PROBLEM CONTRIBUTION

Approximate Identities and Lagrangian Poincaré Recurrence

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Abstract

In this note we discuss three interconnected problems about dynamics of Hamiltonian or, more generally, just smooth diffeomorphisms. The first two concern the existence and properties of the maps whose iterations approximate the identity map with respect to some norm, e.g., C^1- or C^0-norm for general diffeomorphisms and the γ-norm in the Hamiltonian case, and the third problem is the Lagrangian Poincaré recurrence conjecture.

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Достаньте двойные листочки. To Rafail Kalmanovich Gordin on the occasion of his 70th birthday.

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1 Introduction

In this note we focus on three interconnected problems concerning the dynamics of general smooth and, more specifically, Hamiltonian diffeomorphisms. (A Hamiltonian diffeomorphism is the time-one map of the Hamiltonian isotopy generated by a time-dependent Hamiltonian.) The first two problems are about the existence and properties of the maps whose iterations approximate the identity map with respect to some norm, e.g., C^1- or C^0-norm for general diffeomorphisms and the γ-norm in the Hamiltonian case. (The γ-norm is a natural norm on the group of Hamiltonian diffeomorphisms, coming from min-max critical values in Floer theory a.k.a. spectral invariants.) In dynamical systems theory such maps are often called rigid, but we prefer the more intuitive term *approximate identity*, borrowed from analysis. The third problem is the Lagrangian Poincaré recurrence conjecture.

The notion of an approximate identity and several variants of the definition are spelled out in Sect. 2. Approximate identities can have interesting and non-trivial dynamics, and the main examples of such maps are the so-called pseudo-rotations; see (Avila et al. 2015; Bramham 2015; Ginzburg and Gürel 2018a, b; Fayad and Krikorian 2018). Yet, in some way, these maps resemble actions of compact abelian groups on manifolds. For instance, one may expect the fixed point set of an approximate identity to be nowhere dense and the map near isolated fixed points to satisfy some non-degeneracy conditions. This is the essence of the first problem (Question 2.3).

The second problem (Question 3.1) concerns the existence of Hamiltonian diffeomorphisms which are γ-approximate identities. This question appears to be related to the Conley conjecture asserting that for many closed symplectic manifolds every Hamiltonian diffeomorphism has infinitely many un-iterated periodic orbits. The conjecture is known to hold for a broad class of manifolds (see Ginzburg and Gürel 2015, 2017) including all symplectically aspherical closed manifolds, but there are some exceptions. For instance, an irrational rotation of S^2 about the z-axis has exactly two periodic points—the poles. Although there is no established formal connection between the Conley conjecture and approximate identities, all known "counterexamples" to the Conley conjecture are γ-approximate identities. In particular, as proved in (Ginzburg and Gürel 2018a, Thm. 5.1), every pseudo-rotation of $\mathbb{CP}^n$, i.e., a Hamiltonian diffeomorphism with exactly $n+1$ periodic points is a γ-approximate identity. In the spirit of the Conley conjecture, it is reasonable to expect that a symplectically aspherical manifold does not admit compactly supported γ-approximate identities.

Finally, in Sect. 4 we turn to the Lagrangian Poincaré recurrence conjecture according to which the images of a closed Lagrangian submanifold L under the iterates of a Hamiltonian diffeomorphism φ cannot be all disjoint from L, i.e., $L \cap \varphi^{k_i}(L) \neq \emptyset$ for some sequence $k_i \to \infty$; see Conjecture 4.1. Very little is known about this conjecture and, at the time of writing, the only established non-trivial case is when φ is a pseudo-rotation of $\mathbb{CP}^n$, (Ginzburg and Gürel 2018a, Thm. 4.2).

Naturally, in such a short note we cannot possibly spell out all necessary definitions and background results. Sect. 2 can be, perhaps, accessible to a graduate student with sufficient background in manifolds, group actions and basic dynamical systems theory. However, in Sects. 3 and 4 we occasionally make use of fairly advanced notions from symplectic topology (Floer theory) and Hamiltonian dynamics. For these notions and

results we refer the reader to, e.g., (Hofer and Zehnder 1994; McDuff and Salamon 2012; Salamon 1999).

