Fundamental Matrix Factorization in the FJRW-Theory Revisited

Alexander Polishchuk[1,2]

Abstract

We present an improved construction of the fundamental matrix factorization in the FJRW-theory given in Polishchuk and Vaintrob (J Reine Angew Math 714:1–22, 2016). The revised construction makes the independence on choices more apparent and works for a possibly nonabelian finite group of symmetries. One of the new ingredients is the category of dg-matrix factorizations over a dg-scheme.

Keywords FJRW-theory · Matrix factorizations · Cohomological field theory · dg-schemes

Introduction

This short note is supposed to clarify the construction of the cohomological field theory associated with a quasihomogeneous polynomial W and its finite group of symmetries G. Such a cohomological field theory, called the *FJRW-theory* was first proposed in Fan et al. (2013). Then, in Polishchuk and Vaintrob (2016) a different construction, based on categories of matrix factorizations, was given (conjecturally, the two constructions give the same cohomological field theory).

The approach of Polishchuk and Vaintrob (2016) is based on constructing certain *fundamental matrix factorizations* which live over the product of certain finite coverings of $\overline{M}_{g,n}$ (the moduli of Γ-spin structures) with affine spaces. It is this construction that we aim to clarify. More precisely, we would like to present the construction in such

✉ Alexander Polishchuk
  apolish@uoregon.edu

1   University of Oregon, Eugene, OR, USA

2   National Research University Higher School of Economics, Moscow, Russia

a way that it would be analogous to the construction of Ciocan–Fontanine and Kapranov of the virtual fundamental class in Gromov–Witten theory via dg-manifolds (see Ciocan-Fontanine and Kapranov 2009). The second goal that we achieve is to present the construction without using coordinates on the vector space V on which W lives. This has an additional bonus that we can handle the case when the group G is not necessarily commutative (but still finite).

The construction of Polishchuk and Vaintrob (2016) of the fundamental matrix factorization over $\mathcal{S} \times \prod_i V^{\gamma_i}$, where $\mathcal{S}$ is the moduli space of (rigidified) Γ-spin structures with some markings (see Sect. 3.1 for details) roughly has the following two steps. In Step 1 one considers the object $R\pi_*(\mathcal{V})$ in the derived category $D(\mathcal{S})$, where $\pi : \mathcal{C} \to \mathcal{S}$ is the universal curve, $\mathcal{V}$ is the underlying vector bundle of the universal Γ-spin structure, and then equips it with some additional structure. In Step 2 one realizes $R\pi_*(\mathcal{V})$ by a 2-term complex $[A \to B]$, where A and B a vector bundles over $\mathcal{S}$, such that there is a morphism

$$Z : X = \operatorname{tot}(A) \to \prod_i V^{\gamma_i}$$

and a Koszul matrix factorization of $Z^*(\sum W_i)$, where $W_i = W|_{V^{\gamma_i}}$. Then the fundamental matrix factorization is obtained by taking its push-forward with respect to the morphism $(p, Z) : X \to \mathcal{S} \times \prod_i V^{\gamma_i}$, where $p : X \to \mathcal{S}$ is the projection. Note that here the space X is non-canonical, so one has to check independence on the choices made.

The main idea of the present paper is to change the conceptual framework slightly by observing that in fact one gets a *dg-matrix factorization* on a *dg-scheme* over $\mathcal{S} \times \prod_i V^{\gamma_i}$ (the terminology is explained in Sect. 1). Namely, for a non-negatively graded complex of vector bundles $C^\bullet$ over $\mathcal{S}$, one can define the corresponding dg-scheme over $\mathcal{S}$,

$$[C^\bullet] := \operatorname{Spec}(S^\bullet(C^\bullet)^\vee).$$

In our case we consider the dg-scheme

$$\mathcal{X} := [R\pi_*(\mathcal{V})].$$

More concretely, if we realize $\mathcal{V}$ by a 2-term complex $\mathcal{V} = [A \to B]$ then our dg-scheme is realized by the sheaf of dg-algebras

$$\mathcal{O}_{\mathcal{X},[A \to B]} := S^\bullet[B^\vee \to A^\vee],$$

where the complex $[B^\vee \to A^\vee]$ is concentrated in degrees -1 and 0. Then we interpret the additional structure on $R\pi_*(\mathcal{V})$ coming from the universal Γ-spin structure as a structure of a dg-matrix factorization on the structure sheaf of $\mathcal{X}$. More precisely, we get a morphism

$$Z_{\mathcal{X}} : \mathcal{X} \to \prod_i V^{\gamma_i}$$

and a function of degree -1, $f_{-1} \in \mathcal{O}_{\mathcal{X},[A\to B]}^{-1}$, such that

$$d(f_{-1}) = -Z_{\mathcal{X}}^* \left(\sum W_i \right).$$

Now the fundamental matrix factorization is obtained as the push-forward of $(\mathcal{O}_{\mathcal{X}}, d + f_{-1} \cdot \mathrm{id})$ with respect to the morphism $\mathcal{X} \to \mathcal{S} \times \prod_i V^{\gamma_i}$.

