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RESEARCH CONTRIBUTION

Effective Birational Rigidity of Fano Double Hypersurfaces

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Abstract

We prove birational superrigidity of Fano double hypersurfaces of index one with quadratic and multi-quadratic singularities, satisfying certain regularity conditions, and give an effective explicit lower bound for the codimension of the set of non-rigid varieties in the natural parameter space of the family. The lower bound is quadratic in the dimension of the variety. The proof is based on the techniques of hypertangent divisors combined with the recently discovered $4n^2$-inequality for complete intersection singularities.

Keywords Birational rigidity · Maximal singularity · Multiplicity · Hypertangent divisor · Complete intersection singularity

Mathematics Subject Classification 14E05 · 14E07

1 Introduction

1.1 Statement of the Main Result

Fix the integers $M \geqslant 10$, $m \geqslant 2$ and $l \geqslant 2$, satisfying the equality

$$m + l = M + 1.$$

Let $\mathbb{P} = \mathbb{P}^{M+1}$ be the complex projective space. By the symbol $\mathcal{P}_{k,M+2}$ we denote the space of homogeneous polynomials of degree $k \in \mathbb{Z}_+$ in $M + 2$ homogeneous coordinates on $\mathbb{P}$, that is, the linear space $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k))$. Let

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$$(g, h) \in \mathcal{P}_{m,M+2} \times \mathcal{P}_{2l,M+2} = \mathcal{F}$$

be a pair of irreducible polynomials.

Consider the double cover

$$\sigma\colon V \xrightarrow{2:1} G \subset \mathbb{P},$$

where $G = \{g = 0\} \subset \mathbb{P}$ is an irreducible hypersurface of degree m and σ is branched over the divisor $W = \{h|_G = 0\} \subset G$, which is cut out on G by the hypersurface $W_{\mathbb{P}} = \{h = 0\}$. The variety V can be realized as a complete intersection of codimension 2 in the weighted projective space

$$\mathbb{P}(\underbrace{1, \ldots, 1}_{M+2}, l) = \mathbb{P}(1^{M+2}, l)$$

with the homogeneous coordinates $x_0, \ldots, x_{M+1}$ of weight 1 and the new homogeneous coordinate u of weight l:

$$V = \{g = 0,\ u^2 = h\}.$$

If the variety V is factorial and its singularities are terminal, then V is a primitive Fano variety:

$$\operatorname{Pic} V = \mathbb{Z}H, \quad K_V = -H,$$

where H is the class of "hyperplane section", corresponding to $\sigma^*\mathcal{O}_{\mathbb{P}}(1)|_G$. It makes sense now to test V for being birationally (super)rigid. In Pukhlikov (2000) it was shown that a Zariski general non-singular variety V is birationally superrigid. The aim of this paper is to generalize and strengthen that result in the following way.

Let us define the integer-valued function

$$\xi\colon \mathbb{Z}_{\geqslant 10} = \{M \in \mathbb{Z} \mid M \geqslant 10\} \to \mathbb{Z}_+,$$

setting $\xi(M) = \frac{(M-9)(M-8)}{2} + 12$.

For simplicity of notations, we identify a pair of irreducible polynomials $(g, h) \in \mathcal{F}$ with the corresponding Fano double cover V and write $V \in \mathcal{F}$; this can not lead to any confusion. Now we can state the main result of the paper.

Theorem 1 *There exists a Zariski open subset $\mathcal{F}_{\mathrm{reg}} \subset \mathcal{F}$ such that the following claims are true.*

(i) *Every variety $V \in \mathcal{F}_{\mathrm{reg}}$ is factorial and has at most terminal singularities.*
(ii) *The complement $\mathcal{F} \backslash \mathcal{F}_{\mathrm{reg}}$ is of codimension at least $\xi(M)$ in $\mathcal{F}$.*
(iii) *Every variety $V \in \mathcal{F}_{\mathrm{reg}}$ is birationally superrigid.*

Corollary 1 *For every variety $V \in \mathcal{F}_{\mathrm{reg}}$ the following claims are true.*

(i) *Every birational map $V \dashrightarrow V'$ to a Fano variety with $\mathbb{Q}$-factorial terminal singularities and Picard number 1 is a biregular isomorphism.*
(ii) *There are no rational dominant maps $V \dashrightarrow S$ onto a positive-dimensional variety S, the general fibre of which is rationally connected (or has negative Kodaira dimension). In particular, there are no structures of a Mori fibre space over a positive-dimensional base on V.*
(iii) *The variety V is non-rational and its groups of birational and biregular automorphisms are the same:* $\mathrm{Bir}\, V = \mathrm{Aut}\, V$.

Proof of the corollary The claims (i)–(iii) are all the standard implications of the property of being birationally superrigid, see, for instance (Pukhlikov 2013, Chapter 2). □

1.2 The Regularity Conditions

The open subset $\mathcal{F}_{\mathrm{reg}} \subset \mathcal{F}$ is defined by a number of explicit local conditions, to be satisfied at every point, which we now list. Let $o \in V$ be a point, $p = \sigma(o) \in G$ its image on $\mathbb{P}$. We assume, therefore, that $g(p) = 0$. Let $z_1, \dots, z_{M+1}$ be a system of affine coordinates on $\mathbb{P}$ with the origin at p and

$$g = q_1 + q_2 + \dots + q_m, \quad h = w_0 + w_1 + w_2 + \dots + w_{2l}$$

the decomposition of g, h (dehomogenized but for simplicity of notations denoted by the same symbols) into components, homogeneous in z_*. We may assume that $z_i = x_i / x_0$ are coordinates on the affine chart $\{x_0 \neq 0\}$ on $\mathbb{P}$. Adding the new affine coordinate $y = u / x_0^l$, we extend that chart to

$$\mathbb{A}^{M+2}_{z_*, y} \subset \mathbb{P}\left(1^{M+2}, l\right),$$

where the variety V is a complete intersection, given by the system of two equations:

$$\begin{aligned} q_1 + q_2 + \dots + q_m &= 0, \\ -y^2 + w_0 + w_1 + w_2 + \dots + w_{2l} &= 0. \end{aligned}$$

Note that $p \in W$ if and only if $w_0 = 0$.

