CrossMark

Non-avoided Crossings for n-Body Balanced Configurations in $\mathbb{R}^3$ Near a Central Configuration

Alain Chenciner[1,2]

Abstract The balanced configurations are those n-body configurations which admit a relative equilibrium motion in a Euclidean space E of high enough dimension $2p$. They are characterized by the commutation of two symmetric endomorphisms of the $(n-1)$-dimensional Euclidean space of codispositions, the intrinsic inertia endomorphism B which encodes the shape and the Wintner–Conley endomorphism A which encodes the forces. In general, p is the dimension d of the configuration, which is also the rank of B. Lowering to $2(d-1)$ the dimension of E occurs when the restriction of A to the (invariant) image of B possesses a double eigenvalue. It is shown that, while in the space of all $d \times d$ symmetric endomorphisms, having a double eigenvalue is a condition of codimension 2 (the avoided crossings of physicists), here it becomes of codimension 1 provided some condition (H) is satisfied. As the condition is always satisfied for configurations of the maximal dimension (i.e. if $d = n-1$), this implies in particular the existence, in the neighborhood of the regular tetrahedron configuration of four bodies with no three of the masses equal, of exactly three families of balanced configurations which admit relative equilibrium motion in a four dimensional space.

Keywords n-body problem · Central configurations · Balanced configurations · Relative equilibria · Bifurcations

Contents

✉ Alain Chenciner
  Alain.Chenciner@obspm.fr

1 Observatoire de Paris, IMCCE (UMR 8028), ASD, Paris, France

2 Département de mathématique, Université Paris VII, Paris, France

1 Central Configurations, Balanced Configurations and Their Relative Equilibria

Central configurations are the N-body configurations which collapse homothetically on their center of mass when released without initial velocity; they are known since Euler and Lagrange to admit periodic homographic motions of all eccentricities and in particular periodic relative equilibrium motions. More generally, *balanced configurations* (see Albouy and Chenciner 1998; Chenciner 2013b) are the N-body configurations which admit a (in general quasi-periodic) relative equilibrium motion in a Euclidean space of high enough dimension. They are characterized by the commutation of two endomorphisms of the *codisposition space* $\mathcal{D}^*$, the first one A which characterizes the attraction forces between the bodies of the configuration, the second one B, an intrinsic inertia which encodes the shape of the configuration.

1.1 From B to A: Shapes and Forces

An n-body configuration $x = (\vec{r}_1, \cdots, \vec{r}_n)$ up to translation in the Euclidean space[1] (E, ϵ) is a mapping

---

[1] The Euclidean structure is identified with an isomorphism $\epsilon : E \to E^*$ from E to its dual.

$$x : \mathcal{D}^* \to E, \ (\xi_1, \ldots, \xi_n) \mapsto \sum_{i=1}^{n} \xi_i \vec{r}_i,$$

equivalently an element of $\mathcal{D} \otimes E$, where

$$\mathcal{D} := \mathbb{R}^n / (1, \ldots, 1)\mathbb{R}$$

is the *dispositions* space and $\mathcal{D}^* = \{(\xi_1, \ldots, \xi_n) | \sum_{i=1}^{n} \xi_i = 0\}$ is its dual. Fixing the masses naturally endows $\mathcal{D}$ (resp. $\mathcal{D}^*$) with the *mass Euclidean structure* $\mu : \mathcal{D} \to \mathcal{D}^*$ defined by

$$\mu(x_1, \ldots, x_n) = (m_1(x_1 - x_G), \ldots, m_n(x_n - x_G)),$$

where $x_G = (m_1 x_1 + \cdots + m_n x_n)/\sum m_i$ is the center of mass of the x_i (resp. $\mu^{-1} : \mathcal{D}^* \to \mathcal{D}$ defined by $\mu^{-1}(\xi_1, \ldots, \xi_n) = (\frac{\xi_1}{m_1}, \ldots, \frac{\xi_n}{m_n}))$.

The *intrinsic inertia form* (resp. the *(dual) inertia form*) of the configuration x is the quadratic form β on $\mathcal{D}^*$ (resp. the quadratic form b on E^*), defined by

$$\beta = x^*\epsilon = x^{tr} \circ \epsilon \circ x \in Hom_s(\mathcal{D}^*, \mathcal{D}) \equiv Q(\mathcal{D}^*) \equiv \mathcal{D} \odot \mathcal{D},$$
$$\text{resp.} \quad b = (x^{tr})^*\mu = x \circ \mu \circ x^{tr} \in Hom_s(E^*, E) \equiv Q(E^*) \equiv E \odot E.$$

The form β defines the configuration up to a rigid motion (translation and rotation) in E. The *intrinsic inertia endomorphism* (resp. the *(dual) inertia endomorphism*) are respectively the μ^{-1}-symmetric (resp. ϵ-symmetric) endomorphisms

$$B = \mu \circ \beta : \mathcal{D}^* \to \mathcal{D}^* \quad (\text{resp.} \quad S = b \circ \epsilon : E \to E).$$

Let

$$U(x) = \sum_{i<j} m_i m_j \Phi(r_{ij}^2), \quad \text{where} \quad \Phi(s) = \mathcal{G}s^{-\frac{1}{2}},$$

be the potential[2] of the n-body configuration x. Its invariance under isometries implies the factorization $U(x) = \hat{U}(\beta) = \tilde{U}(B)$. The Wintner–Conley endomorphism associated to x_0 is the $(\mu^{-1} - symmetric)$ endomorphism of $\mathcal{D}^*$ defined by

$$A = d\hat{U}(\beta) \circ \mu^{-1} = \mu \circ d\tilde{U}(B) \circ \mu^{-1}.$$

It is characterized (see Albouy and Chenciner 1998; Chenciner 2013b) by the fact that the equations of motion are

$$\ddot{x} = 2x \circ A.$$

---

[2] Nothing essential would change for a more general homogeneous Φ.

In some μ^{-1}-orthonormal basis, the two μ^{-1}-symmetric endomorphisms A and B of $\mathcal{D}^*$, are represented by symmetric matrices. Giving any one of these two matrices is equivalent to giving the squared mutual distances $s_{ij} = r_{ij}^2$, that is defining the configuration up to isometry (see Albouy and Chenciner 1998), hence the mapping $F : B \mapsto A$ is well defined outside of the collisions (that is when all the r_{ij} are strictly positive) and bijective on its image: *the shape of the configuration determines the forces and the forces determine the shape.* Moreover,

Lemma 1 *The mapping F is a diffeomorphism.*

Indeed, in well chosen bases of the space of symmetric matrices, the coordinates of B are the squared mutual distances $s_{ij} = r_{ij}^2$, $1 \le i < j < n$ while the coordinates of A are the $\varphi(s_{ij}) := \Phi'(s_{ij}) = -\frac{1}{2}\mathcal{G}s_{ij}^{-\frac{3}{2}}$.

1.2 Balanced Configurations and Their Relative Equilibria

A *rigid motion* is a solution of the equations of motion along which the mutual distances r_{ij} remain constant. Such a motion is necessarily a *relative equilibrium*, that is an equilibrium of the equations after reduction of their natural symmetry under isometries (Albouy and Chenciner 1998, Proposition 2.5). Moreover, relative equilibria are of the following form (Albouy and Chenciner 1998, Propositions 2.8 and 2.9):

$$x(t) = e^{\Omega t}x,$$

where Ω is a constant ϵ-antisymmetric[3] isomorphism[4] of the (necessarily even dimensional) Euclidean space (E, ϵ) and x belongs to the very special class of *balanced configurations* which we now characterize: it follows from the equations of motion that

$$\Omega^2 \circ x = 2x \circ A.$$

From now on, we shall identify $\mathcal{D}$ and E with their respective duals $\mathcal{D}^*$ and E^* using their Euclidean structures μ and ϵ. Moreover, we shall choose a μ^{-1}-orthonormal basis of $\mathcal{D}^*$ and an ϵ-orthonormal basis of E and represent endomorphisms of $\mathcal{D}^*$ and endomorphisms of E by matrices in such bases.

If X is the $2p \times (n-1)$ matrix representing x_0 in these bases, we have:

$$S = XX^{tr}, \quad B = X^{tr}X, \quad \Omega^2 X = 2XA.$$

From the symmetry of the matrices $2BA = X^{tr}\Omega^2 X$ and $\Omega^2 S = 2XAX^{tr}$, we deduce the vanishing of the following commutators:
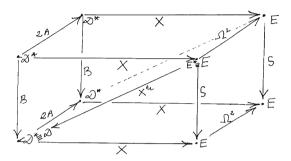
$$[A, B] = 0, \quad [\Omega^2, S] = 0.$$

---

[3] I.e. such that $\varpi = \epsilon \circ \Omega = \Omega^{tr} \circ \epsilon = \varpi^{tr} \in \Lambda^2 E^*$.

[4] It is assumed that E is the space effectively visited by the motion.

Of course, as Ω^2 commutes with $e^{\Omega t}$, one can replace S by the inertia endomorphism $S(t) = X(t)X(t)^{tr}$. The first equation, independent of the dimension of the ambient space, was shown in Albouy and Chenciner (1998) to define the balanced configurations. The following diagram summarizes these relations (on the left, the "side of the bodies", on the right the "side of ambient space"):

In terms of mutual distances, the equations of balanced configurations are (Albouy and Chenciner 1998)

$$P_{ijk} = -\frac{1}{2}\nabla_{ijk} + \frac{1}{2}\sum_{l \neq i,j,k} Y_{ijk}^l = 0, \ i < j < k,$$

with (recalling that $\varphi(s) = -\frac{1}{2}s^{-\frac{3}{2}}$)

$$\nabla_{ijk} = \begin{vmatrix} 1 & 1 & 1 \\ m_i\left(r_{jk}^2 - r_{ki}^2 - r_{ij}^2\right) & m_j\left(r_{ki}^2 - r_{ij}^2 - r_{jk}^2\right) & m_k\left(r_{ij}^2 - r_{jk}^2 - r_{ki}^2\right) \\ \varphi(r_{jk}^2) & \varphi(r_{ki}^2) & \varphi(r_{ij}^2) \end{vmatrix},$$

$$Y_{ijk}^l = m_l \begin{vmatrix} 1 & 1 & 1 \\ r_{jk}^2 + r_{il}^2 & r_{ki}^2 + r_{jl}^2 & r_{ij}^2 + r_{kl}^2 \\ \varphi(r_{il}^2) & \varphi(r_{jl}^2) & \varphi(r_{kl}^2) \end{vmatrix}.$$

As soon as the number n of bodies is greater than three, these equations are not independent: they are obtained by taking the exterior product by $(1, 1, \ldots, 1)$ of the equation $[A, B] = 0$, that is by embedding the space $\wedge^2 \mathcal{D}$, whose dimension is $\binom{n}{2} = \frac{(n-1)(n-2)}{2}$, into the space $\wedge^3 \mathbb{R}^n$, whose dimension is $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$.

Finally, let us recall (see Albouy and Chenciner 1998) that the balanced (resp. central) configurations are those configurations whose intrinsic inertia endomorphism B is a critical point of the potential function $\tilde{U}$ restricted to its *isospectral submanifold*, consisting in all the μ^{-1}-symmetric endomorphisms with the same spectrum (resp. restricted to the submanifold of symmetric μ^{-1}-symmetric endomorphisms which have the same trace I).

1.3 The Invariant Subspaces of Ω and the Minimal Dimension of E

Each n-body balanced configuration x admits relative equilibrium motions in a Euclidean space E of high enough even dimension (twice the rank of x, that is twice the rank of B, suffices, see Albouy and Chenciner 1998, Proposition 2.8). The smallest possible dimension allowing such a motion depends on the multiplicities of the eigenvalues of the Wintner–Conley endomorphism A (Albouy and Chenciner 1998, Remark 2.11). In order to explain this, we study the invariant subspaces of the instantaneous rotation matrix Ω.

