

On the Roots of a Hyperbolic Polynomial Pencil

Victor Katsnelson¹ 

Received: 3 May 2016 / Accepted: 20 July 2016

© Institute for Mathematical Sciences (IMS), Stony Brook University, NY 2016

Abstract Let $v_0(t), v_1(t), \dots, v_n(t)$ be the roots of the equation $R(z) = t$, where $R(z)$ is a rational function of the form

$$R(z) = z - \sum_{k=1}^n \frac{\alpha_k}{z - \mu_k},$$

μ_k are pairwise distinct real numbers, $\alpha_k > 0$, $1 \leq k \leq n$. Then for each real ξ , the function $e^{\xi v_0(t)} + e^{\xi v_1(t)} + \dots + e^{\xi v_n(t)}$ is exponentially convex on the interval $-\infty < t < \infty$.

Keywords Hyperbolic polynomial pencil · Determinant representation · Exponentially convex functions

Mathematics Subject Classification 11C99 · 26C10 · 26C15 · 15A22 · 42A82

1 Roots of the Equation $R(z) = t$ as Functions of t

In the present paper we discuss questions related to properties of roots of the equation

$$R(z) = t \tag{1.1}$$

✉ Victor Katsnelson
victor.katsnelson@weizmann.ac.il; victorkatsnelson@gmail.com

¹ Department of Mathematics, Weizmann Institute, 7610001 Rehovot, Israel

as functions of the parameter $t \in \mathbb{C}$, where R is a rational function of the form

$$R(z) = z - \sum_{1 \leq k \leq n} \frac{\alpha_k}{z - \mu_k}, \quad (1.2)$$

μ_k are pairwise distinct real numbers, $\alpha_k > 0$, $1 \leq k \leq n$. We adhere to the enumeration agreement¹

$$\mu_1 > \mu_2 > \cdots > \mu_n. \quad (1.3)$$

The function R is representable in the form

$$R(z) = \frac{P(z)}{Q(z)}, \quad (1.4)$$

where

$$Q(z) = (z - \mu_1) \cdot (z - \mu_2) \cdot \cdots \cdot (z - \mu_n), \quad (1.5)$$

$$P(z) \stackrel{\text{def}}{=} R(z) \cdot Q(z) \quad (1.6)$$

are monic polynomials of degrees

$$\deg P = n + 1, \quad \deg Q = n. \quad (1.7)$$

Since $P(\mu_k) = -\alpha_k Q'(\mu_k) \neq 0$, the polynomials P and Q have no common roots. Thus the ratio in the right hand side of (1.4) is irreducible. The Eq. (1.1) is equivalent to the equation

$$P(z) - tQ(z) = 0. \quad (1.8)$$

Since the polynomial $P(z) - tQ(z)$ is of degree $n + 1$, the latter equation has $n + 1$ roots for each $t \in \mathbb{C}$.

The function R possess the property

$$\operatorname{Im} R(z) / \operatorname{Im} z > 0 \quad \text{if} \quad \operatorname{Im} z \neq 0. \quad (1.9)$$

Therefore if $\operatorname{Im} t > 0$, all roots of the equation (1.1), which is equivalent to the Eq. (1.8), are located in the half-plane $\operatorname{Im} z > 0$. Some of these roots may be multiple.

However if t is real, all roots of the Eq. (1.1) are real and simple, i.e. of multiplicity one. Thus for real t , the Eq. (1.1) has $n + 1$ pairwise distinct real roots $v_k(t)$: $v_0(t) > v_1(t) > \cdots > v_{n-1}(t) > v_n(t)$. Moreover for each real t , the poles μ_k of the function R and the roots $v_k(t)$ of the Eq. (1.1) are interlacing:

$$v_0(t) > \mu_1 > v_1(t) > \mu_2 > v_2(t) > \cdots > v_{n-1}(t) > \mu_n > v_n(t), \quad \forall t \in \mathbb{R}. \quad (1.10)$$

¹ We assume that $n \geq 1$.