We assume that the hypersurface $G \subset \mathbb{P}$ has at most quadratic singularities: if $q_1 \equiv 0$, then $q_2 \not\equiv 0$. Furthermore, we assume that G is *regular* in the standard sense at very point $p \in G$:

(R0.1) If $q_1 \not\equiv 0$, then the sequence

$$q_1, \ q_2, \dots, q_m$$

is regular in $\mathcal{O}_{p,\mathbb{P}}$.

(R0.2) If $q_1 \equiv 0$, then the sequence

$$q_2, \ldots, q_m$$

is regular in $\mathcal{O}_{p,\mathbb{P}}$.

We will need also some additional regularity conditions for the polynomials g, h at the point p, which depend on whether $p \in W$ or $p \notin W$ and on the type of singularity $o \in V$ that we allow.

We start with the *non-singular case*.

(R1.1) If $w_0 \neq 0$, then we have no additional conditions [only (R0.1) is needed].
(R1.2) If $w_0 = 0$, then

$$q_2|_{\{q_1=w_1=0\}} \not\equiv 0.$$

Note that in the second case as the point $o \in V$ is assumed to be non-singular, the linear forms q_1 and w_1 must be linearly independent.

Now let us consider the *quadratic case*.

Here we have three possible ways of getting a singular point and, accordingly, three types of regularity conditions.

(R2.1) Out side the ramification divisor: if $w_0 \neq 0$, then $q_1 \equiv 0$ and

$$\text{rk}\, q_2 \geqslant 7.$$

(R2.2) On the ramification divisor with G non-singular: $w_0 = 0$, $q_1 \not\equiv 0$, $w_1 \equiv 0$ and

$$\text{rk}\, w_2|_{\{q_1=0\}} \geqslant 6.$$

(R2.3) On the ramification divisor with G singular: $w_0 = 0$, $q_1 \equiv 0$, $w_1 \not\equiv 0$ and

$$\text{rk}\, q_2|_{\{w_1=0\}} \geqslant 7.$$

Apart from non-singular points and quadratic singularities, we allow more complicated points which we call *bi-quadratic*. Assume that $w_0 = 0$ and $q_1 \equiv w_1 \equiv 0$.

(R2^2) For a general 11−dimensional linear subspace $P \subset \mathbb{C}^{M+2}_{z_*,y}$ the closed algebraic set

$$Q_P = \left\{ q_2|_P = (y^2 - w_2)|_P = 0 \right\} \subset \mathbb{P}(P) \cong \mathbb{P}^{10}$$

is a non-singular complete intersection of codimension 2.

We say that a pair $(g, h) \in \mathcal{F}$ is *regular* if the hypersurface $G = \{g = 0\} \subset \mathbb{P}$ is regular at every point in the sense of the conditions (R0.1) and (R0.2) (whichever applies at the given point), and the relevant regularity condition from the list above is satisfied at every point $o \in \{g = u^2 - h = 0\}$.

Note that $(g, h) \in \mathcal{F}$ being regular implies that the closed set

$$V = \{g = u^2 - h = 0\} \subset \mathbb{P}(1^{M+2}, l)$$

is an irreducible complete intersection of codimension 2, the singular points of which are either quadratic singularities of rank $\geqslant 7$ or bi-quadratic singularities satisfying the condition (R2^2). In any case, the singularities of V are complete intersection singularities and the singular locus Sing V has codimension at least 7 in V, so the Grothendieck theorem on parafactoriality (Call and Lyubeznik 1994) applies and V turns out to be a factorial variety. Furthermore, it is easy to check that the property of having at most quadratic singularities of rank $\geqslant r$ is stable with respect to blowing up non-singular subvarieties [see (Pukhlikov 2015, Section 3.1) for a detailed proof and discussion, and the same arguments apply to bi-quadratic singularities satisfying (R2^2)], so that, in particular, the singularities of V are terminal.

Now setting $\mathcal{F}_{\text{reg}} \subset \mathcal{F}$ to be the open subset of regular pairs (g, h) (or, abusing the notations, regular varieties $V = V(g, h)$), we get the claim (i) of Theorem 1.

Therefore, Theorem 1 is implied by the following two claims.

Theorem 2 *The complement $\mathcal{F} \backslash \mathcal{F}_{\text{reg}}$ is of codimension at least $\xi(M)$ in $\mathcal{F}$.*

Theorem 3 *A regular variety $V \in \mathcal{F}_{\text{reg}}$ is birationally superrigid.*

1.3 The Structure of the Paper

We prove Theorem 3 in Sect. 2 and Theorem 2 in Sect. 3. The arguments are independent of each other.

