to s_1, s_2, S_1, S_2. Let l_1 be an interval transverse to the left separatrix connection of s_1 and S_1 of $v_{\rho,\phi}$; let $\alpha(v)$ and $\beta(v)$ be first intersections of separatrices of $s_1(v), S_1(v)$ with l_1, in a local chart on l_1. Define $\delta_1(v) = \alpha(v) - \beta(v)$, which is a smooth function of v. In a similar way, define functions $\delta_2(v), \delta_3(v), \delta_4(v)$ for each of the separatrix connections of v. Note that we have $\delta_k(v_{\rho,\phi}) = 0$, $k = 1, 2, 3, 4$, since $v_{\rho,\phi}$ has four separatrix connections. Define

$$\mathcal{M}^0 = \{v \in \mathcal{V}^2(T^2) \mid \delta_k(v) = 0, k = 1, 2, 3, 4\}.$$

Denote the characteristic numbers of the saddles $s_1(v), s_2(v), S_1(v), S_2(v)$ by $\mu_1(v), \mu_2(v), \nu_1(v), \nu_2(v)$ respectively.

For any $v \in \mathcal{M}^0$, let P_v be the Poincare map along v from M_0 to itself. Formally, P_v is undefined at the first intersections of separatrices $\gamma_1(v)$, $\gamma_2(v)$ with M_0, but it extends continuously to these points. Denote these points $A(v), B(v)$. On the left semi-neighborhood of $A(v)$, the map P_v is a composition of two Dulac maps, from M_0 to l_1 and from l_1 to M_0. Similarly, P_v is a composition of two Dulac maps on the right semi-neighborhood of $A(v)$. Due to Lemma 6.2, P_v has the following form:

$$\begin{aligned} x &\to P_v(A(v)) + C_1(v) \cdot (x - A(v))^{\mu_1(v) \cdot \nu_1(v)}(1 + o(1)) \text{ for } x < A(v) \\ x &\to P_v(A(v)) + C_2(v) \cdot (x - A(v))^{\mu_1(v) \cdot \nu_1(v)}(1 + o(1)) \text{ for } x > A(v) \end{aligned}.$$

Note that constants $C_1(v)$ and $C_2(v)$ do not necessarily coincide. Since the Poincare map from M_0 to itself along $v_{\rho,\phi}$ is identity, we get that $\mu_1(v_{\rho,\phi}) \cdot \nu_1(v_{\rho,\phi}) = 1$; similarly, $\mu_2(v_{\rho,\phi}) \cdot \nu_2(v_{\rho,\phi}) = 1$.

Let $\mathcal{M}^1 \subset \mathcal{U}$ be given by

$$\mathcal{M}^1 = \{v \in \mathcal{M}^0 \mid \mu_1(v) \cdot \nu_1(v) = 1, \mu_2(v) \cdot \nu_2(v) = 1\}.$$

Then $\mathcal{M}^1$ is a codimension-6 smooth submanifold in $\mathcal{V}^2(T^2)$. Condition on characteristic numbers implies that P_v has nonzero one-sided derivatives on both sides of $A(v), B(v)$ for all $v \in \mathcal{M}^1$. So P_v is a C^2-smooth circle homeomorphism with two break points $A(v), B(v)$.

Finally, the set $\mathcal{M}$ is given by

$$\mathcal{M} = \{v \in \mathcal{M}^1, \operatorname{rot}(P_v) = \phi\}.$$

Codimension of $\mathcal{M}$.

We will prove that $\mathcal{M}$ is a codimension-1 continuous submanifold (i.e. the graph of a continuous function) in $\mathcal{M}^1$. Indeed, let $R_t : \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation by the angle t. Let $v = (v^1, v^2)$, and let

$$\mathcal{L} = \{v \in \mathcal{M}^1 \mid \frac{v^1(0)}{v^2(0)} = \frac{v^1_{\rho,\phi}(0)}{v^2_{\rho,\phi}(0)}\}.$$

Since on a neighborhood of a boundary of the unit square, we have $v_{\rho,\phi} = (1, \phi)$, the vector field $R_t^* v$ is well-defined on the torus for small t. Moreover, $v \in \mathcal{M}^1$ implies $R_t^* v \in \mathcal{M}^1$ for small t, therefore the set $\mathcal{M}^1$ can be locally represented as a Cartesian product $\mathbb{R} \times \mathcal{L}$ in the smooth chart $(t, v) \to R_t^* v(x, y)$.

We have $\frac{d}{dt} P_{R_t^* v_{\rho,\phi}} > 0$ due to the construction of $v_{\rho,\phi}$, thus $\frac{d}{dt} P_{R_t^* v} > 0$ for v close to $v_{\rho,\phi}$. Properties of the rotation number imply that for any v, the set $\mathcal{M}$ intersects each fiber $R_t^* v$ on a single point, i.e. $\mathcal{M}$ is a graph in $\mathcal{M}^1$. Since $\operatorname{rot}(\cdot)$ is continuous, $\mathcal{M}$ is the graph of the continuous function. Hence $\mathcal{M}$ is a continuous manifold of codimension 7.

Finding orbitally topologically equivalent $v_{R,\phi}$.

For any $v \in \mathcal{M}$, we will find R such that v is orbitally topologically conjugate to $v_{R,\phi}$. It is an easy generalization of a classical Denjoy theorem that a circle homeomorphism with breaks that has irrational rotation number is continuously conjugate to the rotation $x \to x + \operatorname{rot}(f)$ (see e.g. [3, Theorem 2.4]). Let $\xi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ conjugate P_v to the rotation $x \to x + \phi$. Post-composing ξ with rotation, we may and will assume that $\xi(A(v)) = 0$. Let

$$R = \xi(B(v)).$$

Constructing conjugacy.

Now, we will prove that $v \in \mathcal{M}$ is orbitally topologically conjugate to $v_{R,\phi}$. Let γ_1^R, γ_2^R denote separatrices of the vector field $v_{R,\phi}$.

Lift $v, v_{R,\phi}$ to vector fields $\tilde{v}, \tilde{v}_{R,\phi}$ in closed cylinders $C = \mathbb{R}/\mathbb{Z} \times [0, 1]$. Comparing phase portraits, we can see that $\tilde{v}$ is orbitally topologically equivalent to $\tilde{v}_{R,\phi}$ on C. Choose topological equivalence H to coincide with ξ on the lower boundary of C, $H(x, 0) = (\xi(x), 0)$. This is possible since ξ takes $A(v)$ to 0 and $B(v)$ to R, i.e. matches intersection points of separatrices of $\tilde{v}$ with M_0 to the intersection points of separatrices of $\tilde{v}_{R,\phi}$ with M_0.

Since H maps trajectories of $\tilde{v}$ to trajectories of $\tilde{v}_{R,\phi}$, we get $H(P_v(x), 1) = (\xi(x) + \phi, 1)$ on the upper boundary of C. Hence $H(x, 1) = (\xi(P_v^{-1}(x)) + \phi, 1) = (\xi(x) - \phi + \phi, 1) = (\xi(x), 1)$.

Since $H(x,0) = (\xi(x), 0)$ and $H(x,1) = (\xi(x), 1)$, the map H descends to a continuous map on T^2. This completes the proof.

$\square$

Remark 6.3. We could simplify the family $v_{\rho,\phi}$ by replacing two saddles in U_1 and/or U_2 with a single saddle that has a separatrix loop (cf. the next section), thus improving the codimension. However, in this case, the Poincare map will be necessarily critical with non-symmetric critical points. Such circle maps are not well-studied, and we could not find a reference to the analogue of the Denjoy theorem that applies to this case.

7 Analytic vector fields

In this section, we prove Theorem 1.3. We provide an explicit analytic family with no complete numerical invariants; the proof is computer-assisted.

Consider the family of Hamiltonian vector fields

$$v_{\phi,b,c,d} = (\frac{d}{dy}u_{\phi,b,c,d}, -\frac{d}{dx}u_{\phi,b,c,d})$$

with the Hamiltonian

$$u_{\phi,b,c,d}(x,y) = x - \phi y + (\cos y - 1)(b\sin(x-y) + c\sin(x) + d\cos(y)). \tag{1}$$

We will prove the following theorem; it implies Theorem 1.3 due to Proposition 2.3.

Theorem 7.1. *For some open interval $I \subset \mathbb{R}/\mathbb{Z}$, the equivalence relation E_ϕ on $I \setminus \{n\phi\}_{n\in\mathbb{Z}}$ is Borel reducible to the orbital topological equivalence on a subset of the family $v_{\phi,b,c,d}$.*

Namely, there exist analytic functions D and ρ defined on an open set V in $\mathbb{R}^3$, such that the function ρ is non-constant on c for fixed b, ϕ, and two vector fields $v_1 = v_{\phi_1,b_1,c_1,D(\phi_1,b_1,c_1)}$ and $v_2 = v_{\phi_2,b_2,c_2,D(\phi_2,b_2,c_2)}$ for $(\phi_{1,2}, b_{1,2}, c_{1,2}) \in U$, irrational ϕ_1, ϕ_2, and $\rho(\phi_1, b_1, c_1) \notin \{n\phi_1\}_{n\in\mathbb{Z}}$, $\rho(\phi, b_2, c_2) \notin \{n\phi_2\}_{n\in\mathbb{Z}}$ are orbitally topologically equivalent if and only if $\phi_1 = \phi_2 = \phi$ and $\rho(\phi, b_1, c_1) = \rho(\phi, b_2, c_2) \bmod n\phi$ in $\mathbb{R}/\mathbb{Z}$.