The connection with the original approach is the following: for each presentation $\mathcal{V} = [A \to B]$, for which the first construction works, there is a morphism $q : \mathcal{X} \to X = \mathrm{tot}(A)$, such that $Z \circ q = Z_{\mathcal{X}}$, and an isomorphism of the push-forward $q_*(\mathcal{O}_{\mathcal{X}}, d + f_{-1} \cdot \mathrm{id})$ with the Koszul matrix factorization of $Z^*(\sum W_i)$ constructed through the first approach.

The second technical improvement we present is in the construction of f_{-1}. The idea is to work systematically with the categories of sheaves over pairs (scheme, closed subscheme) to deal with non-functoriality of the cone construction (such categories fit into the framework of Lunts's poset schemes in Lunts (2012)) . Namely, we work with the enhancement of the usual push-forward with respect to the projection $\pi : \mathcal{C} \to \mathcal{S}$ to a morphism of pairs $(\mathcal{C}, \Sigma) \to (\mathcal{S}, \mathcal{S})$, where $\Sigma \subset \mathcal{C}$ is the union of the images of the universal marked points (see Sect. 2).

Recall that in Polishchuk and Vaintrob (2016), we used the fundamental matrix factorizations to construct cohomological field theories associated with (W, G) by viewing them as kernels for Fourier-Mukai functors and passing to Hochschild homology. It seems that the approach via dg-matrix factorizations presented here could also be useful in the development of a more general construction in Gauged Linear Sigma Model, see Fan et al. (2018); Ciocan-Fontanine et al. (2018).

Throughout this work the ground field is $\mathbb{C}$.

1 Matrix Factorizations Over dg-Schemes

1.1 Definition

We consider dg-schemes in the spirit of Ciocan-Fontanine and Kapranov (2009). We fix a space S (a scheme or a stack), and consider the structure sheaf of a dg-scheme over S to be a sheaf $(\mathcal{O}_X^\bullet, d)$ of $\mathbb{Z}_-$-graded commutative dg-algebras over $\mathcal{O}_S$ (one can make a restriction $\mathcal{O}_X^0 = \mathcal{O}_S$, but it is not really necessary).

Given a function $f_0 \in \mathcal{O}_X^0$ we can consider the category of (quasicoherent) *dg-matrix factorizations* of f_0. By definition, these are $\mathbb{Z}/2$-graded complexes of sheaves $P = P^{\overline{0}} \oplus P^{\overline{1}}$ together with a (quasicoherent) $\mathcal{O}_X^\sharp$-module structure, such that $\mathcal{O}_X^i \cdot P^{\overline{a}} \subset P^{\overline{i+a}}$. In addition P is equipped with an odd differential δ satisfying the Leibnitz identity

$$\delta(\phi \cdot p) = d(\phi) \cdot p + (-1)^k \phi \delta(p),$$

for $\phi \in \mathcal{O}_X^k$, $p \in P$, and the equation $\delta^2 = f_0 \cdot \mathrm{id}_P$.

Example 1.1.1 Given an element $f_{-1} \in \mathcal{O}_X^{-1}$, such that $d(f_{-1}) = f_0$, we get a structure of a dg-matrix factorization on $\mathcal{O}_X^\bullet$ by setting

$$\delta(\phi) = d(\phi) + f_{-1} \cdot \phi.$$

(In checking that $\delta^2 = 0$ one has to use the fact that $f_{-1}^2 = 0$.)

The above example can be obtained from the following more general operation. Suppose we are given a function $f_0 \in \mathcal{O}_X^0$ and a dg-matrix factorization (P, δ) of f_0. Then for any $f_{-1} \in \mathcal{O}_X^0$ we can change the differential δ to $\delta + f_{-1} \cdot \mathrm{id}_P$. Then $(P, \delta + f_{-1} \cdot \mathrm{id}_P)$ will be a dg-matrix factorization of $f_0 + d(f_{-1})$.

1.2 Positselski's Framework of Quasicoherent CDG-Algebras

More generally, we can assume that f_0 a section in $\mathcal{O}_X^0 \otimes L$, where L is a locally free $\mathcal{O}_X^0$-module of rank 1. The theory of the corresponding categories of dg-matrix factorizations fits into the framework of quasicoherent CDG-algebras developed by Positselski (see [Efimov and Positselski 2015, Sect. 1]].

With the data $(\mathcal{O}_X^\bullet, L, f_0)$ as above we can associate a quasicoherent CDG-algebra

$$\mathcal{B} := \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X^\bullet \otimes_{\mathcal{O}_X^0} L^{\otimes n}[-2n],$$

with the natural structure of a complex of sheaves (i.e., the $\mathbb{Z}$-grading and the differential d), the natural product and the global curvature element given by $f_0 \in \mathcal{O}^0 \otimes L \subset \mathcal{B}_2$.

