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RESEARCH CONTRIBUTION

Origami, Affine Maps, and Complex Dynamics

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Abstract We investigate the combinatorial and dynamical properties of so-called *nearly Euclidean Thurston maps*, or *NET maps*. These maps are perturbations of many-to-one folding maps of an affine two-sphere to itself. The close relationship between NET maps and affine maps makes computation of many invariants tractable.

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In addition to this, NET maps are quite diverse, exhibiting many different behaviors. We discuss data, findings, and new phenomena.

Keywords Thurston map · Branched covering · Teichmüller theory · Self-similar group

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Contents

1 Introduction

Complex dynamics studies iteration of rational functions $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. An important subclass consists of the *postcritically finite* rational maps: those for which the *postcritical set* $P(f) := \cup_{n>0} f^{\circ n}(C(f))$ is finite; here $C(f)$ is the finite set of points at which f is not locally injective. For example, if $c_R \in \mathbb{C}$ is the unique root of $c^3 + 2c^2 + c + 1$ with $\Im(c_R) > 0$, then the quadratic polynomial $f(z) = z^2 + c_R$, known as *Douady's rabbit*, is postcritically finite: it has one fixed critical point at infinity, and the unique finite critical point at the origin is periodic of period 3. Another example is provided by $f(z) = z^2 + i$. The Julia sets of these maps are shown in Fig. 1. Their three finite postcritical points are marked with tiny circles.

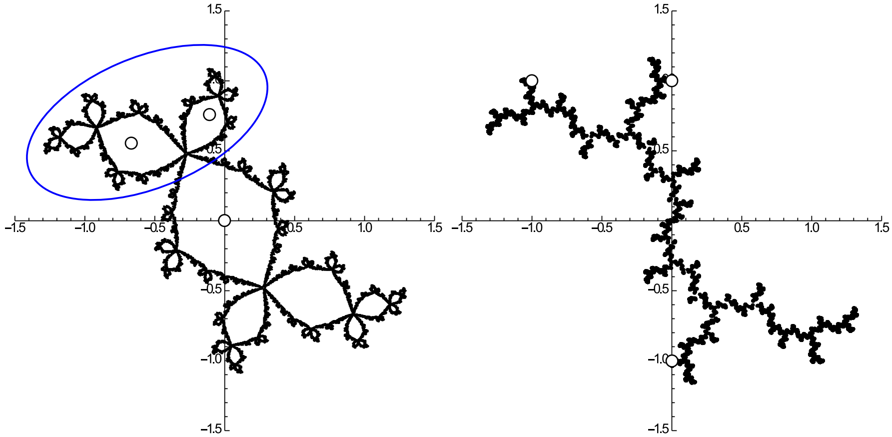


Fig. 1 Julia sets of the rabbit $z \mapsto z^2 + c_R$ and $z \mapsto z^2 + i$

Thurston Maps. A *Thurston map* is a continuous, orientation-preserving branched covering $f : S^2 \to S^2$ of degree at least two for which the set $P(f)$ is finite. For example, if h is a Dehn twist about the blue ellipse in Fig. 1, one may *twist* Douady's rabbit by post-composing f with h to yield a Thurston map $g = h \circ f$. More generally, if $h_0, h_1 : S^2 \to S^2$ are orientation-preserving homeomorphisms such that h_0 agrees with h_1^{-1} on $P(f)$, then we call $h_0 \circ f \circ h_1$ a *twist* of f. We call the resulting collection of maps the *pure Hurwitz class* of f. See Sect. 4 for related definitions and discussion.

Combinatorial Equivalence. Two Thurston maps f, g are *combinatorially equivalent* or *Thurston equivalent* if there are orientation-preserving homeomorphisms $h_0, h_1 : (S^2, P(f)) \to (S^2, P(g))$ for which $h_0 \circ f = g \circ h_1$ and h_0, h_1 are isotopic through homeomorphisms agreeing on $P(f)$. More succinctly: they are conjugate up to isotopy relative to their postcritical sets. This is related to a more familiar notion. For a finite set $P \subset S^2$, denote by $\mathrm{PMod}(S^2, P)$ the pure mapping class group of the pair (S^2, P). Suppose $P(f) = P(g) = P$. The notion of combinatorial equivalence between f and g is analogous to the notion of conjugacy in $\mathrm{PMod}(S^2, P)$, but now the representing maps are branched coverings instead of homeomorphisms.

W. Thurston's Characterization of Rational Maps. W. Thurston (Douady and Hubbard 1993) gave necessary and sufficient combinatorial conditions for a Thurston map f to be equivalent to a rational map g. The statement has two cases, depending on the Euler characteristic $\chi(\mathcal{O}(f))$ of a certain orbifold structure $\mathcal{O}(f)$ on the sphere associated to the dynamics of f on the set $P(f) \cup C(f)$; see Douady and Hubbard (1993). Typical Thurston maps have hyperbolic orbifold ($\chi < 0$) and checking rationality involves ruling out certain families of curves, called *obstructions*. Atypical Thurston maps have Euclidean orbifold ($\chi = 0$)—we call these *Euclidean*—and checking rationality involves examining the eigenvalues of a two by two matrix. Apart from a well-understood subset of Euclidean maps known as *flexible rational Lattès maps*, the rational map g equivalent to a Thurston map f, if it exists, is unique up to holomorphic conjugacy.

Checking that there are no obstructions is often very difficult. To give a sense of the complexity that can occur, consider the following result, Theorem 1. All of the maps involved are typical NET maps—the special class of Thurston maps that is the focus of this paper.

Theorem 1 *Each twist of the rabbit $f(z) = z^2 + c_R$ is combinatorially equivalent to a complex polynomial $z^2 + c$ where $c^3 + 2c^2 + c + 1 = 0$; all three cases arise. In contrast, for twists of $f(z) = z^2 + i$, the problem of determining rationality of a twist $h \circ f$ reduces to checking the image of h under a homomorphism to a finite group of order* 100. *Among these combinatorial classes there are precisely two classes of rational maps, namely $z \mapsto z^2 \pm i$, and a countably infinite family of pairwise inequivalent twists of the form $h^n \circ g, n \in \mathbb{Z}$, where g is a particular obstructed twist and h is a Dehn twist about the obstruction of g.*

The first statement follows from a general result now known as the *Berstein-Levy theorem* (Levy 1985), while the second is more recent and is one of the main results of Bartholdi and Nekrashevych's article (Bartholdi and Nekrashevych 2006, §6).

When all twists of f are equivalent to rational maps, we say its pure Hurwitz class is *completely unobstructed*. Some pure Hurwitz classes are completely obstructed (defined analogously) and some are neither, i.e. contain both obstructed and unobstructed maps.

Induced Dynamics on Curves. A simple closed curve in $S^2 - P(f)$ is *essential* if it is not freely homotopic to a constant curve at a point in $S^2 - P(f)$. An essential curve is *peripheral* if it is homotopic into arbitrarily small neighborhoods of a point of $P(f)$. A Thurston map f and all its iterates are unramified outside the set $P(f)$, so curves in $S^2 - P(f)$ can be iteratively lifted under f. For example, it is easy to see that under iterated pullback the blue ellipse γ in Fig. 1 is periodic of period 3 up to homotopy, and that $\deg(f^3 : \tilde{\gamma} \to \gamma) = 4$, where $\tilde{\gamma}$ is the unique preimage of γ under f^3 that is essential and nonperipheral in $S^2 - P(f)$. There are countably infinitely many simple closed curves up to homotopy in $S^2 - P(f)$, though, and it is harder to see the following:

Theorem 2 (Pilgrim 2012, Theorem 1.6) *Under iterated pullback of the rabbit polynomial $f(z) = z^2 + c_R$, any simple closed curve becomes either inessential or peripheral in $S^2 - P(f)$ or, up to homotopy, falls into the above 3-cycle.*

See the end of Sect. 7 for an outline of another way to prove this result.

Focusing on the behavior of curves under pullback is important. The statement of W. Thurston's characterization theorem for rational maps among Thurston maps says that obstructions to f being rational are multicurves $\Gamma \subset S^2 - P(f)$ with a certain invariance property. Specifically: after deleting inessential and peripheral preimages, we have $f^{-1}(\Gamma) \subset \Gamma$ up to homotopy in $S^2 - P(f)$, and the spectral radius of a certain associated linear map $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ is greater than or equal to 1.

Teichmüller Theory. The proof of W. Thurston's characterization theorem reduces the question "Is f equivalent to a rational map?" to the problem of finding a fixed point for a certain holomorphic self-map $\sigma_f : \mathrm{Teich}(S^2, P(f)) \to \mathrm{Teich}(S^2, P(f))$ of a Teichmüller space, given by pulling back complex structures under f; see Douady and Hubbard (1993). For the precise definition of σ_f, we refer the reader to Buff et al. (2009). Although σ_f is complicated and transcendental, it covers a finite algebraic correspondence on moduli space:

$$
\begin{tikzcd}
\mathrm{Teich}(S^2, P(f)) \arrow[rr, "\sigma_f"] \arrow[dr, "\omega"] \arrow[dd, "\pi"'] & & \mathrm{Teich}(S^2, P(f)) \arrow[dd, "\pi"] \\
& \mathcal{W} \arrow[dl, "Y"'] \arrow[dr, "X"] & \\
\mathrm{Moduli}(S^2, P(f)) & & \mathrm{Moduli}(S^2, P(f))
\end{tikzcd}
$$

See Koch (2013) and Koch et al. (2016). In the above diagram, Y is a finite covering, X is holomorphic, and only σ_f depends on f; up to isomorphism induced by conjugation by impure mapping class elements, the remainder depends only on the pure Hurwitz class of f, cf. Koch (2013) and Koch et al. (2016, §3). In the case of the rabbit, the moduli space is isomorphic to $\mathbb{P}^1 - \{0, 1, \infty\}$, the map X is injective, so that we may regard $\mathcal{W} \subset \mathbb{P}^1 - \{0, 1, \infty\}$, the map Y is given by $x \mapsto 1 - \frac{1}{x^2}$, and $\mathcal{W} = \mathbb{P}^1 - \{\pm 1, 0, \infty\}$; see Bartholdi and Nekrashevych (2006). For quadratics with four postcritical points and hyperbolic orbifold, X is always injective, and the formulas for $Y \circ X^{-1}$ are quite simple. For other maps, the equations defining the correspondence $\mathcal{W}$ may be complicated. This happens even for maps with four postcritical points, including Euclidean quadratics, many cubics, and most NET maps.

Of special interest is the group $G_f < \mathrm{PMod}(S^2, P(f))$ represented by *liftable* homeomorphisms h, i.e. those for which there is a lift $\tilde{h}$ representing an element in $\mathrm{PMod}(S^2, P(f))$ with $h \circ f = f \circ \tilde{h}$; the assignment $h \mapsto \tilde{h}$ gives a homomorphism $\phi_f : G_f \to \mathrm{PMod}(S^2, P(f))$ which we call the *virtual endomorphism* on $\mathrm{PMod}(S^2, P(f))$. If there is a fixed point τ of σ_f, and if $w := \omega(\tau)$, $m := Y(w) = X(w)$, then $\phi_f = X_* \circ Y_*^{-1}$ is the induced map on fundamental groups based at these points. The domain of ϕ_f is the subgroup $Y_*(\pi_1(\mathcal{W}), w)$.

Nearly Euclidean Thurston Maps. The family of *nearly Euclidean Thurston* (NET) maps, introduced in Cannon et al. (2012), provides an extremely rich family of simple examples of Thurston maps for which explicit algorithmic computations are possible. By definition, a Thurston map f is NET if (1) each critical point has local degree 2, and (2) $\#P(f) = 4$. So, both $z \mapsto z^2 + c_R$ and $z \mapsto z^2 + i$ are NET maps. A NET map is Euclidean if and only if $P(f) \cap C(f) = \emptyset$. One thing that makes NET maps so interesting is that each NET map f admits what we call a *NET map presentation*. This means that f is combinatorially equivalent to a map in a very special normal form. See Sect. 3. Conversely, a NET map presentation defines a combinatorial equivalence class of NET maps.

NET Map Presentation for the Rabbit. Figure 2 shows a NET map presentation diagram for the rabbit $f(z) = z^2 + c_R$. With some conventions understood, it is remarkably simple. Here are the details.

Let $\Lambda_2 = \mathbb{Z}^2$, and let $\Lambda_1 < \Lambda_2$ be the lattice generated by $(0, -1)$ and $(2, 1)$. For $i = 1, 2$ let Γ_i be the groups generated by 180° rotations about the elements of Λ_i, and let $S_i^2 = \mathbb{R}^2/\Gamma_i$ be the quotient. A fundamental domain for Γ_1 is shown in Fig. 2. Since $\Gamma_1 < \Gamma_2$ there is an "origami" quotient map $\overline{id} : S_1^2 \to S_2^2$. Let $A = \left[\begin{smallmatrix} 0 & 2 \\ -1 & 1 \end{smallmatrix}\right]$,

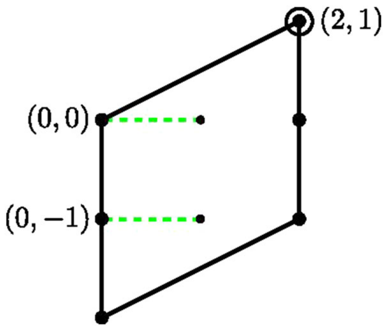


Fig. 2 A NET map presentation diagram for the rabbit

$b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine map $\Phi(x) = Ax + b$. The columns of A are our lattice generators, and b is the circled lattice point in Fig. 2. The map Φ induces an affine homeomorphism $\overline{\Phi} : S^2_2 \to S^2_1$. We set $g = \overline{\Phi} \circ \overline{id}$; it is an affine branched cover of S^2_1 to itself. Finally, we put $f = h \circ g$ where $h : S^2_1 \to S^2_1$ is a point-pushing homeomorphism along the indicated green (dashed horizontal) segments in Fig. 2. Figure 2 completely describes this Thurston map up to combinatorial equivalence.