Proof. First, let us explain how the second part of the theorem implies its first part. Define an analytic function $C(\phi, r)$ implicitly on some open set by a condition $\rho(\phi, b_0, C(\phi, r)) = r$. Then for fixed irrational ϕ, the Borel reduction of E_ϕ on $I \setminus \{n\phi\}_{n\in\mathbb{Z}}$ to the orbital topological equivalence is given by $\rho \to v_{\phi,b_0,C(\phi,\rho),D(\phi,b_0,C(\phi,\rho))}$. This implies the first part of Theorem 1.3.

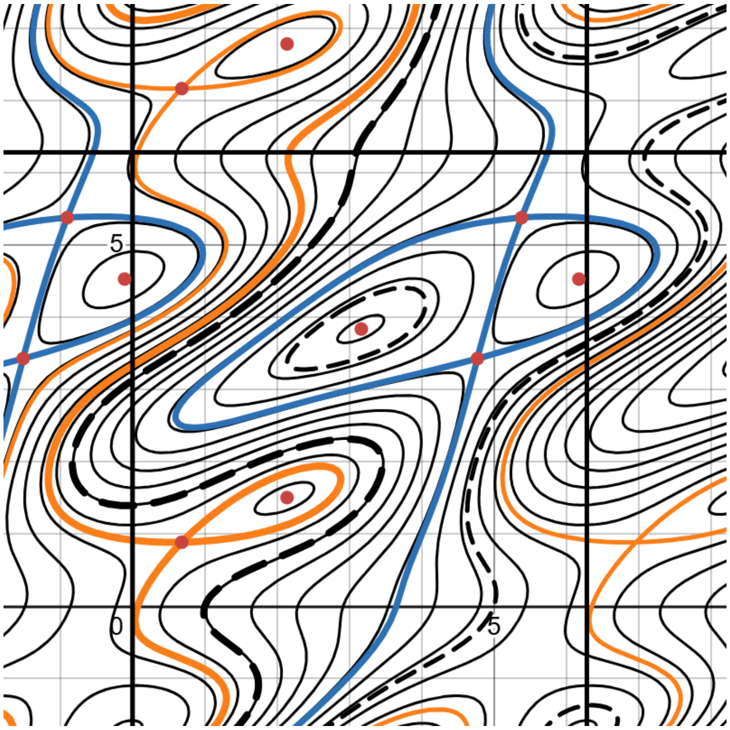

Figure 3: Level curves of $u_{\phi,b,c,d}$ (phase curves of the vector fields $v_{\phi,b,c,d}$) for $\phi = 1/3$, $c = 1, b = 2, d = 1.0016$. The black square has sides of length 2π. Separatrices of the saddles s_1, s_2, s_3 are shown in thick. Thick dots represent singular points of $v_{\phi,b,c,d}$. Dashed lines represent level curves $u_{\phi,b,c,d} = 1, u_{\phi,b,c,d} = 5$.

Clearly, the vector field $v_{\phi,b,c,d}$ is well-defined on the torus $T^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$. For simplicity of notation, we will lift it to the annulus $0 \le y \le 1$. It is easy to check that for $\phi \ne 0$, the vector field is transversal to $y = 0, y = 1$. Its phase curves are level curves of $u_{\phi,b,c,d}$. Since $u_{\phi,b,c,d}(x, 0) = x$ and $u_{\phi,b,c,d}(x, 2\pi) = x - 2\pi\phi$, the correspondence map from $\{y = 0\}$ to $\{y = 1\}$

is $x \to x + 2\pi\phi$ whenever defined.

Fig. 3 shows the level curves of $u_{\phi,b,c,d}$ for $\phi = 1/3, b = 2, c = 1, d \approx 1$. The following statements on vector fields $v_{\phi,b,c,d}$ were verified numerically for $\phi = 1/3, b = 2, c \in [0.7, 1.1]$, $d \in [0.9, 1.3]$.

- Vector fields $v_{\phi,b,c,d}$ have six singular points on the torus.

This was checked by applying Python's fsolve method (that uses the Powell's hybrid method) with the dense mesh of parameter values and initial guesses. Singular points are marked on Fig. 3.

- These six singular points are hyperbolic; there are three saddle points, two minima of $u_{\phi,b,c,d}$ and one maximum of $u_{\phi,b,c,d}$.

This was checked by computing the Jacobians of the linear part of $v_{\phi,b,c,d}$. Jacobians remain greater than 2 in modulus.

Let s_1, s_2, s_3 be the saddles of $v_{\phi,b,c,d}$, numbered left to right in x-coordinate, $x \in [0, 2\pi]$.

- The curves $\{y = 0\}, \{y = 1\}$, the level curve $\{u_{\phi,b,c,d}(x, y) = 1, 0 < y < 1\}$, and a connected component of the level curve $\{u_{\phi,b,c,d}(x, y) = 5, 0 < y < 1\}$ divide the torus into two domains V_1 and V_2; one of them (V_1) contains s_1 and a minimum of $u_{\phi,b,c,d}$, the other (V_2) contains s_2, s_3, and the remaining minimum and maximum of $u_{\phi,b,c,d}$.

This was checked by (1) plotting these level curves for $u_{1/3,2,1,1}$ (see Fig. 3) and (2) verifying that for all $c \in [0.7, 1.1]$, $d \in [0.9, 1.3]$, values of $u_{1/3,2,c,d}$ at its critical points in $0 < y < 1$ are not equal to $1 \pm 2\pi k, 5 \pm 2\pi k$. This implies that critical points cannot move from one strip to another as parameters vary.

Since level curves of $u_{\phi,b,c,d}$ are phase curves for $v_{\phi,b,c,d}$ and $u_{\phi,b,c,d}(x, 0) = x$, we conclude that the correspondence map from $\{y = 0\}$ to $\{y = 1\}$ is defined near $x = 1, 5$. Consider the domain V_1. Since $u_{\phi,b,c,d}$ is monotonic on $y = 0$, out of four separatrices of s_1, only one can intersect $\{y = 0\}$ and only one can intersect $\{y = 1\}$. Since separatrices are the only obstruction from extending Poincare maps, exactly one separatrix must intersect

$\{y = 0\}$ and one must intersect $\{y = 1\}$. Thus the remaining two separatrices of s_1 must form a separatrix loop in V_1 as shown in Fig. 3, and the correspondence map from $\{y = 0\}$ to $\{y = 1\}$ in V_1 is well-defined except a single point of intersection with a separatrix (to which it extends continuously).

Suppose that $u_{\phi,b,c,d}(s_2) = u_{\phi,b,c,d}(s_3)$. Since $u_{\phi,b,c,d}$ is monotonic on $y = 0$ and $y = 1$, in the strip V_2, only two of the eight separatrices of s_2, s_3 can intersect $\{y = 0\}$ and $\{y = 1\}$. Since separatrices are the only obstruction from extending Poincare maps, exactly two of these separatrices intersect $\{y = 0\}$ and $\{y = 1\}$. The remaining six separatrices must form three separatrix connections. There are two possibilities: we either have three separatrix connections between s_2 and s_3, or one connection and two separatrix loops. To check that we always have the first possibility as shown on Fig. 3, we verified the following.

- The function $u_{\phi,b,c,d}$ is monotonic on the straight line segment joining the minimum and the maximum of $u_{\phi,b,c,d}$ in V_2.

This was checked by computing the directional derivative of $u_{\phi,b,c,d}$ at the points of this segment, with step size equal to 0.01 of its length. If separatrices of s_2, s_3 formed separatrix loops, minimum and maximum of u would be inside these loops and the segment $[s_2, s_3]$ would intersect the level set $u_{\phi,b,c,d}(x, y) = u_{\phi,b,c,d}(s_2)$ at least twice, which contradicts monotonicity of $u_{\phi,b,c,d}$ on $[s_2, s_3]$. Hence s_2, s_3 form three separatrix connections. The correspondence map from $\{y = 0\}$ to $\{y = 1\}$ is everywhere defined, except two intersection points with separatrices to which it extends continuously. Thus it coincides with $x \to x + 2\pi\phi$ as noted above.

We claim that the condition $u_{\phi,b,c,d}(s_2) = u_{\phi,b,c,d}(s_3)$ defines a graph of an analytic function $D = D(\phi, b, c)$ in the parameter space for $c \in [0.7, 1.1]$, $d \approx 2$, $\phi \approx 1/3$. This follows from the property below.

- Derivative of $u_{\phi,b,c,d}(s_3) - u_{\phi,b,c,d}(s_2)$ with respect to d is positive for all $c \in [0.7, 1.1]$, $d \in [0.9, 1.3]$.

Derivative was computed via implicit function theorem. It remains between 2.05 and 2.15.

Hence for each $c \in [0.7, 1.1]$, the interval $d \in [0.9, 1.3]$ contains at most one value d such that $u_{\phi,b,c,d}(s_2) = u_{\phi,b,c,d}(s_3)$. This value $d = D(\phi, b, c)$ was determined numerically for $\phi = 1/3, b = 2, c \in [0.7, 1.1]$ and remains in $[0.9, 1.3]$. Since the condition is open, the same holds for all $b \approx 2$, $\phi \approx 1/3$, and $D(\phi, b, c)$ is well-defined. It is analytic since singular points of $v_{\phi,b,c,d}$ depend analytically on parameters.

For any vector field $v_{\phi,b,c,D(\phi,b,c)}$, let U_1 be the domain bounded by the separatrix connection of s_1; let U_2 be the domain bounded by the separatrix connections of s_2, s_3. Let stable separatrices of s_1, s_2 be γ_1, γ_2, let unstable separatrices of s_1, s_3 be Γ_1, Γ_2. Let $\rho(\phi, b, c) = u_{\phi,b,c,d}(s_2) - u_{\phi,b,c,d}(s_1)$ be the distance between the intersection points of γ_1 and γ_2 with the meridian $M_0 = \{y = 0\}$. Numerically, we checked the following.

