



On Algorithms that Effectively Distinguish Gradient-Like Dynamics on Surfaces

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Abstract

In the present paper we survey existing graph invariants for gradient-like flows on surfaces up to the topological equivalence and develop effective algorithms for their distinction (let us recall that a flow given on a surface is called a *gradient-like flow* if its non-wandering set consists of a finite set of hyperbolic fixed points, and there is no trajectories connecting saddle points). Additionally, we construct a parametrized algorithm for the Fleitas's invariant, which will be of linear time, when the number of sources is fixed. Finally, we prove that the classes of topological equivalence and topological conjugacy are coincide for gradient-like flows, so, all the proposed invariants and distinguishing algorithms works also for topological classification, taking in sense time of moving along trajectories. So, as the main result of this paper we have got multiple ways to recognize equivalence and conjugacy class of arbitrary gradient-like flow on a closed surface in a polynomial time.

Keywords Gradient-like dynamics · Effective distinguishing algorithms · Graph invariants · Surface

1 Introduction

It is well-known that the Morse functions exist on any manifolds and, hence, there exist gradient flows as well. Generically, they are structurally stable and a dynamics of such systems is a base for the class of *gradient-like flows*, i.e. flows, whose non-wandering

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set consists of a finite number of hyperbolic fixed points, whose invariant manifolds cross transversally.

Such flows are used everywhere, when modelling regular processes in different natural sciences is used (see, for example, Pesin and Yurchenko 2004). Hence, it is very important to be able to compare dynamics of such models without depending on a nature of their appearing. Meanwhile, depending on research goals, it is important both a qualitative behaviour of a system, i.e. partition into trajectories, and time of moving along trajectories. In dynamical systems theory, the relation, preserving partition into trajectories up to homeomorphism, is called the *topological equivalence* and the relation, preserving in addition time of moving along trajectories, is called the *topological conjugacy*. A finding of invariants uniquely determining an equivalence class for a system is called the *topological classification*.

The finiteness of the set of non-wandering orbits of a gradient-like flow leads to invariants that goes back to the classical works of Andronov and Pontryagin (1937), Leontovich and Mayer (1937, 1955). In the present paper we survey existing graph invariants for gradient-like flows on surfaces up to the topological equivalence and develop effective algorithms¹ (i.e. polynomial-time algorithms) for their distinction. Additionally, we construct a parametrized algorithm for the Fleitas's invariant, which will be of linear time, when the number of sources is fixed. Finally, we prove that the classes of topological equivalence and topological conjugacy are coincide for gradient-like flows. So, all the proposed invariants and distinguishing algorithms work also for the topological conjugacy.

2 Supporting Concepts and Facts

In this part we recall some definitions and facts to effectively distinguish graphs appearing below as invariants for gradient-like flows.

2.1 On Gradient-Like Flows on Surfaces

A C^1 -smooth flow on a closed surface S is a family of diffeomorphisms $f^t : S \rightarrow S$, smoothly depended on $t \in \mathbb{R}$, such that $f^0(x) = x$ and $f^t(f^s(x)) = f^{t+s}(x)$, for any $x \in S$, $t, s \in \mathbb{R}$. Everywhere below f^t is a smooth flow on a closed surface S .

For every point $x \in S$ the set $\mathcal{O}_x = \{f^t(x), t \in \mathbb{R}\}$ is called an *orbit of x* . The flow orbit is homeomorphic to either a point, or a circle, or a line; and is called either a *fixed point*, or a *periodic orbit*, or a *non-singular trajectory*, respectively. Each orbit, other than a fixed point, is oriented in the direction of increasing time t .

Two flows $f^t : S \rightarrow S$, $f'^t : S' \rightarrow S'$ are called *topologically equivalent* if there is a homeomorphism $h : S \rightarrow S'$ sending the trajectories of f^t to trajectories of f'^t and preserving the orientation of them. If, in addition, the homeomorphism h has a

¹ The notion of an effectively solvable problem rises to Cobham (1964), who asserts that a computational problem can be feasibly computed on some device only if it can be computed in time bounded by a polynomial on a parameter representing the length of input data. The complexity status of the general graph isomorphism problem is not proved to be polynomial or not. Graphs in the criteria are not graphs of the general type, they possess peculiar combinatorial properties.

property $hf^t = hf^{t'}$ for every $t \in \mathbb{R}$, then the flows are said to be *topologically conjugate*. An *invariant* of some class of flows is a properties which is preserved up to the topological equivalence or conjugacy.

If the eigenvalues in a fixed point p have a non-zero real part, then p is called a *hyperbolic fixed point*. The hyperbolicity of the fixed point is expressed by the fact that the point have a *stable manifold* W_p^s and an *unstable manifold* W_p^u as following

$$W_p^s = \{x \in S \mid f^t(x) \rightarrow p, t \rightarrow \infty\}, \quad W_p^u = \{x \in S \mid f^t(x) \rightarrow p, t \rightarrow -\infty\},$$

respectively. Besides, $\dim W_p^u$ is equal to the number of eigenvalues with positive real parts. If $\dim W_p^u = 0$, then p is called a *sink*, if $\dim W_p^s = 1$, then p is called a *saddle point*, if $\dim W_p^s = 2$, then p is called a *source*. A connected component l_p^u, l_p^s of $W_p^u \setminus \{p\}, W_p^s \setminus \{p\}$ is called unstable, stable separatrix of p respectively.

Proposition 2.1 (Palis and De Melo 1982, Ch. 2, Theorem 4.10; Kruglov 2018, Lemma 1; Robinson 1995, Ch. 4, Theorem 7.1) *A flow $f^t : S \rightarrow S$ in some neighbourhood of a hyperbolic fixed point p is topologically conjugate to one of the following linear flows*

$$\begin{aligned} a^t(x, y) &= (2^{-t}x, 2^{-t}y), \\ b^t(x, y) &= (2^t x, 2^{-t}y), \\ c^t(x, y) &= (2^t x, 2^t y), \end{aligned}$$

if p is a sink, saddle or source, respectively (see Fig. 1).

A point $p \in S$ is called a *wandering point* of a flow f^t , if there is an open neighbourhood U_p of p , such that $f^t(U_p) \cap U_p = \emptyset$, for all $|t| > 1$. Otherwise p is called a *non-wandering point*. The set of all the non-wandering points of f^t is called the *non-wandering set*, and it is denoted by Ω_{f^t} .

A flow $f^t : S \rightarrow S$ is a *gradient-like* if its non-wandering set consists of a finite set of hyperbolic fixed points, and there are no trajectories connecting saddle points². Everywhere below f^t is a gradient-like flow on S with at least one saddle point³, $\Omega_{f^t}^0, \Omega_{f^t}^1, \Omega_{f^t}^2$ are the sets of all sinks, saddles, and sources of f^t , respectively.

Proposition 2.2 (Grines et al. 2016, Theorem 2.1.1 and Corollary 2.2) *For every gradient-like flow f^t , we have the following properties:*

1. $S = \bigcup_{p \in \Omega_{f^t}} W_p^u = \bigcup_{p \in \Omega_{f^t}} W_p^s$;

² If the non-wandering set includes also a finite number of hyperbolic periodic orbits, then the flow is called a *Morse-Smale flow* and topological classification of such flows on surfaces has been obtained, for example, in Peixoto (1973) and Oshemkov and Sharko (1998). If a flow allows also trajectories connecting saddle points, then such flow is called an Ω -stable flow; classification of Ω -stable flows without limit cycles on surfaces has been obtained in Kruglov et al. (2018a); classification of all Ω -stable flows on surfaces has been obtained in Kruglov et al. (2018b).

³ All gradient-like flows without saddle points are given on a sphere and have only two fixed points: source and sink. All such flows are topologically conjugate.