Fixing the balanced configuration x, we choose a μ^{-1}-orthonormal basis of $\mathcal{D}^*$ and an ϵ-orthonormal basis of E such that the matrices B and S representing the inertia endomorphisms of x be diagonal:

$$\begin{cases} B = diag(b_1, \ldots, b_{n-1}), \\ S = diag(\sigma_1, \ldots, \sigma_{2p}). \end{cases}$$

From the above mentioned commutations, it follows that such bases can be chosen so as to satisfy also

$$\begin{cases} 2A = diag(-\lambda_1, \ldots, -\lambda_{n-1}), \\ \Omega^2 = diag(-\omega_1^2, \ldots, -\omega_{2p}^2). \end{cases}$$

Note that all the eigenvalues of $2A$ are strictly negative because the Newton force is attractive. The non-zero eigenvalues of $S = XX^{tr}$ and $B = X^{tr}X$ are the same; their number is the dimension of the configuration, that is the rank $d = \dim\mathrm{Im}X$ of X. Moreover, as $\mathrm{Im}\,X = \mathrm{Im}\,S$ is generated by vectors of the basis of E, we can suppose, after a possible reordering of the bases of $\mathcal{D}^*$ and E, that X is of the form $X = \begin{pmatrix} V & W \\ 0 & 0 \end{pmatrix}$, where the $d \times d$ upper left block V is invertible. Moreover, as $B = X^{tr}X$ is diagonal, $W = 0$, that is

$$X = \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{with} \quad V : \mathrm{Im}B \to \mathrm{Im}S = \mathrm{Im}X = \mathrm{Im}x \quad \text{an isomorphism,}$$

and $VV^{tr} = diag(\sigma_1, \ldots, \sigma_d)$, $V^{tr}V = diag(b_1, \ldots, b_d)$, hence

$$diag(\sigma_1, \ldots, \sigma_d)V = Vdiag(b_1, \ldots, b_d).$$

Finally, the equation $\Omega^2 X = 2XA$, is equivalent to

$$diag(-\omega_1^2, \ldots, -\omega_d^2)V = Vdiag(-\lambda_1, \ldots, -\lambda_d),$$

hence, after possibly replacing V by its product VP with a permutation matrix, which amounts to permuting the first d vectors of the basis of $\mathcal{D}^*$, which generate $\mathrm{Im}B$, one can suppose that

$$\omega_k^2 = \lambda_k, \quad \text{for} \quad k = 1, \ldots, d,$$

while the b_i, $i = 1, \ldots, d$, are a permutation of the σ_i, $i = 1, \ldots, d$.

This shows in particular, and this comes as no surprise, that it is only the restriction A_B of A to the image of B which plays a role. Recall that a necessary and sufficient condition for x_0 to be a central configuration is that the restriction of A to this subspace be proportional to the Identity, that is $\lambda_1 = \lambda_2 = \cdots = \lambda_d$.

Notation. *Bases of $\mathcal{D}^*$ and E with the properties above will be denoted respectively* $\{u_1, \ldots, u_{n-1}\}$ *and* $\{\rho_1, \ldots, \rho_{2p}\}$. *In particular,* $u_1, \ldots, u_d$ *generate* $\text{Im} B$ *and* $\rho_1, \ldots, \rho_d$ *generate* $\text{Im} x = \text{Im} S$.

The real invariant planes of Ω can be generated either by the couple formed by a vector $\rho_k \in \text{Im} x$ and a vector ρ_{d+l} in the orthogonal $(\text{Im} x)^{\perp}$, or by the couple formed by two vectors $\rho_k, \rho_l \in \text{Im} x$, both associated with the same eigenvalue $\lambda_k = \lambda_l$ of A. Similar descriptions hold for higher dimensional invariant subspaces. It follows that, writing $\tilde{\omega}_1^2, \ldots, \tilde{\omega}_r^2$ the distinct values taken by the $\omega_k^2 = \lambda_k, k = 1, \ldots, d$, a space E of minimal dimension where a relative equilibrium motion with such a configuration may take place decomposes into a direct sum $E_1 \oplus \cdots \oplus E_r$ of eigenspaces of Ω, which are complex[5] spaces (E_l, J_l), and the motion is quasi-periodic of the form

$$x(t) = (x_1(t), \ldots, x_r(t)), \quad \text{with} \quad x_l(t) = e^{\tilde{\omega}_l t} x_l, \quad l = 1, \ldots, r.$$

When $r = 1$, that is when $\lambda_1 = \cdots = \lambda_d = \tilde{\omega}^2$, which means that the configuration x is central, the motion becomes periodic, of the form $x(t) = e^{\tilde{\omega} J t} x$, with J a complex (hermitian) structure on E.

Let us denote by

$$\varpi = \epsilon \circ \Omega \in \wedge^2 E^*$$

the instantaneous rotation bivector. The main possibilities of invariant spaces for Ω can be read on the inverse image of ϖ by x:

$$x^* \varpi = x^{tr} \circ \varpi \circ x = -\rho,$$

which is nothing (up to sign) but the antisymmetric part of $x^{tr} \circ \epsilon \circ y$:

$$\rho = \frac{1}{2} \left(-x^{tr} \circ \epsilon \circ y + y^{tr} \circ \epsilon \circ x \right),$$

which was introduced in Lagrange (1772) in the case of three bodies and in Albouy and Chenciner (1998) in the general case of n bodies. In term of matrices, the endomorphism $R = \mu \circ \rho$ of $\mathcal{D}^*$ is represented by

$$R = \frac{1}{2} \left(-X^{tr} Y + Y^{tr} X \right) = -X^{tr} \Omega X.$$

---

[5] More precisely "hermitian", that is such that each J_l is an isometry.

Indeed, only contribute to ρ the invariant subspaces of Ω entirely contained in $\operatorname{Im} x_0$. In particular, ρ vanishes in the generic case where the eigenvalues λ_i of $A|_{\operatorname{Im} B}$ are all distinct, that is when the motion takes place in a space of dimension twice the one of $\operatorname{Im} x$ (compare Albouy and Chenciner 1998, Remark 2.11).

1.4 A Criterion for Degeneracy

The equalities $\Omega^2 X = 2XA$ and $R = -X^{tr}\Omega X$ imply the commutation of R with the Wintner–Conley matrix: $[A, R] = 0$ (compare to the equations of relative equilibrium in Albouy and Chenciner 1998). This looks quite natural in view of the following lemma, where "degenerate" means "possess a multiple eigenvalue":

Lemma 2 (Lax 1998) *A real symmetric matrix A is degenerate if and only if it commutes with some real antisymmetric matrix $R \neq 0$.*

The proof is obvious in an orthonormal basis where A is diagonal.

Of course, the existence of a double eigenvalue is also equivalent to the vanishing of the *discriminant*, that is the resultant of the characteristic polynomial and its derivative. But already for 3×3 symmetric matrices, the discriminant is a quite long homogeneous degree six polynomial in the six coefficients of the matrix, which can be written as a sum of five squares see Domokos (2011); and it is not even clear on this expression that, as was already known to Von Neuman and Wigner, its regular part (corresponding to the existence of exactly one pair of equal eigenvalues) defines a codimension 2 submanifold (this is the classical phenomenon of *avoided crossings* in quantum mechanics, the obvious geometric proof of which can be found in Arnold 1989).

One is tempted to use this criterion in the study of degeneracies of the Wintner–Conley matrix A but this appears to be impracticable in the general case. Fortunately, another approach succeeds as will be seen in the next section. It is only in the case of four different masses that, using computer algebra, this criterion could be used to find the tangents to the three branches of degenerate balanced configurations stemming from the regular tetrahedron (see Sects. 3.2 and 3.7). We shall also use it when characterizing in Proposition 9 the configurations of four bodies of general type (which are defined in Sect. 1.5).

1.5 Central Configurations of General Type

Definition 1 The balanced configuration x_0 is said to be of general type if

(1) the non-zero eigenvalues of its intrinsic inertia endomorphism B_0 are all distinct;
(2) B_0 is a non-degenerate critical point of the restriction of the potential $\tilde{U}$ to its isospectral submanifold.

Lemma 3 *The intrinsic inertia endomorphisms B of balanced configurations x close enough to a balanced configuration x_0 of general type and of the same rank d, form a d-dimensional submanifold of the space of $(n-1)\times(n-1)$-symmetric endomorphisms of rank d, which intersects transversally the isospectral submanifold of B_0.*

Proof Isospectral submanifolds of endomorphisms B close to B_0 and with the same rank d all have the same codimension d as they may be labelled by their non-zero (and distinct) eigenvalues. As a non-degenerate critical point is isolated and differentiably stable under perturbations, the lemma follows : the non-zero eigenvalues of B define coordinates in the set formed by the inertia matrices B of balanced configurations close to B_0. □

1.5.1 The Case of Maximal Rank ($d = n - 1$)

In this case, one can replace B by A in the parametrization given by Lemma 3 and this implies that, for generic balanced configurations, the degeneracy of A becomes a codimension 1 property (hence the "non avoided crossings"):

Proposition 4 *If x_0 is an n-body central configuration of general type of rank $n-1$, the intrinsic inertia endomorphisms B of the balanced configurations x close enough to x_0 form an $(n - 1)$-dimensional submanifold, which can be parametrized either by their eigenvalues $(b_1, \ldots, b_{n-1})$ or by the eigenvalues [up to the factor -1/2] $(\lambda_1, \ldots, \lambda_{n-1})$ of their Wintner–Conley endomorphisms A.*

Proof The first part is nothing but Lemma 3. For the second part, we represent symmetric endomorphisms of $\mathcal{D}^*$ by symmetric matrices in the unique (up to permutation) μ^{-1}-orthonormal basis of $\mathcal{D}^*$ which diagonalizes B_0, and hence also diagonalizes $A_0 = A(B_0)$; given any balanced configuration B close enough to B_0, there exists a unique rotation $R = R(B) \in O(\mathcal{D}^*)$ such that RBR^{-1} (and hence also RAR^{-1}) is diagonal. The conclusion follows from the □

Lemma 5 *Under the hypotheses of Proposition 4, the map $\mathcal{A}$ from balanced configurations B close enough to B_0 to diagonal matrices defined by*

$$\mathcal{A}(B) = R(B)A(B)R(B)^{-1} = -\frac{1}{2}diag(\lambda_1, \ldots, \lambda_{n-1}),$$

is a diffeomorphism onto its image.

Proof The derivative of $\mathcal{A}$ at the central configuration B_0 is

$$d\mathcal{A}(B_0)\Delta B = dA(B_0)\Delta B + [dR(B_0)\Delta B, A_0] = dA(B_0)\Delta B,$$

because A_0 is proportional to the Identity. The conclusion follows because, by Lemma 1, the map $F : B \mapsto A(B)$ is a diffeomorphism. □

1.5.2 The General Case

If the rank d of the configuration is strictly smaller than $n - 1$, it is natural to look at the balanced configurations x close enough to x_0 *with the same rank d*. For such configurations, the analogue of Proposition 4 now requires that the following condition (automatic if the rank of x_0 is $n - 1$) be satisfied by x_0:

(H) *The balanced configuration x_0 (or its inertia B_0) is said to satisfy condition (H) if the mapping $B \mapsto A|_{ImB}$, which to the intrinsic inertia of a balanced configuration x of the same rank d associates the restriction to its image of the Wintner–Conley endomorphism, is a local diffeomorphism at B_0.*

Proposition 6 *If x_0 is an n-body central configuration of general type of rank d satisfying **(H)**, the intrinsic inertia endomorphisms B of the balanced configurations x close enough to x_0 and of the same rank d form a d-dimensional submanifold, which can be parametrized either by their non-zero eigenvalues $(b_1, \ldots, b_d)$ or by the eigenvalues [up to the factor $-1/2$] $(\lambda_1, \ldots, \lambda_d)$ of the restriction $A|_{ImB}$ to their image of their Wintner–Conley endomorphism. In particular, for generic balanced configurations satisfying (H), the degeneracy of A is a codimension 1 property:*

Proof If B is close to B_0, the unique (up to permutation) isometry of $Im\,B$ onto $Im\,B_0$ sending an eigenbasis of $B|_{ImB}$ onto an eigenbasis of $B_0|_{ImB_0}$ can be extended in a smooth way into an isometry $R(B)$ of $\mathcal{D}^*$ close to Identity. In the eigenbasis of $B_0|_{ImB_0}$, we have

$$RBR^{-1}|_{ImB_0} = diag(b_1, \ldots, b_d), \quad \text{and} \quad RAR^{-1}|_{ImB_0} = -\frac{1}{2}diag(\lambda_1, \ldots, \lambda_d).$$

Now, let $\mathcal{A}$ be as above the map from the set of balanced configurations of rank d near B_0 to the set of diagonal matrices, defined by:

$$\mathcal{A}(B) = R(B)A(B)R(B)^{-1} = -\frac{1}{2}diag(\lambda_1, \ldots, \lambda_d, D),$$

where D is an $(n - 1 - d) \times (n - 1 - d)$ matrix. The difference with the case of maximal rank is that, A_0 being only proportional to the Identity in restriction to the image of B_0, the second term $[dR(B_0)\Delta B, A_0]$ does not vanish. More precisely, If we write

$$dR(B_0)\Delta B = \Delta R = \begin{pmatrix} a & b \\ -b^{tr} & d \end{pmatrix}, \quad \text{and} \quad A_0 = \begin{pmatrix} \lambda_0 Id & 0 \\ 0 & D_0 \end{pmatrix}$$

we have

$$[\Delta R, A_0] = \begin{pmatrix} 0 & b(D_0 - \lambda_0 Id) \\ (D_0 - \lambda_0 Id)b^{tr} & [d, D_0] \end{pmatrix}.$$

Hence, in restriction to $Im\,B_0$, the derivative of $\mathcal{A}$ at the central configuration B_0 again reduces to the derivative at B_0 of $B \mapsto A|_{ImB}$.   □

Remarks. (1) Property **(H)** can be checked to hold in the (much too) simple case of $\mathbb{Z}/2\mathbb{Z}$-symmetric balanced configurations of rank 2 of two pairs of equal masses in the neighborhood of a generic (i.e. if some explicit condition $K \neq 0$ holds) planar rhombus central configuration; moreover, if the two masses are distinct and their ratio

avoids one single value γ, close to 0.575, or its inverse, the inertia ellipsoid of the central configuration is generic; see Sect. 5.2.