- The value $\rho(\phi, b, c)$ is not constant on the graph $(\phi, b, c, D(\phi, b, c))$: namely, for $c = 0.7$ we have $d(c) \approx 1.23$ and $\rho \approx 3.40$ while for $c = 1.1$ we get $d \approx 0.92$ and $\rho \approx 3.62$.

The function ρ remains non-constant on c for $\phi \approx 1/3, d \approx 2$ since it is analytic.

The remaining part of the proof is the same as for Theorem 2.5. While the vector field $v_{\phi,b,c,D(\phi,b,c)}$ with $\rho = \rho(\phi, b, c)$ is not orbitally topologically equivalent to $v_{\rho,\phi}$, the only difference is the explicit shape of the phase portrait inside U_1, U_2; the Poincare map and the behavior of separatrices $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$ is the same. Hence the proof of Lemma 5.1 applies to the family $v_{\phi,b,c,D(\phi,b,c)}$ with $\rho = \rho(\phi, b, c)$, for any fixed b close to 2 and any fixed irrational ϕ close to 1/3, without any modification. In the proof of Lemma 4.1, we also used the fact that an orbital topological equivalence H matches corresponding saddles, $H(s_k) = \tilde{s}_k$. For our family, the latter follows from the fact that s_2, s_3 form three separatrix connections while s_1 has a separatrix loop. Hence the proof of Lemma 4.1 applies for the family $v_{\phi,b,c,D(\phi,b,c)}$ with minor modification. These lemmas imply the second part of Theorem 7.1, which completes its proof. □

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Self-duality of multidimensional continued fractions

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Abstract: F. Schweiger introduced the fibred system in [12], to unify and generalize many known continued fraction algorithms. An advantage of a fibred system is that it often provides a systematic construction of an absolutely continuous invariant density. In this paper, we define and study the self-duality of fibred systems, a strong symmetry of a given system. We show that explicit algebraic self-duality holds in many systems and present a curious system with "partial" self-duality.

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1 Introduction

The classical continued fraction is a self map T on $[0,1]$ defined by

$$T : x \longmapsto \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

Its absolutely continuous invariant probabilistic density is

$$d\nu = \frac{1}{\log 2} \cdot \frac{1}{1+x} dx.$$

The cylinder set $\Delta[a_1, a_2, \dots, a_n]$ is the interval whose elements share an initial fraction:

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

Then we have

$$\nu(\Delta[a_1, \dots, a_n]) = \nu(\Delta[a_n, \dots, a_1]).$$

We say that continued fraction algorithm is **symmetric in measure** if this equality holds for all cylinder sets. To see this symmetry, a standard way is to consider its natural extension:

$$\hat{T} : [0,1]^2 \ni (x,y) \longmapsto \left(\frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \frac{1}{y + \left\lfloor \frac{1}{x} \right\rfloor} \right) \in [0,1]^2$$

with the invariant density

$$\frac{1}{(1+xy)^2} dxdy.$$

The map $\hat{T}$ is invertible:

$$\hat{T}^{-1} : (x,y) \longmapsto \left(\frac{1}{x + \left\lfloor \frac{1}{y} \right\rfloor}, \frac{1}{y} - \left\lfloor \frac{1}{y} \right\rfloor \right)$$

and the restriction of $\hat{T}^{-1}$ to the second coordinate is equal to T. The self-duality immediately follows from this fact.

To make concrete the tractable a natural extension for higher dimensional continued fractions, F. Schweiger constructed the dual algorithm $(B^{\#}, T^{\#})$ of the fibred system (B, T). The pair $(B \times B^{\#}, T \times V^{\#}(k(x)))$ gives the natural extension of (B, T) where $V^{\#}(k)$ is a local inverse branch of $T^{\#}$. In this framework, if there exists an isomorphism ϕ which satisfies:

$$\begin{tikzcd}
B^{\#} \arrow[r, "T^{\#}"] \arrow[d, "\phi"'] & B^{\#} \arrow[d, "\phi"'] \\
B \arrow[r, "T"] & B
\end{tikzcd}$$

then the system is self-dual. We say that self-duality is realized by an intertwining map ϕ. We will define an algebraic self-duality. If such a map ϕ is found we simply say that the system (B, T) is **algebraic self-dual**. See section 2 for details. In this paper we start with an easy observation:

Theorem 1. If the fibred system (B, T) is full and algebraic self-dual, then it is symmetric in measure.

However, we do not know when the self-duality holds in general, nor how to construct the intertwining map ϕ for a given full-branched fibred system. In the later section, we shall construct ϕ for several fibred systems in [12] and also give examples of fibred systems which is not self-dual.

2 Invariant measure and self-duality

In this chapter, we briefly review the concept of higher dimensional continued fractions by F. Schweiger and show Theorem 1.

We say that the dynamical system (B, T) is a fibred system if $\{B(k) : k \in I\}$ is a partition of the set B where I is countable and $T|_{B(k)}$ is injective.

Definition 1. The fibred system (B, T) is multidimensional continued fraction (**m.c.f.**) if

1. $B \subset \mathbb{R}^n$,

2. For every digits $k \in I$, there is a matrix $A_T(k) = ((A_{ij})) \in GL(n+1, \mathbb{Z})$ such that $y = T(x)$, $x \in B(k)$ is given as

$$y_i = \frac{A_{i0} + \sum_{j=1}^{n} A_{ij} x_j}{A_{00} + \sum_{j=1}^{n} A_{0j} x_j}.$$

Remark 1. For all invertible $(n+1) \times (n+1)$-matrix (a_{ij}), we define a transformation $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies

$$f(x)_i = \frac{a_{i0} + \sum_{j=1}^{n} a_{ij} x_j}{a_{00} + \sum_{j=1}^{n} a_{0j} x_j},$$

and we denote by A_f the matrix $((a_{ij}))$. Then, we can verify $A_f A_g = A_{f \circ g}$.

Since $T|_{B(k)}$ is injective, there exists a local inverse branch of T

$$V(k) : T(B(k)) \to B(k), \quad x = V(k)y$$

We denote the inverse matrix of $A_T(k)$ by $((B_{ij}))$. Then $y = Tx$ is equivalent to

$$x_i = \frac{B_{i0} + \sum_{j=1}^{n} B_{ij} y_j}{B_{00} + \sum_{j=1}^{n} B_{0j} y_j}.$$

where B_{ij} satisfies $B_{00} + \sum_{j=1}^{n} B_{0j} y_j > 0$.

Definition 2. Let (B, T) be a m.c.f. with matrices $\{A_T(k) : k \in I\}$. The m.c.f. $(B^\#, T^\#)$ is dual algorithm if the following conditions holds:

1. $B(k_1, k_2, \cdots, k_n) \neq \emptyset$ if and only if $B^\#(k_n, k_{n-1}, \cdots, k_1) \neq \emptyset$,

2. There is a partition $\{B^\#(k), k \in I\}$ of $B^\#$ such that the associated matrices $A_{T^\#}(k) = ((A^\#_{ij}))$ of $T^\#$ restricted $B^\#(k)$ are the **transposed matrices** of $A_T(k)$ such that $y = T^\#(x)$, $x \in B^\#(k)$ is given as

$$y_i = \frac{A^\#_{i0} + \sum_{j=1}^{n} A^\#_{ij} x_j}{A^\#_{00} + \sum_{j=1}^{n} A^\#_{0j} x_j}.$$

Given a multidimensional continued fraction algorithm (B, T), its dual map is formally defined by the transpose of A_T. We then try to find an appropriate dual space $B^\#$ and its decomposition $\{B^\#(k) : k \in I\}$ which satisfies condition 1.

After this construction, given an n-dimensional continued fraction we set

$$K(x, y) := \frac{1}{(1 + x_1y_1 + x_2y_2 + \cdots + x_ny_n)^{n+1}},$$

and we denote by $\omega(k_1, k_2, \cdots, k_s; y)$ the Jacobian of $V(k_1, k_2, \cdots, k_s) = V(k_1)\circ V(k_2)\circ \cdots \circ V(k_s)$. Then, we can see

$$K(V(k_1, \ldots, k_s)x, y)\omega(k_1, \ldots, k_s; x) = K(x, V^\#(k_s, \ldots, k_1)y)\omega^\#(k_s, \ldots, k_1; y) \tag{1}$$

by a straightforward calculation. For any $x \in B$, we define

$$D(x) := \{y \in B^\# : x \in \bigcap_{s=1}^{\infty} T^s B(k_s^\#(y), \cdots, k_1^\#(y))\}.$$

Then, it is known that the following assertion holds (see Chapter 3 in [12]):

Proposition 1.

$$h(x) = \int_{D(x)} K(x, y)dy$$

is invariant density for T.

Definition 3. A fibred system (B, T) is called full if $T(B(\mathring{d})) = \mathring{B}$ for all d $\in I$.

Note that $D(x) = B^\#$ if the system (B, T) is full. By Proposition 1, we can obtain an invariant measure $\mu^\#$ for the dual algorithm $(B^\#, T^\#)$.