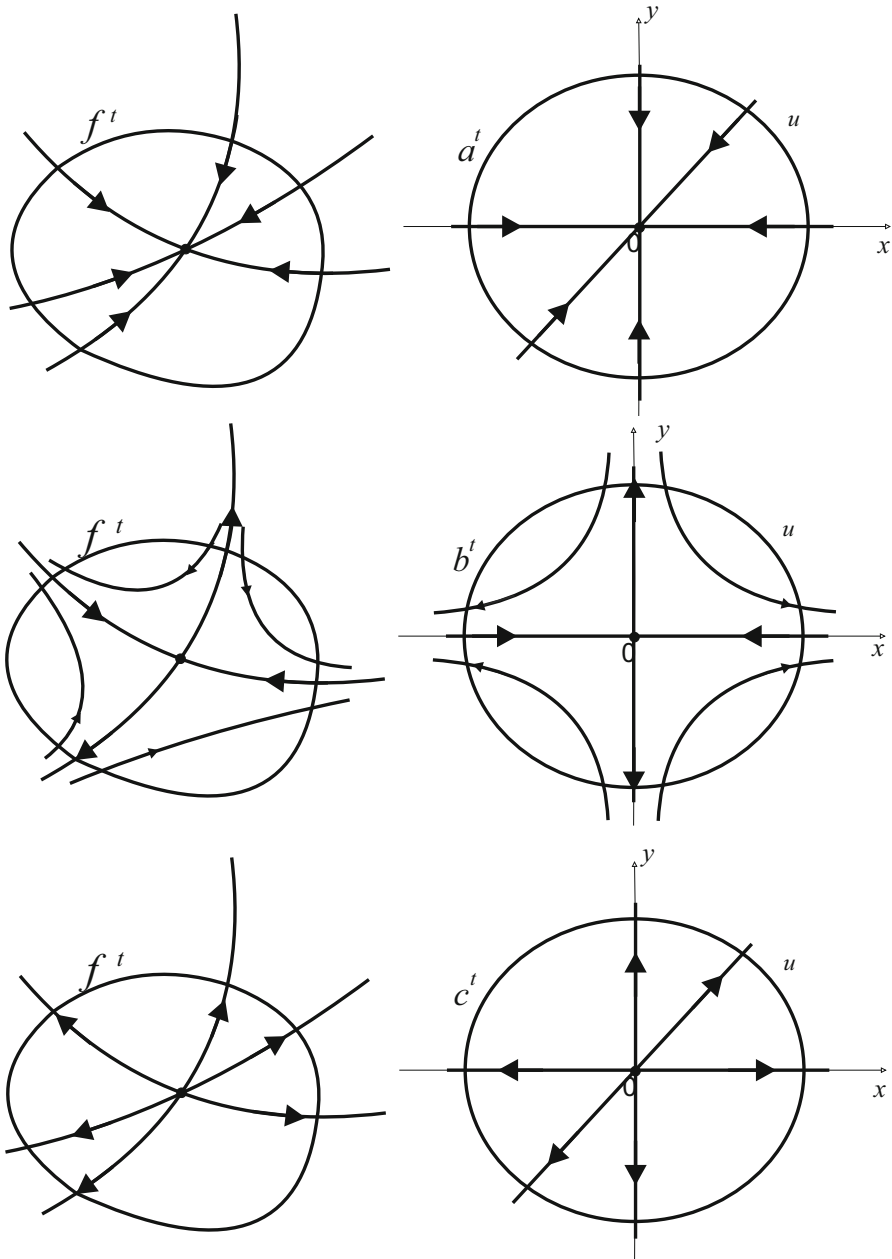


Fig. 1 A local conjugacy between hyperbolic fixed points and fixed points of the linear flows on \mathbb{R}^2 ; f^t is a flow on S , a^t, b^t, c^t are flows from Proposition 2.1, u is a neighbourhood of the Origin O

2. $W_p^u (W_p^s)$ is a smooth submanifold of S diffeomorphic to $\mathbb{R}^i (\mathbb{R}^{2-i})$, for each fixed point $p \in \Omega_{f^t}^i$;

3. for every saddle point σ , there is a unique sink (source) point ω (α), such that

$$cl(l_\sigma^u) \setminus l_\sigma^u = \{\sigma, \omega\} \quad (cl(l_\sigma^s) \setminus l_\sigma^s = \{\sigma, \alpha\});$$

4. for each sink ω (source α), there exists at least one saddle point σ with an unstable (stable) separatrix l_σ^u (l_σ^s), such that $cl(l_\sigma^u) \setminus l_\sigma^u = \{\sigma, \omega\}$ ($cl(l_\sigma^s) \setminus l_\sigma^s = \{\sigma, \alpha\}$).

2.2 Some Basic Definitions and Facts from Graph Theory

A graph is a pair (V, E) , where V is a set of vertices and E is a set of pairs of vertices, which are called edges. If E contains ordered pairs, then the graph is called a directed one. If some of edges is consisted in E more than one time, then it is called a multiple edge, a graph with multiple edges is called a multigraph.

Two vertices are called adjacent if they are connected by an edge (i.e. they constitute the edge), and the edge is incident to each of the vertices. A loop is an edge, whose end-vertices coincide. A simple graph is an undirected graph without loops and multiple edges.

The number of edges incident to a vertex is called degree of the vertex. A non-directed graph of the maximum vertex degree 3 is called a subcubic.

A set $\{b_1, (b_1, b_2), b_2, \dots, b_{k-1}, (b_{k-1}, b_k), b_k\}$ is called a path of the length k . If $b_1 = b_k$, then the path is called a cycle. A graph is called connected if every two its vertices are joined by a path.

A graph (V', E') , where $V' \subset V$ and $E' \subset E$, is called a subgraph of a graph (V, E) .

A graph is called embeddable into a surface if there are sets of points and Jordan curves with pairwise disjoint interiors on the surface, corresponding to the graph's vertices and edges, respectively. The genus of a graph is the minimum among genres of surfaces into which the graph can be embedded. A graph is called planar if it is embeddable into a 2-sphere.

One says that an isomorphism of n vertices graphs can be recognized (tested) in a time $t(n)$ if a distinguishing algorithm contains $t(n)$ operations. The algorithm is called effective if $t(n)$ is restricted by some polynomial function of n .

Proposition 2.3 (Hopcroft and Wong 1974) *Isomorphism of two n -vertex planar simple graphs can be recognized in $O(n)$ time.*

Proposition 2.4 (Miller 1980) *Isomorphism of two n -vertex simple graphs of genus g can be tested in $O(n^{O(g)})$ time.*

Proposition 2.5 (Kawarabayashi 2015, Theorem 1.3) *For every integer g , isomorphism of graphs of genus at most g can be done in linear time⁴.*

Proposition 2.6 (Galil et al. 1987) *Isomorphism of n -vertex simple subcubic graphs can be recognized in $O(n^3 \log(n))$ time.*

⁴ Note that the paper (Kawarabayashi 2015) is a preprint and, to the best of our knowledge, it is not published in a peer-review journal. This is the reason why we will use Propositions 2.3 and 2.4 to obtain our results.

Proposition 2.7 (Luks 1982) *For each fixed ρ , isomorphism of n -vertex simple graphs of the maximum vertex degree at most ρ can be recognized in polynomial time.*

To this end, we will also need the following graph transformations.

A k -subdivision of an edge $e = (a, b)$ is the following modification of an initial graph: to delete e from the graph and add new vertices c_1, c_2, \dots, c_k with edges $ac_1, c_1c_2, \dots, c_{k-1}c_k, c_kb$.

A k^* -subdivision of an edge $e = (a, b)$ is the following modification of an initial graph: to delete e from the graph and add new vertices c_1, c_2, \dots, c_k, d with edges $ac_1, c_1c_2, \dots, c_{k-1}c_k, c_kb, c_1d$.

3 An Algorithm for Peixoto’s Graph

3.1 Peixoto’s Graph

According to Peixoto (1973), the Peixoto’s invariant for a gradient-like flow f^t is a connected directed graph Γ_{f^t} with distinguishing sets. Vertices of Γ_{f^t} correspond to fixed points of the flow f^t and edges correspond to saddle separatrices. The set of vertices is divided into the three parts: vertices for sources, vertices for saddles, and vertices for sinks. Each saddle vertex is incident to four edges, corresponding to four separatrices. There is no an edge, connecting a source vertex with a sink vertex, and all the edges are directed from a source vertex to a saddle vertex or from a saddle vertex to a sink vertex (see Fig. 2).

Figure 3 shows that there are non-equivalent gradient-like flows with isomorphic directed graphs. It is the reason that Γ_{f^t} is equipped with some distinguishing sets to receive an equipped graph $\Gamma_{f^t}^P$, which will be a complete topological invariant.

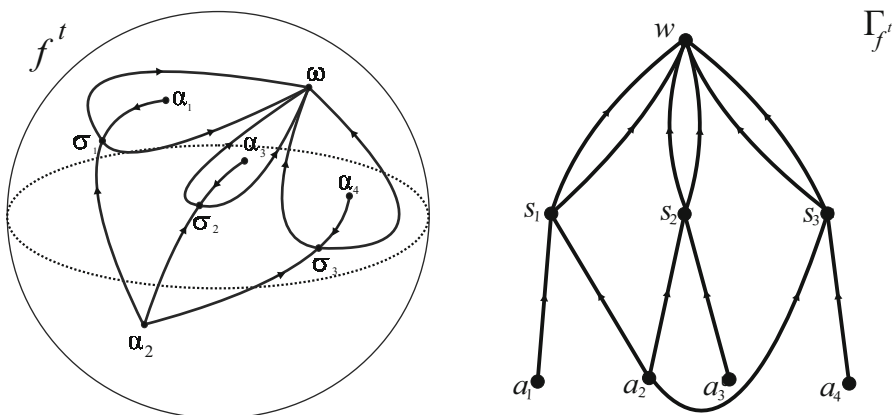


Fig. 2 An example of a gradient-like flow f^t and its directed graph Γ_{f^t} ; α_i is a source, a_i is a vertex on Γ_{f^t} corresponding to $\alpha_i, i = \overline{1, 4}$; σ_i is a saddle, s_i is a vertex on Γ_{f^t} corresponding to $\sigma_i, i = \overline{1, 3}$; ω is a sink, w is a vertex on Γ_{f^t} corresponding to ω

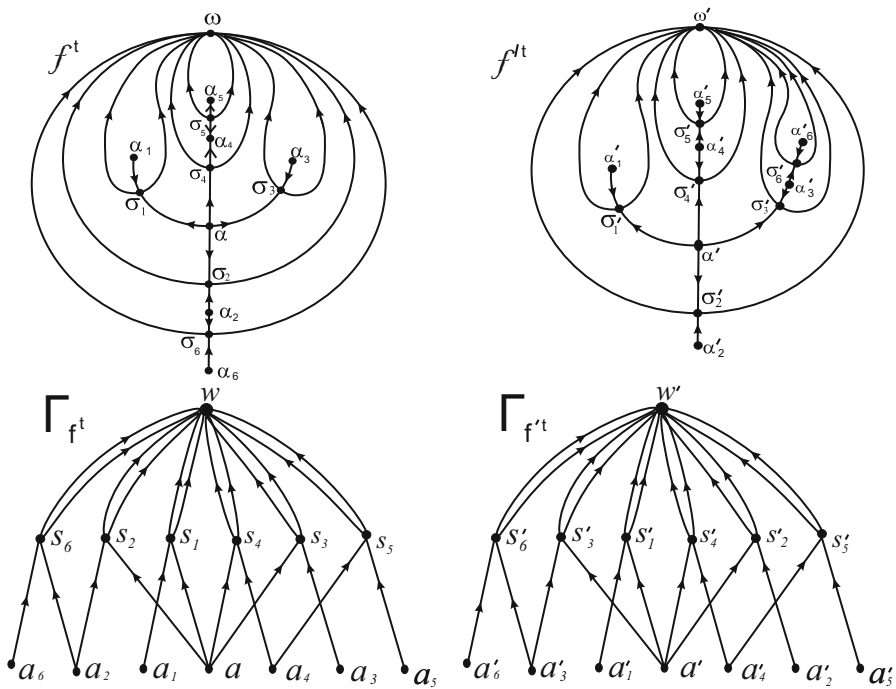


Fig. 3 Two flows f^t and $f^{t'}$ with the same directed graphs Γ_{f^t} and $\Gamma_{f^{t'}}$; α_i is a source, a_i is a vertex on Γ_{f^t} corresponding to $\alpha_i, i = \overline{1, 6}$; σ_i is a saddle, s_i is a vertex on Γ_{f^t} corresponding to $\sigma_i, i = \overline{1, 5}$; ω is a sink, w is a vertex on Γ_{f^t} corresponding to ω ; similar for the primed ones