In order to prove Theorem 3, we assume the converse: V is *not* birationally superrigid. This implies, in a standard way (Pukhlikov 2013, Chapter 2, Section 1) that there is a mobile linear system $\Sigma \subset |nH|$ with a *maximal singularity*. The centre of the maximal singularity is an irreducible subvariety $B \subset V$. There are a number of options for B: it can have a small ($\leqslant 4$) codimension or a higher ($\geqslant 5$) codimension in V, be contained or not contained in the singular locus Sing V (and more specifically, in the locus of bi-quadratic points), be contained or not contained in the ramification divisor. For each of these options, we *exclude* the maximal singularity, that is, we show that its existence leads to a contradiction. After that, we conclude that the initial assumption was incorrect and V is birationally superrigid.

Theorem 2 is shown by different and very explicit arguments. We fix a point $o \in \mathbb{P}(1^{M+2}, l)$ and consider varieties $V \ni o$. For each type of the point o (from the list given in Sect. 1.2) and each regularity condition we estimate the codimension of the closed set of pairs $(g, h) \in \mathcal{F}$ such that $o \in \{g = u^2 - h = 0\}$ and the condition under consideration is violated. Taking the minimum of our estimates, we prove Theorem 2.

The decisive point of this paper is applying the generalized $4n^2$-*inequality* (Pukhlikov 2017) to excluding the maximal singularities, the centre of which is contained in the quadratic or bi-quadratic locus: without it, the task would have been too hard. The regularity conditions make sure that the generalized $4n^2$-inequality applies. Given the new essential ingredient, excluding the maximal singularity becomes straightforward.

1.4 Historical Remarks

We say that a theorem stating birational (super)rigidity is *effective*, if it contains an effective bound for the codimension of the set of non-rigid varieties (in the natural parameter space of the family under consideration). The first effective result was obtained in Eckl and Pukhlikov (2014). For complete intersections see (Evans and Pukhlikov 2017, 2018). The importance of effective results is explained by the problem of birational rigidity of Fano–Mori fibre spaces, see (Pukhlikov 2015), generalizing the famous Sarkisov theorem (Sarkisov 1982) to fibre spaces with higher-dimensional fibres.

Birational rigidity of certain mildly singular Fano double covers was shown in Cheltsov (2006) and Johnstone (2017). The result of Johnstone (2017) was effective in our sense. Iterated double covers and cyclic covers of degree $\geqslant 3$ were considered in Pukhlikov (2003, 2009), respectively (only non-singular varieties were treated in these papers). Triple covers with singularities were shown to be birationally superrigid in Cheltsov (2004). For a study of the question, how many families of higher-dimensional non-singular Fano complete intersections are there in the weighted complete intersections, see (Przyjalkowski and Shramov 2016).

2 Proof of Birational Superrigidity

In this section we prove Theorem 3. First, we remind the definition and some basic facts about maximal singularities, classifying them and excluding the cases of low codimension of the centre (Sect. 2.1). Then we exclude the maximal singularities, the centre of which is not contained in the singular locus of V (Sect. 2.2). Finally, we exclude the cases when the centre of a maximal singularity is contained in the singular locus (Sect. 2.3). The last group of cases, which traditionally was among the hardest to deal with, now becomes the easiest due to the generalized $4n^2$-inequality shown in Pukhlikov (2017).

2.1 Maximal Singularities

Assume that a fixed regular double hypersurface $V \in \mathcal{F}_{\mathrm{reg}}$ is not birationally superrigid. It is well known [see, for instance, (Pukhlikov 2013, Chapter 2, Section 1)], that this assumption implies that there is a mobile linear system $\Sigma \subset |nH|$, a birational morphism $\varphi\colon \widetilde{V} \to V$ and a φ-exceptional prime divisor $E \subset \widetilde{V}$, satisfying the *Noether–Fano inequality*

$$\mathrm{ord}_E\, \varphi^* \Sigma > n \cdot a(E).$$

Here $\widetilde{V}$ is assumed to be non-singular projective, φ a composition of blow ups with non-singular centres, $a(E) = a(E, V)$ is the discrepancy of E with respect to V. The prime divisor E (or the discrete valuation of the field of rational functions $\mathbb{C}(\widetilde{V}) \cong \mathbb{C}(V)$) is called a *maximal singularity* of the system Σ. Equivalently, for any divisor $D \in \Sigma$

the pair $(V, \frac{1}{n}D)$ is not canonical with E a non-canonical singularity of the pair. Set $B = \varphi(E) \subset V$ to be its centre on V and $\overline{B} = \sigma(B) \subset G$ its projection on $\mathbb{P}$. We have the following options:

(1) $\mathrm{codim}(B \subset V) = 2$,
(2) $\mathrm{codim}(B \subset V) = 3$ or 4,
(3) $\mathrm{codim}(B \subset V) \geqslant 5$ and $\overline{B} \not\subset W$, $B \not\subset \mathrm{Sing}\, V$,
(4) $\mathrm{codim}(B \subset V) \geqslant 5$ and $\overline{B} \subset W$, $B \not\subset \mathrm{Sing}\, V$,
(5) B is contained in the (closure of the) locus of quadratic singularities, but not in the locus of bi-quadratic singularities,
(6) B is contained in the locus of bi-quadratic singularities.

We have to show that none of these cases take place. Note that the inequality

$$\mathrm{mult}_B\, \Sigma > n \tag{1}$$

holds. Let $Z = (D_1 \circ D_2)$ be the algebraic cycle of scheme-theoretic intersection of general divisors $D_1, D_2 \in \Sigma$, the *self-intersection* of the system Σ. Note that $Z \sim n^2 H^2$.