In particular for $t = 0$, the roots $\nu_k(0) = \lambda_k$ of the Eq. (1.1) are the roots of the polynomial P :

$$P(z) = (z - \lambda_0) \cdot (z - \lambda_1) \cdot \dots \cdot (z - \lambda_n), \tag{1.11}$$

$$\lambda_0 > \mu_1 > \lambda_1 > \mu_2 > \lambda_2 > \dots > \lambda_{n-1} > \mu_n > \lambda_n. \tag{1.12}$$

Since $R'(x) > 0$ for $x \in \mathbb{R}$, $x \neq \mu_1, \dots, \mu_n$, each of the functions $\nu_k(t)$, $k = 0, 1, \dots, n$, can be continued as a single valued holomorphic function to some neighborhood of \mathbb{R} . However the functions $\nu_k(t)$ can not be continued as single-valued analytic functions to the whole complex t -plane. According to (1.4),

$$R'(z) = \frac{P'(z)Q(z) - Q'(z)P(z)}{Q^2(z)}. \tag{1.13}$$

The polynomial $P'Q - Q'P$ is of degree $2n$ and is strictly positive on the real axis. Therefore this polynomial has n roots ζ_1, \dots, ζ_n in the upper half-plane $\text{Im}(z) > 0$ and n roots $\bar{\zeta}_1, \dots, \bar{\zeta}_n$ in the lower half-plane $\text{Im}(z) < 0$. (Not all roots ζ_1, \dots, ζ_n must be distinct.) The points ζ_1, \dots, ζ_n and $\bar{\zeta}_1, \dots, \bar{\zeta}_n$ are the critical points of the function R : $R'(\zeta_k) = 0$, $R'(\bar{\zeta}_k) = 0$, $1 \leq k \leq n$. The critical values $t_k = R(\zeta_k)$, $\bar{t}_k = R(\bar{\zeta}_k)$, $1 \leq k \leq n$, of the function R are the ramification points of the function $\nu(t)$:

$$R(\nu(t)) = t \tag{1.14}$$

(Even if the critical points ζ' and ζ'' of R are distinct, the critical values $R(\zeta')$ and $R(\zeta'')$ may coincide.) We denote the set of critical values of the function R by \mathcal{V} :

$$\mathcal{V} = \mathcal{V}^+ \cup \mathcal{V}^-, \quad \mathcal{V}^+ = \{t_1, \dots, t_n\}, \quad \mathcal{V}^- = \{\bar{t}_1, \dots, \bar{t}_n\}. \tag{1.15}$$

Not all values t_1, \dots, t_n must be distinct. However $\mathcal{V} \neq \emptyset$. In view of (1.9), $\text{Im } t_k > 0$, $1 \leq k \leq n$. So

$$\mathcal{V}^+ \subset \{t \in \mathbb{C} : \text{Im } t > 0\}, \quad \mathcal{V}^- \subset \{t \in \mathbb{C} : \text{Im } t < 0\}. \tag{1.16}$$

Let G be an arbitrary simply connected domain in the t -plane which does not intersect the set \mathcal{V} . Then the roots of Eq. (1.1) are pairwise distinct for each $t \in G$. We can enumerate these roots, say $\nu_0(t)$, $\nu_1(t)$, \dots $\nu_n(t)$, such that all functions $\nu_k(t)$ are holomorphic in G .

The strip S_h ,

$$S_h = \{t \in \mathbb{C} : |\text{Im } t| < h\}, \quad \text{where } h = \min_{1 \leq k \leq n} \text{Im } t_k, \tag{1.17}$$

does not intersect the set \mathcal{V} . So $n + 1$ single valued holomorphic branches of the function $\nu(t)$, (1.14), are defined in the strip S_h . We choose such enumeration of these branches which agrees with the enumeration (1.10) on \mathbb{R} .

From (1.6) and (1.2) it follows that the polynomial P is representable in the form

$$P(z) = z Q(z) - \sum_{k=1}^n \alpha_k Q_k(z), \tag{1.18a}$$

where

$$Q_k(z) = Q(z)/(z - \mu_k), \quad k = 1, 2, \dots, n. \tag{1.18b}$$

2 Determinant Representation of the Polynomial Pencil $P(z) - t Q(z)$

The polynomial pencil $P(z) - t Q(z)$ is *hyperbolic*: for each real t , all roots of the Eq. (1.8) are real.

Using (1.18), we represent the polynomial $P(z) - t Q(z)$ as the characteristic polynomial $\det(zI - (A + tB))$ of some matrix pencil, where A and B are self-adjoint $(n + 1) \times (n + 1)$ matrices, $\text{rank } B = 1$. We present these matrices explicitly.