2 Approximate Identities and Almost Periodic Maps

Consider a class of compactly supported C^∞-diffeomorphisms φ of a smooth manifold M (e.g., all such diffeomorphisms or compactly supported Hamiltonian diffeomorphisms when M is symplectic, etc.), equipped with some metric, e.g., the C^0- or C^1- or C^r-metric or the γ-metric in the Hamiltonian case—see Sect. 3. The norm $\|\varphi\|$ is by definition the distance from φ to the identity. A map φ is said to be a $\|\cdot\|$-*approximate identity*, or a $\|\cdot\|$-*a.i.* for the sake of brevity, if $\varphi^{k_i} \to id$ with respect to the norm $\|\cdot\|$ for some sequence $k_i \to \infty$. (Strictly speaking, the entire sequence of iterates φ^{k_i} should be called an approximate identity. We believe that a confusion with approximate identities in analysis is unlikely.) Clearly, a C^1-a.i. is automatically a C^0-a.i. The definition extends to flows in an obvious way. Approximate identities have been extensively studied in dynamics, although usually from a perspective different than ours; see, e.g., (Bramham 2015; Fayad and Katok 2004; Avila et al. 2015; Fayad and Krikorian 2018; Kunde 2018) and references therein. In this section we focus on C^0- and C^1-a.i.'s, but first some terminological remarks are due.

In dynamics, approximate identities are often called *rigid* maps. We find this terminology misleading, for the term "rigid" is routinely used in a different sense. Furthermore, rigidity is often associated with structural stability and we are not aware of any situation where an a.i. would be structurally stable. In fact, in many instances it is not hard to show that an a.i. cannot be structurally stable in a suitable class of maps. (Regarding terminology, we also note that in low-dimensional dynamics C^0-rigid maps are sometimes called recurrent, Kolev and Pérouème (1998).)

One should keep in mind that the above definition allows some room for pathological behavior. For example, hypothetically it is possible that $\|\varphi^{k_i'}\| \to \infty$ for some sequence $k_i' \to \infty$ for an a.i. φ. The definition can be and has been refined and amended in several ways. For instance, one can impose conditions on the rate of convergence of $\varphi^{k_i} \to id$ (e.g., Diophantine vs. Liouville) or on the arithmetic properties of the sequence

$$K_\epsilon = \{k \mid \|\varphi^k\| < \epsilon\},$$

e.g., that K_ϵ has positive density or contains infinitely many primes, etc., significantly restricting the class of a.i.'s and their possible dynamics; cf. (Fayad and Krikorian 2018).

One such refinement is of particular relevance to us. Namely, φ is called $\|\cdot\|$-*almost periodic* if for every $\epsilon > 0$ the sequence K_ϵ, where we now take $K_\epsilon \subset \mathbb{Z}$, is *quasi-arithmetic*, i.e., the difference between any two consecutive terms is bounded by a constant (possibly depending on ϵ). Clearly, an almost periodic map is an a.i. Topological dynamics of C^0-almost periodic maps is studied in detail in Gottschalk and Hedlund (1955); see also Rejeb (2014). Almost periodic maps are closely related to compact group actions on M: φ is C^0-almost periodic if and only if the family

$\{\varphi^k\}$ is equicontinuous and thus generates a compact abelian group G of (compactly supported) homeomorphisms, Gottschalk and Hedlund (1955).

Example 2.1 (Actions of compact Lie groups) A translation, $x \mapsto x + \alpha$ where $\alpha \in G$, in the circle or torus G is C^1-almost periodic. More generally, φ is C^1-almost periodic whenever it topologically generates a C^1-action $G \times M \to M$ of a compact abelian Lie group G on M, i.e., the subgroup $\{\varphi^k \mid k \in \mathbb{Z}\}$ is relatively compact in the group of C^1-diffeomorphisms, or, equivalently, φ is an isometry with respect to some metric.

In fact, the converse is also true, although this is ultimately a non-trivial result closely related to the Hilbert–Smith conjecture asserting that a locally compact group acting effectively on a manifold is a Lie group; see, e.g., (Montgomery and Zippin 1955; Pardon 2013; Tao 2014) and references therein. Namely, it is not hard to show that, when φ is C^1-almost periodic, the family $\{\varphi^k\}$ is precompact in the C^1-topology and thus generates an action of a compact abelian group G by compactly supported C^1-diffeomorphisms of M. Then G is necessarily a Lie group (this is a deep result, Repovš and Ščepin (1997)) and the action map $G \times M \to M$ is C^1; see Montgomery and Zippin (1955) and references therein. As a consequence, φ is exactly as in Example 2.1, and the dynamics of C^1-almost periodic maps is rather boring.