Our first observation is that the case (1) does not realize. Indeed, let P be a general 7-dimensional plane in $\mathbb{P}$. Then $V_P = V \cap \sigma^{-1}(P)$ is a non-singular 6-dimensional variety. By the Lefschetz theorem,

$$\mathrm{Pic}\, V_P = \mathbb{Z}H_P \quad \text{and} \quad A^2 V_P = \mathbb{Z}H_P^2,$$

where H_P is the hyperplane section and A^2 the numerical Chow group of codimension 2 cycles. The restriction $Z_P = (Z \circ V_P) \sim n^2 H_P^2$ is an effective cycle. If $\mathrm{codim}(B \subset V) = 2$, then Z contains B as a component with multiplicity at least $(\mathrm{mult}_B\, \Sigma)^2$; therefore, Z_P contains $B_P = (B \circ V_P) = B \cap V_P$ with multiplicity at least $(\mathrm{mult}_B\, \Sigma)^2$. However, $B_P \sim b H_P^2$ for some $b \geqslant 1$ and the inequality (1) can not be true. So we may assume that $\mathrm{codim}(B \subset V) \geqslant 3$.

Proposition 2.1 *The case* (2) *does not realize.*

Proof Assume the converse: $\mathrm{codim}(B \subset V) \in \{3, 4\}$. Then $B \not\subset \mathrm{Sing}\, V$ and so the standard $4n^2$-inequality holds:

$$\mathrm{mult}_B\, Z > 4n^2,$$

see (Pukhlikov 2013, Chapter 2). Again, take a general 7-dimensional plane $P \subset \mathbb{P}$ and let V_P, Z_P, H_P and B_P mean the same as above. We can find an irreducible subvariety $Y \sim d H_P^2$ of codimension 2 in V_P such that

$$\mathrm{mult}_{B_P}\, Y > 4d.$$

Set $G_P = G \cap P$: it is a non-singular hypersurface of degree m in $P \cong \mathbb{P}^7$. Writing H_G for the class of its hyperplane section, we get

$$\mathrm{Pic}\, G_P = \mathbb{Z}H_G \quad \text{and} \quad A^2 G_P = \mathbb{Z}H_G^2.$$

Let $\overline{Y} = \sigma(Y) \subset G_P$ and $\overline{B}_P = \sigma(B_P)$ be the images of Y and B_P, respectively. Then

$$\overline{Y} \sim d^* H_G^2$$

with $d^* = d$ or $\frac{1}{2}d$, and the inequality

$$\mathrm{mult}_{\overline{B}_P} \overline{Y} > 2d^*$$

holds. But $\dim \overline{B}_P \in \{2, 3\}$, so we get a contradiction with (Pukhlikov (2002), Proposition 5) [see also "Pukhlikov's Lemma" in Suzuki (2017)]. □

From now on, we assume that $\mathrm{codim}(B \subset V) \geqslant 5$.

In order to exclude the cases (3–6), we will need the regularity conditions (R0.1, 2), or rather, the facts that are summarized in the proposition below.

Proposition 2.2 *Let $S \subset G$ be an irreducible subvariety of codimension $a \in \{2, 3\}$ and $p \in S$ a point.*

(i) *Assume that G is non-singular at p. Then*

$$\mathrm{mult}_p S \leqslant \frac{a+1}{m} \deg S.$$

(ii) *Assume that G is singular at p. Then*

$$\mathrm{mult}_p S \leqslant \frac{a+2}{m} \deg S.$$

Proof The claims are the standard implications of the regularity conditions (R0.1, 2). see, for instance, (Pukhlikov 2013, Chapter 3) for the standard arguments delivering the estimates for the multiplicity in terms of degree.

2.2 The Non-Singular Case

Let us exclude the options (3) and (4). Here $\overline{B} \not\in \mathrm{Sing}\, G$ and in any case $\overline{B} \not\in \mathrm{Sing}\, W$.

Proposition 2.3 *The case* (4) *does not realize.*

Proof Here we can argue in word for word the same way as in (Pukhlikov 2000, Subsection 3.3, Case 2): take a general point $o \in B$, so that $p = \sigma(o) \in W$ is a non-singular point on W. The tangent hyperplanes

$$T_p G \quad \text{and} \quad T_p W_{\mathbb{P}}$$

are distinct and their σ-preimages on V are singular. Therefore,

$$\Delta = \sigma^{-1}\left(T_p G \cap T_p W_{\mathbb{P}} \cap G\right)$$

is an irreducible subvariety of codimension 2 on V, satisfying the relations

$$\Delta \sim H^2 \quad \text{and} \quad \text{mult}_o \Delta = 4,$$

the second equality is guaranteed by the regularity condition (R1.2).

On the other hand, from the (standard) $4n^2$-inequality we get that there is an irreducible subvariety $Y \subset V$ such that

$$Y \sim dH^2 \quad \text{and} \quad \text{mult}_o Y > 4d$$

for some $d \in \mathbb{Z}_+$. Therefore, $Y \neq \Delta$, which means that Y is not contained in at least one of the two divisors

$$\sigma^{-1}(T_p G) \quad \text{and} \quad \sigma^{-1}(T_p W_{\mathbb{P}}).$$

Taking the scheme-theoretic intersection of Y with that divisor and selecting a suitable irreducible component, we obtain an irreducible subvariety $Y^* \subset V$ of codimension 3 such that

$$\text{mult}_o Y^* > \frac{4}{m} \deg_H Y^*,$$

where the symbol $\deg_H$ stands for the H-degree, that is, $\deg_H Y^* = (Y^* \cdot H^{M-3})$. The image $S = \sigma(Y^*) \subset G$ is an irreducible subvariety of codimension 3, satisfying the inequality