Lemma 1. The multidimensional c.f. (B, T) is full, then one has

$$\mu(B(k_1, k_2, \cdots, k_s)) = \mu^\#(B^\#(k_s, k_{s-1}, \cdots, k_1)).$$

Proof. For all $k_1, \ldots .k_s \in I$, by Proposition 1,

$$\begin{aligned}\mu(B(k_1, k_2, \cdots, k_s)) &= \int_{B(k_1,k_2,\cdots,k_s)} \int_{B^\#} K(x, y)dydx \\ &= \int_B \int_{B^\#} K(V(k_1, k_2, \cdots, k_s)x, y)\omega(k_1, k_2, \cdots, k_s; x)dydx.\end{aligned}$$

By (1), we have

$$\begin{aligned}\mu(B(k_1,k_2,\cdots,k_s)) &= \int_B\int_{B^{\#}} K(x,V^{\#}(k_s,k_{s-1},\cdots,k_1)y)\omega^{\#}(k_s,k_{s-1},\cdots,k_1;y)dydx\\ &= \int_{B^{\#}}\int_B K(V^{\#}(k_s,k_{s-1},\cdots,k_1)y,x)\omega^{\#}(k_s,k_{s-1},\cdots,k_1;y)dxdy\\ &= \mu^{\#}(B^{\#}(k_s,k_{s-1},\cdots,k_1)).\end{aligned}$$

□

Definition 4. A n-dimensional continued fraction (B,T) is "algebraic self-dual" on $\mathcal{D}\subset I$ if the diagram

$$\begin{tikzcd} B^{\#} \arrow[r, "T^{\#}"] \arrow[d, "\phi"] & B^{\#} \arrow[d] \\ B \arrow[r, "T"] & B \end{tikzcd}$$

is commutative and ϕ is a bijective, differentiable, and measurable function map such that $\phi(B^{\#}\mathring{(}k)) = B\mathring{(}k)$ for all $k\in\mathcal{D}$.

For the regular continued fraction algorithm $([0,1),T)$, the matrix $A_T(k)$ is symmetric. Thus, it is clearly self-dual since $T=T^{\#}$.

Proof of Theorem 1. Note that the map ϕ is bijective, since (B,T) is algebraic self-dual (on $\mathcal{D}=I$). By substitution, for all $k_1,k_2,\cdots,k_s\in I$

$$\begin{aligned}\mu^{\#}(\phi^{-1}B(k_1,k_2,\cdots,k_s)) &= \int_{\phi^{-1}B(k_1,k_2,\cdots,k_s)}\int_B K(x,y)dydx\\ &= \int_{B(k_1,k_2,\cdots,k_s)}\int_{B^{\#}} K(X,Y)dYdX\\ &= \mu(B(k_1,k_2,\cdots,k_s)).\end{aligned}$$

Therefore, by Lemma 1, we have

$$\begin{aligned}\mu(B(k_1,k_2,\cdots,k_s)) &= \mu^{\#}(B^{\#}(k_1,k_2,\cdots,k_s))\\ &= \mu(B(k_s,k_{s-1},\cdots,k_1)).\end{aligned}$$

□

Note that the m.c.f. (B, T) is algebraic self-dual on $\mathcal{D} \subset I$, then for all $k_1, k_2, \cdots, k_s \in \mathcal{D}$

$$\mu(B(k_1, k_2, \cdots, k_s)) = \mu(B(k_s, k_{s-1}, \cdots, k_1)).$$

Definition 5. The set function

$$\tau(B(k_1, k_2, \cdots, k_s)) := \mu(B(k_s, k_{s-1}, \cdots, k_1)).$$

is called the polar measure for (B, T).

In fact, τ is also an invariant measure for T. And $\mu = \tau$ means symmetric in measure. F. Schweiger gave an equivalent condition for $\mu = \tau$ under the conditions in [10].

3 Selmer Algorithm

Let $E^{n+1} := \{x \in \mathbb{R}_{>}^{n+1} : x_0 \geq x_1 \geq \cdots \geq x_n \geq 0\}$. Then define

$$x \in E^{n+1} \longmapsto x^{'} = (x_0 - x_n, x_2, \dots, x_n).$$

There is an index $i = i(x)$, $0 \leq i \leq n$ such that

$$S(x) = (x_1, \dots, x_i, x_0 - x_n, \dots, x_n) \in E^{n+1}.$$

We obtain the bottom map $T_S : \Delta \longrightarrow \Delta$ which makes the diagram

$$\begin{tikzcd}
E^{n+1} \arrow[r, "S"] \arrow[d, "p"'] & E^{n+1} \arrow[d] \\
\Delta \arrow[r, "T=T_S"] & \Delta
\end{tikzcd}$$

commutative. Since

$$A_T(i) = \begin{array}{cc} & \begin{array}{ccccccc} & & & i & & & n \end{array} \\ \begin{array}{c} 0 \\ \\ \\ i \\ \\ \\ \\ \end{array} & \begin{pmatrix} & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ 1 & & & & & & -1 \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array} \ \textbf{on } \Delta(i),$$

the 1-time partition is $\Delta(i) = \{x \in \Delta : x_i > 1 - x_n \geq x_{i+1}\}$ where $x_0 = 1$, $x_{n+1} = 0$. We can see $T(\Delta(i)) = \{x \in \Delta : x_i + x_n \geq 1\}$. Therefore, for $i = 0, 1, \dots, n-1$

$$T(\Delta(i)) = \bigcup_{i \leq j} \Delta(j), \quad T(\Delta(n)) = \Delta(n-1) \cup \Delta(n).$$

The system (Δ, T) is not full, but $(X = \Delta(n-1) \cup \Delta(n), T)$ is full-branched system. The dual map of Selmer's algorithm (X, T) is defined on $X^{\#} = \mathbb{R}^n_{\geq}$. It is known that Selmer algorithm (X, T) is ergodic and admits an absolutely continuous invariant measure (see [12]).

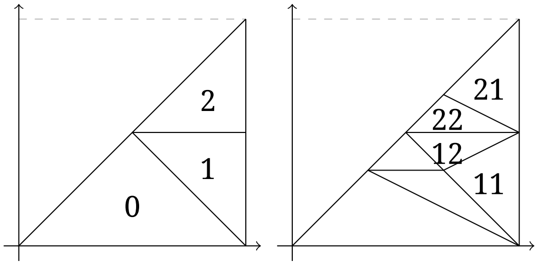

Fig. 1: The 1-time and 2-time partition of (Δ, T_S).

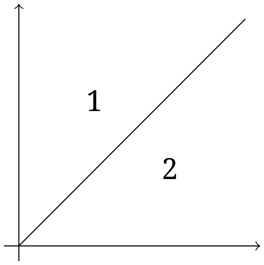

Fig. 2: The 1-time partition of $(X^{\#}, T_S^{\#})$.

Note that the 1-time partition of $X^{\#}$ is

$$X^{\#}(n-1) = \{x \in X^{\#} \,:\, x_{n-1} \leq x_n\},$$
$$X^{\#}(n) = \{x \in X^{\#} \,:\, x_{n-1} \geq x_n\}.$$

We construct the intertwining map ϕ for the Selmer algorithm.

For $n=2$, the same method in the previous chapter gives

$$A_\phi = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

A simple analogy for the n-dimensional case works fine and we obtain

Proposition 2. Selmer's algorithm is algebraic self-dual. And

$$A_\phi = \begin{pmatrix} 2 & \cdots & 2 & 1 & 1 \\ \vdots & \iddots & & \vdots & \vdots \\ 2 & & & & \\ 1 & \cdots & & & 1 \\ 1 & \cdots & & 1 & 0 \end{pmatrix}.$$

Proof. We can see that for all $k \in \{n-1, n\}$

$$A_\phi A_{T^{\#}}(k) = A_T(k) A_\phi.$$

Let $\phi(B_1, \dots, B_n) = (b_1, \dots, b_n)$. Then, for all $(B_1, \dots, B_n) \in X^{\#}(n-1)$,

$$b_{n-1} + b_n - 1 = \frac{B_{n-1}}{2 + 2B_1 + \cdots + 2B_{n-2} + B_{n-1} + B_n} > 0$$

and

$$1 - 2b_n = \frac{B_n - B_{n-1}}{2 + 2B_1 + \cdots + 2B_{n-2} + B_{n-1} + B_n} > 0.$$

Thus, $\phi(X^{\#}(n-1)) \subset \Delta(n-1)$. We show $\phi(X^{\#}(n-1)) \supset \Delta(n-1)$. Since

$$A_{\phi}^{-1} = \begin{pmatrix} & & 1 & -1 & \\ & \iddots & -1 & & \\ 1 & \iddots & & & \\ -1 & & & 1 & 1 \\ & & & 1 & -1 \end{pmatrix},$$

for all $(b_1, \dots, b_n) \in \Delta(n-1)$, we can verify that $B_i \geq 0$, $i = 1, \dots, n$ and

$$B_n - B_{n-1} = \frac{1 - 2b_n}{b_{n-2} - b_{n-1}} > 0.$$

Thus, $\phi(X^{\#}(n-1)) = \Delta(n-1)$. Similarly, we can see $\phi(X^{\#}(n)) = \Delta(n)$. □