Computations for NET Maps. If f is a NET map, then $\#P(f) = 4$. This makes things much easier than for general Thurston maps. After some natural identifications, we have the following.

(1) The Teichmüller space Teich$(S^2, P(f))$ is the upper half-plane $\mathbb{H} \subset \mathbb{C}$.
(2) The pure and ordinary mapping class groups PMod$(S^2, P(f))$ and Mod$(S^2, P(f))$ are the congruence subgroup $\mathrm{P}\Gamma(2)$ and $\mathrm{PSL}(2, \mathbb{Z}) \ltimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ respectively.
(3) The domain of the correspondence $\mathcal{W}$ is a classical modular curve (see Sect. 4).
(4) The map $Y : \mathcal{W} \to \text{Moduli}(S^2, P(f))$ extends to a Belyi map $\overline{Y} : \overline{\mathcal{W}} \to \mathbb{P}^1$.
(5) The homotopy classes of curves in $S^2 - P(f)$ are classified by their slopes, that is, elements of $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\pm 1/0 = \infty\}$; with conventional identifications, each slope p/q corresponds to the ideal boundary point $-q/p \in \partial\mathbb{H}$.
(6) By taking preimages of curves, we obtain a slope function $\mu_f : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}} \cup \{\odot\}$ where $\odot$ denotes the union of inessential and peripheral homotopy classes; this encodes the Weil–Petersson boundary values of σ_f, shown to exist in general by Selinger (2012). More precisely, if $\frac{p}{q} \in \overline{\mathbb{Q}}$ and $\sigma_f(-\frac{q}{p}) \in \overline{\mathbb{Q}}$, then $\mu_f(\frac{p}{q}) = -\sigma_f(-\frac{q}{p})^{-1}$. If $\sigma_f(-\frac{q}{p}) \notin \overline{\mathbb{Q}}$, then $\mu_f(\frac{p}{q}) = \odot$.
(7) Varying the choice of translation term b does not affect the fundamental invariants, such as σ_f above, or the ones given below in Theorem 3 (so long as such choices result in maps with four postcritical points, which is almost always the case). Thus *virtual NET map presentation diagrams*, in which the translation term is omitted, suffice to compute such invariants.

Since NET maps are very close to affine maps, it turns out that explicit computations of what happens to slopes under pullback are possible.

Theorem 3 (Cannon et al. 2012, Theorems 4.1, 5.1, 5.3) *Given a NET map presentation for a Thurston map f and the slope p/q of a curve γ, there is an algorithm which computes*

(1) $c_f(p/q) =$ *the number of essential and nonperipheral preimages* $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_c \subset f^{-1}(\gamma)$,
(2) $d_f(p/q) =$ *the common degree by which these preimages map onto γ, and*
(3) $\mu_f(p/q) =$ *the slope of the common homotopy class of the preimages $\tilde{\gamma}_i$.*

The behavior of the slope function μ_f is rather intricate. For the rabbit, Fig. 3 is a plot of the values $\mu_f(\frac{p}{q})$ with $|p| \leq 50$ and $0 \leq q \leq 50$. In fact, the closure of this graph is all of $\mathbb{R}^2$.

The *multiplier* of p/q under f is $\delta_f(p/q) = c_f(p/q)/d_f(p/q)$. When $\#P(f) = 4$, W. Thurston's characterization theorem reduces to the following. A Thurston map f with hyperbolic orbifold is obstructed if and only if there exists a slope p/q for which

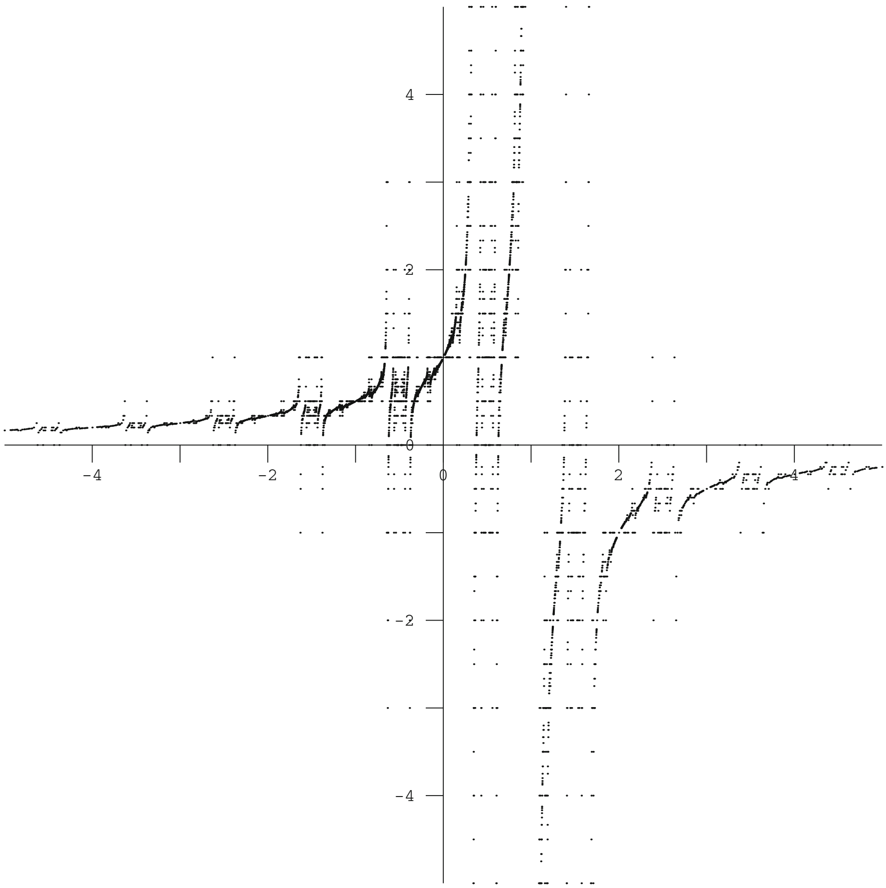


Fig. 3 A portion of the graph of the rabbit's slope function

$\mu_f(p/q) = p/q$ and $\delta_f(p/q) \geq 1$. It turns out that knowledge of data points of the form $(p/q, p'/q', \delta_f(p/q))$, where $p'/q' = \mu_f(p/q)$, restricts the possible slopes of such obstructions:

Theorem 4 (Half-Space Theorem (Cannon et al. 2012, Theorem 6.7)) *Suppose $p'/q' = \mu_f(p/q) \neq p/q$ or $\odot$. There is an algorithm that takes as input the triple $(p/q, p'/q', \delta_f(p/q))$ and computes as output an* excluded open interval *$J = J(p/q, p'/q', \delta_f(p/q)) \subset \overline{\mathbb{Q}}$ containing $-q/p$ such that no point of J is the negative reciprocal of the slope of an obstruction.*

The intervals in Theorem 4 are obtained from half-spaces in $\mathbb{H}$. The boundary of a half-space in $\mathbb{H}$ has a finite part, consisting of points in $\mathbb{H}$, and an infinite part, consisting of points in $\partial\mathbb{H}$. The intervals in Theorem 4 are the infinite boundary points of half-spaces in $\mathbb{H}$ minus endpoints. Figure 4 shows the deployment of some of the half-spaces (in grey, with black boundaries, $|p| \leq 25$, $|q| \leq 25$) for these excluded intervals in the case of the presentation of the rabbit in Fig. 2.

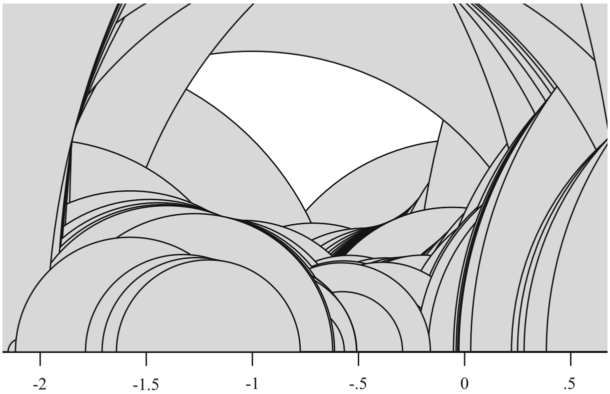


Fig. 4 Half-spaces for the rabbit

It may happen that for some finite set $\{p_1/q_1, \ldots, p_m/q_m\}$ the associated excluded intervals cover all of $\overline{\mathbb{Q}}$. This implies that there are no obstructions and therefore, by W. Thurston's characterization theorem, f is equivalent to a rational map. As indicated by Fig. 4, finitely many excluded intervals cover $\overline{\mathbb{Q}}$ for the rabbit. In fact, careful inspection shows that three half-spaces suffice. An extension of this theorem to the case when $\mu_f(p/q) = p/q$ or $\odot$ is described in Sect. 8. See the discussion of Sect. 8 in the introduction below.

Parry with assistance from Floyd has written and continues to improve a computer program `NETmap` which computes information like the above for a given NET map. Figures 3 and 4 are part of this program's output for the rabbit with the presentation in Fig. 2. That it can do what it does illustrates the tractability of NET maps. Executable files, documentation and more can be found at the NET maps (2016) website.

Summary. Here is a summary of this article.
Findings (Sect. 2). We briefly report on the phenomena observed among the NET maps we have investigated.

NET map presentations (Sect. 3). This section explains presentations of general NET maps of the type given above for the rabbit.

Hurwitz classes (Sect. 4). We first briefly recall some terminology and facts related to Hurwitz classes. We present invariants of Hurwitz classes of NET maps, in particular, a complete set of invariants for impure Hurwitz classes of NET maps. For NET maps, an impure Hurwitz class consists either entirely of Euclidean maps, or of non-Euclidean maps.

Parry has written a computer program which enumerates these impure Hurwitz class invariants and outputs one representative virtual NET map presentation for every impure Hurwitz class of NET maps. For the definition of NET map presentation, see Sect. 3. It organizes these virtual NET map presentations by elementary divisors. (See the discussion of Hurwitz invariants in Sect. 4 for the definition of their elementary divisors.) The NET map web site (2016) contains a catalog of these representative NET map presentations through degree 30. It also contains `NETmap`'s output for every such example. We use the notation mnHClass$k$ to denote the kth virtual NET map presentation with elementary divisors m and n in this catalog.

We prove the following theorem.

Theorem 5 *Suppose f is a non-Euclidean NET map and $\mathcal{H}$ its impure Hurwitz class. There is an algorithm which computes the image of δ_f. This image $\delta(\mathcal{H})$ depends only on $\mathcal{H}$ and not on the choice of representative f. Furthermore*:

(1) $\delta(\mathcal{H}) = \{0\} \iff \sigma_f$ *is constant*;
(2) $\delta(\mathcal{H}) \subset [0, 1) \iff \mathcal{H}$ *is completely unobstructed*;
(3) $\delta(\mathcal{H}) \ni 1 \iff \mathcal{H}$ *contains infinitely many distinct combinatorial classes.*

There are analogous statements for general Thurston maps and pure Hurwitz classes.

We discuss instances of statement 1 in Finding 7 of Sect. 2. We discuss instances of statement 2 in Findings 9 and 10. We discuss instances of when $\mathcal{H}$ contains only finitely many distinct combinatorial classes in Finding 4.

In Sect. 4, we also relate the correspondence $\mathcal{W}$ to classical modular curves.

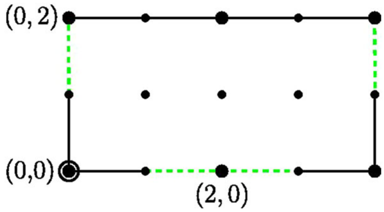
Invariants of degree 2 NET maps (Sect. 5). We discuss invariants of degree 2 NET maps. The complete classification for quadratics has recently been completed by Kelsey and Lodge (2017). In Floyd et al. (2017b), a classification of dynamical portraits for NET maps is given; these classify the corresponding pure Hurwitz classes in degrees 2 and 3.

A conformal description of σ_f for a degree 2 example (Sect. 6). This section demonstrates the tractability of NET maps. It illustrates how numerous invariants of NET maps can be computed by doing so for a specific example. We show for this example that the pullback map is the analytic continuation, via repeated reflection, of a conformal map between ideal hyperbolic polygons.

For this, recall that the slope function μ_f encodes the boundary values of σ_f, and that lifting under f determines a virtual endomorphism ϕ_f : $\mathrm{PMod}(S^2, P(f)) \dashrightarrow \mathrm{PMod}(S^2, P(f))$ with domain G_f. By enlarging G_f to include reflections and so extending ϕ_f, and noting that reflections must lift to reflections, we can sometimes obtain detailed information about both ϕ_f and σ_f. In this way exact calculations of certain values of σ_f are sometimes possible. Along similar lines, a perhaps remarkable feature is that if $\mu_f(p/q) = \odot$, in some circumstances, exploiting the structure of functional equations involving reflections yields exact calculations of the limiting behavior of $\sigma_f(\tau)$ as $\tau \to -q/p$ from within a fundamental domain of G_f. This analysis is done for a particular example in Sect. 6. It is done for the rabbit near the end of Sect. 7.