Now a quasicoherent dg-matrix factorization is a quasicoherent DG-module over $\mathcal{B}$, i.e., a graded $\mathcal{B}$-module $M = \bigoplus_n M_n$, equipped with a differential $\delta = \delta_M$ such that $\delta^2 = f_0 \cdot \mathrm{id}_M$ and δ satisfies the Leibnitz identity with respect to the $\mathcal{B}$-action. Note that such a DG-module necessarily has

$$M_{n+2} \simeq M_n \otimes L,$$

so it is determined by the components M_0 and M_1, and we get the structure of a dg-matrix factorization on $M_0 \oplus M_1$.

There are several exotic derived categories associated to a quasicoherent CDG-algebra. The one that is most relevant for the theory of dg-matrix factorizations is the category

$$\mathrm{qcoh} - \mathrm{MF}_{ffd}(f_0) := D^{co}(\mathcal{B} - \mathrm{qcoh}_{ffd}) \simeq D^{co}(\mathcal{B} - \mathrm{qcoh}_{fl}) \simeq D^{abs}(\mathcal{B} - \mathrm{qcoh}_{fl}),$$

where the superscripts "abs" and "co" refer to "absolute" and "coderived", while the subscripts "fl" and "ffd" mean "flat" and "finite flat dimension" (see [Efimov and Positselski 2015, Sect. 1]).

Assume that $f : (X, \mathcal{O}_X^\bullet) \to (Y, \mathcal{O}_Y^\bullet)$ is a morphism of finite flat dimension, L is a locally free $\mathcal{O}_Y^0$-module of rank 1, W_0 is a section of L. Then we have the induced

section f^*W_0 of f^*L. In this situation we have the push-forward functor (see [Efimov and Positselski 2015, Prop. 1.9])

$$Rf_* : \mathrm{qcoh} - \mathrm{MF}_{ffd}(f^*W_0) \to \mathrm{qcoh} - \mathrm{MF}_{ffd}(W_0).$$

1.3 Koszul Matrix Factorizations as Push-Forwards

Let V be a vector bundle over a scheme X, and suppose we have sections $\alpha \in H^0(X, V^\vee)$, $\beta \in H^0(X, V)$. With these data one associates a Koszul matrix factorization $\{\alpha, \beta\}$ of $W = \langle \alpha, \beta \rangle$, whose underlying super-vector bundle is $\bigwedge^\bullet(V)$. On the other hand, we have the derived zero locus of β, $\mathcal{Z}(\beta) \to X$, which corresponds to the dg-algebra given by the Koszul complex of β:

$$\mathcal{O}_{\mathcal{Z}(\beta)} = \left(\bigwedge{}^\bullet(V), d = \iota_\beta \right).$$

Now we can view α as a function of degree -1 on $\mathcal{Z}(\beta)$ such that $d(\alpha)$ is the pull-back of W. Thus, by definition, $\{\alpha, \beta\}$ is the push-forward of the dg-matrix factorization $(\mathcal{O}_{\mathcal{Z}(\beta)}, d + \alpha \cdot \mathrm{id})$ by the morphism $\mathcal{Z}(\beta) \to X$.

This explains why in the case when β is a regular section of V, the Koszul matrix factorization $\{\alpha, \beta\}$ is equivalent to the push-forward of the structure sheaf on the usual zero locus of β.

2 Trace Maps Via Morphisms of Pairs

2.1 Sheaves on Pairs

Let $\iota : Y \to X$ be a closed embedding.

We consider a very simple poset scheme in the sense of Lunts (2012) for the poset consisting of two elements $\alpha > \beta$, so that $X_\alpha = Y$ and $X_\beta = X$. Then a quasicoherent sheaf on this poset scheme is a triple $(\mathcal{F}_\alpha, \mathcal{F}_\beta, \phi)$, with $\mathcal{F}_\alpha \in \mathrm{Qcoh}(Y)$, $\mathcal{F}_\beta \in \mathrm{Qcoh}(X)$ and $\phi : \mathcal{F}_\beta \to \iota_* \mathcal{F}_\alpha$ is a morphism. We denote by $\mathrm{Qcoh}(X, Y)$ this abelian category, and by $\mathrm{Coh}(X, Y)$ its subcategory corresponding to $\mathcal{F}_\alpha \in \mathrm{Coh}(Y)$, $\mathcal{F}_\beta \in \mathrm{Coh}(X)$. Furthermore, we have a subcategory of locally free coherent sheaves (those with $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ locally free).

The perfect derived category $\mathrm{Perf}(X, Y)$ of bounded complexes of locally free sheaves on (X, Y) has a natural monoidal structure given by the tensor product, so we can also define symmetric powers of objects in $\mathrm{Perf}(X, Y)$.

Given a morphism of pairs $f : (X, Y) \to (X', Y')$ we have a natural derived push-forward morphism

$$Rf_* : D^+ \mathrm{Qcoh}(X, Y) \to D^+ \mathrm{Qcoh}(X', Y'),$$

where D^+ denotes the derived category of bounded below complexes.