A C^0-a.i. can be thought of as a map with simultaneous return times: for every $\epsilon > 0$ there exist infinitely many $k \in \mathbb{N}$ such that $d(x, \varphi^k(x)) < \epsilon$ for all $x \in M$, where d is the distance with respect to some fixed metric on M. (The key point here is that k is independent of x.) In particular, φ is not a C^0-a.i. whenever there exists a non-recurrent point $x \in M$. This observation provides numerous examples of maps which are not C^0-a.i.'s. For instance, φ is not a C^0-a.i. when it has a non-elliptic periodic orbit. However, it is not clear to the authors if, say, a volume-preserving C^1-a.i. must have zero topological entropy and vanishing all Lyapunov exponents, although this seems rather probable; see Katok and Hasselblatt (1995) for the definitions and also Avila et al. (2016) for some related results. (For a C^0-almost periodic diffeomorphism, vanishing of the topological entropy is easy to prove.) Furthermore, the growth rate of $D\varphi^k$ is closely related to a.i.-type features of φ; see (Frączek and Polterovich 2008; Polterovich 2002, 2003; Polterovich and Sodin 2004) and references therein. Finally, note that obviously a C^0-a.i. cannot be mixing or topologically mixing, and a.i.'s are often studied in connection with mixing properties (Avila et al. 2015; Fayad and Krikorian 2018). For instance, the horocycle flow is mixing and hence not a C^0-a.i.; we refer the reader to Parasjuk (1953) for the proof and Marcus (1978) for refinements and generalizations of this result.

There are examples of C^0-a.i.'s with interesting dynamics.

Example 2.2 (Pseudo-rotations in 2D) Let φ be a pseudo-rotation of the sphere $M = S^2$ or the closed disk D^2, i.e., an area-preserving diffeomorphism with exactly two periodic points for S^2 or one periodic point for D^2. (In both cases, all periodic points are necessarily fixed points.) Already in this case, φ can have rather surprising dynamics. For instance, there exist pseudo-rotations φ with exactly three invariant ergodic measures. For S^2 these are the two fixed points and the area form, and for D^2 one has to replace one of the fixed points by the Lebesgue measure on ∂D^2. The linearizations $D\varphi$ at the fixed points (and also $\varphi|_{\partial D^2}$) are rotations in the same angle $\theta \notin \pi\mathbb{Q}$, and

most of the orbits of φ are dense. Such maps are constructed in Anosov and Katok (1970); see also Fayad and Katok (2004). An analytic pseudo-rotation of D^2 is locally, (i.e., near the fixed point) a C^∞-a.i. (Avila et al. 2015). Furthermore, assume that the rotation number θ/π is *exponentially Liouville*, i.e., for any $c > 0$ it can be approximated by rational numbers p/q with error smaller than $\exp(-cq)$. Then φ is also (globally) a C^0-a.i., (Bramham 2015; Ginzburg and Gürel 2018a). At the same time, a pseudo-rotation with a dense orbit is never C^0-almost periodic. Indeed, focusing on $M = S^2$, assume the contrary and let G be the compact topological abelian group generated by φ. Then the action of G on M is transitive since φ has a dense orbit. Thus, by the corollary in [Montgomery and Zippin (1955), Sect. 6.3, p. 243], G is a compact abelian Lie group. However, it is clear that such a group cannot act transitively on S^2.

In some sense, approximate identities resemble actions of compact abelian groups on manifolds, although Example 2.2 shows that an approximate identity can have intricate dynamics and need not literally generate such an action. In any event, our first question is inspired by this analogy. Let $\varphi \colon M \to M$ be a smooth, compactly supported a.i. (in whatever sense) and let $\mathrm{Fix}(\varphi) = \{x \in M \mid \varphi(x) = x\}$ be the set of its fixed points.