Corollary 1. Selmer's algorithm (X, T) is symmetric in measure. i.e., for all $a_1, a_2, \cdots, a_s \in \{n-1, n\}$

$$\mu(\Delta(a_1, a_2, \cdots, a_s)) = \mu(\Delta(a_s, a_{s-1}, \cdots, a_1)).$$

4 Brun Algorithm

Let $E^{n+1} := \{x \in \mathbb{R}_{>}^{n+1} : x_0 \geq x_1 \geq \cdots \geq x_n \geq 0\}$. Define

$$x \in E^{n+1} \longmapsto x' = (x_0 - x_1, x_2, \dots, x_n).$$

Then, there is an index $i = i(x)$, $0 \leq i \leq n$ such that

$$B(x) = (x_1, \dots, x_i, x_0 - x_1, \dots, x_n) \in E^{n+1}.$$

We obtain the bottom map $T : \Delta \longrightarrow \Delta$ which makes the diagram

$$\begin{tikzcd}
E^{n+1} \arrow[r, "B"] \arrow[d, "p"'] & E^{n+1} \arrow[d, "p"'] \\
\Delta \arrow[r, "T=T_B"] & \Delta
\end{tikzcd}$$

commutative, where $p(x) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \dots, \frac{x_n}{x_0}\right)$. The 1-time partition of Δ is

$$\Delta(i) = \{x \in \Delta : x_i \geq 1 - x_1 > x_{i+1}\}$$

where $x_0 = 1$, $x_{n+1} = 0$. For digits $i \in \{0, 1, 2, \dots\}$, the matrix $A_T(i)$ is

$$A_T(i) = \begin{array}{cc}
 & \begin{matrix} & & & i & & & \end{matrix} \\
\begin{matrix} \\ \\ \\ i \\ \\ \\ \end{matrix} & \begin{pmatrix} & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ 1 & -1 & & & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}
\end{array} \quad \textbf{on } \Delta(i).$$

The dual space of Brun's algorithm (Δ, T) is $\Delta^{\#} = \mathbb{R}_{\geq} \times [0,1)^{n-1}$ and the 1-time partition of $\Delta^{\#}$ is

$$\Delta^{\#}(0) = \{x \in \Delta^{\#} : x_1 \geq 1\},$$
$$\Delta^{\#}(i) = \{x \in \Delta^{\#} : 0 \leq x_1 < 1, x_j < x_i, 1 \leq j \leq n\}.$$

We found the intertwining map ϕ of Brun algorithm for only $n = 2$:

$$A_\phi = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proposition 3. The Brun algorithm for 2-dimensions is self-dual.

Now, let (Δ, T_{BM}) be a multidimensional continued fraction with matrices

$$A_{T_{BM}}(i) = \begin{array}{c@{}c} & \begin{array}{ccccccc} & & & i & & & \end{array} \\ \begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ \end{array} & \left(\begin{array}{ccccccc} & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ 1 & -N & & & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{array}\right) \end{array} \quad \textbf{on } \Delta[(i, N)].$$

This algorithm is called the Brun multiplicative algorithm. Since $\Delta^{\#} = [0,1]^n$, we can get the invariant measure μ for T_{BM}. The invariant density when $n = 2$ is

$$\frac{1}{(1+x)(1+x+y)} + \frac{1}{(1+y)(1+x+y)}.$$

For the invariant measure in higher dimensions, see [1]. It is known that the multiplicative version is ergodic with respect to μ, see e.g. [12]. When $n = 2$, we found

$$\mu(\Delta[(1,1),(2,1)]) \neq \mu(\Delta[(2,1),(1,1)]),$$
$$\mu(\Delta[(1,2),(2,1)]) \neq \mu(\Delta[(2,1),(1,2)]),$$
$$\vdots$$

This implies that the polar measure τ doesn't coincide with μ.

5 Garrity-Schweiger Algorithm

Let $E^{n+1} := \{x \in \mathbb{R}^{n+1}_{>} : x_0 \geq x_1 \geq \cdots \geq x_n \geq 0\}$. We define

$$G(x) = (x_1, \dots, x_n, x_0 - x_1 - kx_n) \in E^{n+1}, \quad k = \left\lfloor \frac{x_0 - x_1}{x_n} \right\rfloor.$$

With the help of the projection

$$p : E^{n+1} \longrightarrow \Delta, \quad p(x_0, x_1, \dots, x_n) = \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right),$$

we obtain the bottom map $T : \Delta \longrightarrow \Delta$ which makes the diagram

$$\begin{tikzcd}
E^{n+1} \arrow[r, "G"] \arrow[d, "p"'] & E^{n+1} \arrow[d, "p"'] \\
\Delta \arrow[r, "T=T_G"] & \Delta
\end{tikzcd}$$

commutative. The map T is

$$T(x) = \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \cdots \frac{x_n}{x_1}, \frac{1 - x_1 - kx_n}{x_1}\right), \quad k = k(x) = \left\lfloor \frac{1 - x_1}{x_n} \right\rfloor.$$

The 1-time partition of Δ is

$$\Delta(k) = \{x \in B : 1 - x_1 - kx_n \geq 0 > 1 - x_1 - (k+1)x_n\}, \quad k \in \{0, 1, 2, \dots\}$$

and this fibred system is full.

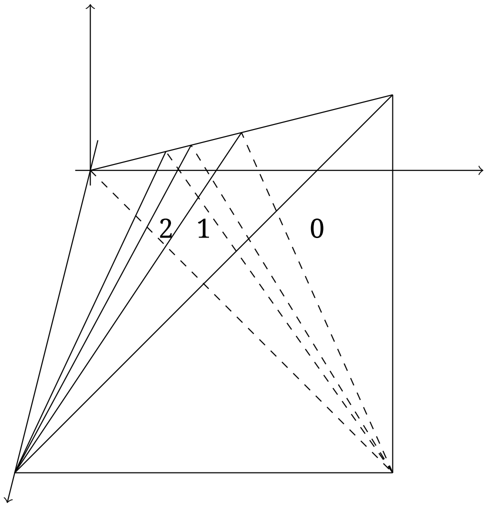

Fig. 3: The 1-time partition of (Δ, T_G) for $n = 3$.

This continued fraction algorithm for $n = 2$ was introduced by Garrity in [4], which is commonly known as the triangle algorithm or Garrity's triangle algorithm. An alternative extension of Garrity's triangle algorithm is studied in [2]. On the other hand, the following map was introduced by Schweiger in [9]:

$$F : x \longmapsto \left(\frac{x_2}{x_1} - \left\lfloor \frac{x_2}{x_1} \right\rfloor, \frac{x_3}{x_1}, \dots, \frac{x_n}{x_1}, \frac{1}{x_1} - 1 \right)$$

on $[0,1) \times \mathbb{R}_{\geq}^{n-1}$. The dynamical system $([0,1) \times \mathbb{R}_{\geq}^{n-1}, F)$ is isomorphic to the dual algorithm of (Δ, T_G). It is known that the (Δ, T_G) is ergodic with respect to the Lebesgue measure for $n = 2$ (see [7]). It was shown that this system is also ergodic with respect to absolutely continuous invariant measures for all dimensions [5]. In this paper, we call the system (Δ, T_G) the Garrity-Schweiger algorithm.

We show that the m.c.f. (Δ, T_G) is algebraic self-dual. Since

$$A_T(k) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & & 1 \\ 1 & -1 & 0 & \cdots & -k \end{pmatrix} \quad \textbf{on } \Delta(k),$$

the dual map $T^{\#}$ is

$$T^{\#}(x) = \left(\frac{1 - x_n}{x_n}, \frac{x_1}{x_n}, \cdots, \frac{x_{n-2}}{x_n}, \frac{x_{n-1} - kx_n}{x_n} \right).$$

This dual map is defined on

$$\Delta^{\#} = \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n-1, 0 \leq x_n < 1\} = \mathbb{R}_{\geq}^{n-1} \times [0,1)$$

and the 1-time partition is given by

$$\Delta^{\#}(k) = \{x \in \Delta^{\#} : x_{n-1} - kx_n \geq 0 > x_{n-1} - (k+1)x_n\}, \quad k = k^{\#}(x) = \left\lfloor \frac{x_{n-1}}{x_n} \right\rfloor.$$

Since the Garrity-Schweiger algorithm is full system, the invariant density h is

$$\int_{\mathbb{R}_{\geq}^{n-1} \times [0,1)} K(x,y)dy \sim \frac{1}{x_1 \cdots x_{n-1}(1 + x_n)}.$$

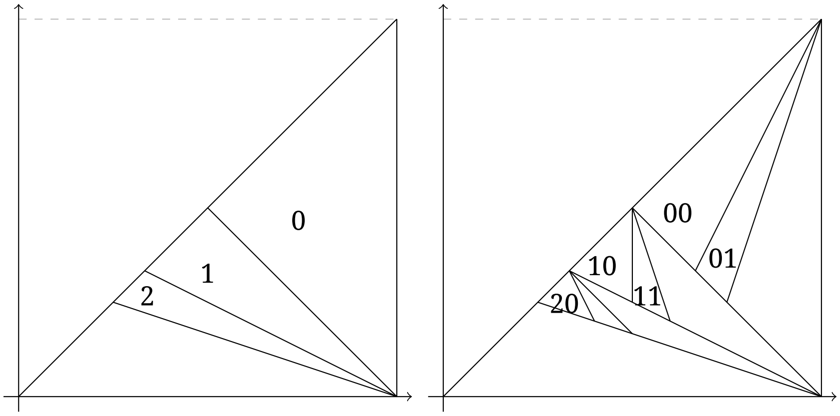

Fig. 4: The 1-time and 2-time partition of (Δ, T_G).

We found that this algorithm is self-dual for $n = 2$ and the matrix A_ϕ is given by

$$A_\phi = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Here, we describe our heuristic method to find such a matrix. First we assume that $A_\phi = ((a_{ij}))$ has integer entries. From $A_\phi A_{T^\#} = A_T A_\phi$, we see A_ϕ is symmetric. Assume that ϕ sends $\Delta^\#(k) \cap \Delta^\#(k+1)$ to $\Delta(k) \cap \Delta(k+1)$. In particular, if $\phi(0,0) = (1,0)$, then we see

$$a_{11} = a_{21}, \quad a_{31} = 0.$$

Put $a_{11} = x$, A_ϕ has to have a form

$$\begin{pmatrix} x & x & 0 \\ x & * & * \\ 0 & * & * \end{pmatrix}.$$

Further if $\phi(k,1) = (\frac{1}{k+1}, \frac{1}{k+1})$, then

$$\frac{x + ka_{22}}{x + kx} = \frac{ka_{32} + a_{33}}{x + kx} = \frac{1}{k+1}.$$

Therefore we have $x = 1$, $a_{22} = a_{32} = 0, a_{33} = 1$ and the condition $A_\phi A_{T^\#}(k) = A_T(k) A_\phi$, $\phi(\Delta^\#(k)) = \Delta(k)$ are guaranteed. Thus, we obtain the following.