More generally, the pullback map of every quadratic NET map is the analytic continuation, via repeated reflection, of a conformal map between ideal hyperbolic polygons. This is also surely true of all pullback maps of cubic NET maps. In all of these cases `NETmap` reports that the subgroup of liftables in the extended modular group (which allows reversal of orientation) acts on $\mathbb{H}$ as a reflection group. However, there are NET maps with degree 4 (41HClass3) for which the extended modular group liftables do not act on $\mathbb{H}$ as a reflection group. This seems to be the predominant behavior in higher degrees. In such cases we do not understand the behavior of σ_f as well.

Fig. 5 A presentation diagram for a NET map whose rationality cannot be detected by the half-space theorem but whose rationality is detected by the extended half-space theorem



Dynamics on curves in degree 2 (Sect. 7). Relying heavily on the results of Sect. 6, we investigate the dynamics on the set of homotopy classes of curves under iterated pullback of quadratic NET maps with one critical postcritical point. For a map $\mu : X \to X$ from a set X to itself, we say a subset A of X is a *finite global attractor* if A consists of finitely many cycles into which each element $x \in X$ eventually iterates. We show that for maps in this class that are rational, there is a finite global attractor containing at most four slopes, while for obstructed maps, there may be either (a) a finite global attractor; (b) an infinite set of fixed slopes with no wandering slopes; or (c) a finite set of fixed slopes coexisting with wandering slopes.

We remark that using techniques from self-similar groups, Kelsey and Lodge (2017) have accomplished this for all quadratic rational maps f with $\#P(f) = 4$ and hyperbolic orbifold.

The extended half-space theorem (Sect. 8). The half-space theorem, Theorem 4, applies to all extended rational numbers r which are mapped to different extended rational numbers by the Thurston pullback map σ_f of a NET map f. The half-space theorem provides an explicit interval about r, called an excluded interval, which contains no negative reciprocals of slopes of obstructions for f. If finitely many such excluded intervals cover a cofinite subset of $\partial\mathbb{H}$, then we have only finitely many remaining slopes to check to determine whether f is combinatorially equivalent to a rational map.

Computations using `NETmap` suggest that there exist many NET maps for which every finite union of excluded intervals omits an interval of real numbers. Under suitable hypotheses, this can be proved rigorously. For example, consider the NET map f with the presentation diagram in Fig. 5. It is rational since the algorithms show $\delta_f(p/q) \in [0, 1)$ for all p, q. However, one can prove that every excluded interval arising from the half-space theorem, Theorem 4, is bounded. This implies that every finite union of such intervals fails to cover all of the boundary.

Saenz Maldonado (2012) establishes rationality of his main example by finding infinitely many excluded intervals. This provides motivation to extend the half-space theorem to extended rational numbers r such that either $\sigma_f(r) = r$ or $\sigma_f(r) \in \mathbb{H}$. This is what the extended half-space theorem does. It does not actually provide any new excluded intervals. Instead it provides a way to construct an explicit union of infinitely many excluded intervals such that this union is a deleted neighborhood of a given extended rational number r such that either $\sigma_f(r) \in \mathbb{H}$ or $\sigma_f(r) = r$ and $-1/r$ is not an obstruction with multiplier 1.

One feature of the computer program `NETmap` is to implement a straightforward algorithm based on the extended half-space theorem. In practice, it almost always determines whether or not a NET map is, or is not, equivalent to a rational map. This leads us to

Conjecture 1 *Suppose f is a NET map. Then the extended half-space algorithm decides, in finite time, whether or not f is equivalent to a rational function.*

For general Thurston maps, that such an algorithm exists in theory is announced in Bonnot et al. (2012). That such an algorithm exists in practice is announced in Bartholdi and Dudko (2017, Algorithm V.8) and indeed this is what Bartholdi's program (Bartholdi 2014) attempts to do.

We outline a proof of the extended half-space theorem in Sect. 8.

2 Findings

We report here on many findings of interest for NET maps.

(1) It is conjectured (see e.g. Lodge 2015, §9) that for non-Lattès rational maps, the pullback relation on curves has a finite global attractor. For NET maps, our evidence suggests that, more generally, this holds if there do not exist obstructions with multiplier equal to 1. The converse, however, is false. More precisely, in Sect. 7 we prove Theorem 8, which shows that a NET map f_0 introduced in Sect. 6 with virtual presentation 21HClass1 is obstructed and its pullback map on curves has a finite global attractor which consists of just the obstruction. Theorem 8 also shows that there exist many obstructed maps without finite global attractors.

(2) By perturbing flexible Lattès maps slightly within the family of NET maps, we can build examples of NET maps whose slope functions have many fixed points. By perturbing other Lattès maps, we can build examples of NET maps whose slope functions have cycles of lengths 2, 3, 4 or 6; other examples yield 5–11 and 13–15, inclusive. We do not know whether all cycle lengths occur.

(3) There are many examples of NET maps with hyperbolic orbifolds for which no curve has all of its preimages trivial: Example 3.1 of Cannon et al. (2012); the main example of Lodge (2015); all NET maps in impure Hurwitz classes represented by the following virtual presentations: 22HClass6; 31HClass 5, 6, 9; 51HClass 14–16, 23, 25. This property is equivalent to surjectivity of X (Koch et al. 2016, Theorem 4.1). Indeed, among such examples there occur those whose pure (even impure) Hurwitz class is completely unobstructed (22HClass6), completely obstructed (31HClass9 with translation term λ_1), and mixed-case obstructed (Example 3.1 of Cannon et al. 2012 and the main example of Lodge 2015).

(4) There exist NET maps f whose impure Hurwitz class $\mathcal{H}$ contains only finitely many Thurston equivalence classes, some of which are obstructed and some of which are not. Statement 3 of Theorem 5 shows that this is equivalent to the existence in $\delta(\mathcal{H})$ of some multipliers which are less than 1, some multipliers which are greater than 1 but none equal to 1. This occurs for 41HClass6, 8, 11, 19, 24.

(5) The operation of *formal mating* takes the dynamics of two polynomials and glues them together to form a Thurston map; see e.g. Milnor (2004). Given a Thurston

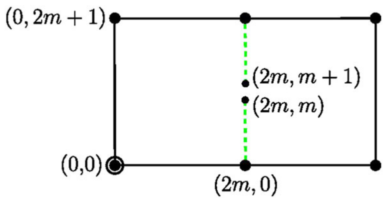


Fig. 6 A presentation diagram for an NET map that arises as a mating in at least $\phi(2m+1)$ ways, where ϕ is Euler's totient function

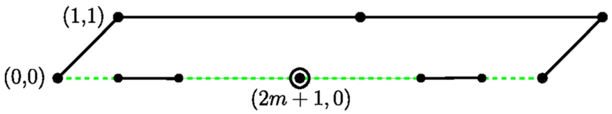


Fig. 7 A presentation diagram for a NET map with degree $2m + 1$ and dynamic portrait 29 which arises as a mating in at least $m + 1$ ways

map, it might be expressible as a mating in multiple ways. The NET map of Fig. 6 arises as a mating in at least $\phi(2m + 1)$ ways, where ϕ is Euler's totient function; that of Fig. 7 in at least $m + 1$ ways. This is established by showing that μ_f has at least the corresponding number of *equators*—fixed-points of maximal multiplier, with the additional condition of preserving orientation—and appealing to Meyer (2014, Theorem 4.2).

(6) There are at least two simple ways to create pairs of combinatorially inequivalent Thurston maps f, g for which $\sigma_f = \sigma_g$:
   (a) Let g be any NET map, let h be a flexible Lattès map with $P(h) = P(g)$ and let $f = g \circ h$. Since σ_h is the identity map, $\sigma_f = \sigma_g$.
   (b) The translation term in the affine map Φ in the definition of NET map presentation does not affect σ_f. For example, it turns out that changing the translation term in a NET map presentation for $f(z) = z^2 + i$ obtains a NET map g whose dynamic portrait is different from that of f, but $\sigma_f = \sigma_g$. Thus typically, a given NET map f has three cousins sharing the same induced map σ_f.

   However, there exist other examples: the NET maps with virtual presentation 41HClass19 have the same pullback maps as those with virtual presentation 41HClass24. These phenomena suggest that non-dynamical, Hurwitz-type invariants might be viewed as more fundamental than dynamic portraits.

(7) It is possible for a NET map f to have a constant pullback map σ_f. Examples are given in Cannon et al. (2012, §10) and Saenz Maldonado (2012, Chap. 5). The property that σ_f is constant depends only on the impure Hurwitz class of f and hence only on its Hurwitz structure set (Sect. 4, Hurwitz invariants).

Proposition 5.1 of Buff et al. (2009) provides a way to construct Thurston maps f with constant pullback maps. Very briefly, the idea here is that if $f = g \circ s$ and the pullback map of s maps to a trivial Teichmüller space, then the pullback map of f is constant. We refer to the hypotheses of this Proposition 5.1 as McMullen's condition.

To describe the NET maps which satisfy McMullen's condition, we define two types of Hurwitz structure sets. Let G be a finite Abelian group generated by two elements such that $G/2G \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. Let $\mathcal{HS}$ be a Hurwitz structure set in G. We say that $\mathcal{HS}$ is an MC2 Hurwitz structure set if $\mathcal{HS} = \{\pm a\} \amalg \{\pm b\} \amalg \{\pm c\} \amalg \{\pm d\}$, where both a and b have order 4 and $2a = 2b = c - d$. We say that $\mathcal{HS}$ is an MC4 Hurwitz structure set if $\mathcal{HS} = \{\pm a\} \amalg \{\pm b\} \amalg \{\pm c\} \amalg \{\pm d\}$, where both a and b have order 4, $2a \neq 2b$, $c = a + b$ and $d = a - b$.

The following theorem essentially answers the question of what NET maps satisfy McMullen's condition. Its proof will appear elsewhere.

Theorem 6 *A NET map is impurely Hurwitz equivalent to a NET map which satisfies McMullen's condition if and only if its Hurwitz structure set is either an MC2 or MC4 Hurwitz structure set.*

Now that we essentially know what NET maps satisfy McMullen's conditions, what NET maps have constant pullback maps? We do not know the answer, but we have the following. We say that a NET map f is imprimitive if there exist NET maps f_1 and f_2 such that f_1 is Euclidean, $f = f_1 \circ f_2$ and the postcritical sets of f, f_1 and f_2 are equal. In this case the pullback map of f is constant if and only if the pullback map of f_2 is constant. We say that f is primitive if it is not imprimitive. These notions extend to Hurwitz structure sets. We have found five equivalence classes of primitive Hurwitz structure sets whose NET maps have constant pullback maps but which do not satisfy McMullen's condition. Here are representatives for them.

$$\{\pm(1,0), \pm(1,1), \pm(7,1), \pm(3,2)\} \subseteq (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}) \quad \deg(f) = 8$$

$$\{\pm(2,0), \pm(0,2), \pm(2,2), \pm(4,2)\} \subseteq (\mathbb{Z}/6\mathbb{Z}) \oplus (\mathbb{Z}/6\mathbb{Z}) \quad \deg(f) = 9$$

$$\{\pm(1,0), \pm(0,1), \pm(5,1), \pm(2,2)\} \subseteq (\mathbb{Z}/6\mathbb{Z}) \oplus (\mathbb{Z}/6\mathbb{Z}) \quad \deg(f) = 9$$

$$\{\pm(1,0), \pm(0,1), \pm(1,2), \pm(4,1)\} \subseteq (\mathbb{Z}/6\mathbb{Z}) \oplus (\mathbb{Z}/6\mathbb{Z}) \quad \deg(f) = 9$$

$$\{\pm(1,0), \pm(1,2), \pm(11,2), \pm(3,3)\} \subseteq (\mathbb{Z}/12\mathbb{Z}) \oplus (\mathbb{Z}/6\mathbb{Z}) \quad \deg(f) = 18$$

The second of these is the degree 9 example in Cannon et al. (2012, §10) and Saenz Maldonado (2012, Chap. 5). This leads us to make the following conjecture.

Conjecture 2 *The Hurwitz structure set of every primitive NET map with constant pullback map is either an MC2 or MC4 Hurwitz structure set or it is equivalent to one of the above five exceptional Hurwitz structure sets.*

This conjecture has been verified by computer for all NET maps with first elementary divisor $m \leq 300$. In particular, it has been verifed for all NET maps with degree at most 300.

(8) Table 1 gives the possibilities for the genus and number of cusps of $\mathcal{W}$ for all NET maps with degree at most 8. See Sect. 4 for a discussion of the relationship between $\mathcal{W}$, classical modular curves, and Teichmüller curves. Using the Riemann–Hurwitz formula, one can show that

$$\deg(Y) = 2(g-1) + n.$$

Note the entries (0, 3) for which $Y : \mathcal{W} \to \mathrm{Moduli}(S^2, P(f))$ is an isomorphism. The one in degree 4 arises from flexible Lattès maps. The one in degree 8 arises

Table 1 A table of all possible ordered pairs (g, n), where g is the genus and n is the number of cusps of $\mathcal{W}$ with degree $d \leq 8$

d	2	3	4	5	6	7	8
(g, n)	(0,4)	(0,6)	(0,3)	(1,6)	(0,6)	(2,6)	(0,3)
			(0,4)	(1,12)	(0,10)	(4,18)	(0,4)
			(0,6)		(1,8)		(0,6)
			(0,10)		(1,16)		(0,10)
							(1,8)
							(1,16)
							(2,14)
							(5,24)

from compositions $g_2 \circ g_1$, where g_1 is a quadratic Thurston map with three postcritical points and g_2 is a flexible Lattès map; note that every mapping class element lifts under g_2.