(2) Let us suppose that x_0 is a balanced configuration of general type of rank $d = n-2$. As there is only one zero eigenvalue in the spectrum of the inertia endomorphism B_0, the dimension of its isospectral manifold is the same as the one of the nearby isospectral manifolds of endomorphisms B of maximal rank $n-1$. One deduces that B_0 is a regular point of the boundary of the set of inertia matrices of balanced configurations, the local equation of this boundary being the equality to zero of the last eigenvalue $b_{n-1} = \sigma_{n-1}$ of B. In terms of the eigenvalues of A, if the property **(H)** is satisfied, this equation reads $\lambda_{n-1} = f(\lambda_1, \ldots, \lambda_{n-2})$. The (too simple) example of colinear configurations of three bodies is illustrated in the figure 2 of Albouy and Chenciner (1998); More significant examples are four bodies in $\mathbb{R}^2$ (see Sect. 5.2) or five bodies in $\mathbb{R}^3$.

2 Bifurcations of a Periodic Relative Equilibrium into a Family of Quasi-periodic Ones

We consider continuous families of quasi-periodic relative equilibria

$$x_s(t) = e^{\Omega_s t} x_s(0), \quad s \geq 0 \text{ small},$$

of a family $s \mapsto x_s(0)$ of balanced configurations in some Euclidean space E, originating from a periodic relative equilibrium of a central configuration x_0 (in particular, $\Omega_0 = \omega J$, where J is a complex structure on the ambient space E). *We suppose that all the x_s, $s \geq 0$ have the same dimension d.* We make the following assumptions (the second one can be satisfied by composing with a well chosen family of rotations):

(1) the spectral type of Ω_s is constant for $s > 0$ small,
(2) the eigenspaces $E_1, \ldots, E_r$ of Ω_s have a limit when $s \to 0$.

We shall study the two extreme cases: the "generic" case where the dimension $2p$ of E is twice the dimension d of the configuration x_0 and the case where $2p = d$ if d is even, $2p = d + 1$ if d is odd.

2.1 The Generic Case

If for $s > 0$ small the first d eigenvalues of the Wintner–Conley matrices A_s of the balanced configuration $x_s(0)$ are all distinct, that is if the quasi-periodic relative equilibria $x_s(t)$ which bifurcate from the periodic relative equilibrium $x_0(t)$ have d frequencies, the eigenspaces of Ω_s are necessarily generated by an eigenvector of Ω_s^2 (and hence of S_s) contained in $\operatorname{Im} x_s$ and a vector orthogonal to $\operatorname{Im} x_s$. Going to the limit when s tends to 0, we get that $\dim E = 2 \dim \operatorname{Im} x_0$ and that the complex structure J sends each eigenvector ρ_k of S_0 onto a vector k orthogonal to $\operatorname{Im} x_0 = \operatorname{Im} S_0$.

Given a central configuration of dimension d in an euclidean space of dimension $2d$, a *basic hermitian structure* is one for which there exists a partition of an eigenbasis of S_0 into d pairs such that the planes generated by the two members of each pair are complex lines (see Chenciner 2013a).

Definition 2 In the case when $dim\, E = 2\, dim\, Im\, x_0$, a basic hermitian structure will be called "of extrinsic type" if the partition is, as above, of the type $\{\rho_1, v_1\} \cup \cdots \cup \{\rho_d, v_d\}$, where $\{\rho_1, \ldots, \rho_d\}$ is a basis of $Im\, x_0$ formed of eigenvectors of S_0 and $\{v_1, \ldots, v_d\}$ is a basis of $Im\, x_0^{\perp}$.

From the above discussion, we get

Proposition 7 *Let* $x(t) = e^{\omega J t} x_0$ *be a relative equilibrium motion of a central configuration* x_0 *of dimension d in a space E of dimension* $2d$. *One supposes that, from this periodic relative equilibrium, stems a one parameter family of quasi-periodic relative equilibria with d frequencies* $x_s(t) = e^{\Omega_s t} x_0$, $\Omega_0 = \omega J$. *Then the hermitian structure defined by J on the Euclidean space E is basic and of extrinsic type.*

2.2 Bifurcations Without Increase of Dimension

Let $x_0 \in \mathcal{D} \otimes E$ be a central configuration of dimension d of n bodies in the Euclidean space E of dimension $2p = d$ if d is even, $2p = d + 1$ if d is odd. Let $x(t) = e^{\omega J t} x_0$ be a relative equilibrium of x_0 in E directed by the complex structure J. From Sect. 1.3, we know that a family of quasi-periodic relative equilibria of balanced configurations can bifurcate *in the same space E* from this periodic relative equilibrium if and only in two conditions are satisfied:

(1) E admits a direct sum decomposition $E = E_1 \oplus \cdots \oplus E_r$ into at least two J-complex subspaces generated (over $\mathbb{R}$) by eigenvectors of the inertia S_0;

(2) The non-zero eigenvalues of the Wintner–Conley matrix of x_0 corresponding to eigenvectors generating any of these subspaces are equal.

Thanks to Proposition 6, one deduces the codimension of the corresponding bifurcations; in particular :

Proposition 8 *Let* x_0 *be a central configuration of general type and satisfying* **(H)**, *of dimension* $d = 2p - 1$ *or* $d = 2p$ *in the Euclidean space E of dimension* $2p$. *In the manifold*[6] *of balanced configurations close enough to* x_0 *and of the same dimension, those which admit a relative equilibrium motion in E form a stratified subset of dimension p. The main stratum, of dimension p, corresponds to quasi-periodic relative equilibria in E with p frequencies, the smallest stratum (if* $p \geq 2$*), of dimension 2, to quasi-periodic relative equilibria in E with 2 frequencies.*

One should not forget that because of the projective invariance, the pertinent dimensions are respectively $p - 1$ *and* 1.

Proof It is enough to notice that the main stratum corresponds to identities of the following type between the eigenvalues λ_i of the endomorphism A_B (see Proposition 4):

$$\lambda_{i_1} = \lambda_{j_1}, \ldots \lambda_{i_{p-1}} = \lambda_{j_{p-1}} \quad \text{for } (2p-1)\text{-dimensional configurations,}$$
$$\lambda_{i_1} = \lambda_{j_1}, \ldots \lambda_{i_p} = \lambda_{j_p} \quad \text{for } 2p\text{-dimensional configurations.}$$

In both cases, the dimension of the stratum is

---

[6] With boundary if $d = 2p - 1$ and $n \geq 2p + 1$.

$$(2p - 1) - (p - 1) = 2p - p = p.$$

In the same way, the smallest stratum corresponds to the equations

$$\lambda_{i_1} = \cdots = \lambda_{i_{2p-2}} \quad \text{for } (2p - 1)\text{-dimensional configurations,}$$
$$\lambda_{i_1} = \cdots = \lambda_{i_{2p-2}} \text{ and } \lambda_{j_1} = \lambda_{j_2} \quad \text{for } 2p\text{-dimensional configurations.}$$

In both cases, the dimension of the stratum is

$$(2p - 1) - (2p - 3) = 2p - (2p - 3) - 1 = 2.$$

$\square$

I leave to the reader the pleasure of describing the intermediate strata.

Examples (1) Central or balanced configurations of three bodies which are not of general type can be observed in figure 2 of Albouy and Chenciner (1998). On the other hand, one notices also on these figures examples of balanced configurations for which B_0 is proportional to the Identity but is nevertheless a regular point of the set of balanced configurations [this case is realized when the center of mass of the configuration coincides with the orthocentre of the triangle (see also ex. 3)].

(2) Three-dimensional four-body central configurations of general type are characterized in Corollary 10.

(3) The three-dimensional balanced configurations of four bodies farthest from being of general type—the ones with B proportional to Identity—are the orthocentric tetrahedra (i.e. the tetrahedra which possess an orthocenter, which is the intersection of the four heights) such that the orthocenter coincides with the center of mass; this is equivalent to the mutual distances being given by $r_{ij}^2 = \text{constant}(\frac{1}{m_i} + \frac{1}{m_j})$.

Remark More generally, the proof of Proposition 9 implies that higher degeneracies with ν_2 pairs, ν_3 triples, etc., of equal eigenvalues, instead of having the generic codimension $\sum \frac{1}{2}(i - 1)(i + 2)\nu_i = 2\nu_2 + 5\nu_3 + 9\nu_4 + \cdots$ have codimension $\sum (i - 1)\nu_i = \nu_2 + 2\nu_3 + 3\nu_4 + \cdots$

3 The Case of Four Bodies

The unique non planar central configuration of four arbitrary masses is the regular tetrahedron. We show that, whatever be the masses, it belongs to at least three (and exactly three in the generic case) 1-parameter (up to rotation and scaling) families of balanced configurations which admit a relative equilibrium motion in $\mathbb{R}^4$. Two cases have to be treated separately, depending of whether or not three of the masses are equal (see Corollary 11 and Sect. 3.6).

We use the following notations for the squared mutual distances (Fig. 1):

$$r_{13}^2 = a, \ r_{14}^2 = b', \ r_{12}^2 = b'', \ r_{34}^2 = d', \ r_{32}^2 = d'', \ r_{24}^2 = f.$$
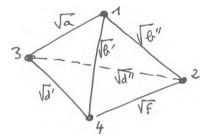
Fig. 1 Four-body configurations

They were chosen because they specialize nicely to the case when the masses m_2 and m_4 are symmetric with respect to the plane containing m_1 and m_3 (see Sect. 3.5 and Chenciner 2014) .

A convenient μ^{-1}-orthonormal basis of $\mathcal{D}^*$ is $\{u_1, u_2, u_3\}$ with

$$
\begin{cases}
u_1 = \kappa_1 \left(m_1(m_2 + m_4), \, -m_2(m_1 + m_3), \, m_3(m_2 + m_4), \, -m_4(m_1 + m_3) \right), \\
u_2 = \kappa_2 (1, 0, -1, 0), \\
u_3 = \kappa_3 (0, 1, 0, -1),
\end{cases}
$$

where

$$
\kappa_1^2 = \frac{1}{M(m_1 + m_3)(m_2 + m_4)}, \quad \kappa_2^2 = \frac{m_1 m_3}{m_1 + m_3}, \quad \kappa_3^2 = \frac{m_2 m_4}{m_2 + m_4}.
$$

3.1 The Intrinsic Inertia Matrix

Expressed in the basis $\{u_1, u_2, u_3\}$, $B = \begin{pmatrix} u & z & y \\ z & v & x \\ y & x & w \end{pmatrix}$, where

$$
\begin{cases}
u = \frac{1}{M} \left[m_2(m_1 b'' + m_3 d'') + m_4(m_1 b' + m_3 d') \right] - \frac{1}{M} \left[m_1 m_3 \frac{Y}{Z} a + m_2 m_4 \frac{Z}{Y} f \right], \\
v = \frac{m_1 m_3}{m_1 + m_3} a, \quad w = \frac{m_2 m_4}{m_2 + m_4} f, \\
x = \frac{1}{2} V (d'' - d' + b' - b''), \\
y = \frac{U}{2MZ} \left[m_1(b' - b'') + m_3(d' - d'') \right] + \frac{U(m_4 - m_2)}{2MY} f, \\
z = \frac{T}{2MY} \left[m_2(b'' - d'') + m_4(b' - d') \right] + \frac{T(m_1 - m_3)}{2MZ} a,
\end{cases}
$$

with the following notations ($X = MYZ = TU/V$ is for future use in Sect. 3.3):

$$
\begin{cases}
X = \frac{M}{(m_1+m_3)(m_2+m_4)}, & Y = \frac{1}{m_1+m_3}, \\[2mm]
Z = \frac{1}{m_2+m_4}, & T = \frac{1}{m_1+m_3}\sqrt{\frac{Mm_1m_3}{m_2+m_4}}, \\[2mm]
U = \frac{1}{m_2+m_4}\sqrt{\frac{Mm_2m_4}{m_1+m_3}}, & V = \sqrt{\frac{m_1m_2m_3m_4}{(m_1+m_3)(m_2+m_4)}}.
\end{cases}
$$

For the regular tetrahedron ($a = b' = b'' = d' = d'' = f = 1$), B becomes the matrix

$$
\begin{pmatrix}
\frac{(m_2+m_4)(m_1^2+m_3^2)}{2M(m_1+m_3)} + \frac{(m_1+m_3)(m_2^2+m_4^2)}{2M(m_2+m_4)} & -\frac{1}{2}\sqrt{\frac{m_1m_3(m_2+m_4)}{M}}\frac{m_3-m_1}{m_1+m_3} & -\frac{1}{2}\sqrt{\frac{m_2m_4(m_1+m_3)}{M}}\frac{m_2-m_4}{m_2+m_4} \\[3mm]
-\frac{1}{2}\sqrt{\frac{m_1m_3(m_2+m_4)}{M}}\frac{m_3-m_1}{m_1+m_3} & \frac{m_1m_3}{m_1+m_3} & 0 \\[3mm]
-\frac{1}{2}\sqrt{\frac{m_2m_4(m_1+m_3)}{M}}\frac{m_2-m_4}{m_2+m_4} & 0 & \frac{m_2m_4}{m_2+m_4}
\end{pmatrix}
$$

which reduces to $B = \frac{1}{2}Id$ when all the masses are equal to 1.