Lemma 2.1 *Let $A = \|a_{p,q}\|$ and $B = \|b_{p,q}\|$, $0 \leq p, q \leq n$, be $(n + 1) \times (n + 1)$ matrices with the entries*

$$\begin{aligned} a_{0,0} &= 0, \quad a_{p,p} = \mu_p \quad \text{for } p = 1, 2, \dots, n, \\ a_{p,q} &= 0 \quad \text{for } p = 1, 2, \dots, n, \quad q = 1, 2, \dots, n, \quad p \neq q, \\ a_{0,p} &= \overline{a_{p,0}} \quad \text{for } p = 1, 2, \dots, n, \end{aligned} \tag{2.1}$$

and

$$b_{0,0} = 1, \quad \text{all other } b_{p,q} \text{ vanish.} \tag{2.2}$$

Then the equality

$$\det(zI - A - tB) = (z - t) \cdot Q(z) - \sum_{k=1}^n |a_{0,k}|^2 Q_k(z). \tag{2.3}$$

holds.

Proof The matrix $zI - (A + tB)$ is of the form

$$zI - (A + tB) = \begin{bmatrix} z - t & -a_{0,1} & -a_{0,2} & \dots & -a_{0,n-1} & -a_{0,n} \\ -\overline{a_{0,1}} & z - \mu_1 & 0 & \dots & 0 & 0 \\ -\overline{a_{0,2}} & 0 & z - \mu_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\overline{a_{0,n-1}} & 0 & 0 & \dots & z - \mu_{n-1} & 0 \\ -\overline{a_{0,n}} & 0 & 0 & \dots & 0 & z - \mu_n \end{bmatrix}$$

We compute the determinant of this matrix using the cofactor formula. □

Comparing (1.18) and (2.3), we see that if the conditions

$$|a_{0,p}|^2 = \alpha_p, \quad p = 1, 2, \dots, n \tag{2.4}$$

are satisfied, then the equality

$$P(z) - tQ(z) = \det(zI - A - tB) \tag{2.5}$$

holds for every $z \in \mathbb{C}, t \in \mathbb{C}$.

The following result is an immediate consequence of Lemma 2.1.

Theorem 2.2 *Let R be a function of the form (1.2), where $\mu_1, \mu_2, \dots, \mu_n$ are pairwise distinct real numbers and $\alpha_1, \alpha_2, \dots, \alpha_n$ are positive numbers. Let Q and P be the polynomials related to the the function R by the equalities (1.5) and (1.18).*

Then the pencil of polynomials $P(z) - tQ(z)$ is representable as the characteristic polynomial of the matrix pencil $A + tB$, i.e. the equality (2.5) holds for every $z \in \mathbb{C}, t \in \mathbb{C}$, where B is the matrix with the entries (2.2), and the entries of the matrix A are defined by by (2.1) with

$$a_{0,p} = \sqrt{\alpha_p} \omega_p, \quad p = 1, 2, \dots, n, \tag{2.6}$$

ω_p are arbitrary² complex numbers of absolute value one:

$$|\omega_p| = 1, \quad p = 1, 2, \dots, n. \tag{2.7}$$

Corollary 2.3 *Let R, A, B be the same as in Theorem 2.2. For each $t \in \mathbb{C}$, the roots $v_0(t), v_0(t), \dots, v_n(t)$ of the Eq. (1.2) are the eigenvalues of the matrix $A + tB$.*

Lemma 2.4 *Let R, A, B be the same as in Theorem 2.2, $v_0(t), v_0(t), \dots, v_n(t)$ be the roots of the Eq. (1.2) and $h(z)$ be an entire function. Then the equality*

$$\sum_{k=0}^n h(v_k(t)) = \text{trace} \{h(A + tB)\} \tag{2.8}$$

holds for every $t \in \mathbb{C}$.

Proof We refer to Corollary 2.3. If v is an eigenvalue of some square matrix M , then $h(v)$ is an eigenvalue of the matrix $h(M)$. In (2.8), we interpret the trace of the matrix $h(A + tB)$ as its *spectral trace*, that is as the sum of all its eigenvalues. □

² We will use the freedom in choosing ω_p to prescribe signs \pm to the entries $a_{0,p}$.