$$\text{mult}_p S > \frac{4}{m} \deg S.$$

We get a contradiction with the claim (i) of Proposition 2.2. □

Proposition 2.4 *The case* (3) *does not realize.*

Proof Assume the converse. Let $o \in B$ be a general point, so that $p = \sigma(o) \notin W$ and $p \notin \text{Sing}\, V$. Note that $\sigma_* \colon T_o V \to T_p G$ is an isomorphism of vector spaces. Let $\lambda \colon V^+ \to V$ be the blow up of the point o and $\lambda_G \colon G^+ \to G$ the blow up of the point p, with the exceptional divisors E^+ and E_G^+, respectively. We have the natural isomorphism

$$E^+ \stackrel{\sigma}{\longrightarrow} E_G^+ \cong \mathbb{P}^{M-1}.$$

It is well known [the "$8n^2$-inequality", see, for instance, (Pukhlikov 2013, Chapter 2, Section 4)], that there is a linear subspace $\Lambda \subset E^+$ of codimension 2 such that

$$\text{mult}_o Z + \text{mult}_\Lambda Z^+ > 8n^2,$$

where Z^+ is the strict transform of the self-intersection Z on V^+. Let $P \subset \mathbb{P}$ be a general hyperplane such that

$$\sigma^{-1}(G \cap P)^+ \supset \Lambda.$$

Set $G_P = G \cap P$; obviously, for a general P none of the irreducible components of Z is contained in $\sigma^{-1}(G_P)$. Therefore, we get a well defined effective cycle

$$Z_P = \left(Z \circ \sigma^{-1}(G_P)\right)$$

of codimension 3 on V.

Lemma 2.1 *The cycle Z_P satisfies the inequality*

$$\text{mult}_o Z_P > 8n^2.$$

Proof Let us compare the two effective cycles: the strict transform Z_P^+ of Z_P on V^+ and the cycle of the scheme-theoretic intersection $(Z^+ \circ \sigma^{-1}(G_P)^+)$. Of course, they are identically the same outside E^+, so that their difference must be supported on E^+ [we argue as in Pukhlikov (2013, Chapter 2, Section 2, Lemma 2.2)]:

$$(Z^+ \circ \sigma^{-1}(G_P)^+) = Z_P^+ + (\text{mult}_\Lambda Z^+)\Lambda + R,$$

where R is some effective cycle supported on E^+ (possibly zero). Therefore,

$$\text{mult}_o Z_P \geqslant \text{mult}_o Z + \text{mult}_\Lambda Z^+ > 8n^2,$$

as required. □

Now, taking a suitable irreducible component Y of Z_P and its image $S = \sigma(Y)$, we obtain an irreducible subvariety $S \subset G$ of codimension 3, satisfying at the non-singular point $p \in G$ the inequality

$$\text{mult}_p S > \frac{4}{m} \deg S.$$

This contradicts the claim (i) of Proposition 2.2. □

2.3 The Singular Case

It remains to exclude the options (5) and (6), where $B \subset \text{Sing}\, V$. It is here that we use the generalized $4n^2$-inequality shown in Pukhlikov (2017).

Proposition 2.5 *The cases* (5) *and* (6) *do not realize.*

Proof Assume that the case (5) takes place. Let $o \in B$ be a point of general position. The singularity $o \in V$ is a quadratic singularity, satisfying the requirements of the main theorem of Pukhlikov (2017). Therefore,

$$\operatorname{mult}_o Z > 4n^2 \cdot \operatorname{mult}_o V = 8n^2.$$

Taking a suitable irreducible component Y of Z and its image $S = \sigma(Y)$, we obtain an irreducible subvariety $S \subset G$ of codimension 2, satisfying at the quadratic point $p \in G$ the inequality

$$\operatorname{mult}_p S > \frac{4}{m} \deg S,$$

which contradicts the claim (ii) of Proposition 2.2.

The case (6) is excluded in a similar way, just for Z we get the inequality

$$\operatorname{mult}_o Z > 4n^2 \cdot \operatorname{mult}_o V = 16n^2.$$

and for S the inequality

$$\operatorname{mult}_p S > \frac{8}{m} \deg S,$$

which can not be satisfied at a quadratic point $p \in G$ by Proposition 2.2. □

Proof of Theorem 3 is now complete.

3 Estimates for the Codimension

In this section we prove Theorem 3. To this purpose, for each $M \geqslant 10$ we construct an algebraic subset $\mathcal{Z} \subset \mathcal{F}$ of codimension $\geqslant \xi(M)$, such that $\mathcal{F} - \mathcal{Z} \subset \mathcal{F}_{\text{reg}}$.

As a first step we reduce the construction to double hypersurfaces containing a fixed point $o \in \mathbb{P}(1^{M+2}, l)$: The point $[(0 : \cdots : 0) :_l 1] \in \mathbb{P}(1^{M+2}, l)$ is contained in no such double hypersurface, by its construction. For all other points $o = [o' :_l u] \in \mathbb{P}(1^{M+2}, l)$ the subset $\mathcal{F}^o \subset \mathcal{F}$ of pairs $(g, h) \in \mathcal{F}$ such that o is contained in

$$V = \{g = 0, u^2 = h\} \subset \mathbb{P}(1^{M+2}, l),$$

the double cover of $G = \{g = 0\}$ associated to (g, h), is equal to $\mathcal{P}^o_{m,M+2} \times \mathcal{P}^o_{2l,M+2}$, with

$$\mathcal{P}^o_{m,M+2} = \left\{g \in \mathcal{P}_{m,M+2} : g(o') = 0\right\} \text{ and } \mathcal{P}^o_{2l,M+2} = \left\{h \in \mathcal{P}_{2l,M+2} : u^2 = h(o')\right\}$$

affine hyperplanes of $\mathcal{P}_{m,M+2}$ resp. $\mathcal{P}_{2l,M+2}$.