Question 2.3 *Assume that M is connected and $\varphi \neq \mathrm{id}$. Must $\mathrm{Fix}(\varphi)$ have empty interior? What can be said about $D_x\varphi \colon T_x M \to T_x M$ at $x \in \mathrm{Fix}(\varphi)$? For instance, can $D_x\varphi$ be degenerate when x is an isolated point in $\mathrm{Fix}(\varphi)$?*

When M is non-compact, the interior of $\mathrm{Fix}(\varphi)$ is necessarily non-empty since φ is compactly supported. Thus the affirmative answer to the first part of the question would simply mean that such a map does not exist in this case. It is an easy exercise to check this for $n = 1$. If φ is C^1-almost periodic and thus generates a compact Lie group G of smooth transformations, the answers readily follow from the discussion above and the standard fact that a G-action can be linearized near a fixed point; see, e.g., Guillemin et al. (2002). When φ is C^0-almost periodic, the negative answer to the first question would follow from the Hilbert–Smith conjecture and a theorem of Newman asserting that the interior of the fixed point set M^Γ is empty whenever Γ is a finite group acting on M by homeomorphisms; see Newman (1931) and also Dress (1969). (The difficulty is that although φ is C^1, we have no control over the "smoothness" of the G-action, e.g., we do not know that G acts by Lipschitz transformations and thus is a Lie group, cf. Repovš and Ščepin (1997).) When M is a closed surface (and in some other instances in dimension two) and φ is a C^0-a.i. homeomorphism homotopic to the identity, the interior of $\mathrm{Fix}(\varphi)$ is empty; this follows from the results in Kolev and Pérouème (1998). The question appears to be completely open in general, even for C^1-a.i.'s and even when $M = \mathbb{R}^n$, $n \geq 3$. Finally, we point out that one can expect a.i.'s (in any sense) to be very non-generic.

3 Approximate Identities in the Hamiltonian Setting

In this section, which requires some background in symplectic topology, we turn to the case where (M^{2n}, ω) is a symplectic manifold and $\varphi = \varphi_H$ is a Hamiltonian diffeomorphism of M, generated by a time-dependent Hamiltonian $H \colon S^1 \times M \to \mathbb{R}$.

For the sake of simplicity, the manifold M is assumed to be (positive) monotone or symplectically aspherical, i.e., $\omega|_{\pi_2(M)} = 0 = c_1(TM)|_{\pi_2(M)}$, throughout the rest of the paper. (We refer the reader to, e.g., (Hofer and Zehnder 1994; McDuff and Salamon 2012; Salamon 1999) for the necessary definitions.) In this setting, there are two other natural norms to consider in addition to the C^1- and C^0-norm. These are the γ-norm and the Hofer norm. We will focus on the former. When M is closed,

$$\gamma(\varphi) = c_{[M]}(H) + c_{[M]}\left(H^{\mathrm{inv}}\right),$$

where H^{inv} is the Hamiltonian generating the flow $\left(\varphi_H^t\right)^{-1}$ and c_w is the spectral invariant associated with a class $w \in H_*(M)$; see, e.g., (Entov and Polterovich 2003; Oh 2005a, b; Schwarz 2000; Viterbo 1992). (For our purposes, it is convenient to think of φ_H as an element of the universal covering of the group of Hamiltonian diffeomorphisms.) When M is symplectically aspherical and also for $M = \mathbb{CP}^n$ equipped hereafter with the standard symplectic form,

$$\gamma(\varphi) = c_{[M]}(H) - c_{[pt]}(H).$$

One can also extend this definition to the case where M is open, provided that its structure at infinity is well-controlled, e.g., M is convex at infinity; cf. (Frauenfelder and Schlenk 2006; Gürel 2008). When M is symplectically aspherical, the γ-norm is continuous with respect to the C^0-norm and thus a Hamiltonian C^0-a.i. is automatically a γ-a.i (Buhovsky et al. 2018). This is also true for $\mathbb{CP}^n$, [Shelukhin (2018), Thm. C].

The authors learned of the following question from L. Polterovich and Seyfaddini:

Question 3.1 *Does a symplectically aspherical manifold M admit compactly supported C^0- or γ-almost periodic Hamiltonian diffeomorphisms $\varphi \neq id$ or, more generally, a.i.'s?*

It is reasonable to conjecture that the answer is negative in all instances. As a warmup, it is easy to see that M does not admit a C^1-a.i. In fact, $\|\varphi^k\|_{C^1} \to \infty$ even when $\varphi \neq id$ is degenerate. Indeed, let $\mathcal{S}(H)$ be the action spectrum of H. This is a compact subset of $\mathbb{R}$, Hofer and Zehnder (1994). Set $\mathrm{width}(\varphi) := \max \mathcal{S}(H) - \min \mathcal{S}(H)$. Then $\mathrm{width}(\varphi) > 0$ whenever $\varphi \neq id$, Schwarz (2000). Hence,

$$\mathrm{width}(\varphi^k) \geq k \, \mathrm{width}(\varphi) \to \infty.$$

On the other hand, $\mathrm{width}(\varphi) \leq const\|\varphi\|_{C^1}$. Hence, $\mathrm{width}(\varphi^k) \leq O\left(\|\varphi^k\|_{C^1}\right)$. (This argument is taken from Polterovich (2002).)