To define a distinguishing set, let us consider the set

$$\tilde{S} = S \setminus \bigcup_{\sigma \in \Omega^1_{f^t}} (cl(W^u_\sigma) \cup cl(W^s_\sigma)).$$

The closure of any its connected component is called a *cell*. Due to Peixoto (1973), all the cells can be of the three types, described in Fig. 4.

A *distinguishing set* is a subgraph corresponding to the boundary of a cell. Hence, distinguishing sets can be of the three types, corresponding to the cell types (see Fig. 5). We call by a *tail* an edge with one degree in one of its ends.

Two Peixoto's graphs $\Gamma^P_{f^t}, \Gamma^P_{f^{t'}}$ of gradient-like flows f^t and $f^{t'}$, respectively, are called *isomorphic* if there is an isomorphism between Γ_{f^t} and $\Gamma_{f^{t'}}$, which preserves distinguishing sets.

Proposition 3.1 (Peixoto 1973, Proposition 4.4) *Two gradient-like flows $f^t, f^{t'}$ are topologically equivalent iff their Peixoto graphs $\Gamma^P_{f^t}, \Gamma^P_{f^{t'}}$ are isomorphic.*

3.2 An Algorithm Distinguishing the Peixoto's Graphs

We assume that gradient-like flows are given on the same surface S and denote by $g(S)$ the genus of S . The idea to construct an algorithm consists of a transformation

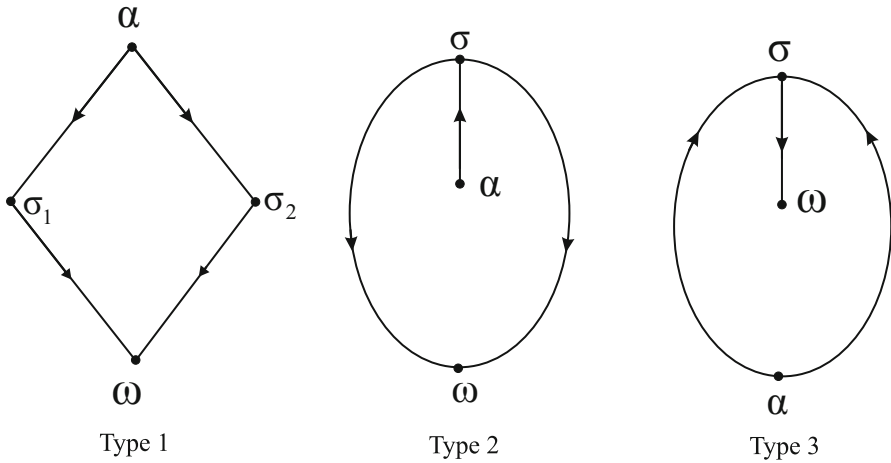


Fig. 4 The types of cells for a gradient-like flow; α is a source, $\sigma, \sigma_1, \sigma_2$ are saddles, ω is a sink

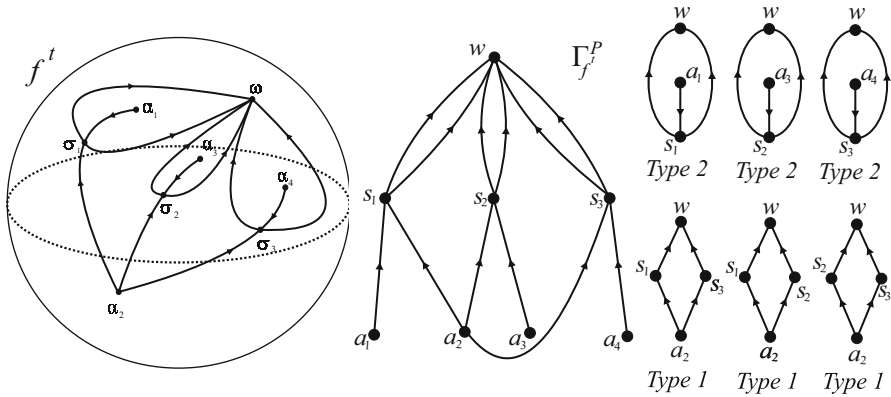


Fig. 5 A flow f^t from G and its Peixoto's graph $\Gamma_{f^t}^P$ with subgraphs; α_i is a source, a_i is a vertex on Γ_{f^t} corresponding to $\alpha_i, i = \overline{1, 4}$; σ_i is a saddle, s_i is a vertex on Γ_{f^t} corresponding to $\sigma_i, i = \overline{1, 3}$; ω is a sink, w is a vertex on Γ_{f^t} corresponding to ω

of the equipped Peixoto's graph $\Gamma_{f^t}^P$ to a simple graph $\tilde{\Gamma}_{f^t}$ such that $\Gamma_{f^t}^P$ and $\Gamma_{f^t}^P$ are isomorphic iff $\tilde{\Gamma}_{f^t}$ and $\tilde{\Gamma}_{f^t}$ are isomorphic. Moreover we will do it preserving the embeddability for $\tilde{\Gamma}_{f^t}$. In more details.

First, we add two degree one neighbours to every saddle vertex of $\Gamma_{f^t}^P$. Next, for any type 1 subgraph of $\Gamma_{f^t}^P$, we add a new vertex and four edges, connecting the new vertex with all the vertices of the subgraph. Finally, we apply the 2^* -subdivision operation to every oriented edge of $\Gamma_{f^t}^P$ the way if they would be oriented otherwise. The resulting graph is $\tilde{\Gamma}_{f^t}$ (see Fig. 6).

Lemma 3.1 *The graphs $\Gamma_{f^t}^P$ and $\Gamma_{f^t}^P$ are isomorphic iff $\tilde{\Gamma}_{f^t}$ and $\tilde{\Gamma}_{f^t}$ are isomorphic.*

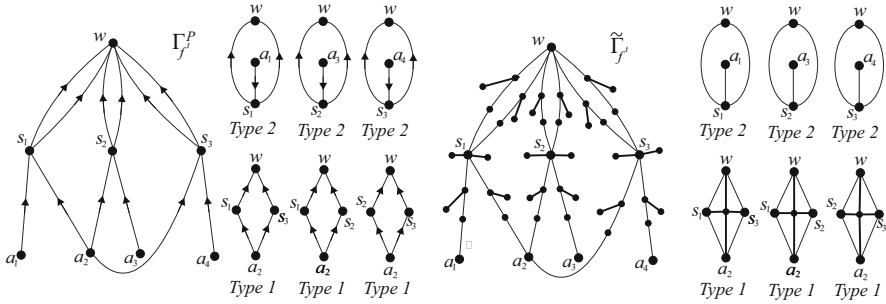


Fig. 6 The Peixoto's graph $\Gamma_{f^t}^P$ of with subgraphs and its simple graph $\tilde{\Gamma}_{f^t}$; a_i is a vertex corresponding to a sink, $i = \overline{1, 4}$; s_i is a vertex corresponding to a saddle, $i = \overline{1, 3}$; w is a vertex corresponding to a sink

Proof Obviously, $\tilde{\Gamma}_{f^t}$ can be uniquely constructed by $\Gamma_{f^t}^P$. We will show that $\Gamma_{f^t}^P$ can also be uniquely restored by $\tilde{\Gamma}_{f^t}$. This fact implies the correctness of this lemma.

A vertex of $\tilde{\Gamma}_{f^t}$ is a saddle in $\Gamma_{f^t}^P$ iff it has two neighbour degree one vertices. Hence, all the saddle vertices can be uniquely determined. Let x be a saddle vertex of $\Gamma_{f^t}^P$. In $\tilde{\Gamma}_{f^t}$, it is adjacent to a degree two vertex y and to a degree three vertex z . The vertex y has a degree three neighbour y' , which has two neighbours and one of them, denoted by y'' , has degree one or more than one. The vertex y'' is a sink vertex and xy'' is an edge in $\Gamma_{f^t}^P$. The vertex z has a degree two neighbour z' , which has a neighbour $z'' \neq z$. The vertex z'' is a source vertex and $z''x$ is an edge in $\Gamma_{f^t}^P$.

Knowing sink, source, and saddle vertices of $\Gamma_{f^t}^P$, it is possible to restore all its tails. Any subgraph of type 2 or 3 is completely defined by its tail. Hence, all subgraphs of these types can be uniquely found. Vertices of type 1 subgraphs are defined as the neighbours of those degree four vertices of $\tilde{\Gamma}_{f^t}$ that belong to four 5-cycles. \square

Theorem 1 Let f^t and f^n be gradient-like flows on S , $\Gamma_{f^t}^P$ and $\Gamma_{f^n}^P$ be their equipped n -vertex Peixoto's graphs with m edges. Then isomorphism of $\Gamma_{f^t}^P$ and $\Gamma_{f^n}^P$ can be tested in $O(n^{O(g(S))})$ time. This problem can be solved in $O(m)$ time if $g(S) = 0$.