The push-forward is compatible with the tensor products in the usual way: we have natural morphisms

$$Rf_*(F) \otimes Rf_*(G) \to Rf_*(F \otimes G), \quad S^\bullet Rf_*(F) \to Rf_* S^\bullet(F). \quad (2.1.1)$$

We have a fully faithful exact embedding $j_! : D\,\mathrm{Qcoh}(X) \to D\,\mathrm{Qcoh}(X, Y)$ sending $\mathcal{G}$ to $\mathcal{F}_\alpha = 0$, $\mathcal{F}_\beta = \mathcal{G}$. There is a right adjoint functor to it (see Lunts 2012),

$$Rj^! : D^+\,\mathrm{Qcoh}(X, Y) \to D^+\,\mathrm{Qcoh}(X),$$

which is defined as the right derived functor of the functor

$$j^! : \mathrm{Qcoh}(X, Y) \to \mathrm{Qcoh}(X) : \mathcal{F}_\bullet \mapsto \ker(\mathcal{F}_\beta \to \iota_* \mathcal{F}_\alpha).$$

Note that objects $\mathcal{F}_\bullet \in \mathrm{Qcoh}(X, Y)$, such that $\mathcal{F}_\beta \to \iota_* \mathcal{F}_\alpha$ is surjective, are acyclic with respect to $j^!$. Furthermore, every object of $\mathrm{Qcoh}(X, Y)$ has a canonical resolutions by such acyclic objects:

$$0 \to (\mathcal{F}_\alpha, \mathcal{F}_\beta) \to (\mathcal{F}_\alpha, \mathcal{F}_\beta \oplus \iota_* \mathcal{F}_\alpha) \to (0, \iota_* \mathcal{F}_\alpha) \to 0$$

Computing $Rj^!$ using these resolutions has a very simple interpretation: given a complex $(\mathcal{F}_\alpha^\bullet, \mathcal{F}_\beta^\bullet)$ over $\mathrm{Qcoh}(X)$, the functor $Rj^!$ sends it to the complex

$$\mathrm{Cone}(\mathcal{F}_\beta^\bullet \to \iota_* \mathcal{F}_\alpha^\bullet)[-1].$$

In particular, there is a natural exact triangle

$$Rj^!(\mathcal{F}_\alpha^\bullet, \mathcal{F}_\beta^\bullet) \to \mathcal{F}_\beta^\bullet \to \iota_* \mathcal{F}_\alpha^\bullet \to \cdots$$

We also have the following compatibility between $Rj^!$ and the push-forward.

Lemma 2.1.1 *Let $f : (X, Y) \to (X', Y')$ be a morphism of pairs. Assume that there exists a finite open covering of X, affine over X'. Then for $\mathcal{F} \in D^+\,\mathrm{Qcoh}(X, Y)$ we have a natural isomorphism*

$$Rj^! Rf_*(\mathcal{F}) \simeq Rf_* Rj^!(\mathcal{F}) \quad (2.1.2)$$

in $D^+\,\mathrm{Qcoh}(X')$.

Proof Let us choose a quasi-isomorphism $\mathcal{F} \to \widetilde{\mathcal{F}}$, such that all $\widetilde{\mathcal{F}}_\alpha^i$ and $\widetilde{\mathcal{F}}_\beta^i$ are f_*-acyclic (this can be done using Cech resolutions). Then the left-hand side of (2.1.2) is represented by the complex

$$\mathrm{Cone}(f_* \widetilde{\mathcal{F}}_\beta \to \iota_* f_* \widetilde{\mathcal{F}}_\alpha)[-1].$$

On the other hand, the terms of $\mathrm{Cone}(\widetilde{\mathcal{F}}_\beta \to \iota_* \widetilde{\mathcal{F}}_\alpha)[-1]$ are also f_*-acyclic, so the right-hand side of (2.1.2) is represented by the complex

$$f_* \mathrm{Cone}(\widetilde{\mathcal{F}}_\beta \to \iota_* \widetilde{\mathcal{F}}_\alpha)[-1],$$

which is isomorphic to the one above. □

2.2 Differentials on Curves

Let $\pi : \mathcal{C} \to \mathcal{S}$ be a family of stable curves, $p_i : \mathcal{S} \to \mathcal{C}, i = 1, \ldots, r$, be sections of π, such that π is smooth along their images, and let $\Sigma = \sqcup_i p_i(\mathcal{S})$. We view $(\mathcal{C}, \Sigma)$ as a poset scheme and consider the corresponding category $\mathrm{Coh}(\mathcal{C}, \Sigma)$ whose objects are collections $(F, (F_i), (f_i))$, where F is a coherent sheaf on $\mathcal{C}$, F_i is a coherent sheaf on $\mathcal{S}$ and $f_i : F \to p_{i*}F_i$ is a morphism. Sometimes we will omit the morphisms (f_i) from the notation and just write $(F, (F_i))$.