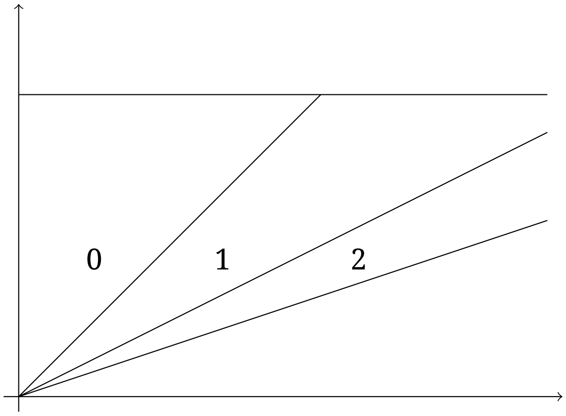

Fig. 5: The 1-time partition of $(\Delta^{\#}, T_G)$.

Proposition 4. The Garrity-Schweiger algorithm is algebraic self-dual. And

$$A_\phi = \begin{pmatrix} 1 & \cdots & 1 & \\ \vdots & \iddots & & \\ 1 & & & \\ & & & 1 \end{pmatrix}.$$

Proof. By a straightforward calculation, we can see that for all $x \in \Delta$

$$\phi \circ T^{\#}(x) = T \circ \phi(x).$$

Let $\phi(B_1, B_2, \dots, B_n) = (b_1, b_2, \dots, b_n)$. Then, since

$$A_\phi^{-1} = \begin{pmatrix} & & & 1 & \\ & & 1 & -1 & \\ & \iddots & \iddots & & \\ 1 & -1 & & & \\ & & & & 1 \end{pmatrix},$$

we have

$$B_{n-1} - \left\lfloor \frac{B_{n-1}}{B_n} \right\rfloor B_n = \frac{1 - b_1 - \left\lfloor \dfrac{1 - b_1}{b_n} \right\rfloor b_n}{b_{n-1}}.$$

Therefore, we can see $\phi(\mathring{\Delta}^{\#}(k)) = \mathring{\Delta}(k)$ for all $k \in \mathbb{Z}_{\geq 0}$. □

Corollary 2. The Garrity-Schweiger algorithm is symmetric in measure.

We introduce the slow version of the Garrity-Schweiger map. F. Schweiger defined the Flip-flop map in [13]. It is known that the jump transformation of the map is Garrity's triangle map (See also [3]). Similarly, we can see that the jump transformation of the n-dimensional Flip-flop map is the Garrity-Schweiger map and it is algebraic self-dual.

Let $\Delta = \{x \in \mathbb{R}^n_> : 1 \geq x_1 \geq \cdots \geq x_n > 0\}$. Let the cylinder set of the Selmer algorithm and Brun algorithm be $\Delta_S(i)$ and $\Delta_B(i)$ respectively. Then, since

$$\Delta_S(i) = \{x \in \Delta : x_i > 1 - x_n \geq x_{i+1}\}, \quad \Delta_B(i) = \{x \in \Delta : x_i > 1 - x_1 \geq x_{i+1}\},$$

and $x_0 = 1$ $x_{n+1} = 0$, we have

$$\Delta = \Delta_S(0) \cup \Delta_B(n).$$

Now, we define the map $T : \Delta \to \Delta$ as

$$A_T = \begin{cases} \begin{pmatrix} 1 & & & -1 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} & \textbf{on } \Delta_S(0), \\ \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & -1 & & \end{pmatrix} & \textbf{on } \Delta_B(n). \end{cases}$$

We consider the jump transformation over the cylinder $\Delta_S(0)$, then we obtain a map with matrices

$$\begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & -1 & & \end{pmatrix} \begin{pmatrix} 1 & & & -1 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}^k = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & -1 & & -k \end{pmatrix}$$

This map is Garrity-Schweiger map T_G. The dual space is $\mathbb{R}^n$, and the invariant density is

$$\frac{1}{x_1 x_2 \cdots x_n}.$$

Proposition 5. The n-dimensional Flip-Flop algorithm is algebraic self-dual. And

$$A_\phi = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \iddots & \\ 1 & & \end{pmatrix}.$$

6 Poincaré Algorithm

Finally, we give an example that is not self-dual. Note that the algorithm below is conjugate to the original Poincaré algorithm. There are maps F, G that the original map is $F \circ G$, but ours is $G \circ F$, where G is the sorting map into non-increasing order. We commonly refer to the projective map of the original map $F \circ G$ is the Parry-Daniel map. See e.g. [8], [12, Chapter 21] or [11, Chapter 14].

Let $E^{n+1} := \{x \in \mathbb{R}^{n+1}_{>} : x_1 \geq x_2 \geq \cdots \geq x_{n+1} \geq 0\}$. Then define

$$F : x \in E^{n+1} \longmapsto x' = (x_1 - x_2, x_2 - x_3, \ldots, x_n - x_{n+1}, x_{n+1}).$$

There is an element σ of symmetric group $\mathcal{S}_{n+1}$ such that

$$P(x) = (x'_{\sigma(1)}, x'_{\sigma(2)}, \ldots, x'_{\sigma(n+1)}) = G \circ F(x) \in E^{n+1}.$$

In this section, we consider the normalized map $T_P : \Delta \longrightarrow \Delta$ which makes the diagram

$$\begin{tikzcd} E^{n+1} \arrow[r, "P"] \arrow[d, "p"'] & E^{n+1} \arrow[d, "p"'] \\ \Delta \arrow[r, "T=T_P"] & \Delta \end{tikzcd}$$

commutative, where $p(x) = \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_{n+1}}{x_1} \right)$.

Then for all digit $\sigma \in \mathcal{S}_{n+1}$, $y = Tx$, $x \in \Delta(\sigma)$ is given by

$$y_i = \frac{\sum_{j=1}^{n+1} A_{\sigma^{-1}(i)j} x_{j-1}}{\sum_{j=1}^{n+1} A_{\sigma^{-1}(1)j} x_{j-1}}$$

where $x_0 = 1$ and

$$A_T(e) = ((A_{i,j})) := \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \\ & & & 1 \end{pmatrix}.$$

The dual map is defined on $\Delta^{\#} = \mathbb{R}^n_>$. Therefore the invariant density is

$$\frac{1}{x_1 x_2 \cdots x_n}.$$

Note that

$$A_T(\sigma) = ((A_{\sigma^{-1}(i)j})) = \begin{array}{cc} & \begin{array}{cccc} & \sigma^{-1}(i) & \sigma^{-1}(i)+1 & n \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} & \vdots & \vdots & \vdots \\ \cdots & 1 & -1 & \\ & & & \\ \cdots & & & 1 \\ & & & \end{pmatrix} \end{array},$$

$$A_{T^{\#}}(\sigma) = \begin{array}{cc} & \begin{array}{ccccc} & \sigma(1) & \sigma(i) & & \sigma(i-1) \end{array} \\ \begin{array}{c} 1 \\ \\ i \\ \\ \end{array} & \begin{pmatrix} \cdots & 1 & \vdots & & \vdots \\ & & & & \\ \cdots & & 1 & \cdots & -1 \\ & & & & \end{pmatrix} \end{array}.$$

Then, we have

$$\Delta(\sigma)=\begin{cases}\{x\in\Delta:\ \ x_{\sigma^{-1}(i)-1}-x_{\sigma^{-1}(i)}>x_{\sigma^{-1}(i+1)-1}-x_{\sigma^{-1}(i+1)},\ \text{for } i=1,2,\dots,n\ (i\neq j-1,j),\\ \qquad\qquad x_{\sigma^{-1}(j-1)-1}-x_{\sigma^{-1}(j-1)}>x_n,\ x_n>x_{\sigma^{-1}(j+1)-1}-x_{\sigma^{-1}(j+1)}\},\ \text{if } j\neq 1,n+1,\\ \\ \{x\in\Delta:\ \ x_n>x_{\sigma^{-1}(2)-1}-x_{\sigma^{-1}(2)},\\ \qquad\qquad x_{\sigma^{-1}(i)-1}-x_{\sigma^{-1}(i)}>x_{\sigma^{-1}(i+1)-1}-x_{\sigma^{-1}(i+1)},\ \text{for } i=2,\dots,n\},\ \text{if } j=1,\\ \\ \{x\in\Delta:\ \ x_{\sigma^{-1}(i)-1}-x_{\sigma^{-1}(i)}>x_{\sigma^{-1}(i+1)-1}-x_{\sigma^{-1}(i+1)},\ \text{for } i=1,2,\dots,n-1,\\ \qquad\qquad x_{\sigma^{-1}(n)-1}-x_{\sigma^{-1}(n)}>x_n\},\ \text{if } j=n+1\end{cases}$$

and

$$\Delta^{\#}(\sigma)=\{x\in\Delta^{\#}\ :\ x_{\sigma(i+1)-1}-x_{\sigma(i)-1}>0\ \text{for } i=1,\dots,n\}$$

where $x_0=1$ and j is a integer satisfies $\sigma(n+1)=j$.