(9) There are many examples of NET maps f for which σ_f is nonconstant and for which the impure Hurwitz class is completely unobstructed: 22HClass1, 4–6; 31HClass7. Indeed, the impure Hurwitz class of almost every NET map which is a push of a flexible Lattès map is completely unobstructed, and the associated pullback maps are nonconstant. The NET map with presentation diagram in Fig. 5 is an example of this. However, all NET maps whose pullback maps are nonconstant and for which the impure Hurwitz class is completely unobstructed seem to have degrees of the form n^2, $2n^2$, $3n^2$ or $6n^2$. We have verified this by computer through degree 100.

(10) Among quadratic pure Hurwitz classes, we observe that being completely unobstructed is equivalent to the condition that the inverse $Y \circ X^{-1}$ of the correspondence extends to a postcritically finite hyperbolic rational map $g_f : \mathbb{P}^1 \to \mathbb{P}^1$ whose postcritical set consists of the three points at infinity in Moduli$(S^2, P(f)$.

3 NET Map Presentations

We next describe NET map presentations. This section expands on the discussion of a NET map presentation for the rabbit in the introduction. Details can be found in Floyd et al. (2017a). We begin with the lattice $\Lambda_2 = \mathbb{Z}^2$, a proper sublattice Λ_1 and an orientation-preserving affine isomorphism $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi(\Lambda_2) = \Lambda_1$. Let Γ_1 be the group of isometries of $\mathbb{R}^2$ of the form $x \mapsto 2\lambda \pm x$ for some $\lambda \in \Lambda_1$. This information determines a Euclidean Thurston map $g\colon \mathbb{R}^2/\Gamma_1 \to \mathbb{R}^2/\Gamma_1$ as in the introduction's discussion of a NET map presentation for the rabbit. The postcritical set P_1 of g is the image of Λ_1 in $\mathbb{R}^2/\Gamma_1$. The image of $\Lambda_2 - \Lambda_1$ in $\mathbb{R}^2/\Gamma_1$ is the set of critical points of g. To describe g, all we need is to express Φ as $\Phi(x) = Ax + b$, where A is a 2×2 matrix of integers and b is an integral linear combination of the columns, λ_1 and λ_2, of A. We may even assume that b is either 0, λ_1, λ_2 or $\lambda_1 + \lambda_2$. Then λ_1, λ_2 and b determine g up to Thurston equivalence.

The parallelogram F_1 with corners $0, 2\lambda_1, \lambda_2$ and $2\lambda_1+\lambda_2$ is a fundamental domain for the action of Γ_1 on $\mathbb{R}^2$. The points of Λ_1 in F_1 are 0, λ_1, $2\lambda_1$, λ_2, $\lambda_1+\lambda_2$ and $2\lambda_1+\lambda_2$. These six points map onto P_1. We choose six line segments (possibly trivial, just a point) whose union is the full inverse image in F_1 of four disjoint arcs β_1, β_2, β_3, β_4 in $\mathbb{R}^2/\Gamma_1$. Each of the six line segments joins one of 0, λ_1, $2\lambda_1$, λ_2, $\lambda_1+\lambda_2$, $2\lambda_1+\lambda_2$ and an element of $\mathbb{Z}^2$. We call them the *point-push line segments*, and we call β_1, β_2, β_3, β_4 the *point-push arcs*; we color them green (dashed) whenever possible.

Recall that if β is an oriented arc in a surface, then a *point-pushing homeomorphism along* β is a homeomorphism which is the terminal homeomorphism of an isotopy of the surface supported in a regular neighborhood of β that pushes the starting point of β to its ending point along β. We have four point-push arcs β_1, β_2, β_3, β_4. Pushing along each β_i determines, up to homotopy rel P_1, a "push map" homeomorphism $h\colon \mathbb{R}^2/\Gamma_1 \to \mathbb{R}^2/\Gamma_1$ which pushes along β_1, β_2, β_3, β_4 in $\mathbb{R}^2/\Gamma_1$ from P_1 to a set P_2 of four points in the image of Λ_2.

Now that we have g and h, we set $f = h \circ g$. This is a Thurston map. It is a NET map if it has four postcritical points, in which case its postcritical set is P_2. This fails only in special situations when the degree of f is either 2 or 4. (See the second paragraph of Section 2 of Cannon et al. 2012 for more on this.) Every NET map can be expressed as a composition of a Euclidean map and a push map in this way. We call this a NET map presentation of f. The result of omitting the translation term b from a NET map presentation is by definition a *virtual NET map presentation*. The program `NETmap` takes as input a virtual NET map presentation.

So every NET map can be described up to Thurston equivalence by a simple diagram. This diagram consists of first the parallelogram F_1. This determines λ_1 and λ_2 and therefore the matrix A. Second, one of the elements 0, λ_1, λ_2, $\lambda_1+\lambda_2$ in F_1 is circled to indicate the translation term b. Third, the (nontrivial) point-push line segments are drawn in F_1. We call this a NET map presentation diagram. Figure 2 is such a diagram for the rabbit.

Note that the group $\mathrm{SL}(2,\mathbb{Z})$ acts naturally on NET map presentation diagrams: given $P \in \mathrm{SL}(2,\mathbb{Z})$, transform the entire diagram by application of P. In Floyd et al. (2017a) it is shown that this corresponds to postcomposition by the element of the modular group determined by P.

4 Hurwitz Classes

Hurwitz Equivalence. Let $f, f'\colon S^2 \to S^2$ be Thurston maps with postcritical sets $P = P(f)$ and $P' = P(f')$. We say that f and f' belong to the same modular group Hurwitz class if there exist orientation-preserving homeomorphisms $h_0, h_1\colon (S^2, P) \to (S^2, P')$ such that $h_0 \circ f = f' \circ h_1$. If in addition h_0 and h_1 agree on P, then we say that f and f' belong to the same pure modular group Hurwitz class. For brevity, we usually speak of pure and impure Hurwitz classes.

Proof of Theorem 5. Statements 1 and 2 of Theorem 3 imply that there is an algorithm which computes $c_f(p/q)$ and $d_f(p/q)$ for every slope p/q. Theorem 4.1 of Cannon et al. (2012) implies that these values depend only on the image of the ordered pair (q, p) in $\mathbb{Z}_{2m} \oplus \mathbb{Z}_{2n}$ once $\Lambda_2/2\Lambda_1$ is appropriately identified with $\mathbb{Z}_{2m} \oplus \mathbb{Z}_{2n}$. This proves that there is an algorithm which computes the image of δ_f.

Let $h\colon (S^2, P(f)) \to (S^2, P(f))$ be an orientation-preserving homeomorphism. Then h induces by pullback a bijection μ_h on slopes. Let s be a slope. Then $c_{h\circ f}(s) = c_f(\mu_h(s))$ and $d_{h\circ f}(s) = d_f(\mu_h(s))$. Also, $c_{f\circ h}(s) = c_f(s)$ and $d_{f\circ h}(s) = d_f(s)$. This proves the second assertion of Theorem 5.

We now establish the three final assertions. Statement 1 follows from Koch et al. (2016, Theorem 5.1). Statement 2 follows from W. Thurston's characterization theorem and the observation that the impure modular group acts transitively on slopes.

We now turn to the necessity in statement 3. Let $\simeq$ denote the equivalence relation on $\mathcal{H}$ determined by isotopy rel $P = P(f)$. So $\mathcal{H}/\simeq$ is the set of isotopy classes of maps in the impure Hurwitz class of f; in what follows, we write equality for equality in this set.

The full modular group $\mathrm{Mod}(S^2, P)$ acts on $\mathcal{H}/\simeq$ both by pre-composition and post-composition. The set of combinatorial classes in $\mathcal{H}$ is in bijective correspondence with the orbits of the induced conjugation action of $\mathrm{Mod}(S^2, P)$ on $\mathcal{H}/\simeq$. Since the pure mapping class group $\mathrm{PMod}(S^2, P)$ has finite index in $\mathrm{Mod}(S^2, P)$, it suffices to show that there are infinitely many orbits under the conjugation action of $\mathrm{PMod}(S^2, P)$.

The assumption $1 \in \delta_f(\mathcal{H})$ implies that there exist h_0, h_1 representing elements of $\mathrm{Mod}(S^2, P)$ for which $f_* := h_0 f h_1$ has an obstruction given by a curve γ with multiplier equal to 1. Let T be a (full, not half) Dehn twist about γ. By Koch et al. (2016, Theorem 9.1) there is a smallest positive integer k such that T^k commutes with f_* up to isotopy relative to $T^k f_* = f_* T^k$ in $\mathcal{H}/\simeq$. Let k' be the smallest positive integer for which $T^{k'}$ lifts under f_* to an element of $\mathrm{Mod}(S^2, P)$. Since Dehn twists must lift to Dehn twists, and f_* leaves γ invariant, we have $T^{k'} f_* = f_* T^{k''}$ for some $k'' \in \mathbb{Z}$. Since k' is minimal, $k = nk'$ for some positive integer n. Thus $f_* T^{nk'} = f_* T^k = T^k f_* = T^{nk'} f_* = f_* T^{nk''}$. The right action of $\mathrm{PMod}(S^2, P)$ on $\mathcal{H}/\simeq$ is free (Pilgrim 2012, §3 or Kameyama 2001, Prop. 4.1) and so $k = k'$.

For $n \in \mathbb{Z}$ let $f_n := f_* T^n$. We claim that for $n \neq m \in \mathbb{Z}$, the maps f_n and f_m are not conjugate via an element of $\mathrm{PMod}(S^2, P)$. We argue somewhat similarly as in Koch et al. (2016, §9). Suppose as elements of $\mathcal{H}/\simeq$ we have $hf_n = f_m h$ for some $h \in \mathrm{PMod}(S^2, P)$. The (class of) curve γ is the unique obstruction for both f_n and f_m, so h must fix the class of γ. Since $h \in \mathrm{PMod}(S^2, P)$, h is a power of T, say T^l. Then $hf_n = f_m h \implies T^l f_* T^n = f_* T^m T^l$. This equation implies that T^l lifts under f_* to a pure mapping class element and so by the previous paragraph $l = qk$ for some q. Continuing, we have $f_* T^m T^l = T^l f_* T^n = T^{qk} f_* T^n = f_* T^n T^{qk} \implies f_* T^{qk} T^n = f_* T^m T^{qk} \implies f_* T^{n+l} = f_* T^{m+l} \implies n = m$, again by freeness of the right action.

To prove sufficiency, suppose $1 \notin \delta(\mathcal{H})$. It suffices to show there are only finitely many combinatorial classes of obstructed maps. We use the combination and decomposition theory developed in Pilgrim (2003) and outline the main ideas.

Suppose $f \in \mathcal{H}$ has an obstruction, γ. Let A_0 be an annulus which is a regular neighborhood of γ and put $\mathcal{A}_0 := \{A_0\}$. By altering f within its homotopy class we may assume that the collection $\mathcal{A}_1$ of preimages of A_0 containing essential nonperipheral preimages of γ is a collection of essential subannuli of A_0 with $\partial\mathcal{A}_1 \supset \partial A_0$; the restriction $f : \mathcal{A}_1 \to \mathcal{A}_0$ is the set of *annulus maps* obtained by decomposing f along $\{\gamma\}$. Similarly, if we set $\mathcal{U}_0$ to be the collection of two components of $S^2 - \mathcal{A}_0$, we obtain a collection of *sphere maps* $f : \mathcal{U}_1 \to \mathcal{U}_0$. Capping the holes of the sphere maps to disks yields a collection of Thurston maps with three or fewer postcritical points, and these are rational, by Thurston's characterization.

As f varies within $\mathcal{H}$, the collection of sphere maps varies over a finite set of combinatorial equivalence classes, since they are rational. The hypothesis that the multiplier of the obstructions are not equal to one implies that the set of combinatorial equivalence classes of annulus maps vary over a finite set as well. The additional set-theoretic gluing data needed to reconstruct f from the sphere and annulus maps also varies over a finite set. By Pilgrim (2003, Theorem 4.5), given two maps f_1, f_2 presented as a gluing of sphere and annulus maps, a combinatorial equivalence between gluing data, a collection of annulus maps, and a collection of sphere maps yields a combinatorial equivalence between f_1 and f_2. We conclude that the set of possible equivalence classes of maps f in $\mathcal{H}$ is finite. □

Modular Groups. From the discussion of the NET map presentation of the rabbit (Sect. 1) and of NET map presentations (Sect. 3), recall the definition of the quotient sphere $S^2_2 := \mathbb{R}^2/\Gamma_2$. Equipped with its Euclidean half-translation structure, it is a "square pillowcase" with four corners given by the images of the lattice $\Lambda_2 = \mathbb{Z}^2$. Let P be this set of corners. In this case the modular group $\mathrm{Mod}(S^2, P)$ and pure modular group $\mathrm{PMod}(S^2, P)$ have the forms

$$\mathrm{Mod}(S^2, P) \cong \mathrm{PSL}(2, \mathbb{Z}) \ltimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$$

and

$$\mathrm{PMod}(S^2, P) \cong P\Gamma(2) = \{A \equiv I \bmod 2\} \subseteq \mathrm{PSL}(2, \mathbb{Z}).$$

The group $\mathrm{Mod}(S^2, P)$ is then the group of orientation-preserving affine diffeomorphisms, and $P\Gamma(2)$ the subgroup fixing the corners pointwise. The group $\mathrm{PSL}(2, \mathbb{Z})$ fixes the corner corresponding to the image of the origin, but permutes the other three corners so as to induce the natural action of the symmetric group S_3.