Furthermore, if we knew that a C^0-a.i. is automatically non-degenerate (see Question 2.3), it would follow that any closed symplectic manifold M such that $H_{odd}(M; \mathbb{Z}) \neq 0$, e.g., a symplectically aspherical manifold, does not admit Hamiltonian C^0-a.i.'s. (Indeed, by Floer theory, every non-degenerate Hamiltonian diffeomorphism of M must have non-elliptic fixed points and thus cannot be a C^0-a.i.) Note also that the closure of the group of Hamiltonian diffeomorphisms with respect to the γ-norm is rather complicated and poorly understood, but it certainly contains

elements other than homeomorphisms and does not act on M in any obvious sense, Humilière (2008).

One can expect few manifolds to admit γ-a.i.'s. Of course, M admits C^1-almost periodic Hamiltonian diffeomorphisms (cf. Example 2.1) when it has a Hamiltonian S^1-action. In particular, all symplectic toric manifolds and coadjoint orbits of compact Lie groups (e.g., $\mathbb{CP}^n$, complex Grassmannians, flag manifolds, etc.) admit C^1-almost periodic Hamiltonian diffeomorphisms. However, there are no other known manifolds having γ-a.i.'s. Overall, the situation seems to be parallel to the Conley conjecture asserting that for many manifolds (but obviously not all, e.g., S^2) every Hamiltonian diffeomorphism has infinitely many un-iterated periodic orbits. The conjecture has been proved for a broad class of manifolds including all symplectically aspherical manifolds; see (Ginzburg and Gürel 2015, 2017). All known Hamiltonian diffeomorphisms with finitely many periodic orbits are γ-a.i.'s. The converse is not quite true: the fixed point set of a Hamiltonian S^1-action can have positive dimension.

Regarding Question 3.1, it is also worth pointing out that conjecturally the γ-diameter of the group of Hamiltonian diffeomorphisms of a symplectically aspherical manifold M is infinite. In some instances, e.g., for surfaces, this is obvious. Moreover, one might expect that $\gamma(\varphi^k)$ is unbounded for generic (all?) maps $\varphi \neq id$ or even that $\gamma(\varphi^k) \to \infty$.

One interesting class of γ-a.i.'s, relevant for what follows, is identified in Ginzburg and Gürel (2018a). These are (Hamiltonian) pseudo-rotations of $\mathbb{CP}^n$, i.e., Hamiltonian diffeomorphisms of $\mathbb{CP}^n$ with minimal possible number of periodic points, equal to $n + 1$ by the Arnold conjecture, (McDuff and Salamon 2012; Salamon 1999). Among pseudo-rotations are the Anosov–Katok pseudo-rotations from Example 2.2 and true rotations (i.e., isometries) of $\mathbb{CP}^n$ with finitely many fixed points. The following theorem is proved in a slightly different form in Ginzburg and Gürel (2018a):

Theorem 3.2 (γ-convergence, Thm. 5.1, Ginzburg and Gürel (2018a)) *Let φ be a pseudo-rotation of $\mathbb{CP}^n$. Then φ is γ-almost periodic. Furthermore, there exist a constant $C > 0$ and a non-negative integer $d \leq n$, both depending only on φ, such that for every ϵ in the range $(0, \pi)$, we have*

$$\liminf_{k \to \infty} \frac{|\{\ell \leq k \mid \gamma(\varphi^\ell) < \epsilon\}|}{k} \geq C\epsilon^d.$$

The proof of the theorem is based on trading the behavior of φ^k with respect to the γ-norm for the dynamics of a certain translation in the d-dimensional torus, which is of course almost periodic; see Example 2.1. We note that φ cannot be C^0-almost periodic when it has a dense orbit—the proof of this fact is identical to the argument in Example 2.2. However, φ is a C^0-a.i. when it meets a certain additional requirement generalizing the condition from that example that the rotation number θ/π is exponentially Liouville, (Ginzburg and Gürel 2018a, Thm. 1.4). It is unknown if every Hamiltonian pseudo-rotation of $\mathbb{CP}^n$ is a C^0-a.i. and if there are γ-a.i.'s on $\mathbb{CP}^n$ which are not pseudo-rotations. (Note that the γ-diameter of the universal covering of the group of Hamiltonian diffeomorphisms of $\mathbb{CP}^n$ is bounded—in fact equal to π—in

contrast with symplectically aspherical manifolds, Entov and Polterovich (2003).) In any event, one can expect γ-a.i.'s to be extremely non-generic for all M; for Hamiltonian diffeomorphisms with finitely many periodic orbits this is proved in Ginzburg and Gürel (2009) for many symplectic manifolds including $\mathbb{CP}^n$.