3 Exponentially Convex Functions

Definition 3.1 A function $f(t)$ on the interval $a < t < b$ is said to belong to the class $W_{a,b}$ if f is continuous on (a, b) and if all forms

$$\sum_{r,s=1}^N f(t_r + t_s) \zeta_r \bar{\zeta}_s \quad (N = 1, 2, 3, \dots) \tag{3.1}$$

are non-negative for every choice of complex numbers $\zeta_1, \zeta_2, \dots, \zeta_N$ and for every choice of real numbers t_1, t_2, \dots, t_N assuming that all sums $t_r + t_s$ are within the interval (a, b) .

The class $W_{a,b}$ was introduced by [Bernstein \(1928\)](#), see Sect. 15 there. Somewhat later, Widder also introduced the class $W_{a,b}$ and studied it. Bernstein called functions $f(x) \in W_{a,b}$ exponentially convex.

Properties of the class of exponentially convex functions

- P1. If $f(t) \in W_{a,b}$ and $c \geq 0$ is a nonnegative constant, then $cf(t) \in W_{a,b}$.
- P2. If $f_1(t) \in W_{a,b}$ and $f_2(t) \in W_{a,b}$, then $f_1(t) + f_2(t) \in W_{a,b}$.
- P3. If $f_1(t) \in W_{a,b}$ and $f_2(t) \in W_{a,b}$, then $f_1(t) \cdot f_2(t) \in W_{a,b}$.
- P4. Let $\{f_n(t)\}_{1 \leq n < \infty}$ be a sequence of functions from the class $W_{a,b}$. We assume that for each $t \in (a, b)$ there exists a limit $f(t) = \lim_{n \rightarrow \infty} f_n(t)$, and that $f(t) < \infty \forall t \in (a, b)$. Then $f(t) \in W_{a,b}$.

From the functional equation for the exponential function it follows that for each real number u , for every choice of real numbers t_1, t_2, \dots, t_N and complex numbers $\zeta_1, \zeta_2, \dots, \zeta_N$, the equality holds

$$\sum_{r,s=1}^N e^{\xi(t_r+t_s)} \zeta_r \bar{\zeta}_s = \left| \sum_{p=1}^N e^{\xi t_p} \zeta_p \right|^2 \geq 0. \tag{3.2}$$

The inequality (3.2) can be stated as

Lemma 3.2 For each real number ξ , the function $e^{\xi t}$ of the variable t belongs to the class $W_{-\infty, \infty}$.

The term *exponentially convex function* is justified by the following integral representation for any function $f(t) \in W_{a,b}$.

Theorem 3.3 (The representation theorem) For the representation

$$f(x) = \int_{\xi \in (-\infty, \infty)} e^{\xi x} \sigma(d\xi) \quad (a < x < b) \tag{3.3}$$

to be valid, where $\sigma(d\xi)$ is a non-negative measure, it is necessary and sufficient that $f(x) \in W_{a,b}$.

The proof of the Representation Theorem can be found in [Akhiezer \(1965\)](#) (Theorem 5.5.4), and in [Widder \(1946\)](#) (Chapter 6, Theorem 21).

Corollary 3.4 *The representation (3.3) shows that $f(x)$ is the value of a function $f(z)$ holomorphic in the strip $a < \operatorname{Re} z < b$.*

4 Herbert Stahl’s Theorem

In the paper [Bessis et al. \(1975\)](#) a conjecture was formulated which is now commonly known as the BMV conjecture:

The BMV conjecture Let U and V be Hermitian matrices. Then the function

$$\varphi(t) = \operatorname{trace} \{e^{U+tV}\} \tag{4.1}$$

of the variable t belongs to the class $W_{-\infty, \infty}$.

If the matrices U and V commute, the exponential convexity of the function φ , (4.1), is evident. In this case, the sum

$$\sum_{r,s=1}^N \varphi(t_r + t_s) \zeta_r \bar{\zeta}_s = \operatorname{trace} \left\{ e^{U/2} \left(\sum_{r=1}^N e^{t_r V} \zeta_r \right) \left(\sum_{s=1}^N e^{t_s V} \zeta_s \right)^* (e^{U/2})^* \right\}$$

is non-negative because this sum is the trace of a non-negative matrix. The measure σ in the integral representation (3.3) of the function φ , (4.1), is an atomic measure supported on the spectrum of the matrix V .