Proof First, construct $\tilde{\Gamma}_{f^t}$ and $\tilde{\Gamma}_{f^n}$ by $\Gamma_{f^t}^P$ and $\Gamma_{f^n}^P$. It can be done in $O(m)$ time. Additionally, $\tilde{\Gamma}_{f^t}$ and $\tilde{\Gamma}_{f^n}$ both have $O(m)$ vertices. Both graphs $\tilde{\Gamma}_{f^t}$ and $\tilde{\Gamma}_{f^n}$ can be embedded into S . Indeed, it is true for Γ_{f^t} and Γ_{f^n} . Putting new vertices, corresponding to type 1 subgraphs, inside type 1 cells does not create an intersection of edges in an interior point. The same is true for adding degree one vertices. Hence, $\tilde{\Gamma}_{f^t}$ and $\tilde{\Gamma}_{f^n}$ are both embeddable into S . By Propositions 2.3 and 2.4, isomorphism $\tilde{\Gamma}_{f^t}$ and $\tilde{\Gamma}_{f^n}$ can be checked in $O(n^{O(g(S))})$ time (in $O(m)$ time if $g(S) = 0$). Thus, the same is true for $\Gamma_{f^t}^P$ and $\Gamma_{f^n}^P$. \square

4 An Algorithm for a Modification of Peixoto’s Graph

4.1 A Modification of Peixoto’s Graph

In 2011, Grines et al. (2016) suggested⁵ another version of the Peixoto’s invariant. Namely, they avoid the distinguishing sets and equipped the directed Peixoto’s graph Γ_{f^t} by orders of some edges to obtain an equipped graph $\Gamma_{f^t}^{GP}$.

Let ω be a sink of f^t , and let L_ω be the set of separatrices containing ω in their closures. It follows, for example, from Proposition 2.1 that there is a 2-disk $B_\omega \ni \omega$, such that each separatrix $l \subset L_\omega$ intersects ∂B_ω at a unique point. For the vertex w , corresponding to ω , let E_w be the set of edges of the graph Γ_{f^t} incident to w . Let N_w be the cardinality of the set E_w . Let us enumerate elements of E_w in the next way. We pick a 2-disk B_ω in the basin of ω and set $c_\omega = \partial B_\omega$. We define the counterclockwise orientation on c_ω with respect to B_ω . We enumerate all the edges e_1, \dots, e_{N_w} from E_w according to the order of the corresponding separatrices as we move along c_ω , starting from some point on c_ω . This enumeration of all the elements of E_w is said to be compatible with the embedding of the separatrices.

Two equipped graphs $\Gamma_{f^t}^{GP}$ and $\Gamma_{f^u}^{GP}$ of flows f^t and f^u (possibly, on different ambient surfaces) are said to be isomorphic if there is an isomorphism ξ between Γ_{f^t} and Γ_{f^u} , such that the permutation $\Theta_{w,w'}$, where $w' = \xi(w)$, induced by ξ is a power of a cyclic permutation⁶, for each vertex w , corresponding to a sink.

Consider Fig. 3 once again and suppose that the vertex w (w') of the graph corresponds to the sink point ω (ω') and the vertex a (a') of the graph corresponds to the source point α (α'). We renumber the separatrices $l^u_1, l^u_2, l^u_3, l^u_4$ ($l^t_1, l^t_2, l^t_3, l^t_4$) of the saddle points belonging to the stable manifold of the point ω (ω') in accordance with the positive orientation on a closed curve around ω (ω'). We denote by $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ ($\sigma'_1, \sigma'_2, \sigma'_3, \sigma'_4$) the saddle points that satisfy the condition that σ_i (σ'_i) belongs to the closure of l^u_i (l^t_i). Introduce a numeration on the set E_ω ($E_{\omega'}$) compatible with the embedding of the separatrices. As we have already noted, the graphs Γ_{f^t} and Γ_{f^u} are isomorphic. There are exactly two isomorphisms of these graphs: the isomorphism ξ_1 of the natural identification of the graph Γ_{f^t} with the graph Γ_{f^u} and the isomorphism ξ_2 that is the composition of the natural identification and the reflection with respect to the axis $a'w'$. One can directly check that the isomorphism ξ_1 induces the permutation $\Theta_{w,w'} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$ and the isomorphism ξ_2 induces the permutation $\Theta_{w,w'} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$. None of the permutations is a power of a cyclic permutation and the equipped graphs $\Gamma_{f^t}^{GP}$ and $\Gamma_{f^u}^{GP}$ are not isomorphic.

Proposition 4.1 (Grines et al. 2016, Theorem 3.2.1) *Gradient-like flows f^t and f^u are topologically conjugate iff their equipped graphs $\Gamma_{f^t}^{GP}$ and $\Gamma_{f^u}^{GP}$ are isomorphic.*

⁵ The classification results in Grines et al. (2016) have been obtained for gradient-like diffeomorphisms, but they are also valid for flows.

⁶ It can be directly checked that the property of the permutation to be a power of a cyclic permutation is independent on a choice of the curves c_ω and $c_{\omega'}$.

4.2 An Algorithm Distinguishing the Modifying Peixoto's Graphs

We will use an idea which is similar to one for Peixoto's graphs and modify the graph $\Gamma_{f^t}^{GP}$ up to a graph $\bar{\Gamma}_{f^t}$ by the following way.

For any sink vertex w of $\Gamma_{f^t}^{GP}$, we consider the cyclically ordered set E_w of all the edges, incident to w . We delete w from $\Gamma_{f^t}^{GP}$ (and all the edges, incident to w), add a cycle with N_w vertices to the resulting graph, whose vertices are numbered with $1, \dots, N_w$. For any $1 \leq i \leq N_w$, we add a non-oriented edge between i -th vertex of the cycle and the second end of i -th element of E_w distinct to w . Next, we add two degree one neighbours to every saddle vertex of $\Gamma_{f^t}^{GP}$ and apply the 2^* -subdivision operation to every its oriented edge.

Lemma 4.1 *The graphs $\Gamma_{f^t}^{GP}$ and $\Gamma_{f^n}^{GP}$ are isomorphic iff $\bar{\Gamma}_{f^t}$ and $\bar{\Gamma}_{f^n}$ are isomorphic.*

Proof Similar to the reasonings from the proof of Lemma 3.1, knowing $\bar{\Gamma}_{f^t}$, it is possible to restore all the saddles of $\Gamma_{f^t}^{GP}$ and all the edges of $\Gamma_{f^t}^{GP}$. Hence, all the cycles of $\bar{\Gamma}_{f^t}$, corresponding to all the sinks of $\Gamma_{f^t}^{GP}$, can be determined. A cyclical order of E_w is determined under rounding of the cycles. Hence, knowing $\bar{\Gamma}_{f^t}$, it is possible to uniquely restore $\Gamma_{f^t}^{GP}$. The opposite possibility is obvious. \square

Theorem 2 *Let f^t and f^n be gradient-like flows on S , $\Gamma_{f^t}^{GP}$ and $\Gamma_{f^n}^{GP}$ be their equipped n -vertex graphs with m edges. Then isomorphism of $\Gamma_{f^t}^{GP}$ and $\Gamma_{f^n}^{GP}$ can be tested in $O(n^{O(g(S))})$ time. This problem can be solved in $O(m)$ time if $g(S) = 0$.*

Proof We start from constructing the graphs $\bar{\Gamma}_{f^t}$ and $\bar{\Gamma}_{f^n}$ by $\Gamma_{f^t}^{GP}$ and $\Gamma_{f^n}^{GP}$. It can be done in $O(m)$ time. Additionally, $\bar{\Gamma}_{f^t}$ and $\bar{\Gamma}_{f^n}$ both have $O(m)$ vertices. Since $\bar{\Gamma}_{f^t}$ and $\bar{\Gamma}_{f^n}$ can be embedded into S , \ll inscribing \gg cycles into sinks of $\Gamma_{f^t}^{GP}$ and $\Gamma_{f^n}^{GP}$ keeps embeddability into S , then $\bar{\Gamma}_{f^t}$ and $\bar{\Gamma}_{f^n}$ can also be embedded into S . By Propositions 2.3 and 2.4, isomorphism $\bar{\Gamma}_{f^t}$ and $\bar{\Gamma}_{f^n}$ can be checked in $O(n^{O(g(S))})$ time (in $O(m)$ time if $g(S) = 0$). Thus, the same is true for $\Gamma_{f^t}^{GP}$ and $\Gamma_{f^n}^{GP}$. \square

5 An Algorithm for Wang's Graph

5.1 Wang's Graph

The Wang's invariant (Wang 1990) is an invariant for gradient-like flows on oriented surfaces S . The invariant is a graph dual to the unequipped Peixoto's graph: for the Wang's graph $\Gamma_{f^t}^W$ for f^t correspond to the cells of f^t its edges of the colour s correspond to the stable separatrices and edges of the colour u correspond to the unstable separatrices (see Fig. 7). This graph consists of 4-cycles around each saddle point, its edges are coloured in two colours u and s , the opposite edges in all the 4-cycles have the same colour. Hence, each its vertex has degree two or four. In addition, all the

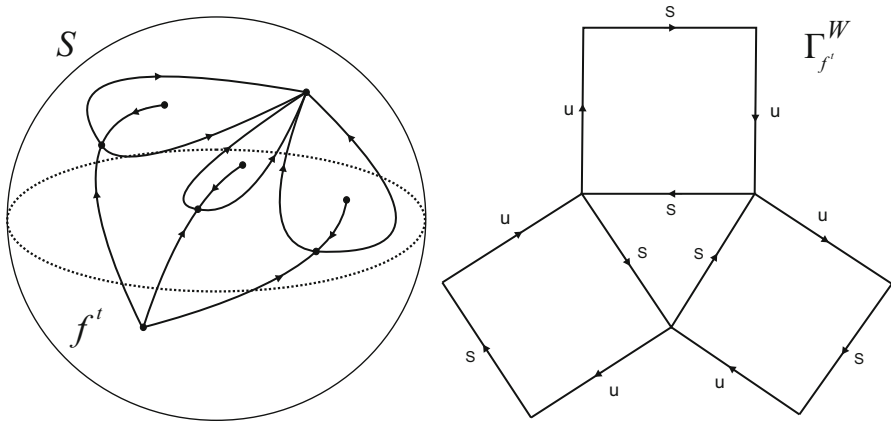


Fig. 7 A flow f^t from G on a surface S and its Wang's invariant $\Gamma_{f^t}^W$; s, u are the colours for edges

edges are directed so that the direction of any edge is consistent with the clockwise orientation of its 4-cycle.