Hurwitz Invariants. Useful invariants of a NET map are its *elementary divisors*, which we now define. Suppose f is a NET map given by a NET map presentation with affine map $x \mapsto Ax + b$, where A is an integral matrix whose determinant equals the degree of f. There are $P, Q \in \mathrm{SL}(2, \mathbb{Z})$ and positive integers m and n such that $n|m$ and $PAQ = \left[\begin{smallmatrix} m & 0 \\ 0 & n \end{smallmatrix}\right]$. The integers m, n are unique and are the *elementary divisors* of A; the matrix on the right is the *Smith normal form*. They form an impure Hurwitz invariant. In fact, according to Floyd et al. (2017b, Theorem 5.5), NET maps f and g have equal elementary divisors if and only if g is Thurston equivalent to a NET map of the form $\varphi \circ f$ for some homeomorphism $\varphi\colon S^2 \to S^2$. (Note that φ need not stabilize $P(f)$.)

For NET maps, a complete invariant of impure Hurwitz classes can be given in terms of the *Hurwitz structure set*, $\mathcal{HS}$. To define this, we use the usual data and Euclidean groups associated to a NET map f. The torus $\mathbb{R}^2/2\Lambda_1$ is a double cover of the sphere $\mathbb{R}^2/\Gamma_1$. The pullback of $P(f)$ in $\mathbb{R}^2/2\Lambda_1$ is a subset of the finite group $\mathbb{Z}^2/2\Lambda_1$, and it is a disjoint union of the form $\mathcal{HS} = \{\pm h_1\} \amalg \{\pm h_2\} \amalg \{\pm h_3\} \amalg \{\pm h_4\}$. This is a Hurwitz structure set. More generally, let G be a finite Abelian group such that $G/2G \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. A Hurwitz structure set in G is a disjoint union of four sets of the form $\{\pm h\}$, where $h \in G$. Returning to $\mathcal{HS}$, if Λ_1' is a sublattice of $\mathbb{Z}^2$, then we say that $\mathcal{HS}$ is equivalent to a Hurwitz structure set $\mathcal{HS}'$ in $\mathbb{Z}^2/2\Lambda_1'$ if and only if there exists an orientation-preserving affine isomorphism $\Psi\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ such that $\Psi(\Lambda_1) = \Lambda_1'$ and the map which Ψ induces from $\mathbb{Z}^2/2\Lambda_1$ to $\mathbb{Z}^2/2\Lambda_1'$ takes $\mathcal{HS}$ to $\mathcal{HS}'$. Theorem 5.1 of Floyd et al. (2017b) states that the equivalence class of $\mathcal{HS}$ under this equivalence relation is a complete invariant of the impure Hurwitz class of f.

Relating $\mathcal{W}$ to Classical Modular Curves. Let f be a NET map with postcritical set $P(f)$. Recall the correspondence $X, Y : \mathcal{W} \to \mathrm{Moduli}(S^2, P(f))$ from Sect. 1. This correspondence is essentially an impure Hurwitz invariant; see Koch (2013, §2). In this section we explicitly relate the space $\mathcal{W}$ to classical modular curves.

Corollary 5.3 of Floyd et al. (2017b) states that every impure Hurwitz class of NET maps is represented by a NET map whose presentation matrix is diagonal. So to understand $\mathcal{W}$, we may assume that the presentation matrix of f has the form $A = \left[\begin{smallmatrix} m & 0 \\ 0 & n \end{smallmatrix}\right]$, where m and n are positive integers with $n|m$ and $mn = \deg(f)$. By definition, m and n are the elementary divisors of f. So the presentation of f has lattices $\Lambda_2 = \mathbb{Z}^2$ and $\Lambda_1 = \langle (m, 0), (0, n)\rangle$. It also has a Hurwitz structure set $\mathcal{HS} \subseteq \Lambda_2/2\Lambda_1$. The discussion at the end of §2 of Floyd et al. (2017b) shows that the group G_f of pure liftables is isomorphic to the image in $\mathrm{PSL}(2, \mathbb{Z})$ of the group $\widehat{G}_f$ of all elements M in $\mathrm{SL}(2, \mathbb{Z})$ such that $M\Lambda_1 = \Lambda_1$, $M \equiv 1 \bmod 2$ and the automorphism of $\mathbb{Z}^2/2\Lambda_1$ induced by M fixes $\mathcal{HS}$ pointwise up to multiplication by ± 1.

We interrupt this discussion to define some subgroups of $\mathrm{SL}(2, \mathbb{Z})$. Let N be a positive integer. The principle congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$ with level N is

$$\Gamma(N) = \left\{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathbb{Z}) : \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \equiv \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \bmod N\right\}.$$

A congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$ is a subgroup which contains $\Gamma(N)$ for some N. Two such subgroups are

$$\Gamma_0(N) = \left\{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathbb{Z}) : c \equiv 0 \bmod N\right\}$$

and

$$\Gamma^0(N) = \left\{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathbb{Z}) : b \equiv 0 \bmod N\right\}.$$

Two others are

$$\Gamma_1(N) = \left\{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_0(N) : a \equiv d \equiv 1 \bmod N\right\}$$

and

$$\Gamma^1(N) = \left\{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma^0(N) : a \equiv d \equiv 1 \bmod N\right\}.$$

We next relate $\widehat{G}_f$ to these congruence subgroups. The condition that $M \equiv 1 \bmod 2$ simply says that $\widehat{G}_f \subseteq \Gamma(2)$. We next interpret the condition that $M\Lambda_1 = \Lambda_1$. Let $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Since $(0, n) \in \Lambda_1$, we need that $M \cdot (0, 1) \in \Lambda_1$. Equivalently, $(bn, dn) \in \Lambda_1$. This amounts to requiring that $bn \equiv 0 \bmod m$, that is, $b \equiv 0 \bmod \frac{m}{n}$. Since $n|m$, the condition that $M \cdot (m, 0) \in \Lambda_1$ is satisfied by every $M \in \mathrm{SL}(2, \mathbb{Z})$.

Hence the condition that $M\Lambda_1 = \Lambda_1$ is equivalent to the condition that $M \in \Gamma^0(\frac{m}{n})$. Therefore $\widehat{G}_f \subseteq \Gamma^0(\frac{m}{n}) \cap \Gamma(2)$. On the other hand, the group $\Gamma^1(2m) \cap \Gamma_1(2n)$ stabilizes Λ_1 and acts trivially on $\Lambda_2/2\Lambda_1$, and so it fixes $\mathcal{HS}$ pointwise. Thus

$$\Gamma(2m) \subseteq \Gamma^1(2m) \cap \Gamma_1(2n) \subseteq \widehat{G}_f \subseteq \Gamma^0(\tfrac{m}{n}) \cap \Gamma(2).$$

In conclusion, let $\mathbb{H}^*$ be the Weil-Petersson completion of $\mathbb{H}$. Then $\mathcal{W}$ is a modular curve such that $\mathbb{H}^*/(\Gamma^1(2m)\cap\Gamma_1(2n))$ maps onto $\mathcal{W}$ and $\mathcal{W}$ maps onto $\mathbb{H}^*/(\Gamma^0(\frac{m}{n})\cap\Gamma(2))$.

5 Invariants of Degree 2 NET Maps

We begin with a discussion of invariants of general Thurston maps and then specialize to NET maps with degree 2.

The *dynamic portrait* is the directed graph with vertex set $C(f)\cup P(f)$ and weighted edges

$$x \xrightarrow{\deg_x(f)} f(x).$$

The *static portrait* is the bipartite directed graph whose vertex set is the disjoint union of $A := C(f) \cup P(f)$ and $B := P(f)$, directed edges

$$x \xrightarrow{\deg_x(f)} f(x)$$

with $x \in A$, and the elements of A that lie in $P(f)$ are marked so as to distinguish them from those elements of A that do not lie in $P(f)$. The *augmented branch data* records, for each $y \in P(f)$, the partition of $d = \deg(f)$ given by the collection of local degrees $\{\deg_x(f) : f(x) = y\}$. For example, the dynamic portrait of the rabbit is $(a \xrightarrow{2} b \to c \to a, d \xrightarrow{2} d)$, the static portrait is $(p_1 \xrightarrow{2} q_1, p_2 \xrightarrow{2} q_2, p_3 \to q_3, p_4 \to q_4)$, and the branch data is ([2], [2], [1, 1], [1, 1]). The static portrait, and hence branch data, are impure Hurwitz invariants. The dynamic portrait is a pure Hurwitz invariant but not, in general, an impure Hurwitz invariant.

For NET maps, dynamic and static portraits are completely classified Floyd et al. (2017b). Table 2 gives the number of dynamic portraits as a function of the degree.

Degree 2 NET Maps. Recall that a quadratic Thurston map is NET if and only if it has four postcritical points. In degree 2 there are 3 impure Hurwitz classes, completely

Table 2 The number n of dynamic portraits among NET maps of degree d

d	2	3	4	5	6	7	8
n	16	94	272	144	338	152	476
$d \mod 4, d \geq 9$	0	1	2	3			
n	483	153	353	153			

classified by static portrait or, equivalently, by the number of critical points in the postcritical set; see Theorem 7. Here are the three static portraits. We label marked points in the domain by p_i, we label marked points in the codomain by q_i, and we label unmarked critical points in the domain by c_i.

$$p_1 \to q_1,\, p_2 \to q_1,\, p_3 \to q_2,\, p_4 \to q_2,\, c_1 \overset{2}{\to} q_3,\, c_2 \overset{2}{\to} q_4$$
$$p_1 \to q_1,\, p_2 \to q_1,\, p_3 \overset{2}{\to} q_2,\, p_4 \to q_3,\, c_1 \overset{2}{\to} q_4$$
$$p_1 \overset{2}{\to} q_1,\, p_2 \overset{2}{\to} q_2,\, p_3 \to q_3,\, p_4 \to q_4$$

Pure Hurwitz classes are completely classified by the corresponding dynamic portraits. There are 16 of them. All but one is represented by rational functions; the exception is $a \overset{2}{\to} b \to a, c \overset{2}{\to} d \to c$.

Kelsey and Lodge (2017) have completed the classification of quadratic NET combinatorial classes. They generalize the methods of Bartholdi–Nekrashevych on the twisted rabbit problem (Bartholdi and Nekrashevych 2006), analyzing wreath recursions on the pure mapping class group. These wreath recursions are derived from the 16 correspondences on moduli space.

Impure Hurwitz classes in degree 2. In this section we prove Theorem 7. It shows that two NET maps with degree 2 belong to the same impure Hurwitz class if and only if they have the same number of critical postcritical points. These maps all have two critical points and four postcritical points. So there might be either 0, 1 or 2 critical postcritical points. Hence there are three impure Hurwitz classes of NET maps with degree 2. The case in which there are no critical postcritical points is exactly the case of the Euclidean NET maps with degree 2. They form one impure Hurwitz class. Another is represented by $f(z) = z^2 + i$ and the other is represented by the rabbit, corabbit and airplane. Here is the theorem.

Theorem 7 *Two degree 2 NET maps belong to the same impure Hurwitz class if and only if they have the same number of critical postcritical points.*

Proof Let f be a degree 2 NET map. The elementary divisors of f are $m = 2$ and $n = 1$ because their product is $\deg(f) = 2$ and the second divides the first. The group $\Lambda_2/2\Lambda_1$ in the definition of Hurwitz structure set is then isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_2$. The Hurwitz structure set $\mathcal{HS}$ of f can thus be identified with a subset of

$$\mathbb{Z}_4 \oplus \mathbb{Z}_2 = \{(0,0), \pm(1,0), (2,0), (0,1), \pm(1,1), (2,1)\}.$$

Elements of order 1 or 2 in $\mathcal{HS}$ correspond to postcritical points of f which are not critical. The other elements of $\mathcal{HS}$ are paired by multiplication by -1, and these pairs correspond to critical postcritical points. It is shown in Floyd et al. (2017b) that two NET maps with equal Hurwitz structure sets belong to the same impure Hurwitz class. Moreover, transforming $\mathcal{HS}$ either by an automorphism of $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ which lifts to $\mathrm{SL}(2, \mathbb{Z})$ or by a translation by an element of order 2 preserves the impure Hurwitz class of f.

It is easy to see that if two NET maps belong to the same impure Hurwitz class, then they have the same number of critical postcritical points. The converse statement is what must be proved.

Suppose that f is a NET map with degree 2 and no critical postcritical points. Then the first paragraph of this proof shows that there is only one possibility for the Hurwitz structure set of f after identifying it with a subset of $\mathbb{Z}_4 \oplus \mathbb{Z}_2$; it must be $\{(0, 0), (2, 0), (0, 1), (2, 1)\}$. So there is just one impure Hurwitz class of such maps.