3.2 Degeneracies of the Inertia of the Regular Tetrahedron

As explained in Sect. 1.4, we shall use the criterion given by Lemma 2, which leads to much more transparent equations than the resultant: B is degenerate if and only if there exists a non trivial antisymmetric matrix

$$
R = \begin{pmatrix} 0 & \zeta & -\eta \\ -\zeta & 0 & \xi \\ \eta & -\xi & 0 \end{pmatrix},
$$

which commutes with B, that is such that $[B, R] = BR - RB = 0$. Writing the six coefficients of this symmetric matrix in the order (u, v, w, x, y, z), this is equivalent to the following linear equation:

$$
\begin{pmatrix}
0 & 2y & -2z \\
-2x & 0 & 2z \\
2x & -2y & 0 \\
v-w & -z & y \\
z & w-u & -x \\
-y & x & u-v
\end{pmatrix}
\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
$$

This equation has a non trivial solution if and only if the above 6×3 matrix is not of maximal rank, that is if its rows generate a subspace of dimension 2 or less. We shall distinguish two cases:

(1) two of the off-diagonal coefficients x, y, z of B are equal to zero. Up to a reordering of the basis, we may suppose that $x = y = 0$. Then the condition is reduced

to the vanishing of two minors: $(u-v)[z^2+(v-w)(w-u)]$ and $z[z^2+(v-w)(w-u)]$, that is

$$(C_1) \begin{cases} \text{either} & z = 0 \quad \text{and} \quad (u-v)(v-w)(w-u) = 0, \\ \text{or} & z \neq 0 \quad \text{and} \quad z^2 + (v-w)(w-u) = 0. \end{cases}$$

(2) at most one of the off-diagonal coefficients is equal to zero. Then the first two lines are linearly independent while the sum of the first three is equal to zero (because the trace of a commutator vanishes). Writing that the last three lines belong to the plane generated by the first two leads to the following equations

$$(C_2) \begin{cases} x(y^2 - z^2) + (v - w)yz = 0, & (C_2') \\ y(z^2 - x^2) + (w - u)zx = 0, & (C_2'') \\ z(x^2 - y^2) + (u - v)xy = 0. & (C_2''') \end{cases}$$

As expected, these equations are not independent: multiplying the first by x, the second by y, the third by z and adding, one gets 0. Recall that they are not valid if two off-diagonal coefficients vanish.

Proposition 9 *The intrinsic inertia matrix of the regular tetrahedron configuration of four masses m_1, m_2, m_3, m_4 is degenerate if and only if at least three of the masses are equal.*

Proof The if part follows without calculation from the fact that an ellipsoid with $\mathbb{Z}/3\mathbb{Z}$-symmetry is necessary of revolution. For the converse, as the off-diagonal coefficient x is equal to 0, equations (C_2) imply that another off-diagonal coefficient must be equal to zero, that is $m_1 = m_3$ or $m_2 = m_4$. Hence equations (C_2) are no more pertinent to decide of the degeneracy. If we suppose $m_2 = m_4$, they must be replaced by equations (C_1):

(1) if all off-diagonal coefficients are zero, that is if $m_2 = m_4$ and $m_1 = m_3$, we have

$$B = \text{diag}\left(\frac{m_1 m_2}{m_1 + m_2}, \frac{m_1}{2}, \frac{m_2}{2}\right),$$

hence degeneracy occurs only if the four masses are equal;

(2) if $m_1 \neq m_3$, a direct computation shows that (C_1) reads

$$-\frac{m_2}{2M}(m_1 - m_2)(m_3 - m_2) = 0,$$

which implies that $m_1 = m_2 = m_4$ or $m_2 = m_3 = m_4$. Starting with $m_1 = m_3$ but $m_2 \neq m_4$ we would have found the remaining possibilities $m_1 = m_2 = m_3$ and $m_1 = m_3 = m_4$. $\qquad\square$

If, for example, $m_2 = m_3 = m_4$, we have

$$B = \begin{pmatrix} \frac{m_2(3m_1^2+3m_2^2+2m_1m_2)}{2(m_1+3m_2)(m_1+m_2)} & -\frac{m_2}{2}\sqrt{\frac{2m_1}{m_1+3m_2}}\frac{m_2-m_1}{m_1+m_2} & 0 \\ -\frac{m_2}{2}\sqrt{\frac{2m_1}{m_1+3m_2}}\frac{m_2-m_1}{m_1+m_2} & \frac{m_1m_2}{m_1+m_2} & 0 \\ 0 & 0 & \frac{m_2}{2} \end{pmatrix}.$$

whose eigenvalues are

$$\lambda_1 = \frac{2m_1m_2}{m_1+3m_2}, \quad \lambda_2 = \lambda_3 = \frac{m_2}{2}.$$

Corollary 10 *The unique three-dimensional central configuration of four bodies—the regular tetrahedron—is of general type if and only if no three of the masses are equal.*

Proof The regular tetrahedron is the unique critical point of the restriction of the potential to the 4-body configurations with fixed moment of inertia with respect to the centre of mass (i.e. fixed trace of B). Using the squared mutual distances r_{ij}^2 as independent variables, it is easy to prove that it is a non-degenerate minimum. Hence its restriction to the isospectral manifold is also a non degenerate minimum. $\qquad\square$

From Proposition 4 on then deduces

Corollary 11 *If no three masses are equal, the balanced 4-body configurations (up to rotation and scaling) close to the regular tetrahedron form a two-dimensional manifold; the balanced configurations which admit a relative equilibrium motion in* $\mathbb{R}^4$ *form three regular curves intersecting at the regular tetrahedron.*

In what follows we give a direct proof of Corollary 11, computing in particular the curves of degenerate balanced configurations, first in case 2 masses are equal (Sect. 3.5), then, at first order, in the general case (Sect. 3.7).

3.3 The Wintner–Conley Matrix

A tedious but straightforward computation gives the following expression of the Wintner–Conley endomorphism in this basis:

$$A = \begin{pmatrix} \alpha & \phi & \epsilon \\ \phi & \beta & \delta \\ \epsilon & \delta & \gamma \end{pmatrix}, \qquad \text{where (with the notations of 3.1)}$$

$$\begin{cases} \alpha = X\left[m_2\left(m_1\varphi(b'') + m_3\varphi(d'')\right) + m_4\left(m_1\varphi(b') + m_3\varphi(d')\right)\right], \\ \beta = (m_1+m_3)\varphi(a) + Y\left[m_2\left(m_3\varphi(b'') + m_1\varphi(d'')\right) + m_4\left(m_3\varphi(b') + m_1\varphi(d')\right)\right], \\ \gamma = (m_2+m_4)\varphi(f) + Z\left[m_1\left(m_2\varphi(b') + m_4\varphi(b'')\right) + m_3\left(m_2\varphi(d') + m_4\varphi(d'')\right)\right], \\ \delta = V\left[\varphi(d'') - \varphi(d') + \varphi(b') - \varphi(b'')\right], \\ \epsilon = U\left[m_1\left(\varphi(b') - \varphi(b'')\right) + m_3\left(\varphi(d') - \varphi(d'')\right)\right], \\ \phi = T\left[m_2\left(\varphi(b'') - \varphi(d'')\right) + m_4\left(\varphi(b') - \varphi(d')\right)\right]. \end{cases}$$

For the regular tetrahedron with unit sides, we find $A = M\varphi(1)Id = -\frac{M}{2}Id$.

3.4 The Equations of Balanced Configurations

With the above notations, the four equations of balanced configurations take the following form, where one checks that, in accordance with Sect. 1.2, they satisfy the relation $P_{123} - P_{124} + P_{134} - P_{234} \equiv 0$.

$$(P_{123}) \begin{cases} m_1(d'' - a - b'')[\varphi(a) - \varphi(b'')] - m_4(d'' + b')[\varphi(f) - \varphi(d')] \\ +m_2(a - b'' - d'')[\varphi(b'') - \varphi(d'')] - m_4(a + f)[\varphi(d') - \varphi(b')] \\ +m_3(b'' - d'' - a)[\varphi(d'') - \varphi(a)] - m_4(b'' + d')[\varphi(b') - \varphi(f)] = 0, \end{cases}$$

$$(P_{124}) \begin{cases} m_1(f - b' - b'')[\varphi(b') - \varphi(b'')] - m_3(f + a)[\varphi(d'') - \varphi(d')] \\ +m_2(b' - b'' - f)[\varphi(b'') - \varphi(f)] - m_3(b' + d'')[\varphi(d') - \varphi(a)] \\ +m_4(b'' - f - b')[\varphi(f) - \varphi(b')] - m_3(b'' + d')[\varphi(a) - \varphi(d'')] = 0, \end{cases}$$

$$(P_{134}) \begin{cases} m_1(d' - b' - a)[\varphi(b') - \varphi(a)] - m_2(d' + b'')[\varphi(d'') - \varphi(f)] \\ +m_3(b' - a - d')[\varphi(a) - \varphi(d')] - m_2(b' + d'')[\varphi(f) - \varphi(b'')] \\ +m_4(a - d' - b')[\varphi(d') - \varphi(b')] - m_2(a + f)[\varphi(b'') - \varphi(d'')] = 0, \end{cases}$$

$$(P_{234}) \begin{cases} m_2(d' - f - d'')[\varphi(f) - \varphi(d'')] - m_1(d' + b'')[\varphi(a) - \varphi(b')] \\ +m_3(f - d'' - d')[\varphi(d'') - \varphi(d')] - m_1(f + a)[\varphi(b') - \varphi(b'')] \\ +m_4(d'' - d' - f)[\varphi(d') - \varphi(f)] - m_1(d'' + b')[\varphi(b'') - \varphi(a)] = 0. \end{cases}$$

Let us use these equations to give a direct proof of the first part of Corollary 11: linearized at the regular tetrahedron whose sides have length 1, the equations of balanced configurations take a particularly simple form, *independent of the precise form of φ*:

$$\varphi'(1)K \begin{pmatrix} \delta a \\ \delta b' \\ \delta b'' \\ \delta d' \\ \delta d'' \\ \delta f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{where}$$

$$K = \begin{pmatrix} m_3 - m_1 & 0 & m_1 - m_2 & 0 & m_2 - m_3 & 0 \\ 0 & m_4 - m_1 & m_1 - m_2 & 0 & 0 & m_2 - m_4 \\ m_1 - m_3 & m_4 - m_1 & 0 & m_3 - m_4 & 0 & 0 \\ 0 & 0 & 0 & m_3 - m_4 & m_2 - m_3 & m_4 - m_2 \end{pmatrix}.$$

The rank of the matrix K is 0 if the four masses are equal, 2 if three of the masses are equal and 3 otherwise. In this last case, its kernel is generated by the three vectors

$$\begin{cases} E_1 = (1, 1, 1, 1, 1, 1), \\ E_2 = (m_2 m_4, m_2 m_3, m_3 m_4, m_1 m_2, m_1 m_4, m_1 m_3), \\ E_3 = (m_2 + m_4, m_2 + m_3, m_3 + m_4, m_1 + m_2, m_1 + m_4, m_1 + m_3). \end{cases}$$

As the rank at neighboring points cannot be higher than three because the equations we have used of the set of balanced configurations are not independent, the rank of the matrix is locally constant if no three of the masses are equal.