Two Wang's graphs $\Gamma_{f^t}^W, \Gamma_{f^n}^W$ for gradient-like flows f^t, f^n , respectively, are called *isomorphic* if there is an isomorphism between $\Gamma_{f^t}^W$ and $\Gamma_{f^n}^W$ preserving the colours.

Proposition 5.1 (Wang 1990) *The gradient-like flows f^t, f^n are topologically equivalent iff their Wang's graphs $\Gamma_{f^t}^W, \Gamma_{f^n}^W$ are isomorphic.*

5.2 An Algorithm Distinguishing the Wang's Graphs

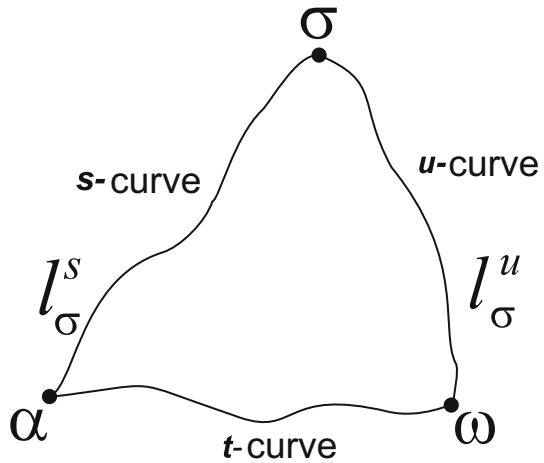
By the graph $\Gamma_{f^t}^W$, we construct a graph $\check{\Gamma}_{f^t}$ as follows. We apply 2^* -subdivision to every oriented edge of the colour s and 3^* -subdivision to every oriented edge of the colour u . The same operation is done for the graph $\Gamma_{f^n}^W$ to obtain a graph $\check{\Gamma}_{f^n}$. These operations take linear time on the numbers of vertices.

Lemma 5.1 *The graphs $\Gamma_{f^t}^W$ and $\Gamma_{f^n}^W$ are isomorphic iff $\check{\Gamma}_{f^t}$ and $\check{\Gamma}_{f^n}$ are isomorphic.*

Proof To prove the lemma, we only need to show that $\Gamma_{f^t}^W$ can be uniquely reconstructed by $\check{\Gamma}_{f^t}$. Indeed, in $\check{\Gamma}_{f^t}$, we find all degree 4 vertices, which will be degree 4 vertices in $\Gamma_{f^t}^W$, remove them to obtain the disjunctive sum of path-like remaining subgraphs. By degree 3 vertices in these subgraphs, it is possible restore all edges in 4-cycles and their colours in $\Gamma_{f^t}^W$. \square

Theorem 3 *Let f^t, f^n be gradient-like flows and $\Gamma_{f^t}^W, \Gamma_{f^n}^W$ be their n -vertex Wang's graphs. Then isomorphism of $\Gamma_{f^t}^W$ and $\Gamma_{f^n}^W$ can be tested in a polynomial time on n . If, additionally, f^t and f^n are given on the same surface S , then isomorphism of $\Gamma_{f^t}^W$ and $\Gamma_{f^n}^W$ can be tested in $O(n^{O(g(S))})$ time (in $O(n)$ time, if $g(S) = 0$).*

Fig. 8 A triangular domain; α is a source, ω is a sink, σ is a saddle, l_σ^s is a stable separatrix, l_σ^u is an unstable separatrix



Proof Firstly, construct $\check{\Gamma}_{f^t}$ and $\check{\Gamma}_{f''}$ by $\Gamma_{f^t}^W$ and $\Gamma_{f''}^W$. It can be done in linear time. The sizes of $\check{\Gamma}_{f^t}$ and $\check{\Gamma}_{f''}$ linearly depend on the sizes of $\Gamma_{f^t}^W$ and $\Gamma_{f''}^W$, respectively. Notice that the maximum vertex degrees of $\check{\Gamma}_{f^t}$ and $\check{\Gamma}_{f''}$ are 4. From this fact, Lemma 5.1, and Propositions 2.3, 2.4, 2.7 the statement of this lemma follows. \square

6 Algorithms for Oshemkov–Sharko’s Graph

6.1 Oshemkov–Sharko’s Graph

The Oshemkov–Sharko’s invariant (Oshemkov and Sharko 1998) for a gradient-like flow f^t is a three-colour graph $\Gamma_{f^t}^{OS}$ defined as follows. We were already speaking about *cells* and their three types in Sect. 3 (see Fig. 4). We remove a trajectory from each cell that connects its source and sink. These curves are said to be *t-curves*. A connected component of the cell that we have got is called a *triangular domain*. Its boundary is the union of closures of an unstable saddle separatrix, i.e. *u-curve*, a stable saddle separatrix, i.e. *s-curve*, and a *t-curve*. We call these coloured curves as *sides of the triangular domain*. Also the boundary of any triangular domain contains exactly one source, one sink, and one saddle (see Fig. 8).

Vertices of the graph $\Gamma_{f^t}^{OS}$ correspond to triangular domains of f^t , edges correspond to their sides and have the same colour – *u*, *s* or *t* (see Fig. 9).

Two graphs $\Gamma_{f^t}^{OS}$ and $\Gamma_{f''}^{OS}$ of gradient-like flows f^t, f'' , respectively, are called *isomorphic* if there is an isomorphism between $\Gamma_{f^t}^{OS}$ and $\Gamma_{f''}^{OS}$ preserving colours.

Proposition 6.1 (Oshemkov and Sharko 1998, Theorem 1.10) *Two gradient-like flows f^t, f'' are topologically equivalent iff their graphs $\Gamma_{f^t}^{OS}$ and $\Gamma_{f''}^{OS}$ are isomorphic.*

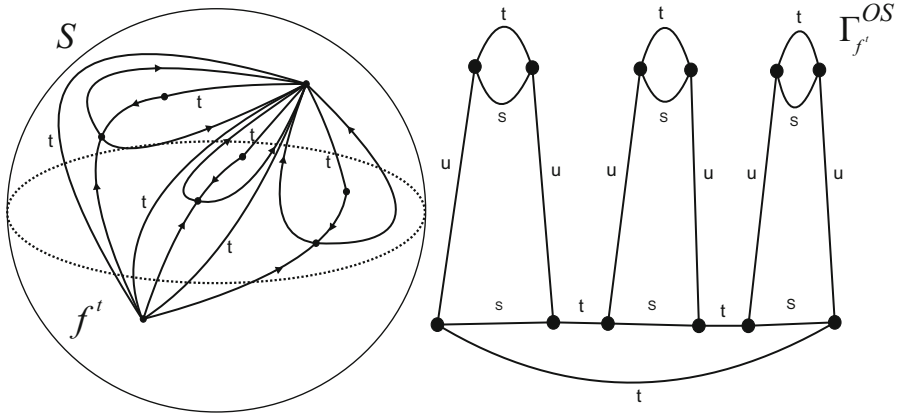


Fig. 9 A gradient-like flow f^t on a surface S and its Oshemkov-Sharko's invariant – three-colour graph $\Gamma_{f^t}^{OS}$; u, s, t are the colours for edges and colour curves

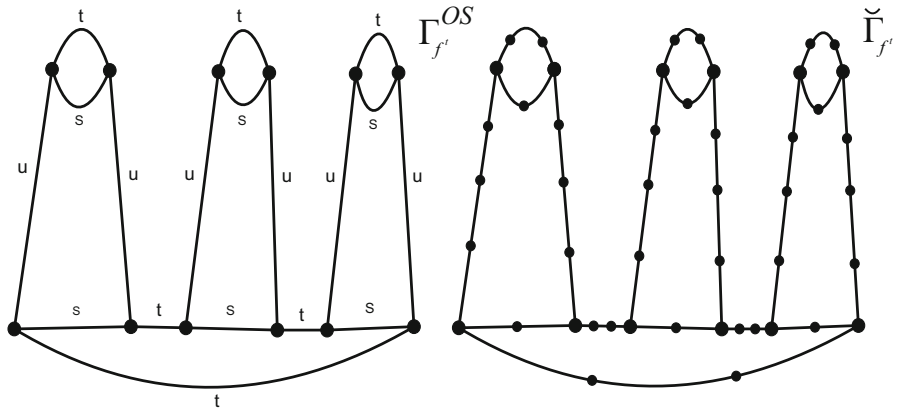


Fig. 10 The three-colour graph $\Gamma_{f^t}^{OS}$ and its simple graph $\check{\Gamma}_{f^t}$; u, s, t are the colours for edges and colour curves

6.2 An Effective Algorithm to Distinguish the Oshemkov–Sharko's Graphs

In the paper (Oshemkov and Sharko 1998), the authors proposed an algorithm for distinguishing their graphs. Unfortunately, it is not polynomial-time and below we suggest an effective algorithm.