Now choose a point $o_1 = [o_1' :_l u_1] \in \mathbb{P}(1^{M+2}, l) \backslash \{[0 : \cdots : 0 :_l 1]\}$ with $u_1 \neq 0$ and a point $o_2 = [o_2' :_l 0] \in \mathbb{P}(1^{M+2}, l)$.

Proposition 3.1 *For $i = 1, 2$ let $\mathcal{Z}_{o_i} \subset \mathcal{F}^{o_i}$ be algebraic subsets such that $\mathcal{F}^{o_i} \backslash \mathcal{Z}_i \subset \mathcal{F}^{o_i}_{\mathrm{reg}}$. Then there exists an algebraic subset $\mathcal{Z} \subset \mathcal{F}$ such that $\mathcal{F} \backslash \mathcal{Z} \subset \mathcal{F}_{\mathrm{reg}}$ and*

$$\mathrm{codim}_{\mathcal{F}} \mathcal{Z} \geqslant \min \left(\mathrm{codim}_{\mathcal{F}^{o_1}} \mathcal{Z}_{o_1} - M, \mathrm{codim}_{\mathcal{F}^{o_2}} \mathcal{Z}_{o_2} - M + 1 \right).$$

Proof $\mathrm{PGL}(M+2)$ acts on $\mathbb{P}(1^{M+2}, l)$ by transforming the first $M+2$ homogeneous coordinates in the standard way. This action has the three orbits $\{[0 : \cdots : 0 :_l 1]\}$, $\{[o' :_l u] \in \mathbb{P}(1^{M+2}, l) : u \neq 0\} \backslash \{[0 : \cdots : 0 :_l 1]\}$ and $\{[o' :_l u] \in \mathbb{P}(1^{M+2}, l) : u = 0\}$. Thus, for each point $o_1 \in \{u \neq 0\} \backslash \{[0 : \cdots : 0 :_l 1]\}$ resp. $o_2 \in \{u = 0\}$ we can find isomorphic algebraic subset $\mathcal{Z}_{o_1}$ resp. $\mathcal{Z}_{o_2}$ such that $\mathcal{F}^{o_1} \backslash \mathcal{Z}_{o_1} \subset \mathcal{F}^{o_1}_{\mathrm{reg}}$ resp. $\mathcal{F}^{o_2} \backslash \mathcal{Z}_{o_2} \subset \mathcal{F}^{o_2}_{\mathrm{reg}}$.

The closure $\mathcal{Z}_1$ of the union of all the $\mathcal{Z}_{o_1}$ has dimension $\leqslant \dim \mathcal{Z}_{o_1} + M + 2$, whereas the closure $\mathcal{Z}_2$ of the union of all the $\mathcal{Z}_{o_2}$ has dimension $\leqslant \dim \mathcal{Z}_2 + M + 1$. Since $\mathrm{codim}_{\mathcal{F}} \mathcal{F}^o = 2$ this implies the bound on the codimension of $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2$. □

Note that a point $o \in \{u \neq 0\}$ can only lie outside the ramification locus of a Fano double cover V, whereas a point $o \in \{u = 0\}$ must lie on the ramification locus.

3.1 Codimension Estimates for Points Outside the Ramification Locus

Choose a point $o \in \{u \neq 0\} \backslash \{[0 : \cdots : 0 :_l 1]\}$. We first treat the cases when the regularity conditions on the hypersurface $G = \{g = 0\} \subset \mathbb{P}$ fail.

Using the notation in the Introduction assume that $q_1 \not\equiv 0$. The set $S_{R0.1}$ of pairs (g, h) in $\mathcal{F}^o$ such that $q_1, \ldots, q_m$ is not a regular sequence in $\mathcal{O}_{p,\mathbb{P}}$ is a closed algebraic subset of the Zariski-open subset $\{q_1 \not\equiv 0\} \subset \mathcal{F}^o$. It is stratified according to the position where $q_1, \ldots, q_m$ is not any longer regular: Since $q_1 \not\equiv 0$ this can only happen from q_2 on, so $S_{R0.1} = S^2_{R0.1} \cup \ldots \cup S^m_{R0.1}$ with

$$S^d_{R0.1} = \{(q_1, \ldots, q_m; h) : q_2, \ldots, q_{d-1} \text{ is regular, but not } q_2, \ldots, q_d\} \subset \mathcal{F}^o.$$

for $d = 2, \ldots, m$. The set $S^d_{R0.1}$ is closed algebraic in $S_{R0.1} \backslash \bigcup_{i=2}^{d-1} S^i_{R0.1}$, thus the codimension of its Zariski closure in $\mathcal{F}^o$ is $\geqslant$ to the codimension of its intersection with the fiber in $\mathcal{F}^o$ over a fixed regular sequence $q_1, \ldots, q_{d-1}$ in this fiber, under the natural projection. By the methods in Pukhlikov (1998) this codimension is $\geqslant \binom{M+1}{d}$ for $2 \leq d \leq m$. Since $m + 2l \leqslant M + 1$ this implies:

$$\mathrm{codim}_{\mathcal{F}^o} S_{R0.1} \geqslant \binom{M+1}{2}. \tag{2}$$

If $q_1 \equiv 0$ and $S_{R0.2}$ denotes the set of pairs (g, h) in $\mathcal{F}^o$ such that $q_2, \ldots, q_m$ is not a regular sequence in $\mathcal{O}_{p,\mathbb{P}}$, we find as before a lower bound for the codimension of the closed algebraic subset $S_{R0.2}$ in $\mathcal{F}^o$:

$$\mathrm{codim}_{\mathcal{F}^o} S_{R0.2} \geqslant M + 1 + \binom{M+1}{2} = \binom{M+2}{2}. \tag{3}$$

Here, the summand $M+1$ counts the codimensions given by the vanishing of q_1.