3.5 The $\mathbb{Z}/2\mathbb{Z}$-Symmetric Case

In this section, we suppose that at least two masses are equal, say $m_2 = m_4$, which makes pertinent the use of the basis $\{u_1, u_2, u_3\}$ introduced at the beginning of Sect. 3. We check directly that in this case the degeneracy of the Wintner–Conley matrix becomes a codimension 1 condition (Corollary 11) among balanced configurations close to the regular tetrahedron. We start with a nice corollary of the fact that the intrinsic inertia matrices of these balanced configurations form a three-dimensional manifold:

Corollary 12 *If two masses and not three are equal, any balanced configuration close enough to the regular tetrahedron is symmetric with respect to a plane separating these two masses and containing the other two.*

To distinguish from a purely geometric symmetry, we shall call *dynamically symmetric* such a symmetric configuration for which symmetric masses are equal.

Proof Supposing $m_2 = m_4$, let us consider the naturally associated symmetric configurations, which satisfy (Fig. 2)
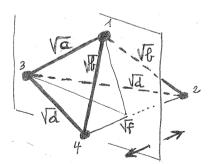
$$b' = b'' = b, \quad d' = d'' = d.$$

Fig. 2 Symmetric 4-body configurations

The four equations P_{ijk} reduce to a single one in the four variables (a, b, d, f). Indeed, P_{234} and P_{124} are identically satisfied, while $P_{123} = -P_{134} = 0$ becomes

$$
\begin{cases}
m_1(d - b - a)[\varphi(b) - \varphi(a)] - m_2(d + b)[\varphi(d) - \varphi(f)] \\
\quad + m_3(b - a - d)[\varphi(a) - \varphi(d)] - m_2(b + d)[\varphi(f) - \varphi(b)] \\
\quad + m_2(a - d - b)[\varphi(d) - \varphi(b)] - m_2(a + f)[\varphi(b) - \varphi(d)] = 0.
\end{cases}
$$

Linearized at the regular tetrahedron, the equation becomes

$$
(m_1 - m_3)\delta a + (m_2 - m_1)\delta b + (m_3 - m_2)\delta d = 0.
$$

Hence the equation is a submersion at the regular tetrahedron, except in case all masses are equal. It follows that, under our hypotheses, the set of symmetric balanced configurations in the neighborhood of the regular tetrahedron is a submanifold whose dimension is the same as the one of all balanced configurations. Hence the two coincide locally.   □

With the same proof, based on a dimension count, we get

Corollary 13 *If the masses form two equal pairs, any balanced configuration close enough to the regular tetrahedron is a* rhombus configuration, *i.e. it is symmetric with respect to two orthogonal planes respectively separating two equal masses and containing the other two. Supposing $m_2 = m_4 \neq m_1 = m_3$, this means that $b' = b'' = d' = d''$.*

Corollary 12 makes it easy to get a good understanding of the degenerate balanced configurations near the regular tetrahedron when two masses and not three are equal. Indeed, suppose as above that $m_2 = m_4$, $b' = b'' = b$, $d' = d'' = d$. In the μ^{-1}-orthonormal basis $\{u_1, u_2, u_3\}$ of $\mathcal{D}^*$ defined at the beginning of Sect. 3, *the Wintner–Conley matrix decomposes into two blocks:*

$$
A = \begin{pmatrix} \alpha & \phi & 0 \\ \phi & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad \text{with} \quad
\begin{cases}
\alpha = \frac{M}{m_1 + m_3}(m_1\varphi(b) + m_3\varphi(d)), \\
\beta = \frac{2m_2}{m_1 + m_3}(m_3\varphi(b) + m_1\varphi(d)) + (m_1 + m_3)\varphi(a), \\
\gamma = m_1\varphi(b) + m_3\varphi(d) + 2m_2\varphi(f), \\
\phi = \frac{\sqrt{2m_1 m_2 m_3 M}}{m_1 + m_3}(\varphi(b) - \varphi(d)).
\end{cases}
$$

Degeneracy occurs if (compare to Sect. 3.2, condition (C_1)) either the upper block degenerates, i.e.

$$
\phi = 0, \ \alpha = \beta,
$$

or the lower right coefficient is equal to an eigenvalue of the upper block, i.e.

$$
\phi^2 + \left(\frac{\alpha - \beta}{2}\right)^2 - \left(\gamma - \frac{\alpha + \beta}{2}\right)^2 = 0.
$$

(i) The first case is equivalent to $a = b = d$ and the equation of balanced configurations is automatically satisfied.

(ii) In order to study the second case, we introduce the following coordinates:

$$\alpha, \quad \theta = \gamma - \frac{\alpha + \beta}{2}, \quad \psi = \frac{\alpha - \beta}{2}, \quad \phi,$$

that is

$$\beta = \alpha - 2\psi, \quad \gamma = \alpha + \theta - \psi.$$

In these coordinates, the degenerate matrices other than the ones defined by $\psi = \phi = 0$ are defined by the equation

$$\phi^2 + \psi^2 - \theta^2 = 0.$$

On the other hand, the Wintner–Conley matrices of balanced configurations are defined by the equation $P_{124} = P_{234}$, where a, b, c, d are expressed in terms of the new coordinates $\alpha, \theta, \psi, \phi$ via the inverse equations (where for saving space we have noted $s_{13} = m_1 + m_3$; thanks to Jacques Féjoz for the computation of the inverse matrix),

$$\begin{cases} \varphi(a) = \left(\frac{1}{M} - \frac{1}{s_{13}}\right)\alpha + \frac{1}{s_{13}}\beta - \frac{m_3 - m_1}{s_{13}^2 T}\phi = \frac{1}{M}\alpha - \frac{2}{s_{13}}\psi - \frac{m_3 - m_1}{s_{13}^2 T}\phi, \\ \varphi(b) = \frac{1}{M}\alpha + \frac{m_3}{2m_2 s_{13} T}\phi, = \frac{1}{M}\alpha + \frac{m_3}{2m_2 s_{13} T}\phi, \\ \varphi(d) = \frac{1}{M}\alpha - \frac{m_1}{2m_2 s_{13} T}\phi = \frac{1}{M}\alpha - \frac{m_1}{2m_2 s_{13} T}\phi, \\ \varphi(f) = \left(\frac{1}{M} - \frac{1}{2m_2}\right)\alpha + \frac{1}{2m_2}\gamma = \frac{1}{M}\alpha + \frac{1}{2m_2}\theta - \frac{1}{2m_2}\psi. \end{cases}$$

We have seen that, if no three masses are equal, the balanced configurations near the regular tetrahedron ($a = b = d = f = l$) form a three-dimensional manifold whose tangent space at this point is defined by the equation

$$(m_1 - m_3)\delta a + (m_2 - m_1)\delta b + (m_3 - m_2)\delta d = 0.$$

On the side of the Wintner–Conley matrices, that is setting

$$\begin{cases} \varphi'(l)\delta a = \frac{1}{M}\delta\alpha - \frac{2}{s_{13}}\delta\psi - \frac{m_3 - m_1}{s_{13}^2 T}\delta\phi, \\ \varphi'(l)\delta b = \frac{1}{M}\delta\alpha + \frac{m_3}{2m_2 s_{13} T}\delta\phi, \\ \varphi'(l)\delta d = \frac{1}{M}\delta\alpha - \frac{m_1}{2m_2 s_{13} T}\delta\phi, \\ \varphi'(l)\delta f = \frac{1}{M}\delta\alpha + \frac{1}{2m_2}\delta\theta - \frac{1}{2m_2}\delta\psi, \end{cases}$$

this equation becomes

$$-\frac{2(m_1 - m_3)}{m_1 + m_3}\delta\psi + \frac{3m_2(m_1^2 + m_3^2) - 2m_1 m_3(m_1 + m_2 + m_3)}{2m_2(m_1 + m_3)^2 T}\delta\phi = 0.$$
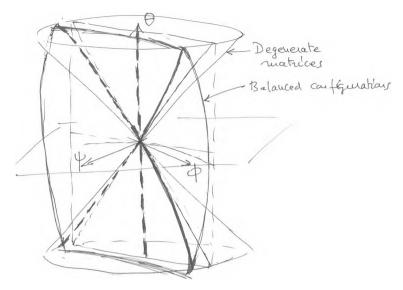
Fig. 3 Degenerate balanced configurations in the symmetric case

It defines a linear susbspace "transversal" to the cone defined by the degenerate matrices. Indeed, It is enough to look in the space of coordinates (θ, ψ, ϕ) obtained by going to the quotient by the α axis : in the coordinates $(\alpha, \beta, \gamma, \phi)$, this corresponds to going to the quotient by the line generated by $(1, 1, 1, 0)$, that is by the addition to the Wintner–Conley matrix of a multiple of the identity, which does not change its degeneracy type and leaves invariant the linearized equations of balanced configurations (see Sect. 3.7). Hence we have identified the three (projective) directions of bifurcation to relative equilibria in $\mathbb{R}^4$ guaranteed by Corollary 11 (Fig. 3).

Remark The degenerate rhombus configurations are explicitly given by one of the following three equations:

$$a = b, \quad \text{or} \quad f = b, \quad \text{or} \quad (m_1 - m_2)\varphi(b) = m_1\varphi(a) - m_2\varphi(f).$$

3.6 The Case of Three Equal Masses

Suppose for example that $m_2 = m_3 = m_4$; we have the following explicit (projective) curves of degenerate symmetric balanced configurations:

(1) *The ($\mathbb{Z}/2\mathbb{Z}$-symmetric) configurations:* we have seen in Sect. 3.5 that, supposing only $m_2 = m_4$, the configurations such that $a = b' = b'' = d' = d''$ are balanced and degenerate. In the basis $\{u_1, u_2, u_3\}$, their Wintner–Conley matrix is

$$diag\left(M\varphi(a), M\varphi(a), (m_1 + m_3)\varphi(a) + 2m_2\varphi(f)\right).$$

From the two other equalities $m_2 = m_3$ (resp. $m_3 = m_4$), one gets two other families of $\mathbb{Z}/2\mathbb{Z}$-symmetric degenerate balanced configurations, namely the configurations such that $a = b' = b'' = d' = f$ (resp. $a = b' = b'' = d'' = f$).

(2) *The ($\mathbb{Z}/3\mathbb{Z}$-symmetric) configurations:* if $a = b' = b''$, $d' = d'' = f$ and assuming only $m_2 = m_4$, the unique equation of balanced configurations reduces to $a(m_2 - m_3)(\varphi(a) - \varphi(f)) = 0$. Supposing moreover $m_2 = m_3$, it is natural to choose the following μ^{-1}-orthonormal basis of $\mathcal{D}^*$:

$$v_1 = \sqrt{\frac{m_1 m_2}{3(m_1 + 3m_2)}}(-3, 1, 1, 1), \;\; v_2 = \sqrt{\frac{m_2}{6}}(0, -1, 2, -1),$$

$$v_3 = \frac{m_2}{2}(0, 1, 0, -1).$$

In this base, the Wintner–Conley matrix is

$$diag\left((m_1 + 3m_2)\varphi(a), m_1\varphi(a) + 3m_2\varphi(f), m_1\varphi(a) + 3m_2\varphi(f)\right).$$

Remarks.

(1) When three and not four masses are equal, non symmetric planar balanced configurations do exist: the idea, communicated to me by Alain Albouy is to start with the central configuration consisting in an isosceles triangle with the fourth mass a little above the center of mass and break the symmetry by slightly changing one mass in the basis of the isosceles triangle. It is not known whether non-symmetric balanced configurations exist in the neighborhood of the regular tetrahedron with three equal masses.

(2) The case of four equal masses is studied in Chenciner (2014). The set of symmetric balanced configurations is singular at the regular tetrahedron: fixing one of the six symmetry planes of the regular tetrahedron, the subset of balanced configurations with this symmetry plane is the union of two (projective) surfaces with transversal intersection: one of these consists in the rhombus configurations and contains three (projective) curves of degenerate configurations; the other one contains the curve of $\mathbb{Z}/3\mathbb{Z}$-symmetric degenerate configurations; also, fixing one of the three symmetry lines of the regular tetrahedron, the subset of balanced configurations with this symmetry line (for example $b' = d''$, $d' = b''$) which are close enough to the regular tetrahedron is the (projective) surface formed of "twisted rectangles" ($b' = d''$, $d' = b''$, $a = f$) which possess in fact the three line symmetries. It is not known whether non-symmetric 4-body balanced configurations exist in the case of equal masses.

3.7 The General Case: TRIP Comes into Play

We have seen in Sect. 3.4 that, as soon as $\varphi'(1) \neq 0$, the linearization at the regular tetrahedron of the equations of balanced configurations takes the form

$$K \begin{pmatrix} \delta a \\ \delta b' \\ \delta b'' \\ \delta d' \\ \delta d'' \\ \delta f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{where } K \text{ is defined in 3.4.}$$

We suppose that the masses are all different, hence the rank of K equals 3 and its kernel is generated by the three vectors E_1, E_2, E_3 defined in Sect. 3.4.