We conclude this section by pointing out that Question 2.3 can be meaningfully restricted to Hamiltonian transformations. For C^0-almost periodic compactly supported Hamiltonian diffeomorphisms, just as in the general case of the Hilbert–Smith conjecture, it boils down to the question whether the action of the group of p-adic integers on M can be generated by such a transformation; cf., e.g., (Pardon 2013; Tao 2014).

4 Lagrangian Poincaré Recurrence

In this section we will concentrate on an apparently different question, which we prefer to state as a conjecture. The question is connected to the γ-convergence theorem (Theorem 3.2), but this connection might be purely accidental. As in Sect. 3, let φ be a compactly supported Hamiltonian diffeomorphism of a symplectic manifold M^{2n}. The following conjecture was put forth by the first author and independently by Claude Viterbo around 2010.

Conjecture 4.1 (Lagrangian Poincaré Recurrence) *For any closed Lagrangian submanifold $L \subset M$ there exists a sequence of iterations $k_i \to \infty$ such that*

$$\varphi^{k_i}(L) \cap L \neq \emptyset.$$

Moreover, the density of the sequence k_i is related to a symplectic capacity of L.

We refer the reader to [Ginzburg and Gürel (2018a), Sect. 5.1.2] for a detailed discussion of this conjecture. Here we only mention that the conjecture is mainly interesting when L is small (e.g., contained in a small ball or more generally displaceable), and that the condition that φ is Hamiltonian is essential. In dimension two, Conjecture 4.1 is an easy consequence of the standard Poincaré recurrence. In general, very little is known about the problem. It is not even clear if this conjecture is a dynamics question or a packing problem: it is possible that there is an upper bound on the number of disjoint Lagrangian submanifolds, Hamiltonian diffeomorphic to each other and embedded into a compact domain in M. However, at the time of writing, the only non-trivial result along the lines of the conjecture is on the dynamics side. This is the following theorem proved in Ginzburg and Gürel (2018a) and establishing a strong form of the Lagrangian Poincaré recurrence for pseudo-rotations of $\mathbb{CP}^n$.

Theorem 4.2 (Thm. 4.2, Ginzburg and Gürel (2018a)) *Let φ be a pseudo-rotation of $\mathbb{CP}^n$ and let $L \subset \mathbb{CP}^n$ be a closed Lagrangian submanifold, which admits a Riemannian metric without contractible closed geodesics (e.g., a torus). Then $\varphi^{k_i}(L) \cap L \neq \emptyset$ for some quasi-arithmetic sequence $k_i \to \infty$. Furthermore, there exists a constant $C > 0$ and a non-negative integer $d \leq n$, both depending only on φ but not L, and a constant $a > 0$ depending only on L such that*

$$\liminf_{k \to \infty} \frac{|\{\ell \le k \mid \varphi^{\ell}(L) \cap L \neq \emptyset\}|}{k} \ge C \cdot a^d.$$

The constant a is a certain homological capacity of L. Theorem 4.2 is an easy consequence of Theorem 3.2 and the standard fact that $\varphi(L) \cap L \neq \emptyset$ when $\gamma(\varphi) < a$. In fact, the condition on L in Theorem 4.2 can be removed; see [Kislev and Shelukhin (2018), Thm. D and Rmk. 13]. Note also that here and in Theorem 3.2 one could slightly refine the quantitative result by replacing the lower bound on the density of the sequence of "returns" by the upper bound $1/(Ca^d)$ on the step of the quasi-arithmetic sequence.

Conjecture 4.1 has deep applications to dynamics. For instance, as has been pointed out by Viterbo, once established in dimension four, it would imply (via a non-trivial argument) that the group of area-preserving transformations of the closed disk is not simple—a well-known open question in two-dimensional dynamics.

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