Next suppose that f is a NET map with degree 2 and exactly one critical postcritical point. Then the Hurwitz structure set of f contains three of the four elements of order 1 or 2 in $\mathbb{Z}_4 \oplus \mathbb{Z}_2$. Since translation by an element of order 2 preserves the impure Hurwitz class of f, we may assume that $\mathcal{H}$ contains $(0, 0)$, $(2, 0)$, $(0, 1)$ and either $\pm(1, 0)$ or $\pm(1, 1)$. Now we verify that $\left[\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right] \in \mathrm{SL}(2, \mathbb{Z})$ induces an automorphism of $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ which fixes $(0, 0)$, $(2, 0)$, $(0, 1)$ and interchanges $\pm(1, 0)$ and $\pm(1, 1)$. Thus there is only one equivalence class of these Hurwitz structure sets and only one impure Hurwitz class of such maps.

Finally, suppose that f is a NET map with degree 2 and two critical postcritical points. In this case $\mathcal{HS}$ must contain $\pm(1, 0)$ and $\pm(1, 1)$ in addition to two of the four elements of order 1 or 2. Since translating $\mathcal{HS}$ by an element of order 2 preserves the impure Hurwitz class of f, we may assume that $\mathcal{HS}$ contains $\pm(1, 0)$, $\pm(1, 1)$ and $(0, 0)$.

Suppose in addition that $(2, 0) \in \mathcal{HS}$. Then $\mathcal{HS} = \{(0, 0), \pm(1, 0), (2, 0), \pm1, 1)\}$. Example 10.3 of Cannon et al. (2012) shows that $\mathcal{HS}$ is never separating (nonseparating in the language there). Between Lemma 10.1 and Theorem 10.2 of Cannon et al. (2012) it is shown that this implies that the Thurston pullback map of f is constant. But Theorem 10.10 of Cannon et al. (2012) shows that there does not exist a NET map with degree 2 whose Thurston pullback map is constant. (The Thurston maps for this choice of $\mathcal{HS}$ have fewer than four postcritical points.) Thus $(2, 0) \notin \mathcal{HS}$.

So $\mathcal{HS}$ contains either $(0, 1)$ or $(2, 1)$. One verifies that $\left[\begin{smallmatrix}1 & 2\\ 0 & 1\end{smallmatrix}\right] \in \mathrm{SL}(2, \mathbb{Z})$ induces an automorphism on $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ which fixes $(0, 0)$, $\pm(1, 0)$, $\pm(1, 1)$ and interchanges $(0, 1)$ and $(2, 1)$. Hence there is only one equivalence class of these Hurwitz structure sets and only one impure Hurwitz class of such maps.

This proves Theorem 7. □

6 A Conformal Description of σ_f for a Degree 2 Example

In this section we discuss how in many, but not all, cases it is possible in a sense to determine the pullback map $\sigma_f : \mathbb{H} \to \mathbb{H}$ of a NET map f. This description is like that of the classical triangle functions, as discussed in Chapter 1 of Lehner (1964). For the triangle functions, we begin with a conformal equivalence between a hyperbolic triangle and the upper half plane. Here we begin with a conformal equivalence between two hyperbolic triangles or, more generally, two hyperbolic polygons. The map is then extended to the entire hyperbolic plane using the reflection principle.

We will focus on a particular example map f_0. Our discussion involves several features of f_0. For each feature, we provide first a brief general discussion for arbitary NET maps f, and then illustrate it using our example map f_0.

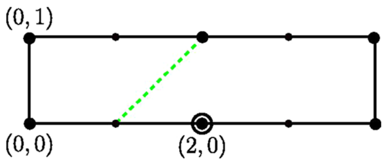


Fig. 8 A presentation diagram for a NET map f_0; a *vertical curve* is an obstruction to rationality

Example **The NET map** f_0. Our example map f_0 is determined up to Thurston equivalence by the presentation diagram in Fig. 8. We have lattices $\Lambda_2 = \mathbb{Z}^2$, $\Lambda_1 = \langle (2,0), (0,1) \rangle$ and Hurwitz structure set

$$\mathcal{HS} = \{(0,0), \pm(1,0), (2,0), (0,1)\} \subseteq \mathbb{Z}_4 \oplus \mathbb{Z}_2 \cong \Lambda_2/2\Lambda_1.$$

We also have a Euclidean NET map g and a push map h such that $f_0 = h \circ g$.

The subgroup G_f of liftables in the extended modular group G. Suppose f now is an arbitrary NET map. We work with the extended modular group $G = \mathrm{EMod}(S^2, P(f))$, which is defined in the same way as the modular group $\mathrm{Mod}(S^2, P(f))$ except that it allows reversal of orientation. §2 of Floyd et al. (2017b) shows that G is isomorphic to the group $\mathrm{Aff}(2, \mathbb{Z})$ of all affine isomorphisms $\Psi \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Psi(\mathbb{Z}^2) = \mathbb{Z}^2$ modulo the subgroup Γ_2 of all maps of the form $x \mapsto 2\lambda \pm x$ for some $\lambda \in \Lambda_2 = \mathbb{Z}^2$.

A map $\varphi \colon (S^2, P(f)) \to (S^2, P(f))$ representing a homotopy class in G is liftable if there exists another such map $\widetilde{\varphi}$ such that $\varphi \circ f$ is homotopic to $f \circ \widetilde{\varphi}$ rel $P(f)$. The subgroup of liftables for f is the subgroup G_f of G represented by all such liftable maps φ. §2 of Floyd et al. (2017b) shows that G_f is isomorphic to the subgroup of G whose elements lift to elements $\Psi \in \mathrm{Aff}(2, \mathbb{Z})$ such that $\Psi(\Lambda_1) = \Lambda_1$ and the map induced by Ψ on $\Lambda_2/2\Lambda_1$ stabilizes $\mathcal{HS}$ setwise. Turning to our example map f_0, we let

$$\Psi_1(x) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] x, \quad \Psi_2(x) = \left[\begin{smallmatrix} -1 & 0 \\ 2 & 1 \end{smallmatrix}\right] x, \quad \Psi_3(x) = \left[\begin{smallmatrix} 1 & 2 \\ 0 & -1 \end{smallmatrix}\right] x.$$

These are elements of $\mathrm{Aff}(2, \mathbb{Z})$ which stabilize Λ_1 and $\mathcal{HS}$. So Ψ_1, Ψ_2 and Ψ_3 determine elements ρ_1, ρ_2 and ρ_3 of G_{f_0}.

In this paragraph we show that ρ_1, ρ_2 and ρ_3 generate G_{f_0}. Let $G_{f_0}^+$ denote the subgroup of orientation-preserving elements of G_{f_0}. The discussion in Sect. 4 which relates $\mathcal{W}$ to classical modular curves shows that if $\Psi(x) = Ax + b \in \mathrm{Aff}(2, \mathbb{Z})$ is the lift of an element of $G_{f_0}^+$, then $A \in \Gamma^0(2)$. Moreover, one easily verifies that there is no such Ψ with $A = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$, an element of $\Gamma^0(2)$. So the set of these matrices arising as lifts of elements of $G_{f_0}^+$ is a proper subgroup of $\Gamma^0(2)$. On the other hand, we will soon see from Fig. 10 that the images of ρ_1, ρ_2 and ρ_3 in $\mathrm{PGL}(2, \mathbb{Z})$ generate a subgroup with index 6. (Its intersection with $\mathrm{PSL}(2, \mathbb{Z})$ equals the image of $\Gamma(2)$.) It easily follows that G_{f_0} is generated by ρ_1, ρ_2, ρ_3 together with the elements of G_{f_0} which lift to translations in $\mathrm{Aff}(2, \mathbb{Z})$. One finally verifies that only the identity element of G_{f_0} lifts to a translation. Therefore ρ_1, ρ_2 and ρ_3 generate G_{f_0}.

Evaluation of μ_f. Let μ_{f_0} denote the usual slope function which f_0 induces on slopes of simple closed curves in $S^2 - P(f_0)$. We want to evaluate μ_{f_0} at -1, $-\frac{1}{2}$, 1 and ∞.

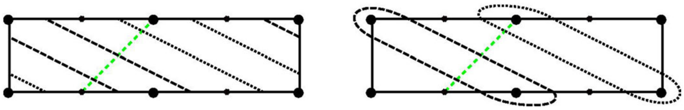


Fig. 9 The pullback of a simple closed curve with slope $-\frac{1}{2}$

This can be done using Theorem 5.1 of Cannon et al. (2012), which provides a method suitable for computer implementation. In the next paragraph we evaluate $\mu_{f_0}(-\frac{1}{2})$ in a more topological way. The other three evaluations can be made similarly.

The left side of Fig. 9 shows the pullback to the diagram in Fig. 8 of a simple closed curve γ in $S^2 - P(f_0)$ with slope $-\frac{1}{2}$; it is straightforward to check this by taking the image of the curves under f_0. The right side of Fig. 9, while actually meaningless, might be helpful. We see that $f_0^{-1}(\gamma)$ has two connected components, one drawn with dots and the other drawn with dashes. The dotted connected component is peripheral. On the other hand, the image in S^2 of the line segment joining (0, 0) and (1, 0) is a core arc for the other connected component. Recall that $f = h \circ g$ where g is Euclidean and h is a push map. The slope of this core arc relative to $P(f_0)$ equals the slope of its image under h^{-1} relative to $P(g)$. This image under h^{-1} is homotopic rel $P(g)$ to the line segment joining (0, 0) and $(2, 1) = 1 \cdot (2, 0) + 1 \cdot (0, 1)$. Hence $\mu_{f_0}(-\frac{1}{2}) = \frac{1}{1} = 1$. In the same way, we find that $\mu_{f_0}(-1) = 0$, $\mu_f(1) = 2$ and $\mu_{f_0}(\infty) = \infty$.

Evaluation of μ_φ. We continue with some generalities on affine maps. For an extended modular group element $\varphi \in G$ let μ_φ denote the induced map on slopes. Suppose that φ lifts to $\Phi \in \mathrm{Aff}(2, \mathbb{Z})$ with linear part given by the matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Then A^{-1} maps the line through (0, 0) and (q, p) to the line through (0, 0) and

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot (q, p) = (-bp + dq, ap - cq).$$

So $\mu_\varphi(s) = \frac{as-c}{-bs+d}$, where $s = \frac{p}{q}$.

Evaluation of σ_φ. Still focusing on generalities, let σ_φ denote the map on $\mathbb{H}$ induced by an extended modular group element $\varphi \in G$. To determine σ_φ, it suffices to determine the action of σ_φ on $\partial\mathbb{H}$. We use the expression for μ_φ in the previous paragraph. Because the slope s corresponds to $x = -\frac{1}{s} \in \partial\mathbb{H}$, the map which φ induces on $\partial\mathbb{H}$ is

$$\sigma_\varphi(x) = -\left(\frac{a(-1/x) - c}{-b(-1/x) + d}\right)^{-1} = \frac{dx + b}{cx + a}.$$

So if $z \in \mathbb{H}$, then

$$\sigma_\varphi(z) = \frac{dz + b}{cz + a} \quad \text{if} \quad \varphi \in G^+ \quad \text{and} \qquad \sigma_\varphi(z) = \frac{d\bar{z} + b}{c\bar{z} + a} \quad \text{if} \quad \varphi \notin G^+.$$

The map f_0 **induces a virtual endomorphism** $\phi_{f_0} : G \dashrightarrow G$. For a general NET map f, given an extended modular group element $g \in G$, a lift of g under f might not be unique. Thus lifting under f maps liftable elements to cosets of $\mathrm{DeckMod}(f)$, the subgroup of G represented by deck transformations of f. Proposition 2.4 of Floyd et al. (2017b) implies that DeckMod(f) is isomorphic to the group of translations in $2\Lambda_2$ which stabilize the Hurwitz structure set $\mathcal{HS}$ modulo the group of translations in $2\Lambda_1$. For our example f_0, this quotient group is trivial; we obtain a well-defined virtual endomorphism $\phi_{f_0} : G \dashrightarrow G$ induced by f_0.

The extended modular group virtual endomorphism maps reflections to reflections. A NET map f preserves orientation. Therefore, if $g \in G$ reverses orientation and $\tilde{g}$ is any lift of g under f, then $\tilde{g}$ must reverse orientation.

For our example map f_0, since ϕ_{f_0} is a homomorphism, if $g \in G$ is a reflection, then g has order two, therefore $\phi_{f_0}(g)$ both has order 2 and reverses orientation, and is therefore again a reflection. Thus in terms of the action of liftable extended mapping class elements G_{f_0} on $\mathbb{H}$, reflections map to reflections under ϕ_{f_0}.