In the general case, if the matrices U and V do not commute, the BMV conjecture remained an open question for longer than 40 years. In 2011, Herbert Stahl proved the BMV conjecture.

Theorem 4.1 (H. Stahl) *Let U and V be Hermitian matrices.*

Then the function $\varphi(t)$ defined by (4.1) belongs to the class $W_{-\infty, \infty}$ of functions exponentially convex on $(-\infty, \infty)$.

The first arXiv version of Stahl’s Theorem appeared in [Stahl \(2011\)](#), the latest arXiv version—in [Stahl \(2012\)](#), the journal publication—in [Stahl \(2013\)](#).

The proof of Herbert Stahl is based on ingenious considerations related to Riemann surfaces of algebraic functions. In [Eremenko \(2015\)](#), a simplified version of the Herbert Stahl proof is presented.

We present a toy version of Theorem 4.1 which is enough for our goal.

Theorem 4.2 *Let U and V be Hermitian matrices. We assume moreover that*

1. *All off-diagonal entries of the matrix U are non-negative.*
2. *The matrix V is diagonal.*

Then the function $\varphi(t)$ defined by (4.1) belongs to the class $W_{-\infty, \infty}$.

Proof For $\rho \geq 0$, let $U_\rho = U + \rho I$, where I is the identity matrix. If ρ is large enough, then all entries of the matrix U_ρ are non-negative. Let us choose and fix such ρ . It is clear that

$$e^{U+tV} = e^{-\rho} e^{U_\rho+tV}. \tag{4.2}$$

We use the Lie product formula

$$e^{U_\rho+tV} = \lim_{m \rightarrow \infty} (e^{U_\rho/m} e^{tV/m})^m. \tag{4.3}$$

All entries of the matrix $e^{U_\rho/m}$ are non-negative numbers. Since matrix V is Hermitian, its diagonal entries are real numbers. Thus

$$e^{tV/m} = \text{diag}(e^{tv_1/m}, e^{tv_2/m}, \dots, e^{tv_m/m}),$$

where v_1, v_2, \dots, v_m are real numbers. The exponentials $e^{tv_j/m}$ are functions of t from the class $W_{-\infty, \infty}$. Each entry of the matrix $e^{U_\rho/m} e^{tV/m}$ is a linear combination of these exponentials with non-negative coefficients. According to the properties P1 and P2 of the class $W_{-\infty, \infty}$, the entries of the matrix $e^{U_\rho/m} e^{tV/m}$ are functions of the class $W_{-\infty, \infty}$. Each entry of the matrix $(e^{U_\rho/m} e^{tV/m})^m$ is a sum of products of some entries of the matrix $e^{U_\rho/m} e^{tV/m}$. According to the properties P2 and P3 of the class $W_{-\infty, \infty}$, the entries of the matrix $(e^{U_\rho/m} e^{tV/m})^m$ are functions of t belonging to the class $W_{-\infty, \infty}$. From the limiting relation (4.3) and from the property P4 of the class $W_{-\infty, \infty}$ it follows that all entries of the matrix $e^{U_\rho+tV}$ are functions of t belonging to the class $W_{-\infty, \infty}$. From (4.2) it follows that all entries of the matrix e^{U+tV} belong to the class $W_{-\infty, \infty}$. All the more, the function $\varphi(t) = \text{trace}\{e^{U+tV}\}$, which is the sum of diagonal entries of the matrix e^{U+tV} , belongs to the class $W_{-\infty, \infty}$. \square

5 Exponential Convexity of the Sum $e^{\xi v_0(t)} + \dots + e^{\xi v_n(t)}$

Let ξ be a real number. Taking $h(z) = e^{\xi z}$ in Lemma 2.4, we obtain

Lemma 5.1 *Let R be the rational function of the form (1.2), $v_0(t), v_1(t), \dots, v_n(t)$ be the roots of the Eq. (1.1). Let A and B be the matrices (2.1), (2.6), (2.2) which appear in the determinant representation (2.5) of the matrix pencil $P(z) - tQ(z)$.*

Then the equality

$$\sum_{k=0}^n e^{\xi v_k(t)} = \text{trace}\{e^{\xi A+t(\xi B)}\} \tag{5.1}$$

holds.

Now we choose ω_p in (2.6) so that all off-diagonal entries of the matrix $U = \xi A$ are non-negative: if $\xi > 0$, then $\omega_p = +1$, if $\xi < 0$, then $\omega_p = -1$, $1 \leq p \leq n$.