Computing a Taylor expansion of A in the neighborhood of the regular tetrahedron with unit sides, one gets

$$
A = -\frac{M}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \delta\alpha & \delta\phi & \delta\epsilon \\ \delta\phi & \delta\beta & \delta\delta \\ \delta\epsilon & \delta\delta & \delta\gamma \end{pmatrix}, \quad \text{with} \quad \begin{pmatrix} \delta\alpha \\ \delta\beta \\ \delta\gamma \\ \delta\delta \\ \delta\epsilon \\ \delta\phi \end{pmatrix} = \varphi'(1)L \begin{pmatrix} \delta a, \\ \delta b' \\ \delta b'' \\ \delta d' \\ \delta d'' \\ \delta f \end{pmatrix},
$$

$$
\text{and} \quad L = \begin{pmatrix} 0 & Xm_1m_4 & Xm_1m_2 & Xm_3m_4 & Xm_2m_3 & 0 \\ m_1 + m_3 & Ym_3m_4 & Ym_2m_3 & Ym_1m_4 & Ym_1m_2 & 0 \\ 0 & Zm_1m_2 & Zm_1m_4 & Zm_2m_3 & Zm_3m_4 & m_2 + m_4 \\ 0 & V & -V & -V & V & 0 \\ 0 & Um_1 & -Um_1 & Um_3 & -Um_3 & 0 \\ 0 & Tm_4 & Tm_2 & -Tm_4 & -Tm_2 & 0 \end{pmatrix},
$$

where we have used the notations of Sect. 3.3. Hence, the tangent space at the regular tetrahedron of the manifold of Wintner–Conley matrices of balanced configurations is generated by the three vectors in $\mathbb{R}^6$ (coordinates $(\alpha, \beta, \gamma, \delta, \epsilon, \phi)$)

$$
L(E_1) = M\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad L(E_2) = \begin{pmatrix} 4XM \\ m_2m_4\left[m_1 + m_3 + 2Y(m_1^2 + m_3^2)\right] \\ m_1m_3\left[m_2 + m_4 + 2Z(m_2^2 + m_4^2)\right] \\ V(m_2 - m_4)(m_3 - m_1) \\ 2Um_1m_3(m_2 - m_4) \\ 2Tm_2m_4(m_3 - m_1) \end{pmatrix},
$$

$$
L(E_3) = \begin{pmatrix} 2X\sum_{i<j<k} m_im_jm_k \\ (m_1 + m_3)(m_2 + m_4) + \frac{m_2+m_4}{m_1+m_3}(m_1^2 + m_3^2) + 2m_2m_4 \\ (m_2 + m_4)(m_1 + m_3) + \frac{m_1+m_3}{m_2+m_4}(m_2^2 + m_4^2) + 2m_1m_3 \\ 0 \\ U(m_1 + m_3)(m_2 - m_4) \\ T(m_2 + m_4)(m_3 - m_1) \end{pmatrix}.
$$

Remark The fact that E_1 belongs to the kernel of the linearized equations K of balanced configurations can be seen directly from the equation $[A, B] = 0$. Indeed, the linearized equation at a central configuration, $[A, \Delta B] + [\Delta A, B] = 0$, is satisfied because A is a multiple of Identity as is $\Delta A = L(E_1)$.

As the equations of degenerate quadratic forms are also invariant by the addition of a multiple of the identity matrix, that is a multiple of $L(E_1)$ in the $(\alpha, \beta, \gamma, \delta, \epsilon, \phi)$ space,

in order to understand the tangent spaces at the regular tetrahedron to the degenerate balanced configurations, it is enough to substitute $L(E_2) + x L(E_3)$ to $(\alpha, \beta, \gamma, \delta, \epsilon, \phi)$ in the three equations (C_2) of Sect. 3.2 (after replacing u, v, w, x, y, z respectively by $\alpha, \beta, \gamma, \delta, \epsilon, \phi$). and this is what I had asked Jacques Laskar to do.

Using the computer algebra software TRIP developed by Gastineau and Laskar (2011), he discovered that the three equations became proportional, namely he obtained

$$
\begin{cases}
\frac{(m_1-m_3)(m_2-m_4)}{(m_1+m_3)(m_2+m_4)} \sqrt{\frac{m_1 m_2 m_3 m_4}{(m_1+m_3)(m_2+m_4)}} \times \cdots \\
\qquad \cdots 2M(-m_3 m_2 m_4 - m_1 m_2 m_4 + m_1 m_2 m_3 + m_1 m_3 m_4)E = 0, \\
-\frac{(m_1-m_3)^2(m_2-m_4)}{m_1+m_3} m_1 m_3 \sqrt{\frac{M m_2 m_4}{m_1+m_3}} \, E = 0, \\
-\frac{(m_1-m_3)(m_2-m_4)^2}{m_2+m_4} m_2 m_4 \sqrt{\frac{M m_1 m_3}{m_2+m_4}} \, E = 0,
\end{cases}
$$

where

$$
E(x) := \left(\sum_i m_i\right) x^3 + 2\left(\sum_{i<j} m_i m_j\right) x^2 + 3\left(\sum_{i<j<k} m_i m_j m_k\right) x + 4\prod_i m_i = 0.
$$

The equation E has three real roots, all negative. Indeed, if we set

$$
\begin{aligned}
F(y) &= (1+m_1 y)(1+m_2 y)(1+m_3 y)(1+m_4 y) \\
&= \left(\prod_i m_i\right) y^4 + \left(\sum_{i<j<k} m_i m_j m_k\right) y^3 + \left(\sum_{i<j} m_i m_j\right) y^2 + \left(\sum_i m_i\right) y + 1,
\end{aligned}
$$

we have

$$
F'(y) = 4\left(\prod_i m_i\right) y^3 + 3\left(\sum_{i<j<k} m_i m_j m_k\right) y^2 + 2\left(\sum_{i<j} m_i m_j\right) y + \left(\sum_i m_i\right),
$$

and hence

$$
E(x) = x^3 F'(1/x).
$$

One concludes because F has the four real roots $-1/m_1, -1/m_2, -1/m_3, -1/m_4$, all negative. This computation gives the tangents to the three (projective) curves of degenerate balanced configurations which intersect at the regular tetrahedron. As I already said in the abstract, this came as a surprise as I had asked Jacques to show that, for masses without any symmetry, no other solution than the regular tetrahedron existed locally, in accordance with the generic crossing of eigenvalues of symmetric matrices being of codimension two. This surprise was the incentive to prove Proposition 4 and Corollary 11.

Remark The equation $E(x) = 0$ remains pertinent in the symmetric case studied in Sect. 3.5, where $m_2 = m_4$, $b' = b''$, $d' = d''$, (hence $\delta = \epsilon = 0$), and no three

masses are equal : the polynome F has a double root $-1/m_2 = -1/m_4$, hence one
of the roots of E is $-m_2 = -m_4$. The corresponding sides

$$E_2 - m_2 E_3 = (-m_2^2, -m_2^2, -m_2^2, -m_2^2, -m_2^2, m_1 m_3 - m_1 m_2 - m_2 m_3),$$

are such that $a = b = d$, hence their Wintner–Conley matrix is diagonal with $\alpha = \beta$.
This was the first case in the study of Sect. 3.5, and the corresponding direction satisfies
the equations of balanced configurations (and not only at the first order).

In the even more special case of rhombus relative equilibria, that is when

$$m_2 = m_4 \neq m_1 = m_3, \quad b' = b'' = d' = d'',$$

the three roots

$$-m_1, \quad -m_2, \quad -\frac{2m_1 m_2}{m_1 + m_2},$$

of the equation $E(x) = 0$ give back the three directions of degeneracy of the Wintner–
Conley matrix; indeed:

$$\begin{cases}
\text{if } x = -m_1, & a = m_2^2 - 2m_1 m_2, \quad b = f = -m_1^2, \\
\text{if } x = -m_2, & a = b = -m_2^2, \quad f = m_1^2 - 2m_1 m_2, \\
\text{if } x = -\frac{2m_1 m_2}{m_1 + m_2}, & a = \frac{m_2^3 - 3m_1 m_2^2}{m_1 + m_2}, \quad b = -m_1 m_2, \quad f = \frac{m_1^3 - 3m_1^2 m_2}{m_1 + m_2}.
\end{cases}$$

The first two cases correspond to actual lines of degeneracy of the Wintner–Conley
matrix, while in the third case, $(m_1 - m_2)b = m_1 a - m_2 f$ gives only the tangent to
the actual degeneracy curve $(m_1 - m_2)\varphi(b) = m_1\varphi(a) - m_2\varphi(f)$.

In case 3 masses are equal, say $m_2 = m_3 = m_4 := m$, one has $E_3 = mE_1 + \frac{1}{m}E_2$
and, indeed, E_2 is the line of degenerate $\mathbb{Z}/3\mathbb{Z}$-symmetric balanced configurations
described in Sect. 3.6.

Finally, when the four masses are equal, the three vectors E_1, E_2, E_3 are propor-
tional and hence they yield only trivial information.

4 The Angular Momentum

To a point (x, y) in the phase space (more accurately the tangent space to the con-
figuration space) of the n-body problem, is attached its *angular momentum bivector*
(Albouy and Chenciner 1998; Chenciner 2013b)

$$c = -x \circ \mu \circ y^{tr} + y \circ \mu \circ x^{tr} \in \wedge^2 E.$$

Transformed to the endomorphism $\mathcal{C} = c \circ \epsilon$ of E, it is represented in an orthonormal
basis by the antisymmetric matrix

$$\mathcal{C} = -XY^{tr} + YX^{tr}.$$

If $Y = \Omega X$ as this is the case for a relative equilibrium motion $X(t) = e^{t\Omega}X(0)$, we get

$$\mathcal{C} = -S\Omega^{tr} + \Omega S = S\Omega + \Omega S,$$

where the inertia matrix S of the configuration is defined in Sect. 1.2.

4.1 The Frequency Polytope and Its Subpolytopes

We now fix a central configuration x_0. In this case, the relative equilibria are all periodic, of the form $x(t) = e^{\tilde{\omega}Jt}x_0$, where J is a complex structure on $E \equiv \mathbb{R}^{2p}$ (see Sect. 1.3). By scaling the configuration, one may even assume that the frequency $\tilde{\omega}$ is equal to 1. As soon as the dimension of the ambient space is 4 or more, the same central configuration admits a whole family of relative equilibrium motions parametrized by the J's. In case the intrinsic inertia B of the configuration is generic, these motions can be characterized by their angular momentum.

To the central configuration $x_0 = \{\vec{r}_1, \ldots, \vec{r}_n\}$ is naturally attached (see Chenciner 2013a; Chenciner and Jiménez-Pérez 2013; Heckman and Zhao 2015) a convex polytope $\mathcal{P}$ contained in the $(p-1)$ simplex

$$\left\{ (v_1, \ldots, v_p) \in (\mathbb{R}_+)^p, v_1 \geq \cdots \geq v_p, \sum_{i=1}^{p} v_i = \frac{1}{\sum_{i=1}^{p} m_i} \sum_{1 \leq i < j \leq p} m_i m_j r_{ij}^2 \right\},$$

where we recall that the $r_{ij} = ||\vec{r}_i - \vec{r}_j||$ are the mutual distances between the bodies. This polytope is the set of ordered p-tuples $(v_1 \geq \cdots \geq v_p)$ of positive real numbers such that $\{\pm i v_1, \ldots, \pm i v_p\}$ is the spectrum of the J-skew-hermitian matrix $SJ + JS$ representing the angular momentum of the relative equilibrium motion of x_0 defined by some J. Once chosen an orthonormal basis of $E \equiv \mathbb{R}^{2p}$, $\mathcal{P}$ can be described as the image of the *frequency map*

$$\mathcal{F} : U(p)/SO(2p) \rightarrow W_p^+\{(v_1, \ldots, v_p), v_1 \geq \cdots \geq v_p\}$$

which, to a complex structure $J \in U(p)/SO(2p)$, that is to an identification of E with $\mathbb{C}^p$ such that the multiplication by i is an isometry, associates the ordered spectrum of the J-hermitian matrix $J^{-1}S_0J + S_0$, where S_0 is the inertia matrix of the chosen configuration. Up to some zeroes coming from the difference in dimensions, the inertia $S_0 = X_0 X_0^{tr}$ and the intrinsic inertia $B_0 = X_0^{tr} X_0$ have the same spectrum. In particular, recall that their common trace is the moment of inertia with respect to the center of mass; by a formula of Leibniz, it is equal to

$$I_0 = trace\ S_0 = \frac{1}{\sum_{i=1}^{p} m_i} \sum_{1 \leq i < j \leq p} m_i m_j r_{ij}^2.$$

As it depends only on the inertia matrix S_0 of the configuration, the polytope $\mathcal{P}$ is defined as well for any solid body,

It was proved in Chenciner (2013a) and Chenciner and Jiménez-Pérez (2013) that $\mathcal{P}$ is a *Horn polytope*, more precisely that if $\{\sigma_1 \geq \cdots \geq \sigma_{2p}\}$ is the ordered spectrum of S_0, $\mathcal{P}$ is the set of ordered spectra of real symmetric $p \times p$ matrices of the form $c = a + b$, where

$$\text{spectrum}\,(a) = \{\sigma_1, \sigma_3, \ldots, \sigma_{2p-1}\}, \quad \text{spectrum}\,(b) = \{\sigma_2, \sigma_4, \ldots, \sigma_{2p}\}.$$

Choosing other partitions Π of the spectrum of S_0 into two subsets with p elements, one defines in the same way Horn polytopes $\mathcal{P}_\Pi$ which turn out to be subpolytopes of $\mathcal{P}$ (this is a non trivial fact[7] which is proved in Fomin et al. 2005). It was noticed in Chenciner (2013a) that, given some (periodic) relative equilibrium of a central configuration, a bifurcation to a family of (quasi-periodic) relative equilibria of balanced configurations can occur only if the corresponding point in the frequency polytope $\mathcal{P}$ lies in some face of one of these subpolytopes $\mathcal{P}_\Pi$. We are interested in identifying the faces which actually correspond to such bifurcations.