Again, we use the idea of reduction of the isomorphism problem for three-colour graphs to the same problem for simple graphs of bounded degree or embedded into the ambient surface. We construct a graph $\check{\Gamma}_{f^t}$ by 1-subdividing each s -edge, 2-subdividing each t -edge, and 3-subdividing each u -edge of $\Gamma_{f^t}^{OS}$ (see Fig. 10).

Lemma 6.1 *The graphs $\check{\Gamma}_{f^t}$ and $\check{\Gamma}_{f^n}$ are isomorphic iff $\Gamma_{f^t}^{OS}$ and $\Gamma_{f^n}^{OS}$ are isomorphic.*

Proof The graphs $\check{\Gamma}_{f^t}$ and $\check{\Gamma}_{f^n}$ can be uniquely constructed by the graphs $\Gamma_{f^t}^{OS}$ and $\Gamma_{f^n}^{OS}$. Let us show that the opposite fact is true. To this end, we may consider the graphs $\check{\Gamma}_{f^t}$ and $\Gamma_{f^t}^{OS}$ only. Degree 3 vertices of $\check{\Gamma}_{f^t}$ are exactly degree 3 vertices of $\Gamma_{f^t}^{OS}$. A colour of any edge of $\Gamma_{f^t}^{OS}$ is determined by length of a path between the edge's end vertices in $\check{\Gamma}_{f^t}$. This finishes the proof of this lemma. \square

Theorem 4 *Let f^t and f^n be gradient-like flows, $\Gamma_{f^t}^{OS}$ and $\Gamma_{f^n}^{OS}$ be their n -vertex Oshemkov-Sharko's graphs. Then isomorphism of $\Gamma_{f^t}^{OS}$ and $\Gamma_{f^n}^{OS}$ can be tested in $O(n^3 \log(n))$ time. If f^t and f^n are given on the same surface S , then isomorphism of $\Gamma_{f^t}^{OS}$ and $\Gamma_{f^n}^{OS}$ can be checked in $O(n^{O(g(S))})$ time (in $O(n)$ time, if $g(S) = 0$).*

Proof The graphs $\check{\Gamma}_{f^t}$ and $\check{\Gamma}_{f^n}$ can be obtained in $O(n)$ time and both have at most $5n$ vertices. The graphs $\check{\Gamma}_{f^t}$ and $\check{\Gamma}_{f^n}$ are subcubic and both embeddable into S whenever f^t and f^n are on S . Hence, by Propositions 2.3, 2.4, 2.6, and Lemma 6.1 this lemma holds. \square

7 Algorithms for the Fleitas' Graph

7.1 Fleitas' Graph

The Fleitas' invariant $\Gamma_{f^t}^F$ (Fleitas 1975) for a gradient-like flow f^t is constructed in the next way. Choose around each source a circle inside the cell's closure, transversal to f^t 's trajectories. Then label all the intersections of each circle and saddle separatrices with *marks* so that intersections with separatrices of the same saddle have the same mark. After that let us equip each chosen intersection point by a *spin*, i.e. an arrow directed along the circle so that if we move intersection points with the same mark in direction of their arrows, then their trajectories would converge to the same separatrix (see Fig. 11).

Two Fleitas' invariants $\Gamma_{f^t}^F$ and $\Gamma_{f^n}^F$ of gradient-like flows f^t, f^n , respectively, are called *isomorphic* if there is an isomorphism mapping circles of $\Gamma_{f^t}^F$ into circles of $\Gamma_{f^n}^F$ preserving marks and spins.

The Fleitas' invariant was created to classify not all gradient-like flows on surfaces but only *polar* flows (only one sink and one source).

Proposition 7.1 (Fleitas 1975) *Two polar flows f^t, f^n (possibly, on distinct surfaces) are topologically equivalent iff their Fleitas' invariants $\Gamma_{f^t}^F$ and $\Gamma_{f^n}^F$ are isomorphic.*

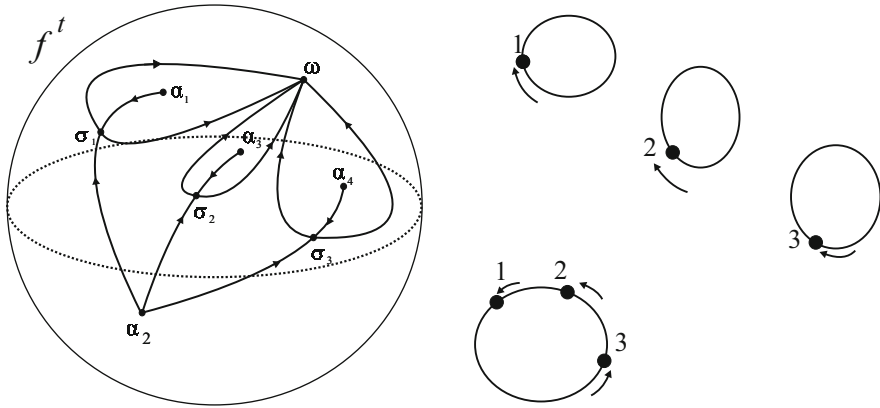


Fig. 11 A gradient-like flow f^t and its Fleitas' invariant $\Gamma_{f^t}^F$; α_i is a source, $i = \overline{1, 4}$; σ_i is a saddle, $i = \overline{1, 3}$, ω is a sink; 1, 2 and 3 are the labels

7.2 An Algorithm Distinguishing the Fleitas' Graphs

7.2.1 Efficient Algorithms Based on Boundedness of Vertex Degrees and Embeddability into the Ambient Surface

For the graph $\Gamma_{f^t}^F$, we define a graph $\hat{\Gamma}_{f^t}$ as follows. We apply 2^* -subdivision to every oriented edge (including loops) and put an edge between any pair of vertices with the same marks. The same operation is done with $\Gamma_{f^n}^F$ to obtain a graph $\hat{\Gamma}_{f^n}$.

Lemma 7.1 *The graphs $\hat{\Gamma}_{f^t}$ and $\hat{\Gamma}_{f^n}$ are isomorphic iff $\Gamma_{f^t}^F$ and $\Gamma_{f^n}^F$ are isomorphic.*

Proof The graphs $\hat{\Gamma}_{f^t}$ and $\hat{\Gamma}_{f^n}$ can be uniquely constructed by the graphs $\Gamma_{f^t}^F$ and $\Gamma_{f^n}^F$. We will show that the opposite fact is also true. One may consider the graphs $\hat{\Gamma}_{f^t}$ and $\Gamma_{f^t}^F$ only. Two vertices x and y of $\hat{\Gamma}_{f^t}$ constitute the oriented edge xy in $\Gamma_{f^t}^F$ iff there is a path (x, x', y', y) in $\hat{\Gamma}_{f^t}$, such that y' has a degree one neighbour in $\hat{\Gamma}_{f^t}$. Therefore, all the cycles around source vertices can be restored. By these cycles and edges between them it is possible to uniquely restore the marks. \square

Theorem 5 *Let f^t and f^n be gradient-like flows, $\Gamma_{f^t}^F$ and $\Gamma_{f^n}^F$ be their n -vertex Fleitas' graphs. Then isomorphism of $\Gamma_{f^t}^F$ and $\Gamma_{f^n}^F$ can be tested in $O(n^3 \log(n))$ time. If f^t and f^n lye on the same surface S , then isomorphism of $\Gamma_{f^t}^F$ and $\Gamma_{f^n}^F$ can be checked in $O(n^{O(g(S))})$ time (in $O(n)$ time, if $g(S) = 0$).*

Proof The graphs $\hat{\Gamma}_{f^t}$ and $\hat{\Gamma}_{f^n}$ can be obtained in $O(n)$ time and both have at most $5n$ vertices. The graphs $\hat{\Gamma}_{f^t}$ and $\hat{\Gamma}_{f^n}$ are subcubic and both embeddable into S whenever f^t and f^n are on S . Hence, by Propositions 2.3, 2.4, 2.6, and Lemma 7.1 this lemma holds. \square

7.2.2 A Parameterized Algorithm for the Fleitas' Invariant

We consider n -vertex subcubic graphs (possibly, with loops and multiple edges), whose vertex sets were partitioned into k clockwise oriented cycles and suppose that the partition into the cycles is explicitly given. We assume that a loop gives one to degree of its ends. For short, we will refer these graphs to as (n, k) -graphs. In this subsection, we present an algorithm to solve the isomorphism problem for (n, k) -graphs with the computational complexity $O((k+1) \cdot n)$. Therefore, if the number of sources is fixed, then the Fleitas' criterion can be applied in linear time.

The main idea will be a reduction of the graph isomorphism problem to the following stringology task. Given two strings $X = X_1 X_2 \dots X_l$ and $Y = Y_1 Y_2 \dots Y_l$, the task is to verify that Y is a cyclic permutation of X . It can be solved in $O(l)$ time by considering the word $Z = X_1 X_2 \dots X_l X_1 X_2 \dots X_l$, containing all cyclic permutations of X , and by subsequent applying any linear-time string matching algorithm (like algorithms in Aho and Corasick (1975) Apostolico and Giancarlo (1986) Donald et al. (1977)) to find an entering of Y to Z . Further, it will be explained how to construct, by two given (n, k) -graphs, two sets of at most $k!$ strings, each with at most $3 \cdot n$ characters in the alphabet $\{-n, -n+1, \dots, -1, 0, 1, \dots, n-1, n\}$. The time-complexity of this process will be $O((k+1)! \cdot n)$. The graphs will be isomorphic iff there are two strings from distinct sets, one of which is a cyclic permutation of another.