Next, we study the case when the point o is too singular on the double cover V, that is when condition (R2.1) fails. This happens when $q_1 \equiv 0$ and $\operatorname{rk} q_2 \leqslant 6$, and we denote the closed algebraic subset of pairs (g, h) in $\mathcal{F}^o$ satisfying these conditions by $S_{R2.1}$.

Quadratic forms in $M+1$ variables correspond to symmetric $(M+1) \times (M+1)$ matrices parametrised by a $\binom{M+2}{2}$-dimensional affine space Sym_{M+1}, and the rank of a quadratic form q_2 equals the rank of the corresponding symmetric matrix A. But $\operatorname{rk} A \leqslant r$ if and only if there exists an $(M+1-r)$-dimensional vector subspace $\Lambda \subset \mathbb{C}^{M+1}$ spanned by 0-eigenvectors of A. Such matrices $A \in \operatorname{Sym}_{M+1,\leqslant r} = \{A \in \operatorname{Sym}_{M+1} : \operatorname{rk} A \leqslant r\}$ lie in the image of the incidence variety

$$\Phi = \{(A, \Lambda) : A \cdot v = 0 \text{ for all } v \in \Lambda\} \subset \operatorname{Sym}_{M+1} \times \operatorname{Gr}(M+1-r, M+1)$$

under the projection to Sym_{M+1}. This projection has 0-dimensional general fibers, for matrices of rank r, so $\operatorname{codim}_{\operatorname{Sym}_{M+1}} \operatorname{Sym}_{M+1,\leqslant r} = \dim \operatorname{Sym}_{M+1} - \dim \Phi$. On the other hand, the projection of Φ onto the Grassmannian $\operatorname{Gr}(M+1-r, M+1)$ has fibers of dimension $\binom{r+1}{2}$, so $\dim \Phi = \binom{r+1}{2} + r(M+1-r)$ and

$$\begin{aligned} \operatorname{codim}_{\operatorname{Sym}_{M+1}} \operatorname{Sym}_{M+1,\leqslant r} &= \binom{M+2}{2} - \binom{r+1}{2} - r(M+1-r) \\ &= \frac{(M+2-r)(M+1-r)}{2}. \end{aligned}$$

Setting $r = 6$ and adding the $M+1$ codimensions given by $q_1 \equiv 0$ we obtain

$$\operatorname{codim}_{\mathcal{F}^o} S_{R2.1} = M + 1 + \frac{(M-4)(M-5)}{2} = \frac{(M-4)(M-3)}{2} + 5. \qquad (4)$$

3.2 Codimension Estimates for Points on the Ramification Locus

Choose a point $o \in \{u = 0\}$. Using the notation in the Introduction o will lie on a double cover given by a pair $(g, h) \in \mathcal{F}^o$ only if $w_0 = 0$.

As for points outside the ramification locus we obtain the following two codimension bounds for subsets $S_{R0.1} \subset \mathcal{F}^o$ and $S_{R0.2} \subset \mathcal{F}^o$ where the regularity conditions on the hypersurface $G = \{g = 0\} \subset \mathbb{P}$ fail:

$$\operatorname{codim}_{\mathcal{F}^o} S_{R0.1} \geqslant \binom{M+1}{2} \qquad (5)$$

and

$$\operatorname{codim}_{\mathcal{F}^o} S_{R0.2} \geqslant \binom{M+2}{2}. \qquad (6)$$

Next, we study the set $S_{R1.2} \subset \mathcal{F}^o$ of pairs (g, h) such that o is non-singular on the associated double cover but condition (R1.2) fails. That is the case when $q_1 \not\equiv 0$,

$w_1 \not\equiv \lambda q_1$ for all $\lambda \in \mathbb{C}$ and $q_{2|\{q_1=w_1=0\}} \equiv 0$. The last identity is equivalent to $q_2 \equiv q_1 \cdot q_1' + w_1 \cdot w_1'$ for two linear forms q_1', w_1'. Since the first two conditions are open in $\mathcal{F}^o$ it is enough to determine the codimension of the set of q_2 in the space of all quadratic forms in $M+1$ variables that are of the above form for given q_1, w_1: By a change of coordinates q_1 and w_1 may be identified with two of the $M+1$ variables, thus the requested codimension equals the dimension of quadratic forms in $M-1$ variables. So we have

$$\operatorname{codim}_{\mathcal{F}^o} S_{R1.2} \geqslant \binom{M}{2}. \tag{7}$$