Set $\omega_{\mathcal{C}/\mathcal{S}}^{\log} = \omega_{\mathcal{C}/\mathcal{S}}(\Sigma)$. Recall that we have natural residue maps

$$\mathrm{Res}_\Sigma : \omega_{\mathcal{C}/\mathcal{S}}^{\log}|_\Sigma \xrightarrow{\sim} \mathcal{O}_\Sigma,$$

so that $\ker(\mathrm{Res}_\Sigma)$ is identified with $\omega_{\mathcal{C}/\mathcal{S}}$. Thus, we can view the triple

$$\left[\omega_{\mathcal{C}/\mathcal{S}}^{\log}, \Sigma \right] := \left(\omega_{\mathcal{C}/\mathcal{S}}^{\log}, \mathcal{O}_\Sigma, \mathrm{Res}_\Sigma \right)$$

as an object of the category $\mathrm{Coh}(\mathcal{C}, \Sigma)$. Furthermore, we have

$$Rj^! \left[\omega_{\mathcal{C}/\mathcal{S}}^{\log}, \Sigma \right] \simeq \omega_{\mathcal{C}/\mathcal{S}}.$$

Note that we have a morphism of pairs

$$\pi : (\mathcal{C}, \Sigma) \to (\mathcal{S}, \mathcal{S}). \tag{2.2.1}$$

By Lemma 2.1.1, the object $R\pi_*[\omega_{\mathcal{C}/\mathcal{S}}^{\log}, \Sigma]$ satisfies

$$Rj^! R\pi_* \left[\omega_{\mathcal{C}/\mathcal{S}}^{\log}, \Sigma \right] \simeq R\pi_* \omega_{\mathcal{C}/\mathcal{S}}. \tag{2.2.2}$$

Note also that we have a morphism of exact triangles (which will be used later)

$$
\begin{array}{ccccccc}
\oplus_{i=1}^r \mathcal{O}_\mathcal{S}[-1] & \to & R\pi_*(\omega_{\mathcal{C}/\mathcal{S}}) & \to & R\pi_*(\omega_{\mathcal{C}/\mathcal{S}}^{\log}) & \to & \oplus_{i=1}^r \mathcal{O}_\mathcal{S} \\
\downarrow & & \downarrow & & \downarrow & & t\downarrow \\
\mathcal{O}_\mathcal{S}[-1] & \xrightarrow{\mathrm{id}} & \mathcal{O}_\mathcal{S}[-1] & \longrightarrow & 0 & \longrightarrow & \mathcal{O}_\mathcal{S}
\end{array}
\tag{2.2.3}
$$

where t is given by the summation.

The above constructions also work in the case of a family of orbicurves with stable coarse moduli spaces.

3 Fundamental Matrix Factorization

3.1 Setup and the Moduli Spaces of Γ-Spin Structures

Let us recall the setup of the FJRW theory (see Fan et al. 2013; Polishchuk and Vaintrob 2016), or rather its slight generalization to noncommutative finite groups of symmetries [as in Fan et al. (2018)].

We start with a finite-dimensional vector space V equipped with an effective $\mathbb{G}_m$-action called the *R-charge*, such that all the weights of this action on V are positive. We denote the corresponding subgroup in $\mathrm{GL}(V)$ by $\mathbb{G}_{m,R}$. and let W be a function of weight d on V. Also, we fix a finite subgroup $G \subset \mathrm{GL}(V)$ such that W is G-invariant, G commutes with $\mathbb{G}_{m,R}$ and G contains a fixed element $J \in \mathbb{G}_{m,R}$ of order d.

We define $\Gamma \subset \mathrm{GL}(V)$ to be the algebraic subgroup generated by G and by $\mathbb{G}_{m,R}$. There is a canonical exact sequence

$$1 \to G \to \Gamma \xrightarrow{\chi} \mathbb{G}_m \to 1,$$

where χ restricts to the subgroup $\mathbb{G}_{m,R}$ as $\lambda \mapsto \lambda^d$.

As in Polishchuk and Vaintrob (2016), we consider the moduli space of Γ-spin structures: it classifies stable orbicurves $(C, p_1, \ldots, p_n)$ equipped with Γ-principal bundle P (our convention is that we have a right action of Γ on P), together with an isomorphism $\chi_* P \xrightarrow{\sim} \omega_C^{\log} \setminus 0$. We can think of the latter isomorphism as a morphism $\chi_P : P \to \omega_C^{\log} \setminus 0$ satisfying

$$\chi_P(x\gamma) = \chi(\gamma) \cdot \chi_P(x)$$

for $\gamma \in \Gamma$.

In addition to requiring the coarse moduli of C to be Deligne-Mumford stable, we require that for each marked point p_i the morphism $B\mathrm{Aut}(p_i) \to B\Gamma$ induced by P is representable. By looking at the corresponding embedding $\mathrm{Aut}(p_i) \simeq \mathbb{Z}/m_i \to \Gamma$ defined up to a conjugacy, we get a conjugacy class γ_i in Γ. Thus, we get a decomposition of our moduli stack into a disjoint union of open and closed substacks $\mathcal{S}_g(\gamma_1, \ldots, \gamma_n)$. As in [Fan et al. 2013, Sect. 2.2], one shows that these are smooth and proper DM stacks with projective coarse moduli.