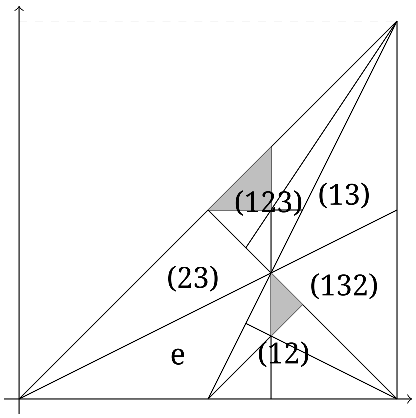

Fig. 6: $\Delta[(12),(123)]$ and $\Delta[(123),(12)]$ for $n=2$

In the case $n=2$, the domain Δ has six partitions $\Delta[e]$, $\Delta[(12)]$, $\Delta[(23)]$, $\Delta[(13)]$, $\Delta[(123)]$, and $\Delta[(132)]$ (see Fig. 6). By direct computation we have

$$\mu(\Delta[(12),(123)]) = \int_{[\frac{2}{3},\frac{3}{4})}\int_{[x-\frac{1}{2},-x+1)} \frac{1}{xy}dxdy = \frac{\log 2}{2}\log\frac{9}{8} = 0.0408\cdots,$$

$$\mu(\Delta[(123),(12)]) = \int_{[\frac{1}{2},\frac{2}{3})}\int_{[\frac{1}{2},x)} \frac{1}{xy}dxdy = \frac{1}{2}\left(\log\frac{4}{3}\right)^2 = 0.0413\cdots.$$

Thus,

$$\mu(\Delta[(12),(123)]) \neq \mu(\Delta[(123),(12)]).$$

From Theorem 1, this algorithm is not algebraic self-dual for $n = 2$.

All the same, we shall prove that this algorithm is algebraic self-dual on $\{e,(13),(123),(132)\}$ with the intertwining map ϕ defined as

$$A_\phi = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let us explain our empirical method to find this intertwining map. At first, we follow the heuristic way as before. Assume that $A_\phi = ((a_{ij}))$ has integer entries. From $A_\phi A_{T^\#}(\sigma) = A_T(\sigma)A_\phi$, we see it is symmetric. If $\phi(1,1) = (\frac{2}{3},\frac{1}{3})$, then we see

$$a_{11}+a_{12}+a_{13} : a_{21}+a_{22}+a_{23} : a_{31}+a_{32}+a_{33} = 3 : 2 : 1. \tag{2}$$

Assume for now that $\Delta^\#[e]\cap\Delta^\#[(12)]$ is mapped to $\Delta[e]\cap\Delta[(12)]$. Then from

$$\phi(1,y) = \left(\frac{a_{21}+a_{22}+a_{23}y}{a_{11}+a_{12}+a_{13}y}, \frac{a_{31}+a_{32}+a_{33}y}{a_{11}+a_{12}+a_{13}y}\right),$$

if $\lim_{y\to\infty}\phi(1,y) = (\frac{1}{2},0)$, then there exists an integer k that we have

$$A_\phi = \begin{pmatrix} * & * & 2k \\ * & * & k \\ 2k & k & 0 \end{pmatrix}.$$

However in this case, it is natural to assume $\lim_{x\to\infty}\phi(x,x)=(0,0)$, and then we have

$$a_{22}+a_{23}=a_{32}=0.$$

This implies $k=0$ and clearly we have $\phi(\Delta^{\#}(\sigma))\neq\Delta(\sigma)$ which does not fit our purpose. After this wrong trial, we reach the correct assumption that $\Delta^{\#}[e]\cap\Delta^{\#}[(12)]$ is mapped to $\Delta[e]\cap\Delta[(23)]$ (See Fig. 7). Indeed if $\lim_{y\to\infty}\phi(1,y)=(0,0)$, then A_ϕ has the form

$$A_\phi=\begin{pmatrix} * & * & * \\ * & * & 0 \\ * & 0 & 0 \end{pmatrix}.$$

From $\lim_{x\to\infty}\phi(x,x)=(\frac{1}{2},0)$, we obtain

$$\frac{a_{22}}{a_{12}+a_{13}}=\frac{1}{2}.$$

Considering (2), by several trials we found

$$A_\phi=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

which satisfies all the conditions on $\Delta[e]\cup\Delta[(13)]\cup\Delta[(123)]\cup\Delta[(132)]$. (See Appendix A.)

We define an involution on $\mathcal{S}_n$.

Definition 6. We denote the set of involutions of the symmetric group by

$$Inv(\mathcal{S}_n)=\{\sigma\in\mathcal{S}_n\,:\,\sigma^2=e\}.$$

The cardinality of this set $\#Inv(\mathcal{S}_n)$:$1,2,4,10,26,76,\cdots$ are also known as telephone numbers and various studies have been made on these numbers (see Section 5.1.4 of [6]).

Theorem 2. The n-dimensional Poincaré algorithm is algebraic self-dual on $w_0Inv(\mathcal{S}_{n+1})$ where

$$w_0=\begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}.$$

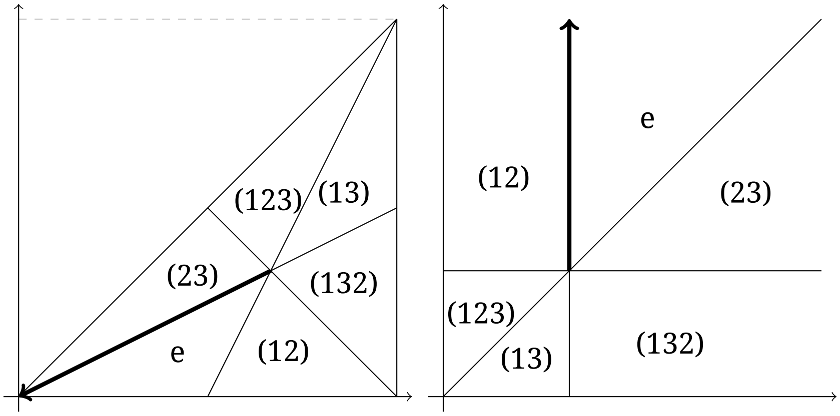

Fig. 7: The 1-time partition of (Δ, T_P) and $(\Delta^{\#}, T_P^{\#})$.

And

$$A_{\phi} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \iddots & \\ 1 & & \end{pmatrix}.$$

Proof. Let $M = ((a_{i,j}))$ be a monomial $(0,1)$–matrix with $a_{i,j} = 1$. i.e., there is exactly one “1” in each row and each column. This has the one-to-one correspondence with permutation, we denote

$$M \longleftrightarrow \begin{pmatrix} \cdots & i & \cdots \\ \cdots & j & \cdots \end{pmatrix}.$$

Then, we have

$$w_0\sigma^{-1} = \begin{pmatrix} \sigma(1) & \sigma(2) & \cdots & \sigma(n+1) \\ n+1 & n & \cdots & 1 \end{pmatrix} \longleftrightarrow (A_{\sigma^{-1}(i)j})B,$$

$$\sigma w_0 = \begin{pmatrix} 1 & 2 & \cdots & n+1 \\ \sigma(n+1) & \sigma(n) & \cdots & \sigma(1) \end{pmatrix} \longleftrightarrow B(A_{\sigma^{-1}(i)j})^t.$$

Therefore, we have

$$\begin{aligned}\{\sigma \in \mathcal{S}_{n+1} : B(A_{\sigma^{-1}(i)j})^t = (A_{\sigma^{-1}(i)j})B\} &= \{\sigma \in \mathcal{S}_{n+1} : (w_0\sigma)^2 = e\} \\ &= w_0 Inv(\mathcal{S}_{n+1}).\end{aligned}$$

Let $\phi(B_1, B_2, \dots, B_n) = (b_1, b_2, \dots, b_n)$ and $\sigma \in w_0 Inv(\mathcal{S}_{n+1})$. By the definition of $Inv(\mathcal{S}_{n+1})$,

$$\begin{aligned}\sigma^{-1}(i) &= w_0\sigma w_0(i) \\ &= w_0\sigma(n+1-i+1) \\ &= n+1-\sigma(n+1-i+1)+1\end{aligned}$$

and we have

$$n - \sigma(n-i+2) + 1 = \sigma^{-1}(i) - 1. \tag{3}$$

We show $\phi(\Delta^{\#}(\sigma)) \supset \Delta(\sigma)$. Let $(b_1, b_2, \dots, b_n) \in \Delta(\sigma)$.