Evaluation of the extended modular group virtual endomorphism. We continue to focus on our example map f_0. If $\varphi \in G_{f_0}$, then we let $\widetilde{\varphi} = \phi_{f_0}(\varphi)$. In this paragraph we evaluate μ_{ρ_i}, σ_{ρ_i}, $\mu_{\widetilde{\rho}_i}$ and $\sigma_{\widetilde{\rho}_i}$ for $i \in \{1, 2, 3\}$. Using the formulas for μ_{ρ_1} and σ_{ρ_1} above, we obtain the leftmost two equations in line 1. We next apply the identity $\mu_f \circ \mu_\varphi = \mu_{\widetilde{\varphi}} \circ \mu_{f_0}$ for $\varphi \in G_{f_0}$. Combining this with μ_{ρ_1} and our values for μ_{f_0}, we obtain the commutative diagram in line 1. Using the bottom map of the commutative diagram and the fact that the extended modular group virtual endomorphism maps reflections to reflections, we easily obtain the rightmost two equations in line 1. We verify the information in lines 2 and 3 similarly.

$$
\begin{array}{c} \mu_{\rho_1}(s) = -s \\ \sigma_{\rho_1}(z) = -z \end{array}
\qquad
\begin{tikzcd}
\infty, -1 \arrow[r, "\mu_{\rho_1}"] \arrow[d, "\mu_{f_0}"'] & \infty, 1 \arrow[d, "\mu_{f_0}"] \\
\infty, 0 \arrow[r, "\mu_{\widetilde{\rho}_1}"] & \infty, 2
\end{tikzcd}
\qquad
\begin{array}{c} \mu_{\widetilde{\rho}_1}(s) = -s + 2 \\ \sigma_{\widetilde{\rho}_1}(z) = \dfrac{-z}{2z+1} \end{array}
\tag{1}
$$

$$
\begin{array}{c} \mu_{\rho_2}(s) = -s - 2 \\ \sigma_{\rho_2}(z) = \dfrac{z}{2z-1} \end{array}
\qquad
\begin{tikzcd}
\infty, -1 \arrow[r, "\mu_{\rho_2}"] \arrow[d, "\mu_{f_0}"'] & \infty, -1 \arrow[d, "\mu_{f_0}"] \\
\infty, 0 \arrow[r, "\mu_{\widetilde{\rho}_2}"] & \infty, 0
\end{tikzcd}
\qquad
\begin{array}{c} \mu_{\widetilde{\rho}_2}(s) = -s \\ \sigma_{\widetilde{\rho}_2}(z) = -z \end{array}
\tag{2}
$$

$$
\begin{array}{c} \mu_{\rho_3}(s) = \dfrac{-s}{2s+1} \\ \sigma_{\rho_3}(z) = -z + 2 \end{array}
\qquad
\begin{tikzcd}
\infty, -1 \arrow[r, "\mu_{\rho_3}"] \arrow[d, "\mu_{f_0}"'] & -\frac{1}{2}, -1 \arrow[d, "\mu_{f_0}"] \\
\infty, 0 \arrow[r, "\mu_{\widetilde{\rho}_3}"] & 1, 0
\end{tikzcd}
\qquad
\begin{array}{c} \mu_{\widetilde{\rho}_3}(s) = \dfrac{s}{s-1} \\ \sigma_{\widetilde{\rho}_3}(z) = -z - 1 \end{array}
\tag{3}
$$

Fundamental domains for G_{f_0}, $G^+_{f_0}$, $\widetilde{G}_{f_0}$ and $\widetilde{G}^+_{f_0}$. Still focusing on our example map f_0, we now have explicit expressions for σ_{ρ_i} and $\sigma_{\widetilde{\rho}_i}$ for $i \in \{1, 2, 3\}$. These maps are all reflections. Let α_i and $\widetilde{\alpha}_i$ denote the reflection axes of σ_{ρ_i} and $\sigma_{\widetilde{\rho}_i}$ (their fixed point sets) for $i \in \{1, 2, 3\}$. One verifies that the unshaded triangle in the left side of Fig. 10 is a fundamental domain for the action of G_{f_0} on $\mathbb{H}$. The two triangles in the left side of Fig. 10 form a fundamental domain for the action of $G^+_{f_0}$ on $\mathbb{H}$. This

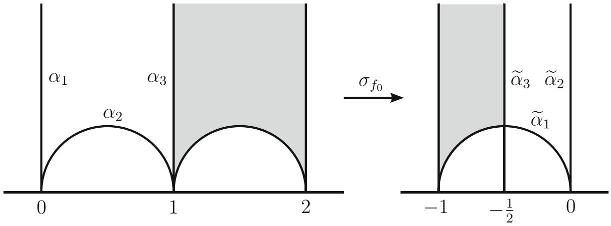


Fig. 10 The pullback map σ_{f_0}

shows that the image of G_{f_0} in $\mathrm{PGL}(2,\mathbb{Z})$ and the image of $G^+_{f_0}$ in $\mathrm{PSL}(2,\mathbb{Z})$ both have index 6. The right side of Fig. 10 shows fundamental domains for $\widetilde{G}_{f_0}$ and $\widetilde{G}^+_{f_0}$, the images of G_{f_0} and $G^+_{f_0}$ under the extended modular group virtual endomorphism.

Images of cusps under σ_{f_0}. For our example map f_0, we have that $\sigma_{f_0}\circ\sigma_{\rho_i} = \sigma_{\widetilde{\rho}_i}\circ\sigma_{f_0}$ for $i\in\{1,2,3\}$. It follows that $\sigma_{f_0}(\alpha_i)\subseteq\widetilde{\alpha}_i$ for $i\in\{1,2,3\}$. Using the continuity of σ_{f_0} on the Weil–Petersson completion $\mathbb{H}^*$ of $\mathbb{H}$, it follows that σ_{f_0} maps the common endpoint 0 of α_1 and α_2 to the common endpoint 0 of $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$. (We already essentially knew this through evaluation of μ_{f_0}.) Similarly, $\sigma_f(1)=\infty$ and $\sigma_{f_0}(\infty)=-\frac{1}{2}+\frac{1}{2}i$. This last equation shows that in special cases such as this, it is possible to determine the image in $\mathbb{H}$ of a cusp under the pullback map of a NET map.

The degree of the induced map $\hat{\sigma}_f\colon\mathbb{H}^*/G^+_f\to\mathbb{H}^*/\widetilde{G}^+_f$. Suppose f is a general NET map. We make the further assumption that σ_f is nonconstant. In this case, the map $\sigma_f\colon\mathbb{H}^*\to\mathbb{H}^*$ induces a nonconstant map $\hat{\sigma}_f\colon\mathbb{H}^*/G^+_f\to\mathbb{H}^*/\widetilde{G}^+_f$ of compact Riemann surfaces. The degree of $\hat{\sigma}_f$ can be calculated as follows. Let y be a cusp in $\mathbb{H}^*/\widetilde{G}^+_f$. Then

$$\deg(\hat{\sigma}_f)=\sum_{\hat{\sigma}_f(x)=y}\deg_x(\hat{\sigma}_f).$$

Returning to our example map f_0, we take y to be the image in $\mathbb{H}^*/\widetilde{G}^+_{f_0}$ of 0. Then there is only one value for x, the image of 0 in $\mathbb{H}^*/G^+_{f_0}$. Because the multiplier for slope ∞ $(=-\frac{1}{0})$ is 1, we see that a generator of the stabilizer of 0 in $G^+_{f_0}$ maps to a generator of the stabilizer of 0 in $\widetilde{G}^+_{f_0}$. Thus $\deg(\hat{\sigma}_{f_0})=1$.

Construction of σ_{f_0}. Finally, we determine σ_{f_0}. We first construct a candidate $\sigma\colon\mathbb{H}\to\mathbb{H}$, which we will eventually see is σ_{f_0}. By the Riemann mapping theorem there exists a unique analytic bijection σ which maps the hyperbolic triangle with vertices 0, 1 and ∞ to the hyperbolic triangle with vertices 0, ∞ and $-\frac{1}{2}+\frac{1}{2}i$ so that $\sigma(0)=0$, $\sigma(1)=\infty$ and $\sigma(\infty)=-\frac{1}{2}+\frac{1}{2}i$. We then extend the definition of σ to all of $\mathbb{H}$ using the reflection principle. This defines $\sigma\colon\mathbb{H}\to\mathbb{H}$.

Just as σ_{f_0} induces the map $\hat{\sigma}_{f_0}\colon\mathbb{H}^*/G^+_{f_0}\to\mathbb{H}^*/\widetilde{G}^+_{f_0}$, the map σ induces a map $\hat{\sigma}\colon\mathbb{H}^*/G^+_{f_0}\to\mathbb{H}^*/\widetilde{G}^+_{f_0}$. These Riemann surfaces both have genus 0. Both $\hat{\sigma}_{f_0}$ and $\hat{\sigma}$ have degree 1 and they agree at the three cusps. Thus they are equal. Therefore the restriction of σ_{f_0} to α_1, for example, agrees with the restriction of σ to α_1. It follows that $\sigma_{f_0}=\sigma$.

7 Dynamics on Curves in Degree 2

In this section we investigate the dynamics on curves for the set of NET maps with degree 2 and exactly one critical postcritical point. We will then indicate how to extend this result to all NET maps with degree 2 and hyperbolic orbifolds.

Theorem 8 *Let g be a NET map with degree 2 and exactly one critical postcritical point. Let μ_g be the slope function of g. Then we have the following.*

(1) *If g is combinatorially equivalent to a rational map, then μ_g has a global attractor containing at most four slopes.*
(2) *Suppose that g is not combinatorially equivalent to a rational map. Let $s \in \overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ be the slope of the obstruction of g. Let η be a generator of the cyclic group of g-liftable elements in the modular group of g which stabilize s. Then under iteration, a slope either becomes undefined (that is, the corresponding curve is trivial or peripheral), or lands in either*
   (a) *$\{s\}$, or*
   (b) *$\{s\} \cup \{\eta^m(r) : m \in \mathbb{Z}\}$ for some $r \in \overline{\mathbb{Q}}$ such that $\mu_g(\eta^m(r)) = \eta^{m+n}(r)$ for every integer m and some integer n.*

All three cases occur as well as all possible values of n in case 2b.

Proof Let f be the NET map f_0 of Sect. 6. As in Sect. 6, let G_f denote the subgroup of liftables for f in the extended modular group $\mathrm{EMod}(S^2, P(f))$.

Figure 10 in effect describes the pullback map σ_f. On the left are two triangles in a tesselation T_f of the Weil–Petersson completion $\mathbb{H}^*$ of $\mathbb{H}$ by fundamental domains for the action of G_f. The pullback map σ_f maps the unshaded, respectively shaded, triangle on the left to the unshaded, respectively shaded, triangle on the right. Extending by the reflection principle, we see that σ_f maps every triangle of T_f into a triangle of T_f.

Theorem 7 implies that g lies in the same impure Hurwitz class as f. If g is conjugate to f, then clearly there is a tesselation T_g of $\mathbb{H}^*$ by fundamental domains for the action of G_g on $\mathbb{H}^*$ such that σ_g maps every triangle of T_g into a triangle of T_g. Suppose that $g = f \circ \varphi$ for some map φ representing an element of $\mathrm{Mod}(S^2, P(f))$. Then $\sigma_g = \sigma_\varphi \circ \sigma_f$. One easily verifies that $G_g = G_f$ and that σ_φ acts as an automorphism of T_f. Thus σ_g maps every triangle of T_g into a triangle of T_g in this case also. We conclude that the map g of Theorem 8 maps every triangle of T_g into a triangle of T_g. It follows that every iterate of σ_g maps every triangle of T_g into a triangle of T_g.

Now suppose that g is combinatorially equivalent to a rational map. Then σ_g has a fixed point $\tau \in \mathbb{H}$. Let $r \in \overline{\mathbb{Q}}$. Let t be a triangle of T_g which has r as a vertex. Let z be a point in the interior of t. Then the points $z, \sigma_g(z), \sigma_g^2(z), \ldots$ converge to τ. So they eventually enter the star of τ in T_g (the union of triangles containing τ). Because iterates of σ_g map t into triangles of T_g, it follows that t eventually enters the star of τ in T_g. Because σ_g is continuous on $\mathbb{H}^*$, it follows that r eventually enters the star of τ in T_g. This star has at most two triangles and at most four vertices. Since σ_g and μ_g are conjugate on $\overline{\mathbb{Q}}$ via $p/q \mapsto -q/p$, this proves statement 1.

Now suppose that g is not combinatorially equivalent to a rational map. So g has an obstruction. The pullback map σ_g fixes the negative reciprocal of the slope of this

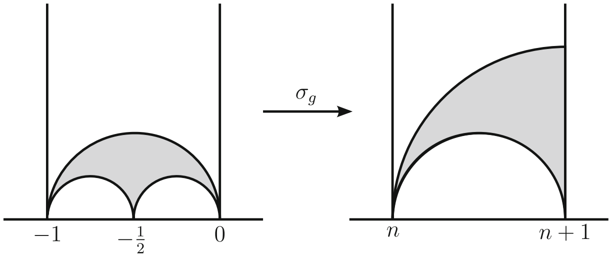


Fig. 11 The pullback map σ_g

obstruction. We find it convenient for this fixed point to be ∞. So we replace the map f two paragraphs above by a conjugate so that the new pullback map is σ_f conjugated by $z \mapsto -1/z$. Arguing as two paragraphs above, we find that Fig. 11 describes σ_g in the same way that Fig. 10 describes σ_f. The n in Fig. 11 is an integer. Any integer is possible. The case $n = 0$ is the case in which g is conjugate to f.

Arguing as in the case in which g is unobstructed, we find that every element of $\overline{\mathbb{Q}}$ eventually enters the star in T_g of ∞ under the iterates of σ_g. Hence it only remains to determine the action of σ_g on integers.

We have that $\sigma_g(-1) = n$. Using the reflection principle, we see that $\sigma_g(m) = m + n + 1$ for every odd integer m. Similarly, $\sigma_g(m) = m + n + 1 + i$ for every even integer m. Furthermore, the stabilizer of ∞ in the subgroup of modular group liftables for g has a generator which acts on $\mathbb{H}$ as $z \mapsto z + 2$.

Now suppose that n is even. Then σ_g maps odd integers to even integers, and it maps even integers into $\mathbb{H}$. So every integer eventually leaves $\overline{\mathbb{Q}}$. We are in the situation of case 2a.