4.2 The Generic Bifurcation Vertex

From Proposition 7 we deduce

Corollary 14 *In the situation of Proposition* 7, *the ordered frequencies of the angular momentum of $x(t)$ are $(\nu_1 = \sigma_1 \geq \nu_2 = \sigma_2 \geq \cdots \geq \nu_d = \sigma_d)$. They correspond to a vertex on the boundary of the frequency polytope* (Chenciner 2013a; Chenciner and Jiménez-Pérez 2013).

The identification of the frequencies $\nu_i = \sigma_i + 0$ of the angular momentum is an immediate consequence of the nature of J in the proposition. It remains to prove that the corresponding vertex is a boundary vertex and not an interior one. It is enough to prove that it is a vertex of the polytope $\mathcal{P}$ associated to the partition $\sigma_- \cup \sigma_+$ of the spectrum of the inertia matrix S_0 with $\sigma_- = \{\sigma_1, \sigma_3, \ldots, 0 \ldots, 0\}$ and $\sigma_+ = \{\sigma_2, \sigma_4, \ldots, 0, \ldots, 0\}$, where $\sigma_1 \geq \cdots \sigma_d$ (see Chenciner 2013a; Chenciner and Jiménez-Pérez 2013). This comes from the fact that whatever be d, odd or even, the number of non zero terms in σ_- (resp. σ_+) is the same as the number of zeroes in σ_+ (resp. σ_-); hence there is a vertex of $\mathcal{P}$ which corresponds to a permutation coupling each σ_i with a 0.

Example: Three bodies.

In the equal mass three-body problem, the bifurcation from an equilateral periodic relative equilibrium of a family of isosceles quasi-periodic relative equilibria in $\mathbb{R}^4$ with two frequencies cannot originate from the planar Lagrange solution but only from an equilateral relative equilibrium whose angular momentum is equivalent to the complex structure J_0 (Fig. 4).

---

[7] Intuitively, each piece of the partition defining $\mathcal{P}$ is as "separated" as possible; in contrast, in Corollary 14 the polytope associated to the partition $\{\nu_1, \nu_2, \ldots, \nu_d\} \sqcup \{0, 0, \ldots, 0\}$ is reduced to a point.
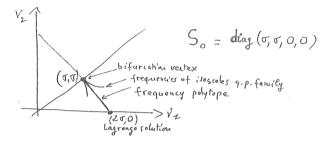
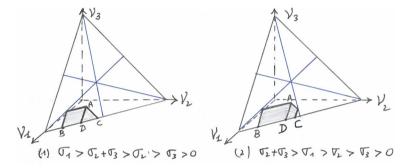**Fig. 4** Bifurcation from periodic equilateral to quasi-periodic isosceles

Fig. 5 Bifurcation loci in the generic cases

4.3 Bifurcation Locus in the Frequency Polytope

Figure 5 depicts the frequency polytope of the regular tetrahedron configuration in $\mathbb{R}^6$. Generically, only two possibilities exist for the inertia eigenvalues $\sigma_1, \sigma_2, \sigma_3$:

$$(1) \ \sigma_1 > \sigma_2 + \sigma_3 > \sigma_2 > \sigma_3 > 0 \quad \text{or} \quad (2) \ \sigma_2 + \sigma_3 > \sigma_1 > \sigma_2 > \sigma_3 > 0.$$

The first case is what becomes the example depicted in Chenciner (2013a) under the assumption that $\sigma_4 = \sigma_5 = \sigma_6 = 0$. An example is the regular tetrahedron with one of the masses much smaller than the three others. Another one is the regular tetrahedron with masses $m_1 = m_3 >> m_2 = m_4$. An example of the second one is the regular tetrahedron with almost equal masses.

Figure 6 indicates the angular momentum frequencies corresponding to the sizes and vertices when the frequency of the corresponding relative equilibrium equals 1 (if not, all the frequencies should be multiplied by this frequency ω).

There are four distinct partitions Π of the spectrum $\{\sigma_1, \sigma_2, \sigma_3, 0, 0, 0\}$:

(1) $\Pi_0 = \{\sigma_1, \sigma_2, \sigma_3\} \cup \{0, 0, 0\}$; the corresponding Horn polytope is reduced to one point, the "generic bifurcation vertex" A, which is the only place where a periodic relative equilibrium in $\mathbb{R}^6$ of the given central configuration could bifurcate
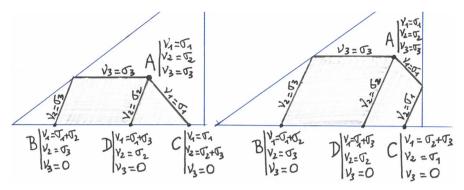
Fig. 6 Angular momentum frequencies (when rotation frequency = 1)

into a family of quasi-periodic relative equilibria in $\mathbb{R}^6$ with three frequencies of balanced configurations with $\lambda_1, \lambda_2, \lambda_3$ all distinct[8].

(2) $\Pi_i = \{\sigma_i, 0, 0\} \cup \{\sigma_j, \sigma_k, 0\}$, $i = 1, 2, 3$, the frequency polytope, which contains all the others, corresponding to $i = 2$.

The vertex B corresponds to a periodic relative equilibrium motion in $\mathbb{R}^4$ which could bifurcate into a family of quasi-periodic relative equilibria with two frequencies in $\mathbb{R}^4$ of balanced configurations with $\lambda_1 = \lambda_2 \neq \lambda_3$. The eigenplanes of the instantaneous rotation Ω would tend respectively to $\{\rho_1, \rho_2\}$ and $\{\rho_3, v_1\}$, where in agreement with the notations in Sect. 1.3, v_1 is any non-zero vector orthogonal to the image of the balanced configuration in question. It follows that the bifurcation happens at the vertex B, which corresponds to a relative equilibrium directed by the complex structure having the planes $\{\rho_1, \rho_2\}$ and $\{\rho_3, v_1\}$ as complex lines.

Analogous descriptions hold for the vertex C ($\lambda_1 \neq \lambda_2 = \lambda_3$) and the vertex D ($\lambda_1 = \lambda_3 \neq \lambda_2$).

The broken edge AB ($v_3 = \sigma_3$ reflected in $v_2 = \sigma_3$) corresponds to periodic relative equilibria in $\mathbb{R}^6$ which could bifurcate into a family of quasi-periodic relative equilibria with two frequencies in $\mathbb{R}^6$ of balanced configurations with $\lambda_1 = \lambda_2 \neq \lambda_3$ and whose instantaneous rotation Ω would have a four-dimensional eigenspace $\{\rho_1, \rho_2, v_1, v_2\}$ and a two-dimensional eigenplane $\{\rho_3, v_3\}$. The edge is parametrized by the choice of a complex structure in the four-dimensional eigenspace.

In the same way, the (possibly broken) edge AC ($v_1 = \sigma_1$) corresponds to periodic relative equilibria in $\mathbb{R}^6$ which could bifurcate into a family of quasi-periodic relative equilibria with two frequencies in $\mathbb{R}^6$ of balanced configurations with $\lambda_1 \neq \lambda_2 = \lambda_3$ and whose instantaneous rotation Ω would have a four-dimensional eigenspace $\{\rho_2, \rho_3, v_2, v_3\}$ and a two-dimensional eigenplane $\{\rho_1, v_1\}$. The edge is, as above, parametrized by the choice of a complex structure in the four-dimensional eigenspace. Finally, the same description holds for the interior side AD.

On the contrary, apart from the vertices B, D, C, the edge BC ($v_3 = 0$) does not correspond to possible bifurcations. This is because it is the interior of the frequency polytope when the dimension of E goes down to 4. This remark indicates in more gen-

---

[8] The notations A, B, C, D for the vertices have, of course, no relation with the matrices A, B, C.

eral situations what faces of the frequency polytope (and subpolytopes) are bifurcation values.

5 Rhombus 4-Body Relative Equilibria

5.1 The 3D Case

According to Corollary 13, any balanced configuration close enough to the regular tetrahedron with only two different masses, say $m_1 = m_3 \neq m_2 = m_4$, is a rhombus configuration:

$$r_{12} = r_{14} = r_{32} = r_{34} = \sqrt{b}, \quad r_{13} = \sqrt{a} \ , r_{24} = \sqrt{f}.$$

In such a simple case, it is possible to give explicit formulæ for the bifurcating families. Given real numbers $(\alpha, \beta, \gamma_1, \gamma_2)$, which we may suppose all positive, such that $m_1\gamma_1 = m_2\gamma_2$, we define a configuration x_0 of four bodies in $\mathbb{R}^3$ with center of mass at the origin by

$$x_0 = \begin{pmatrix} \alpha & 0 & -\alpha & 0 \\ 0 & \beta & 0 & -\beta \\ \gamma_1 & -\gamma_2 & \gamma_1 & -\gamma_2 \end{pmatrix}.$$

The mutual distances are

$$\begin{cases} \sqrt{a} = r_{13} = 2\alpha, \ \sqrt{f} = r_{24} = 2\beta, \\ \sqrt{b} = \sqrt{d} = r_{12} = r_{14} = r_{32} = r_{34} = (\alpha^2 + \beta^2 + (\gamma_1 + \gamma_2)^2)^{1/2}, \end{cases}$$

In the μ^{-1}-orthonormal basis $\{u_1, u_2, u_3\}$ of $\mathcal{D}^*$ formed by the vectors

$$u_1 = \sqrt{\frac{m_1 m_2}{2(m_1 + m_2)}}(1, -1, 1 - 1), \ u_2 = \sqrt{\frac{m_1}{2}}(1, 0, -1, 0),$$

$$u_3 = \sqrt{\frac{m_2}{2}}(0, 1, 0, -1),$$

x_0 is represented by the 3×3 matrix whose columns are Jacobi vectors

$$X_0 = \begin{pmatrix} 0 & \sqrt{2m_1}\alpha & 0 \\ 0 & 0 & \sqrt{2m_2}\beta \\ \sqrt{\frac{2m_1 m_2}{m_1 + m_2}}(\gamma_1 + \gamma_2) & 0 & 0 \end{pmatrix}.$$

The corresponding inertia matrices are

$$B_0 = X_0^{tr} X_0 = \begin{pmatrix} \sigma_3 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & \sigma_2 \end{pmatrix}, \quad S_0 = X_0 X_0^{tr} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix},$$

with

$$\begin{cases} \sigma_1 = 2m_1\alpha^2 = \frac{m_1}{2}a, \ \sigma_2 = 2m_2\beta^2 = \frac{m_2}{2}f, \\ \sigma_3 = \frac{2m_1m_2}{m_1+m_2}(\gamma_1+\gamma_2)^2 = 2(m_1\gamma_1^2 + m_2\gamma_2^2) = \frac{m_1m_2}{2(m_1+m_2)}(4b - a - f), \end{cases}$$

while the Wintner–Conley endomorphism $A : \mathcal{D}^* \to \mathcal{D}^*$ is

$$A = \begin{pmatrix} (m_1 + m_2)\varphi(b) & 0 & 0 \\ 0 & (m_1\varphi(a) + m_2\varphi(b)) & 0 \\ 0 & 0 & (m_1\varphi(b) + m_2\varphi(f)) \end{pmatrix}.$$

5.1.1 Bifurcations in $\mathbb{R}^6$

(1) *Bifurcations from the generic vertex.* We embed the configuration x in $\mathbb{R}^6$ by equaling to 0 the last three coordinates of each body and identify $(q_1, \ldots, q_6) \in \mathbb{R}^6$ with $(z_1 = q_1 + iq_4, \ z_2 = q_2 + iq_5, \ z_3 = q_3 + iq_6) \in \mathbb{C}^3$. A relative equilibrium is defined by making each column of X move according to

$$(Z_1, Z_2, Z_3) \mapsto (e^{i\omega_1 t} Z_1, \ e^{i\omega_2 t} Z_2, \ e^{i\omega_3 t} Z_3),$$

with

$$\omega_1^2 = -2(m_1 + m_2)\varphi(b), \ \omega_2^2 = -2(m_1\varphi(a) + m_2\varphi(b)),$$

$$\omega_3^2 = -2(m_1\varphi(b) + m_2\varphi(f)),$$

where $(Z_1, Z_2, Z_3) \in C^3$ are the columns of X, considered as belonging to $\mathcal{D} \otimes \mathbb{R}^6$ (that is with three zeros added). Its angular momentum $\mathcal{C}$ is (compare to Chenciner 2013a)

$$\begin{pmatrix} 0 & 0 & 0 & -\sigma_1\omega_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sigma_2\omega_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sigma_3\omega_3 \\ \sigma_1\omega_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2\omega_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3\omega_3 & 0 & 0 & 0 \end{pmatrix}.$$

This gives a 2-parameter[9] family bifurcating at the "generic vertex"

$$A = \left(\frac{m_1}{2}, \ \frac{m_1 m_2}{m_1 + m_2}, \ \frac{m_2}{2} \right)$$

from the regular tetrahedron with unit sides and masses m_1, m_2, m_1, m_2 such that $m_1 + m_2 = 1$ (and hence $\omega^2 = 1$) and $m_1 > m_2$.