Let $\pi : \mathcal{C} \to \mathcal{S}_g(\gamma_1, \ldots, \gamma_n)$ be the universal curve over $\mathcal{S}_g(\gamma_1, \ldots, \gamma_n)$, and let $\mathcal{V} = \mathcal{P} \times_\Gamma V$ be the vector bundle over $\mathcal{C}$ associated with the universal Γ-spin structure $\mathcal{P}$ via the embedding $\Gamma \subset \mathrm{GL}(V)$. Note that $\mathcal{V}$ is equipped with a $\mathbb{G}_{m,R}$-action (through its action on V).

As in Polishchuk and Vaintrob (2016), we also consider a Galois covering $\mathcal{S}_g^{\mathrm{rig}}(\gamma_1, \ldots, \gamma_n) \to \mathcal{S}_g(\gamma_1, \ldots, \gamma_n)$ corresponding to choices of a rigidification at every marked point. A *rigidification* is an isomorphism of the restriction of P to

$p_i / \operatorname{Aut}(p_i) \simeq B\langle \gamma_i \rangle$ with $\Gamma / \langle \gamma_i \rangle$ (viewed as a bundle over $B\langle \gamma_i \rangle$). There is a natural simply transitive action of the group $\prod_i C_G(\gamma_i)$ on the set of rigidifications at $p_1, \ldots, p_n$, where $C_G(\gamma) \subset G$ is the centralizer of $\gamma \in G$.

3.2 Construction

Let us set for now $\mathcal{S} = \mathcal{S}_g^{\mathrm{rig}}(\gamma_1, \ldots, \gamma_n)$ and consider the pull-back of all the objects to $\mathcal{S}$ (denoting them by the same symbols).

Note that we have a natural projection $V/\langle \gamma_i \rangle \to V^{\gamma_i}$. Thus, from rigidification structures we get morphisms

$$Z_i : p_i^* \mathcal{V} \to V^{\gamma_i} \otimes \mathcal{O}_{\mathcal{S}}. \tag{3.2.1}$$

Hence, by adjunction we can extend $\mathcal{V}$ to an object

$$[\mathcal{V}, \Sigma] := (\mathcal{V}, (V^{\gamma_i} \otimes \mathcal{O}_{\mathcal{S}}), (Z_i))$$

of $\operatorname{Coh}(\mathcal{C}, \Sigma)$.

On the other hand, we can combine χ_P with W into a polynomial morphism

$$W_{\mathcal{V}} : \mathcal{V} = \mathcal{P} \times_\Gamma V \to \omega_{\mathcal{C}/\mathcal{S}}^{\log} : (x, v) \mapsto W(v) \cdot \chi_P(x).$$

We can view it as a linear morphism of vector bundles on $\mathcal{C}$,

$$W_{\mathcal{V}} : S^\bullet(\mathcal{V})_d \to \omega_{\mathcal{C}/\mathcal{S}}^{\log},$$

where we grade the symmetric algebra of $\mathcal{V}$ using the $\mathbb{G}_{m,R}$-action on $\mathcal{V}$. Furthermore, this morphism is compatible with the morphisms (3.2.1), so that the following diagram is commutative

$$
\begin{array}{ccc}
p_i^* S^\bullet(\mathcal{V})_d & \xrightarrow{\ p_i^* W_{\mathcal{V}}\ } & p_i^* \omega_{\mathcal{C}/\mathcal{S}}^{\log} \\
{\scriptstyle S^\bullet(Z_i)}\Big\downarrow & & \Big\downarrow \\
S^\bullet(V^{\gamma_i})_d \otimes \mathcal{O}_{\mathcal{S}} & \xrightarrow{\ W_i\ } & \mathcal{O}_{\mathcal{S}}
\end{array}
$$

where $W_i = W|_{V^{\gamma_i}}$. This means that we have a morphism

$$(W_{\mathcal{V}}, (W_i)) : S^\bullet[\mathcal{V}, \Sigma]_d \to \left[\omega_{\mathcal{C}/\mathcal{S}}^{\log}, \Sigma \right] \tag{3.2.2}$$

in the category $\operatorname{Qcoh}(\mathcal{C}, \Sigma)$ (where again we take the part of weight d with respect to $\mathbb{G}_{m,R}$). Next, we can take the derived push-forward with respect to the morphism of pairs (2.2.1). Together with (2.1.1) this gives us a morphism

$$S^\bullet(R\pi_*[\mathcal{V}, \Sigma])_d \to R\pi_* S^\bullet[\mathcal{V}, \Sigma]_d \to R\pi_* \left[\omega_{\mathcal{C}/\mathcal{S}}^{\log}, \Sigma\right] \tag{3.2.3}$$

in $D\operatorname{Qcoh}(\mathcal{S}, \mathcal{S})$.