Applying Theorem 4.2 to the matrices $U = \xi A$, $V = \xi B$, we obtain the following result

Theorem 5.2 Let R be the rational function of the form (1.2), $v_0(t), v_1(t), \dots, v_n(t)$ be the roots of the Eq. (1.1). Then for each $\xi \in \mathbb{R}$, the function

$$g(t, \xi) \stackrel{\text{def}}{=} \sum_{k=0}^n e^{\xi v_k(t)} \tag{5.2}$$

of the variable t belongs to the class $W_{-\infty, \infty}$.

Theorem 5.3 Let $f \in W_{u, v}$, where $-\infty \leq u < v \leq +\infty$. Let R be the rational function of the form (1.2), $v_0(t), v_1(t), \dots, v_n(t)$ be the roots of the Eq. (1.1). Assume that for some $a, b, -\infty \leq a < b \leq +\infty$, the inequalities

$$u < v_k(t) < v, \quad a < t < b, \quad k = 0, 1, \dots, n \tag{5.3}$$

hold.

Then the function

$$F(t) \stackrel{\text{def}}{=} \sum_{k=0}^n f(v_k(t)) \tag{5.4}$$

belongs to the class $W_{a, b}$.

Proof According to Theorem 3.3, the representation

$$f(x) = \int_{\xi \in (-\infty, \infty)} e^{\xi x} \sigma(d\xi), \quad \forall x \in (u, v)$$

holds, where σ is a non-negative measure. Substituting $x = v_k(t)$ to the above formula, we obtain the equality

$$f(v_k(t)) = \int_{\xi \in (-\infty, \infty)} e^{\xi v_k(t)} \sigma(d\xi), \quad \forall t \in (a, b), \quad k = 0, 1, \dots, n.$$

Hence

$$F(t) = \int_{\xi \in (-\infty, \infty)} g(t, \xi) \sigma(d\xi), \quad \forall t \in (a, b). \tag{5.5}$$

Theorem 5.4 is a consequence of Theorem 5.2 and of the properties P1, P2, P4 of the class of exponentially convex functions. □

Example For $\gamma > 0$, the function $f(x) = e^{\gamma x^2}$ is exponentially convex on $(-\infty, \infty)$: $e^{\gamma x^2} = \int_{\xi \in (-\infty, \infty)} e^{\xi x} \sigma(d\xi)$, where $\sigma(d\xi) = \frac{1}{2\sqrt{\pi\gamma}} e^{-\xi^2/4\gamma} d\xi$.

Thus the function $F(t) = \sum_{k=0}^n e^{\gamma(v_k(t))^2}$ is exponentially convex on $(-\infty, \infty)$.

Remark 5.4 Familiarizing himself with our proof of Theorem 5.2, Alexey Kuznetsov (<http://www.math.yorku.ca/~akuznets/>) gave a new proof of a somewhat weakened version of this theorem. His proof is based on the theory of stochastic Lévy processes.

References

- Ахиезер, Н.И. *Классическая: проблема моментов*. Физматгиз, Москва (1965) (in Russian). English Transl.: Akhiezer, N.I.: *The Classical Moment Problem*. Oliver and Boyd, Edinburgh (1965)
- Bernstein, S.N.: Sur les fonctions absolument monotones. *Acta Math.* **52**, 1–66 (1928). (in French)
- Bessis, D., Moussa, P., Villani, M.: Monotonic converging variational approximations to the functional integrals in quantum statistical mechanics. *J. Math. Phys.* **16**(11), 2318–2325 (1975)
- Eremenko, A.: Herbert Stahl's proof of the BMV conjecture. *Sb. Math.* **206**(1), 87–92 (2015)
- Stahl, H.: Proof of the BMV conjecture. [arXiv:1107.4875v1](https://arxiv.org/abs/1107.4875v1), pp. 1–56, 25 July 2011 (2011)
- Stahl, H.: Proof of the BMV conjecture. [arXiv:1107.4875v3](https://arxiv.org/abs/1107.4875v3), pp. 1–25, 17 August 2012
- Stahl, H.: Proof of the BMV conjecture. *Acta Math.* **211**, 255–290 (2013)
- Widder, D.V.: *Laplace Transform*. Princeton University Press, Princeton (1946)