For $i \neq n-j+1, n-j+2$, by (3),

$$\begin{aligned}B_{\sigma(i+1)-1} - B_{\sigma(i)-1} &= \frac{b_{n-\sigma(i+1)+1} - b_{n-\sigma(i+1)+2}}{b_n} - \frac{b_{n-\sigma(i)+1} - b_{n-\sigma(i)+2}}{b_n} \\ &= \frac{b_{\sigma^{-1}(n-i+1)-1} - b_{\sigma^{-1}(n-i+1)} - b_{\sigma^{-1}(n-i+2)-1} + b_{\sigma^{-1}(n-i+2)}}{b_n} \\ &> 0.\end{aligned}$$

For $i \neq n-j+1, n-j+2$, since $\sigma(n+1) = j$ and $\sigma \in w_0 Inv(\mathcal{S}_{n+1})$, by (3), $\sigma(n-j+2) = 1$. Then we have

$$\begin{aligned}B_{\sigma(n-j+2)-1} - B_{\sigma(n-j+1)-1} &= 1 - \frac{b_{n-\sigma(n-j+1)+1} - b_{n-\sigma(n-j+1)+2}}{b_n} \\ &= \frac{b_n - b_{\sigma^{-1}(j+1)-1} + b_{\sigma^{-1}(j+1)}}{b_n} \\ &> 0\end{aligned}$$

and

$$\begin{aligned}B_{\sigma(n-j+3)-1} - B_{\sigma(n-j+2)-1} &= \frac{b_{n-\sigma(n-j+3)+1} - b_{n-\sigma(n-j+3)+2}}{b_n} - 1\\ &= \frac{b_{\sigma^{-1}(j-1)-1} - b_{\sigma^{-1}(j-1)} - b_n}{b_n}\\ &> 0.\end{aligned}$$

Similarly, we have $\phi(\Delta^{\#}(\sigma)) \subset \Delta(\sigma)$. $\square$

Corollary 3. The n-dimentional Poincaré algorithm is symmetric in measure on $w_0 Inv(\mathcal{S}_{n+1})$, i.e., for all $\sigma_1, \sigma_2, \dots, \sigma_s \in w_0 Inv(\mathcal{S}_{n+1})$,

$$\mu(\Delta[\sigma_1, \sigma_2, \dots, \sigma_s]) = \mu(\Delta[\sigma_s, \sigma_{s-1}, \dots, \sigma_1]).$$

A Self-duality of 2-dimensional Poincaré Algorithm

We see that the 2-dimensional Poincaré Algorithm (Δ, T_P) is algebraic self-dual on $\{e, (13), (123), (132)\}$ with the intertwining map ϕ defined as

$$A_\phi = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \\ 1 & & \end{pmatrix}.$$

Note that $\Delta = \{\mathbf{x} \in \mathbb{R}^2 : 1 \geq x_1 \geq x_2 > 0\}$, $\Delta^{\#} = \mathbb{R}^2_{\geq}$.

$\underline{\Delta[e]}$; Since

$$A_T(e) = \begin{pmatrix} 1 & -1 & \\ & 1 & -1 \\ & & 1 \end{pmatrix}, \ A_{T^{\#}}(e) = \begin{pmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \end{pmatrix},$$

$A_T(e)A_\phi = A_\phi A_{T^{\#}}(e)$ holds. And

$$T(b_1, b_2) = \left(\frac{b_1 - b_2}{1 - b_1}, \frac{b_2}{1 - b_1}\right), \quad T^{\#}(B_1, B_2) = (-1 + B_1, -B_1 + B_2).$$

Thus, the cylinder set $\Delta[e]$ and $\Delta^{\#}[e]$ are

$$\begin{aligned}\Delta[e] &= \{(b_1, b_2) \in \Delta \,:\, 1 - b_1 > b_1 - b_2,\ b_1 - b_2 > b_2\} \\ &= \{(b_1, b_2) \in \Delta \,:\, 1 - 2b_1 + b_2 > 0,\ b_1 > 2b_2\}, \\ \Delta^{\#}[e] &= \left\{(B_1, B_2) \in \Delta^{\#} \,:\, -1 + B_1 > 0,\ -B_1 + B_2 > 0\right\}.\end{aligned}$$

We show $\phi(\Delta^{\#}[e]) = \Delta[e]$. Let $\phi(B_1, B_2) = (b_1, b_2)$. Then,

$$b_1 = \frac{1 + B_1}{1 + B_1 + B_2},\ b_2 = \frac{1}{1 + B_1 + B_2}$$

and

$$B_1 = \frac{b_1}{b_2} - 1,\ B_2 = \frac{1}{b_2} - \frac{b_1}{b_2}.$$

Thus, we can see $\phi(\Delta^{\#}[e]) = \Delta[e]$ by the following calculation

$$1 - 2b_1 + b_2 = \frac{-B_1 + B_2}{1 + B_1 + B_2}, \quad b_1 - 2b_2 = \frac{-1 + B_1}{1 + B_1 + B_2}$$

and

$$B_1 - 1 = \frac{b_1}{b_2} - 2, \quad B_2 - B_1 = \frac{1 - 2b_1 + b_2}{b_2}.$$

$\underline{\Delta[(12)]}$;

$$A_T((12))A_\phi = \begin{pmatrix} & 1 & -1 \\ 1 & -1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \\ 1 & & \end{pmatrix} = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix},$$

$$A_\phi A_T^{\#}((12)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \\ 1 & & \end{pmatrix}\begin{pmatrix} & 1 & \\ 1 & -1 & \\ -1 & & 1 \end{pmatrix} = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}.$$

$\underline{\Delta[(23)]}$;

$$A_T((23))A_\phi = \begin{pmatrix} 1 & -1 & \\ & & 1 \\ & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \\ 1 & & \end{pmatrix} = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix},$$

$$A_\phi A_T^{\#}((23)) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \\ 1 & & \end{pmatrix}\begin{pmatrix} 1 & & \\ -1 & & 1 \\ & 1 & -1 \end{pmatrix} = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}.$$

$\underline{\Delta[(13)]}$; Since

$$A_T((13)) = \begin{pmatrix} & & 1 \\ & 1 & -1 \\ 1 & -1 & \end{pmatrix},$$

$A_T((13))A_\phi = A_\phi A_{T^{\#}}((13))$ holds.

$$\begin{aligned} \Delta[(13)] &= \{(b_1, b_2) \in \Delta \,:\, b_2 > b_1 - b_2,\ b_1 - b_2 > 1 - b_1\} \\ &= \{(b_1, b_2) \in \Delta \,:\, 2b_2 - b_1 > 0,\ 2b_1 - b_2 - 1 > 0\}, \\ \Delta^{\#}[(13)] &= \left\{(B_1, B_2) \in \Delta^{\#} \,:\, B_1 - B_2 > 0,\ 1 - B_1 > 0\right\}. \end{aligned}$$

We show $\phi(\Delta^{\#}[e]) = \Delta[e]$. Let $\phi(B_1, B_2) = (b_1, b_2)$. Then, we can see $\phi(\Delta^{\#}[(13)]) = \Delta[(13)]$ by the following calculation

$$2b_2 - b_1 = \frac{1 - B_1}{1 + B_1 + B_2}, \quad 2b_1 - b_2 - 1 = \frac{B_1 - B_2}{1 + B_1 + B_2}$$

and

$$B_1 - B_2 = \frac{2b_1 - b_2 - 1}{b_2}, \quad 1 - B_1 = 2 - \frac{b_1}{b_2}.$$

$\underline{\Delta[(123)]}$; Since

$$A_T((123)) = \begin{pmatrix} & & 1 \\ 1 & -1 & \\ & 1 & -1 \end{pmatrix}, \quad A_{T^\#}((123)) = \begin{pmatrix} & 1 & \\ & -1 & 1 \\ 1 & & -1 \end{pmatrix},$$

$A_T(123)A_\phi = A_\phi A_{T^\#}(123)$ holds.

$$\begin{aligned} \Delta[(123)] &= \{(b_1, b_2) \in \Delta : b_2 > 1 - b_1,\ 1 - b_1 > b_1 - b_2\} \\ &= \{(b_1, b_2) \in \Delta : b_1 + b_2 - 1 > 0,\ 1 - 2b_1 + b_2 > 0\}, \\ \Delta^\#[(123)] &= \left\{(B_1, B_2) \in \Delta^\# : -B_1 + B_2 > 0,\ 1 - B_2 > 0\right\}. \end{aligned}$$

We show $\phi(\Delta^\#[(123)]) = \Delta[(123)]$. Let $\phi(B_1, B_2) = (b_1, b_2)$. Then, we can see $\phi(\Delta^\#[(123)]) = \Delta[(123)]$ by the following calculation

$$b_1 + b_2 - 1 = \frac{1 - B_2}{1 + B_1 + B_2}, \quad 1 - 2b_1 + b_2 = \frac{-B_1 + B_2}{1 + B_1 + B_2}$$

and

$$-B_1 + B_2 = \frac{1 - 2b_1 + b_2}{b_2}, \quad 1 - B_2 = \frac{b_1 + b_2 - 1}{b_2}.$$

$\underline{\Delta[(132)]}$; Since

$$A_T((132)) = \begin{pmatrix} & 1 & -1 \\ & & 1 \\ 1 & -1 & \end{pmatrix}, \quad A_{T^\#}((132)) = \begin{pmatrix} & & 1 \\ 1 & & -1 \\ -1 & 1 & \end{pmatrix},$$

$A_T(132)A_\phi = A_\phi A_{T^\#}(132)$ holds.

$$\begin{aligned} \Delta[(132)] &= \{(b_1, b_2) \in \Delta : b_1 - b_2 > b_2,\ b_2 > 1 - b_1\} \\ &= \{(b_1, b_2) \in \Delta : b_1 - 2b_2 > 0,\ b_1 + b_2 - 1 > 0\}, \\ \Delta^\#[(132)] &= \left\{(B_1, B_2) \in \Delta^\# : 1 - B_2 > 0,\ -1 + B_1 > 0\right\}. \end{aligned}$$

We show $\phi(\Delta^{\#}[(132)]) = \Delta[(132)]$. Let $\phi(B_1, B_2) = (b_1, b_2)$. Then, we can see $\phi(\Delta^{\#}[(132)]) = \Delta[(132)]$ by the following calculation

$$b_1 - 2b_2 = \frac{-1 + B_1}{1 + B_1 + B_2}, \quad b_1 + b_2 - 1 = \frac{1 - B_2}{1 + B_1 + B_2}$$

and

$$1 - B_2 = \frac{b_1 + b_2 - 1}{b_2}, \quad -1 + B_1 = \frac{b_1}{b_2} - 2.$$

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