Now let us set

$$E := Rj^! S^\bullet(R\pi_*[\mathcal{V}, \Sigma])_d.$$

Applying $Rj^!$ to morphism (3.2.3), we obtain a morphism

$$E = Rj^! S^\bullet(R\pi_*[\mathcal{V}, \Sigma])_d \to Rj^! R\pi_*[\omega_{\mathcal{C}/\mathcal{S}}^{\log}, \Sigma] \simeq R\pi_* \omega_{\mathcal{C}/\mathcal{S}},$$

where the last isomorphism is (2.2.2). It is easy to see that it fits into a morphism of exact triangles

$$
\begin{array}{ccccccc}
E & \longrightarrow & S^\bullet(R\pi_*(\mathcal{V}))_d & \to & \oplus_{i=1}^r S^\bullet(V^{\gamma_i})_d \otimes \mathcal{O}_\mathcal{S} & \longrightarrow & E[1] \\
\downarrow & & \downarrow & & (W_i)\downarrow & & \downarrow \\
R\pi_*(\omega_{\mathcal{C}/\mathcal{S}}) & \longrightarrow & R\pi_*(\omega_{\mathcal{C}/\mathcal{S}}^{\log}) & \longrightarrow & \oplus_{i=1}^r \mathcal{O}_\mathcal{S} & \longrightarrow & R\pi_*(\omega_{\mathcal{C}/\mathcal{S}})[1]
\end{array}
\tag{3.2.4}
$$

Combining it with the morphism of triangles (2.2.3), we get a commutative diagram with the exact triangle in the first row

$$
\begin{array}{ccccc}
S^\bullet(R\pi_*(\mathcal{V}))_d & \to & \oplus_{i=1}^r S^\bullet(V^{\gamma_i})_d \otimes \mathcal{O}_\mathcal{S} & \longrightarrow & E[1] \\
& & \sum W_i \downarrow & & \tau \downarrow \\
& & \mathcal{O}_\mathcal{S} & \xrightarrow{\text{id}} & \mathcal{O}_\mathcal{S}
\end{array}
$$

Dualizing we get a commutative diagram

$$
\begin{array}{ccccc}
E^\vee[-1] & \to & \oplus_{i=1}^r S^\bullet(V^{\gamma_i})_d^\vee \otimes \mathcal{O}_\mathcal{S} & \to & S^\bullet(R\pi_*(\mathcal{V}))_d^\vee \\
\tau^\vee \uparrow & & \sum W_i \uparrow & & \\
\mathcal{O}_\mathcal{S} & \xrightarrow{\text{id}} & \mathcal{O}_\mathcal{S} & &
\end{array}
$$

This implies that the pull-back $Z^*(\bigoplus_i W_i)$ with respect to the morphism

$$Z : [R\pi_*(\mathcal{V})] \to \prod_i V^{\gamma_i} \tag{3.2.5}$$

induced by (3.2.1), gives the zero morphism from the structure sheaf to itself in the derived category of quasicoherent sheaves on $[R\pi_*(\mathcal{V})]$.

In fact, we can realize this functon by an explicit coboundary. For this we need a realization of the above diagram in the homotopy category of complexes. As in [Polishchuk and Vaintrob 2016, Sect. 4.2], the starting point is that $R\pi_*(\mathcal{V})$ can realized ($\mathbb{G}_{m,R}$-equivariantly) by a complex of the form $[A \to B]$ in such a way that the morphism (3.2.5) is realized by a surjective morphism $A \to \bigoplus_{i=1}^r V^{\gamma_i} \otimes \mathcal{O}_S$. Then the first line of the diagram (3.2.4) can be realized by a short exact sequence of complexes

$$0 \to \ker(S^\bullet(Z)_d) \to S^\bullet(A \to B)_d \xrightarrow{S^\bullet(Z)_d} \bigoplus_{i=1}^r S^\bullet(V^{\gamma_i})_d \otimes \mathcal{O}_S \to 0$$

where the complex $S^\bullet(A \to B)_d$, concentrated in degrees $[0, \mathrm{rk}(B)]$, has form

$$S^\bullet(A)_d \to (S^\bullet(A) \otimes B)_d \to (S^\bullet(A) \otimes \wedge^2 B)_d \to \ldots$$

Using this we get a canonical quasi-isomorphism of E with the bounded complex of vector bundles

$$K^\bullet := \mathrm{Cone}(S^\bullet(R\pi_*(\mathcal{V}))_d \to \oplus_{i=1}^r S^\bullet(V^{\gamma_i})_d \otimes \mathcal{O}_S)[-1].$$

Now we want to realize the morphism $\tau : E \to \mathcal{O}_S[-1]$ in the derived category by a morphism $K^\bullet \to \mathcal{O}_S[-1]$ in the homotopy category of complexes.

By changing $[A \to B]$ to a quasi-isomorphic complex $[\overline{A} \to \overline{B}]$ one can achieve that for $i \geq 1$ the terms K^i satisfy $\mathrm{Ext}^{>0}(K^i, \mathcal{O}_S) = 0$ (see [Polishchuk and Vaintrob 2016, Lem. 4.2.5]). This implies that morphisms $K \to \mathcal{O}_S[-1]$ in the homotopy category of complexes and in the derived category are the same.