Theorem 6 *The isomorphism problem for (n, k) -graphs can be solved in $O((k+1)! \cdot n)$ time.*

Proof Let Γ be a connected (n, k) -graph. We number all the cycles by numbers from 1 to k , such that, for any $2 \leq i \leq k$, there is an edge between $\bigcup_{j=1}^{i-1} V(C_j)$ and C_i . There are at most $k!$ such numerations, and all the appropriate numerations can be computed in $O(k! \cdot n)$ time. Denote by C_1, \dots, C_k all the numbered, oriented cycles. For any $i \geq 2$, some vertices of C_i will be added to a $\ll \text{sum} \gg \tilde{C}_{i-1}$ of C_1, \dots, C_{i-1} . The $\ll \text{sum} \gg \tilde{C}_{i-1}$ will be a clockwise oriented cycle and every vertex of \tilde{C}_{i-1} will have one, two or three ordered labels. Every degree two vertex of Γ , belonging to $\bigcup_{j=1}^i V(C_j)$, will obligatory be on \tilde{C}_i and will have exactly one label. Any adjacent degree three vertices of Γ , both belonging to one of the cycles C_1, \dots, C_i , will obligatory be on \tilde{C}_{i-1} and will have exactly two labels. Vertices of \tilde{C}_i , having exactly three labels, correspond to edges between distinct elements of the set $\{C_1, \dots, C_i\}$. Initially, $\tilde{C}_1 = C_1$ and all its vertices are labelled with the label 1.

The transformation from \tilde{C}_{i-1} to \tilde{C}_i is as follows. Let $x_1 y_1, x_2 y_2, \dots, x_{j_i} y_{j_i}$ be all the edges between \tilde{C}_{i-1} and C_i , i.e., for any $1 \leq j \leq j_i$, we have $x_j \in V(\tilde{C}_{i-1})$ and $y_j \in V(C_i)$. Suppose that y_1, \dots, y_{j_i} are clockwise located on C_i . Let z_j be the right neighbour of x_j on \tilde{C}_{i-1} . For any $1 \leq j \leq j_i$, if y_{j+1} is not a neighbour of y_j in C_i , we delete the edge $x_j z_j$ from \tilde{C}_{i-1} , add a path, strictly clockwise located between y_j and y_{j+1} on C_i , where $y_{j+1} = y_1$. We also add two edges, one of which connects x_j and the right neighbour of y_j on C_i , the another one connects z_j and the left neighbour of y_{j+1} on C_i .

We arrange i as the second label to all the vertices x_1, \dots, x_{j_i} and as the first label to all vertices of all the added paths. Second labels are also given to any two adjacent

vertices a and b of C_i . If $a = b$, then zero is the second label of a . If $a \neq b$, then let n_a and n_b be the numbers of vertices on clockwise oriented arcs between a, b and between b, a , correspondingly. The vertex a has the second label $n_a + 1$, and the vertex b has the second label $n_b + 1$. Clearly, \tilde{C}_i can be uniquely constructed by \tilde{C}_{i-1} and C_i . Moreover, by \tilde{C}_i , it is possible to uniquely reconstruct \tilde{C}_{i-1} , $V(C_i)$, all the edges between \tilde{C}_{i-1} , and internal paths between y_j and y_{j+1} , for any $1 \leq j \leq j_i - 1$. But, we do not know an order for assembling paths between y_j and y_{j+1} , where $1 \leq j \leq j_i - 1$, to obtain the cycle C_i . It is easy to see that the procedure, described in this paragraph, can be done in $O(|V(\tilde{C}_{i-1})| + |V(C_i)|)$ time.

The current information, stored in \tilde{C}_i , and the j_i -link oriented broken line, composed of the vectors $\overrightarrow{x_1x_2}, \overrightarrow{x_2x_3}, \dots, \overrightarrow{x_{j_i}x_1}$, completely define \tilde{C}_{i-1}, C_i and all the edges between them, as \tilde{C}_i and the j_i -link define an order for assembling the paths between y_j and y_{j+1} , where $1 \leq j \leq j_i - 1$. To encode the line, let us consider a clockwise oriented cycle C on the vertices x_1, \dots, x_{j_i} , lied on C in the same order as on \tilde{C}_{i-1} . We clockwise number all the vertices of C . By this numeration, for any $1 \leq j \leq j_i$, it is possible to compute in $O(1)$ time the number n_j of vertices on the clockwise oriented arc between x_j and x_{j+1} . Note that the vector $(n_1, n_2, \dots, n_{j_i})$ does not depend on the numeration of vertices and completely defines the line together with some its point. If arc's orientation coincides with the orientation of $\overrightarrow{x_jx_{j+1}}$, then the third label of x_j is $n_j + 1$; otherwise it is $-n_j - 1$. Clearly that the total computation time for finding all these third labels is $O(|V(\tilde{C}_{i-1})| + |V(C_i)|)$.

Therefore, for any i , \tilde{C}_{i-1}, C_i , and all the edges between them uniquely define \tilde{C}_i and vice versa. Hence, Γ uniquely defines a cycle \tilde{C}_k and vice versa. The complexity for obtaining \tilde{C}_k by the graph G is

$$O\left(\sum_{i=1}^k (k - i) \cdot |V(C_i)|\right) = O\left(k \cdot \sum_{i=1}^k |V(C_i)|\right) = O(k \cdot n).$$

By \tilde{C}_k , a string is obtained as follows. Started from any vertex of \tilde{C}_k , we clockwise read all its vertices, adding one, two or three characters to the current word, depending on a vertex's type. So, the total complexity for obtaining the whole set of words is $O(k! \cdot k \cdot n) = O((k + 1)! \cdot n)$. Two given (n, k) -graphs are isomorphic iff there are two words from distinct sets, such that one of them is a cyclic permutation of another. Thus, the isomorphism problem for (n, k) -graphs can be solved in $O((k + 1)! \cdot n)$ time. □

8 Topological Conjugacy of Gradient-Like Flows

The next theorem gives an opportunity to use all instruments for recognizing an equivalence class of a gradient-like flow also to recognize its conjugacy class. This theorem has been schematically proved as the main theorem in Kruglov (2018), but we give here a modified and most detailed version of its proof.

Theorem 7 *If two gradient-like flows on a closed surface are topologically equivalent, then they are topologically conjugate.*

Proof Let S be a closed surface and $f^t : S \times \mathbb{R} \rightarrow S$ be a C^1 gradient-like flow. Then, for every wandering trajectory ℓ of f^t , there are exactly two different fixed points p, q of f^t , such that the boundary of the trajectory has the form

$$\text{cl}(\ell) \setminus \ell = \{p, q\},$$

and the trajectory is directed from p to q . In this case we will denote the trajectory by $\ell_{p,q}$ assuming that the trajectory is directed from p to q .

Let f'^t be a C^1 gradient-like flow topologically equivalent to f^t , i.e. there is a homeomorphism $h : S \rightarrow S$, sending trajectories of f^t into trajectories of f'^t and preserving orientations of trajectories. It implies that h sends the fixed points of f^t to the fixed points of f'^t , we will denote $p' = h(p)$ for $p \in \Omega_{f^t}$. Then

$$h(\ell_{pq}) = \ell'_{p'q'},$$

for every wandering trajectory ℓ_{pq} of f^t .

Let $U_0 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. By Proposition 2.1, there exist neighbourhoods $u_p, u_{p'}$ of p, p' respectively and homeomorphisms $\psi_p : u_p \rightarrow U_0, \psi_{p'} : u_{p'} \rightarrow U_0$ such that restrictions $f^t|_{u_p}$ and $f'^t|_{u_{p'}}$ are conjugate by means of a homeomorphism $h_p = \psi_{p'}^{-1} \psi_p : u_p \rightarrow u_{p'}$, i.e. $h_p \circ f^t|_{u_p} = f'^t|_{u_{p'}} \circ h_p$ for any t not moving points outward $u_p, u_{p'}$ respectively.

Let σ be a saddle point of f^t . Consider u_σ and U_0 . Let $\tilde{U}_0 \subset U_0$ and its boundary be constructed the next way. Let $l_{++}, l_{-+}, l_{--}, l_{+-}$ be the four trajectories of the flow $b^t(x, y) = (2^{-t}x, 2^t y) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ lying in different quadrants. Let \tilde{U}_0 be the part of U_0 restricted by these trajectories (see Fig. 12). Denote $\tilde{u}_\sigma = \psi_\sigma^{-1}(\tilde{U}_0)$ and $\tilde{u}_{\sigma'} = \psi_{\sigma'}^{-1}(\tilde{U}_0)$. Note that obviously $u_{\sigma'} = h_\sigma(u_\sigma)$, and a map $h^{-1}h_\sigma$ preserves separatrix of σ . Also note that h_σ sends boundary of \tilde{u}_σ to boundary of $\tilde{u}_{\sigma'}$, which means that points of $\partial\tilde{u}_\sigma$ and points of $\partial\tilde{u}_{\sigma'}$ are conjugate.