---

[9] a, b, f modulo scaling.

(2) *Bifurcations from the sides of the frequency polytope.* The relative equilibria bifurcating as above from the generic vertex have three frequencies except when one of the equalities $a = b$ or $b = f$ of $(m_1 - m_2)\varphi(b) = m_1\varphi(a) - m_2\varphi(f)$ holds, in which case, only two distinct frequencies survive. The missing frequency is in some sense replaced by the parameter along one side of the frequency polytope (or subpolytope), which corresponds to the latitude of choice (in fact a 2-sphere) of the complex structure which directs the relative equilibrium from which the family bifurcates. Namely, supposing $a = b = 1$ and $m_1 + m_2 = 1$, and embedding the configuration x in $\mathbb{R}^6$ via the embedding $(q_1, q_2, q_3) \mapsto (q_1, 0, q_2, 0, q_3, 0)$ of $\mathbb{R}^3$ in $\mathbb{R}^6$, we define a relative equilibrium by setting

$$x(t) = \begin{pmatrix} \cos \omega t & 0 & -\cos \theta \sin \omega t & -\sin \theta \sin \omega t & 0 & 0 \\ 0 & \cos \omega t & \sin \theta \sin \omega t & -\cos \theta \sin \omega t & 0 & 0 \\ \cos \theta \sin \omega t & -\sin \theta \sin \omega t & \cos \omega t & 0 & 0 & 0 \\ \sin \theta \sin \omega t & \cos \theta \sin \omega t & 0 & \cos \omega t & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos \omega_3 t & -\sin \omega_3 t \\ 0 & 0 & 0 & 0 & \sin \omega_3 t & \cos \omega_3 t \end{pmatrix} x_0,$$

where $\omega^2 = -2(m_1 + m_2)\varphi(b) = 1$, $\omega_3^2 = -2(m_1\varphi(b) + m_2\varphi(f)) = m_1 + m_2 f^{-\frac{3}{2}}$. Then

$$S_0 = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 0 & 0 & -(\sigma_1 + \sigma_2)\omega \cos \theta & -\sigma_1\omega \sin \theta & 0 & 0 \\ 0 & 0 & \sigma_2\omega \sin \theta & 0 & 0 & 0 \\ (\sigma_1 + \sigma_2)\omega \cos \theta & -\sigma_2\omega \sin \theta & 0 & 0 & 0 & 0 \\ \sigma_1\omega \sin \theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sigma_3\omega_3 \\ 0 & 0 & 0 & 0 & \sigma_3\omega_3 & 0 \end{pmatrix},$$

whose frequencies are $\sigma_3\omega_3$ and the square roots of the solutions of

$$\nu^2 + [\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2 \cos^2 \theta]\omega^2\nu + \sigma_1^2\sigma_2^2\omega^4 \sin^4 \theta = 0.$$

At the bifurcation, when θ varies from $\pi/2$ to 0, they vary from $(\sigma_1, \sigma_2, \sigma_3)$ to $(\sigma_1 + \sigma_2, 0, \sigma_3)$, which makes (ν_1, ν_2, ν_3) travel the broken side AB in Fig. 6.

Remark In agreement with the results of Chenciner (2013a) and Chenciner and Jiménez-Pérez (2013), it is an "adapted" family of complex structures which has been

chosen to direct the relative equilibria of the regular tetrahedron at the bifurcation: they send the plane $\{\rho_1, v_1\}$ onto the plane $\{\rho_2, v_2\}$ and ρ_3 onto v_3.

5.1.2 Bifurcations in $\mathbb{R}^4$

(i) *The case $a = b$.* This is the limit $\theta = 0$ of the above family: the complex structure sends ρ_1 to ρ_2, ρ_3 to v_3 (and v_1 to v_2, which implies that the motion does not visit the corresponding dimensions). After embedding the configuration in $\mathbb{R}^4$ by equaling to 0 the last coordinate of each body and identifying $(q_1, q_2, q_3, q_4) \in \mathbb{R}^4$ with $(z_1 = q_1 + iq_2, z_2 = q_3 + iq_4) \in \mathbb{C}^2$, such a relative equilibrium motion is defined by each column of X moving according to $(Z_1, Z_2) \mapsto (e^{i\omega_1 t} Z_1, e^{i\omega_3 t} Z_2)$, with $\omega_1^2 = -2(m_1 + m_2)\varphi(b)$, $\omega_3^2 = -2(m_1\varphi(b) + m_2\varphi(f))$. Its angular momentum $\mathcal{C}$ is

$$
\begin{pmatrix}
0 & -(\sigma_1 + \sigma_2)\omega_1 & 0 & 0 \\
(\sigma_1 + \sigma_2)\omega_1 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sigma_3\omega_3 \\
0 & 0 & \sigma_3\omega_3 & 0
\end{pmatrix}.
$$

(ii) *The case $(m_1 - m_2)\varphi(b) = m_1\varphi(a) - m_2\varphi(f)$ and the case $b = f$.* The situation is analogous, the only difference being the identification of $\mathbb{R}^4$ with $\mathbb{C}^2$ which is respectively $z_1 = q_2 + iq_3, z_2 = q_1 + iq_4$ and $z_1 = q_1 + iq_3, z_2 = q_2 + iq_4$. The angular momentum spectra are respectively $\{(\sigma_2 + \sigma_3)\omega_1, \sigma_1\omega_2\}$ and $\{(\sigma_1 + \sigma_3)\omega_1, \sigma_2\omega_2\}$.

5.2 The 2D Case

This is the case when $\gamma_1 = \gamma_2 = 0$, that is $4b - a - f = 0$. We check property **(H)** (see Sect. 1.5.2) for the symmetric ($m_1 = m_3$, $b' = b'' = b$, $d' = d'' = d$, see Sect. 3.5) balanced configurations in the neighborhood of the planar rhombus central configuration x_0. As $Im B_0$ is the plane $x = 0$, x_0 is characterized by the equality of the last two eigenvalues of A_0, that is

$$
m_1\varphi(a) - m_2\varphi(f) = (m_1 - m_2)\varphi(b) = (m_1 - m_2)\varphi\left(\frac{a + f}{4}\right).
$$

Lemma 15 *Supposing $m_1 = m_3$ and $m_2 = m_4$, let x_0 be a planar rhombus (hence balanced) configuration and let K be defined by*

$$
K = 2m_1 (\varphi(a_0) - \varphi(b_0)) - (m_1 + m_2)a_0\varphi'(b_0).
$$

*If $K \neq 0$, the set of planar $\mathbb{Z}/2\mathbb{Z}$-symmetric 4-body balanced configurations close to x_0 coincide with the set of planar rhombus configurations and the condition **(H)** of Sect. 1.5.2 is satisfied at x_0 for these planar $\mathbb{Z}/2\mathbb{Z}$-symmetric balanced configurations.*

Proof One linearizes at $(a, b, d, f) = (a_0, b_0, b_0, f_0)$ the couple formed by the unique equation of $\mathbb{Z}/2\mathbb{Z}$-symmetric balanced configurations (see Sect. 3.5) and the Cayley–Menger determinant, proportional to the squared volume of the configuration; the first

assertion of the lemma follows because the 2×4 matrix one gets is of the form

$$det \begin{pmatrix} 0 & K & -K & 0 \\ \dots & \dots & \dots & -2a_0 f_0 \end{pmatrix}.$$

Hence, if $K \neq 0$, the set inertia matrices B of balanced configurations close to x_0 is a two dimensional submanifold parametrized by a and f (or by the non-zero eigenvalues $\frac{1}{2}m_1 a$ and $\frac{1}{2}m_2 f$ of B). The mapping sending the non zero eigenvalues of the inertia matrix of a planar balanced configuration B to the spectrum of $A|_{ImB}$ then reduces to

$$(a, f) \mapsto \left(2m_1\varphi(a) + 2m_2\varphi \left(\frac{a+f}{4} \right), 2m_2\varphi(f) + 2m_1\varphi \left(\frac{a+f}{4} \right) \right),$$

whose derivative at B_0 is always invertible because its determinant

$$\left(m_1^2\varphi'(a_0) + m_2^2\varphi'(f) \right) \varphi' \left(\frac{a_0 + f_0}{4} \right) + 2m_1 m_2 \varphi'(a_0)\varphi'(f_0)$$

is strictly positive. Hence condition **(H)** is satisfied (note that at this point we have not to suppose that x_0 is central). $\qquad\square$

Finally, the ellipsoid of inertia B_0 of a rhombus planar configuration is degenerate (i.e. round) if and only if $m_1 a = m_2 f$. Taking the Newtonian value $\varphi(s) = -\frac{1}{2}s^{-\frac{3}{2}}$ and supposing that x_0 is central, the condition of degeneracy becomes

$$m_1 - m_1^{-\frac{3}{2}} m_2^{\frac{5}{2}} + 8m_2^{\frac{3}{2}}(m_2 - m_1)(m_1 + m_2)^{-\frac{3}{2}} = 0,$$

or, normalizing the masses by setting $m_2 = 1$,

$$m_1 - m_1^{-\frac{3}{2}} + 8(1 - m_1)(1 + m_1)^{-\frac{3}{2}} = 0,$$

which defines three values $\gamma, 1, 1/\gamma$, with $\gamma \simeq 0.575$. Hence,

Lemma 16 *Except when m_1/m_2 equals γ, $1/\gamma$ or 1, the inertia ellipsoid of the planar rhombus central configuration is not round.*

References

Albouy, A., Chenciner, A.: Le Problème des *N* corps et les distances mutuelles. Invent. Math. **131**, 151–184 (1998)

Arnold, V.I.: Mathematical Methods of Classical Mechanics, 2nd edn. Springer, New York (1989)

Chenciner, A.: The angular momentum of a relative equilibrium, Discrete and Continuous Dynamical Systems (numéro dédié à Ernesto Lacomba), vol. 33, number 3 (2013a) . arXiv:1102.0025

Chenciner, A.: The Lagrange reduction of the *N*-body problem: a survey. Acta Math. Vietnam. **38**, 165–186 (2013b). arXiv:1111.1334

Chenciner, A.: Symmetric balanced configurations of four equal masses (2014, manuscript)

Chenciner, A., Jiménez-Pérez, H.: Angular momentum and Horn's problem. Moscow Math. J. **13**(4), 621–630 (2013). arXiv:1110.5030

Domokos, M.: Discriminant of symmetric matrices as a sum of squares and the orthogonal group. Commun. Pure Appl. Math. **64**(4), 443–465 (2011). arXiv:1003.0475

Fomin, S., Fulton, W., Li, C.K., Poon, Y.T.: Eigenvalues, singular values, and Littlewood–Richardson coefficients. Am. J. Math. **127**(1), 101–127 (2005)

Gastineau, M., Laskar, J.: TRIP: a computer algebra system dedicated to celestial mechanics and perturbation series. ACM Commun. Comput. Algebra **44**, 194–197 (2011)

Heckman, G., Zhao, L.: Angular Momenta of Relative Equilibrium Motions and Real Moment Map Geometry (2015, preprint). arXiv:1505.07331

Lagrange, J.L.: Essai sur le problème des trois corps (1772)

Lax, P.D.: On the discriminant of real symmetric matrices. Commun. Pure Appl. Math. **LI**, 1387–1396 (1998)