The dual of this morphism can be interpreted as a canonical homotopy (up to a homotopy between homotopies) f_{-1} between the function $Z^*(\bigoplus_i W_i)$ on $[R\pi_*\mathcal{V}]$ and 0. As we have seen in Example 1.1.1, this corresponds to a structure $\delta = d - f_{-1} \cdot \mathrm{id}$ of a dg-matrix factorization of $-Z^*(\bigoplus_i W_i)$ on the structure sheaf of $[R\pi_*\mathcal{V}]$.

Furthermore, it carries an equivariant structure with respect to the action of the center $Z(\Gamma)$ of Γ (acting trivially on the base) and with respect to $\prod_i C_G(\gamma_i)$ (changing the rigidifications).

3.3 Properties

The first important property is that our dg-matrix factorization over $[R\pi_*\mathcal{V}]$ is supported on the zero section in $[R\pi_*\mathcal{V}]$. Indeed, first, we recall that any matrix factorization is supported on the critical locus of the potential. Since each W_i is non-degenerate, we get that the support belongs to the zero locus of $Z^*(\bigoplus_i W_i)$. Note also that the support can be calculated pointwise (see [Polishchuk and Vaintrob 2016, Sect. 1.4]), so it is enough to deal with the case of a single curve with a Γ-spin structure. Thus, we are reduced to considering the following situation. Let C be a curve, and let $\mathcal{V}$ be a vector bundle over C. Assume also we have a polynomial morphism $W_\mathcal{V} : \mathcal{V} \to \omega_C$, such that over an open dense subset of C there exists a

trivialization $\mathcal{V} \simeq V \otimes \mathcal{O}_C$ such that $W_{\mathcal{V}}$ is induced by our polynomial W on V. Then we have the induced polynomial function of degree -1 on the dg-affine space $[H^0(C, \mathcal{V}) \oplus H^1(C, \mathcal{V})[-1]]$ (recall that the base is now a point), induced by $W_{\mathcal{V}}$ and by the identification $H^1(C, \omega_C) \simeq \mathbb{C}$. We claim that it is supported at the origin. Indeed, we start by observing that the preimage of the origin under the gradient morphism $\Delta W : V \to V^\vee$ is still the origin (since W is non-degenerate). From this we get the similar assertion about the preimage of the zero section under the relative gradient morphism $\Delta W_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}^\vee \otimes \omega_C$. Finally, we note that the support of our function on $[H^0(C, \mathcal{V}) \oplus H^1(C, \mathcal{V})[-1]]$ coincides with the vanishing locus of the polynomial morphism

$$H^0(C, \mathcal{V}) \to H^0(\mathcal{V}^\vee \otimes \omega_C) \simeq H^1(C, \mathcal{V})^\vee$$

induced by the relative gradient map. This implies our claim.

Next, the key gluing property satisfied by the fundamental matrix factorizations (cf. [Polishchuk and Vaintrob 2016, Sect. 5.2, 5.3]) holds in the situation when we consider two natural families of orbicurves $\widetilde{C} \xrightarrow{\widetilde{\pi}} S$, $C \xrightarrow{\pi} S$, over

$$S := S_{g_1}^{\mathrm{rig}}(\gamma_1, \ldots, \gamma_{n_1}, \gamma) \times S_{g_2}^{\mathrm{rig}}(\gamma_1', \ldots, \gamma_{n_2}', \gamma^{-1}),$$

where C is obtained by gluing two smooth points on $\widetilde{C}$ into a node. We denote by $f : \widetilde{C} \to C$ the gluing morphism.

In this setting there are natural Γ-spin structures $\widetilde{P}$ (resp., P) over $\widetilde{C}$ (resp., C), where P is obtained by gluing fibers of $\widetilde{P}$ over the two points that are glued into a node, using the rigidifications and the square root of J, $J^{1/2} \in \mathbb{G}_{m,R}$ such that $\chi(J^{1/2}) = -1$ (see [Polishchuk and Vaintrob 2016, Sect. 5.2]). The main compatibility between the push-forwards of the corresponding vector bundles $\widetilde{\mathcal{V}}$ and $\mathcal{V}$ is given by the cartesian diagram

$$\begin{array}{ccc} [R\pi_* \mathcal{V}] & \longrightarrow & V^\gamma \\ \downarrow & \Delta^{J^{1/2}} \downarrow & \\ [R\widetilde{\pi}_* \widetilde{\mathcal{V}}] & \rightarrow & V^\gamma \times V^{\gamma^{-1}} \end{array}$$

where $\Delta^{J^{1/2}} : V^\gamma \to V^\gamma \times V^{\gamma^{-1}}$ is the twisted diagonal map: $x \mapsto (x, J^{1/2}x)$. Furthermore, the analysis of [Polishchuk and Vaintrob 2016, Sect. 5.2] shows that the natural dg-matrix factorization on $[R\pi_* \mathcal{V}]$ is identified with the pull-back of the one on $[R\widetilde{\pi}_* \widetilde{\mathcal{V}}]$.

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