For a point $x \in S$, we denote by \mathcal{O}_x (\mathcal{O}'_x) the orbit of the flow f^t (f'^t) passing through the point x . Let

$$V_\sigma = \bigcup_{x \in \text{cl}(\tilde{u}_\sigma)} \mathcal{O}_x, \quad V_{\sigma'} = \bigcup_{x \in \text{cl}(\tilde{u}_{\sigma'})} \mathcal{O}'_x.$$

Let us extend h_σ up to a homeomorphism $h_{V_\sigma} : V_\sigma \rightarrow V_{\sigma'}$ by the following rule (see Fig. 13). First let

$$h_{V_\sigma}|_{\tilde{u}_\sigma} = h_\sigma.$$

Second, let for a point $z \in (\text{int}V_\sigma \setminus \text{cl}(\tilde{u}_\sigma))$ be $\{z_0\} = \mathcal{O}_z \cap \partial\tilde{u}_\sigma$ and $f^{t_z}(z_0) = z$, for $t_z \in \mathbb{R}$; for a point $z \in \partial V_\sigma \setminus \partial\tilde{u}_\sigma$ let t_z be such that $|t_z| = \min_{|t|} \{t \in \mathbb{R} \mid f^{-t}(z) \in \partial\tilde{u}_\sigma\}$ and $z_0 = f^{-t_z}(z)$; then

$$h_{V_\sigma}(z) = f'^{t_z}(h_\sigma(z_0)).$$

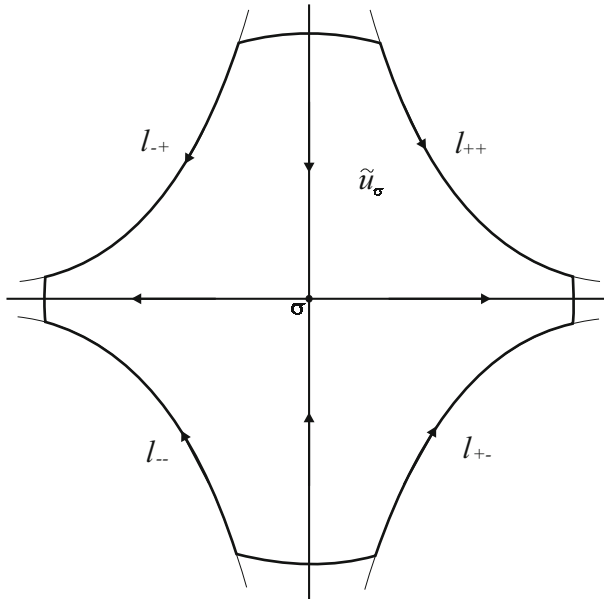


Fig. 12 Neighbourhood \tilde{u}_σ of a saddle σ ; $l_{++}, l_{+-}, l_{-+}, l_{--}$ are trajectories of f^t

Let us show that h_{V_σ} is a conjugating homeomorphism. Note that for each point of $\partial\tilde{u}_\sigma$ the homeomorphism h_σ is well defined. Let $\hat{z} \in \text{cl}\tilde{u}_\sigma$. Note that $\hat{z}' = h_\sigma(\hat{z}) \in \text{cl}\tilde{u}_{\sigma'}$. Let $\bar{z} \in \mathcal{O}_{\hat{z}} \cap \text{cl}\tilde{u}_\sigma$. Then there is a point $\bar{z}' = h_\sigma(\bar{z})$ such that $\bar{z}' \in \text{cl}\tilde{u}_{\sigma'}$ and a value \bar{t} such that $h_\sigma(f^{\bar{t}}(\hat{z})) = f^{\bar{t}'}(h_\sigma(\hat{z}))$, because h_σ is the conjugating one.

Let $\hat{z}, \bar{z} \in \mathcal{O}_{z_0} = \mathcal{O}_z$, where z and z_0 are defined as before. Let \hat{t} be such that $f^{\hat{t}}(\hat{z}) = z$ and let \bar{t} be such that $f^{\bar{t}}(\hat{z}) = z_0$; note that $\hat{t} = \bar{t} + t_z$, where t_z is defined as before. Then $h_{V_\sigma}(f^{\hat{t}}(\hat{z})) = h_{V_\sigma}(f^{t_z}(f^{\bar{t}}(\hat{z}))) = [def] = f^{t_z}(h_\sigma(f^{\bar{t}}(\hat{z}))) = f^{t_z}(f^{\bar{t}'}(h_\sigma(\hat{z}))) = f^{\bar{t}'+t_z}(h_\sigma(\hat{z})) = f^{\hat{t}'}(h_\sigma(\hat{z})) = f^{\hat{t}'}(h_{V_\sigma}(\hat{z}))$, because $h_{V_\sigma}|_{\tilde{u}_\sigma} = h_\sigma$.

Let $V (V')$ be the union of all $V_\sigma (V_{\sigma'})$ and let $h_V : V \rightarrow V'$ be a homeomorphism such that $h_V(z) = h_{V_\sigma}(z)$ if $z \in V_z$.

Next we want to extend the homeomorphism h_V up to ambient conjugating homeomorphism. Note that there exists a sink ω such that the closure T of any connected component of the set $S \setminus (V \cup \Omega_{f^t})$ belongs to the basin of ω . As $h^{-1}h_\sigma$ preserves a separatrix of σ then there is the closure $T' \subset W_{\omega'}^s$ of an unique connected component of the set $S \setminus (V' \cup \Omega_{f^{t'}}$), such that $h(T) \cap T' \neq \emptyset$, also $\omega' = h(\omega)$. Let us extend h_V to T by means of conjugating homeomorphism h_T .

By Proposition 2.1, flows $f^t|_{u_\omega}$ and $f^{t'}|_{u_{h(\omega)}}$ are conjugate by means of ψ_ω and $\psi_{\omega'}$, respectively, to the restriction of the flow $a^t|_{U_0}$. Let $\gamma_0 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = \frac{1}{2}\}$ be some closed curve without a contact, transversally crossing all trajectories of the linear flows, and let $\gamma = \psi_\omega^{-1}(\gamma_0)$, $\gamma' = \psi_{\omega'}^{-1}(\gamma_0)$. So, we correctly constructed a closed curve without a contact around ω and ω' .

Let $J_T = \gamma \cap T$ and let a_0, a_1 be the endpoints of the arc J_T . Then there are saddle points σ_0, σ_1 (possible $\sigma_0 = \sigma_1$), such that $a_i \in (J_T \cap V_{\sigma_i}), i = 0, 1$. Similarly, the

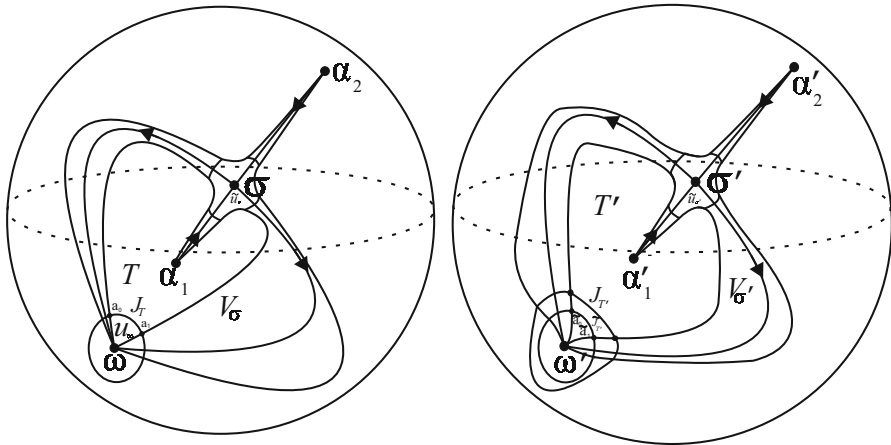


Fig. 13 Main constructions for f^t (on the left sphere) and for f'^t (on the right sphere); $\alpha_1, \alpha_2, \alpha'_1, \alpha'_2$ are sources, σ, σ' are saddles, \tilde{u}_σ is the neighbourhood of $\sigma, V_\sigma, V_{\sigma'}, T, T'$ are the domains, ω, ω' are sinks, \tilde{u}_ω is the ω 's neighbourhood, $J_T, J_{T'}, \tilde{J}_T$ are the cross-sections, a_0, a_1 are the endpoints of $J_T, \tilde{a}_0, \tilde{a}_1$ are the endpoints of \tilde{J}_T

arc $\tilde{J}_{T'} = \gamma' \cap T'$ is bounded by the points \tilde{a}_0, \tilde{a}_1 belonging to $V_{\sigma'_0}, V_{\sigma'_1}$, accordingly. Let $t_0, t_1 \in \mathbb{R}$, so that $f'^{t_i}(\tilde{a}_i) = h_V(a_i), i = 0, 1$ and $\rho: \tilde{J}_{T'} \rightarrow [0, 1]$ be a homeomorphism, such that $\rho(\tilde{a}_i) = i, i = 0, 1$. Let

$$J_{T'} = \{f'^{t_z}(\tilde{z}) \mid \tilde{z} \in \tilde{J}_{T'}, t_z = t_0 + (t_1 - t_0)\rho(\tilde{z})\}.$$

Define an arbitrary homeomorphism $h_J: J_T \rightarrow J_{T'}$, so that $h_J(a_i) = h_V(a_i), i = 0, 1$. Then every point z in T is uniquely defined by the point $z_0 = \mathcal{O}_z \cap J_T$ and the value $t_z \in \mathbb{R}$, such that $f^{t_z}(z_0) = z$. Let us define a homeomorphism $h_T: T \rightarrow T'$ by the formula

$$h_T(f^{t_z}(z_0)) = f'^{t_z}(h_J(z_0)).$$

Let us define the conjugating homeomorphism $h_c: S \rightarrow S$, so that $h_c|_V = h_V, h_c|_T = h_T$, and $h_c|_{\Omega_{f^t}} = h|_{\Omega_{f^t}}$. Thus, the conjugating homeomorphism is constructed and Theorem is proved. □

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