Finally suppose that n is odd. Then σ_g maps odd integers to odd integers, and it maps even integers into $\mathbb{H}$. It follows that we are in case $2b$ with $n + 1$ here being 2 times n there.

The only thing left to prove is that case 1 actually occurs, namely, that there exists a rational NET map with degree 2 and exactly one critical postcritical point. An example of such a map is $f(z) = z^2 + i$.

This proves Theorem 8. □

It was noted in the above proof that the map f_0 of Sect. 6 corresponds to the case $n = 0$. Since 0 is even, the map f_0 falls into case 2a. Thus f_0 provides an example of an obstructed Thurston map with hyperbolic orbifold whose pullback map on curves has a finite global attractor consisting of just the obstruction.

We next indicate how Theorem 8 can be extended to all NET maps with degree 2 and hyperbolic orbifold. Theorem 7 and the paragraph preceding it imply that there are two impure Hurwitz classes of NET maps with hyperbolic orbifolds. The impure Hurwitz class of maps with one critical postcritical point is represented by the map f_0 of Sect. 6, and the proof of Theorem 8 uses f_0. The impure Hurwitz class of maps with two critical postcritical points is represented by the rabbit, and in the same way it is possible to use the rabbit to prove the corresponding result for these maps.

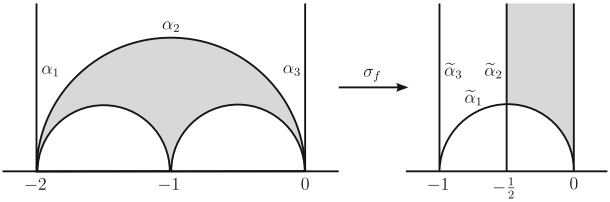


Fig. 12 The rabbit's pullback map σ_f

We discuss this extension in this paragraph. Let $f(z) = z^2 + c_R$ denote the rabbit polynomial of the introduction. Figure 2 gives a NET map presentation diagram for f. Arguing as in Sect. 6, we find that Fig. 12 provides an analog to Fig. 10 for the pullback map σ_f of f. More precisely, σ_f maps the unshaded, respectively shaded, triangle in the left side of Fig. 12 bijectively to the unshaded, respectively shaded, triangle in the right side of Fig. 12 with $\sigma_f(0) = \infty$, $\sigma_f(\infty) = -1$, $\sigma_f(-1) = 0$ and $\sigma_f(-2) = -\frac{1}{2} + \frac{1}{2}i$. The last equation provides another example of an exact evaluation of a pullback map at an element of $\overline{\mathbb{Q}}$ when that value lies in $\mathbb{H}$. Whereas before we worked with a tesselation by triangles, now we work with a tesselation by quadrilaterals, those determined by the union of the two triangles in the left side of Fig. 12. From here the argument proceeds as before. The result is essentially the same, although the statement must be modified a bit. The main difference is that case 2a does not occur here.

We say a few more words about the pullback map σ_f for the rabbit in this paragraph. The blue curve in the left side of Fig. 1 has slope 0 relative to Fig. 2. This curve lies in a 3-cycle of curves for the pullback map on curves. In terms of slopes, $\mu_f(0) = 1$, $\mu_f(1) = \infty$ and $\mu_f(\infty) = 0$. Because points of $\partial\mathbb{H}$ correspond to negative reciprocals of slopes, these equations correspond to the equations $\sigma_f(\infty) = -1$, $\sigma_f(-1) = 0$ and $\sigma_f(0) = \infty$, just as in the previous paragraph. Because σ_f maps the quadrilateral with vertices -2, -1, 0 and ∞ into itself, its fixed point is in this quadrilateral. Figure 4 further shows that this fixed point has small imaginary part (within the Euclidean circle centered at $(-1, 0)$ with radius $\sqrt{2}$).

Since the fixed point τ of σ_f is in the interior of the quadrilateral which is the union of the two triangles on the left side of Fig. 12, the star of τ in this tesselation consists of just this quadrilateral. So for any $t \in \overline{\mathbb{Q}}$, either there exists a positive integer n with $\mu_f^{\circ n}(t) = \odot$, or for n sufficiently large $\sigma_f^{\circ n}(t) \in \{0, -1, \infty\}$. Thus there is a finite global attractor consisting of points whose slopes correspond to 0, 1 and ∞. This gives another proof of Theorem 2.

8 The Extended Half-Space Theorem

The goal of this section is to sketch a proof of the extended half-space theorem. After filling in the details, the intervals mentioned in the theorem can be explicitly determined. See the discussion at the end of Sect. 1. Here is a qualitative statement of the theorem.

Theorem 9 (Extended Half-Space Theorem) *Let f be a NET map with slope function μ_f. Let $\frac{p}{q} \in \overline{\mathbb{Q}}$, and suppose that either $\mu_f(\frac{p}{q}) = \frac{p}{q}$ and $\delta_f(\frac{p}{q}) \neq 1$ or that $\mu_f(\frac{p}{q}) = \odot$. Then there exists an interval in $\mathbb{R} \cup \{\infty\}$ containing $-\frac{q}{p}$ which contains no negative reciprocals of obstructions for f other than possibly $-\frac{q}{p}$.*

To begin a sketch of the proof of this, we recall the setting of the half-space theorem. Let s_1 be the slope of a simple closed curve in $S^2 - P(f)$ whose preimage under f contains a connected component which is essential and nonperipheral (if no such s_1 exists, f is unobstructed, by Thurston's characterization theorem). Let $s_1' = \mu_f(s_1)$, and suppose that $s_1' \neq s_1$. In this situation the half-space theorem supplies an open half-space H_1 in the upper half-plane which contains no fixed point of σ_f and whose boundary's interior contains $-1/s_1$ but no negative reciprocal of a obstruction for f. The half-space H_1 depends only on s_1, s_1' and the multiplier $\delta_f(s_1)$. We call such a half-space an excluded half-space.

Let t be an extended rational number which is not mapped to a different extended rational number by σ_f. We will use functional equations satisfied by σ_f to produce excluded half-spaces near t so that the collection of all extended real numbers excluded by these half-spaces together with t forms an open interval about t in $\mathbb{R} \cup \{\infty\}$.

We consider the simplest case, the case in which $t = \infty$. The general case can be gotten from this by applying an element of $\mathrm{PSL}(2, \mathbb{Z})$ to t. Keep in mind that points of $\overline{\mathbb{Q}}$ in the boundary of $\mathbb{H}$ are to be viewed as negative reciprocals of slopes. So either $\mu_f(0) = 0$ or $\mu_f(0) = \odot$. Let γ be a simple closed curve in $S^2 - P(f)$ with slope 0. Let d be the degree with which f maps every connected component of $f^{-1}(\gamma)$ to γ. Let c be the number of these connected components which are neither inessential nor peripheral. Theorem 7.1 of Cannon et al. (2012), for example, yields the functional equation $\sigma_f \circ \varphi^d = \varphi^c \circ \sigma_f$, where $\varphi(z) = z + 2$. We have that $\delta_f(0) = \frac{c}{d}$. By hypothesis, $\frac{c}{d} \neq 1$. For convenience we consider the case that $\frac{c}{d} < 1$, so $c < d$.

Let $t_1 = -\frac{1}{s_1}$ and $t_1' = -\frac{1}{s_1'}$, where s_1 and s_1' are as above. Since $t = \infty$, it is natural to assume that $t_1 \neq \infty$. For simplicity, we also assume that $t_1' \neq \infty$. Let B_1 and B_1' be closed horoballs at t_1 and t_1' as in the statement of the half-space theorem in Cannon et al. (2012, Theorem 5.6). Let r be the Euclidean radius of B_1, and let r' be the Euclidean radius of B_1'.

If $r > r'$, then our excluded half-space H_1 is unbounded in the Euclidean metric, and so we already have an open neighborhood of ∞ in $\mathbb{R} \cup \{\infty\}$ which contains no negative reciprocals of obstructions. So we assume that $r \leq r'$. The case in which $r = r'$ can be handled as follows. In this case H_1 is bounded by a vertical Euclidean ray with endpoint the average value of t_1 and t_1'. This gives us an unbounded interval of real numbers which contains no negative reciprocals of obstructions. Using the fact that $c < d$, we replace t_1 and t_1' by their images under an appropriate power of φ^d and φ^c (possibly negative) so that the order of these images is opposite to the order of t_1 and t_1'. The resulting excluded half-space and H_1 combine to produce an open neighborhood of ∞ in $\mathbb{R} \cup \{\infty\}$ containing no negative reciprocals of obstructions. This establishes the existence of such an interval. Hence we assume that $r < r'$. In this case H_1 lies within a Euclidean semicircle. Let C_1 and R_1 be the center and radius of this semicircle. See Fig. 13, which assumes that $t_1 > t_1'$.

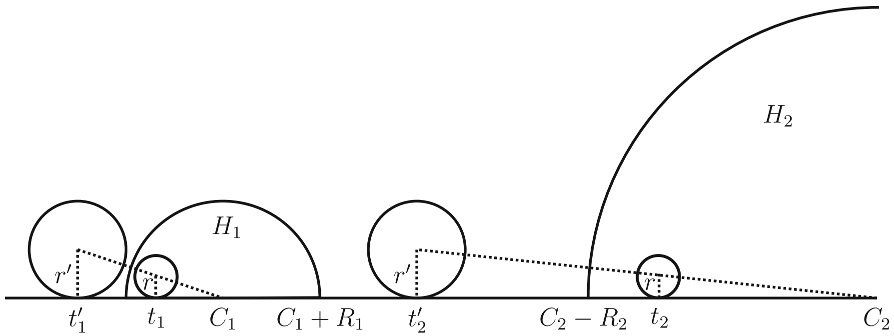


Fig. 13 The basic diagram for the extended half-space theorem

Now we apply the functional equation $\sigma_f \circ \varphi^d = \varphi^c \circ \sigma_f$. Set $t_2 = \varphi^d(t_1)$ and $t_2' = \varphi^c(t_1')$. Then the equation $\mu_f(s_1) = s_1'$ implies that $\sigma_f(t_1) = t_1'$, and so $\sigma_f(t_2) = t_2'$. Because $t_1 > t_1'$, and $d > c$, we have that $t_2 > t_2'$.

In this paragraph we show that the half-space theorem applies to t_2 and t_2' using the horoballs $B_2 = \varphi^d(B_1)$ and $B_2' = \varphi^c(B_1')$. We have the equation $\sigma_f \circ \varphi^d = \varphi^c \circ \sigma_f$. This is induced by a homotopy equivalence of the form $\phi^d \circ f \sim f \circ \phi^c$, where ϕ is a Dehn twist about a curve with slope 0. The equation $t_2 = \varphi^d(t_1)$ implies that if γ_1 is a simple closed curve in $S^2 - P(f)$ with slope $s_1 = -1/t_1$, then $\gamma_2 = \phi^{-d}(\gamma_1)$ is a simple closed curve in $S^2 - P(f)$ with slope $s_2 = -1/t_2$. A corresponding statement holds for ϕ^{-c}. Now the homotopy equivalence $\phi^d \circ f \sim f \circ \phi^c$ shows that if c_2 is the number of connected components of $f^{-1}(\gamma_2)$ which are neither inessential nor peripheral and if d_2 is the degree with which f maps these components to γ_2, then $c_2 = c_1$ and $d_2 = d_1$. So $\delta_f(s_2) = \delta_f(s_1)$. Combining this with Corollary 6.2 of Cannon et al. (2012), which shows how elements of $\mathrm{PGL}(2, \mathbb{Z})$ map horoballs to horoballs, it follows that the half-space theorem applies to t_2 and t_2' using the horoballs $B_2 = \varphi^d(B_1)$ and $B_2' = \varphi^c(B_1')$. Hence we obtain another excluded half-space H_2 corresponding to the horoballs B_2 and B_2' at t_2 and t_2' with Euclidean radii r and r'.

We want $H_1 \cap H_2 \neq \emptyset$ because then the open intervals in $\mathbb{R}$ determined by H_1 and H_2 can be combined to form a larger interval. Since $t_2 > t_1$ as in Fig. 13, $H_1 \cap H_2 \neq \emptyset$ if and only if $C_1 + R_1 > C_2 - R_2$. We make an explicit computation based on this and find that

$$H_1 \cap H_2 \neq \emptyset \iff t_1 - t_1' > d\left(\sqrt{r'/r} - 1\right) + c\left(1 - \sqrt{r/r'}\right).$$

Suppose that the last inequality is satisfied. Then because

$$t_2 - t_2' = t_1 + 2d - t_1' - 2c = t_1 - t_1' + 2(d - c) > t_1 - t_1',$$

the inequality in the next-to-last display is satisfied with $t_1 - t_1'$ replaced by $t_2 - t_2'$. Inductively, we conclude that if $H_1 \cap H_2 \neq \emptyset$, then f has no obstruction s with $-\frac{1}{s} > C_1 - R_1$. Furthermore, the last display shows that the differences $t_1 - t_1'$ increase without bound under iteration, and so it is possible to find t_1 such that $H_1 \cap H_2 \neq \emptyset$.

This obtains an unbounded interval of positive real numbers which contains no negative reciprocals of obstructions. Symmetry yields a corresponding interval of negative real numbers. This is the gist of the extended half-space theorem. It remains to make the estimates explicit for computation. This is a bit tedious, but straightforward.

Theorem 9 is false if $\delta_f(p/q) = 1$; counterexamples are found among maps in 21HClass3 and 31Hclass5, 6, 9.

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