Pairs (g, h) for which o is a singular point on the associated double cover mapped to a non-singular point on the hypersurface $G \subset \mathbb{P}$ fail condition (R2.2) if and only if $q_1 \not\equiv 0$, $w_1 = \lambda q_1$ for some $\lambda \in \mathbb{C}$ and $\mathrm{rk}(w_2 - \lambda q_{2|\{q_1=0\}}) \leqslant 5$. The codimension of the set $S_{\mathrm{R}2.2} \subset \mathcal{F}^o$ of such pairs equals the sum of M (from $w_1 \equiv \lambda q_1$) and the codimension of quadratic forms of rank $\leqslant 5$ when restricted to a given linear form, in the space of all quadratic forms in $M+1$ variables. Since by a coordinate change we can assume that q_1 is one of the $M+1$ variables it is enough to calculate the codimension of quadratic forms of rank $\leqslant 5$ in the space of all quadratic forms in M variables. Imitating the calculations in Sect. 3.1 we obtain a lower bound for this codimension as $\frac{(M-4)(M-5)}{2}$. Adding up this leads to

$$\operatorname{codim}_{\mathcal{F}^o} S_{R2.2} \geqslant M + \frac{(M-4)(M-5)}{2} = \frac{(M-4)(M-3)}{2} + 4. \tag{8}$$

Pairs (g, h) for which o is a singular point on the associated double cover mapped to a singular point on the hypersurface $G \subset \mathbb{P}$ fail condition (R2.3) if and only if $q_1 \equiv 0$, $w_1 \not\equiv 0$ and $\mathrm{rk}(q_{2|\{w_1=0\}}) \leqslant 6$. As before we obtain a lower bound for the codimension of the set $S_{\mathrm{R}2.3} \subset \mathcal{F}^o$ of such pairs as

$$\operatorname{codim}_{\mathcal{F}^o} S_{R2.3} \geqslant (M+1) + \frac{(M-5)(M-6)}{2} = \frac{(M-5)(M-4)}{2} + 6. \tag{9}$$

Finally, we need to look at the set $S_{\mathrm{R}2.2^2} \subset \mathcal{F}^o$ of pairs (g, h) where o is a biquadratic singular point on the associated double cover failing condition R2.2^2. This is the case if and only if $q_1 \equiv 0$, $w_1 \equiv 0$ and $\{q_{2|P} = y^2 - w_{2|P} = 0\} \subset \mathbb{P}(P) \cong \mathbb{P}^{10}$ is not a non-singular 8-dimensional complete intersection for a general 11-plane $P \subset \mathbb{C}^{M+2}$. To obtain a lower bound for the codimension of $S_{\mathrm{R}2.2^2}$ in $\mathcal{F}^o$ we follow the strategy in Evans and Pukhlikov (2017, Sects. 2.2 and 2.3); our situation is much simpler but requires some adjustments.

Proposition 3.2 *If $Q = \{q_2 = y^2 - w_2 = 0\} \subset \mathbb{P}^{M+1}$ is an irreducible and reduced complete intersection with* $\operatorname{codim}_Q \mathrm{Sing}(Q) \geqslant 9$ *then $Q \cap \mathbb{P}(P)$ is non-singular for a general 11-dimensional hyperplane $P \subset \mathbb{C}^{M+2}$.*

Proof This follows from a version of Bertini's Theorem implying that $\mathrm{Sing}(Q \cap \mathbb{P}(P)) \subset \mathrm{Sing}(Q)$ for a general hyperplane $P \subset \mathbb{C}^{M+2}$ (see Hartshorne 1978, II.Thm. 8.18), and the fact that a general 10-dimensional hyperplane $\mathbb{P}(P)$ will not intersect the $\geqslant 11$-codimensional algebraic subset $\mathrm{Sing}(Q) \subset \mathbb{P}^{M+1}$.

The proposition shows that is enough to find lower bounds for the codimension of the set of pairs $(g, h) \in \mathcal{F}^o$ such that $q_1 \equiv 0$, $w_1 \equiv 0$ and $Q = \{q_2 = y^2 - w^2 = 0\} \subset \mathbb{P}^{M+1}$ is reducible or non-reduced, and the (Zariski closure of the) set of pairs (g, h) such that $q_1 \equiv 0$, $w_1 \equiv 0$ and $\mathrm{codim}_Q \mathrm{Sing}(Q) \leqslant 8$. In both cases we have $\mathrm{codim}_{\mathbb{P}^{M+1}} Q = 2$ as long as $q_2 \not\equiv 0$ since then q_2 cannot have a factor in common with $y^2 - w_2$.

We split up the first set into pairs where the quadric $Q_2 = \{q_2 = 0\} \subset \mathbb{P}^{M+1}$ is reducible or non-reduced, and pairs where Q_2 is irreducible and reduced and Q not. Q_2 is reducible or non-reduced if and only if q_2 is a product of two linear forms. The set of such quadrics has codimension $\binom{M+2}{2} - 2(M+1)$ in $\mathcal{P}_{2,M+1}$, so the codimension of this component of the first set in $\mathcal{F}^o$ is

$$2(M+1) + \binom{M+2}{2} - 2(M+1) = \binom{M+2}{2}. \tag{10}$$

Next we assume that Q_2 is irreducible and reduced. By Grothendiecks Parafactoriality Theorem (Call and Lyubeznik 1994) and the Lefschetz Theorem for Picard groups (Lazarsfeld 2004, Ex. 3.1.35) classes of Weil divisors on Q_2 are classes of restrictions of hypersurfaces in $\mathbb{P}^{M+1}$. Furthermore,

$$H^0\left(\mathbb{P}^{M+1}, \mathcal{O}_{\mathbb{P}^{M+1}}(a)\right) \to H^0\left(Q_2, \mathcal{O}_{Q_2}(a)\right)$$

is surjective for all integers $a \geq 0$, bijective for $a \neq 2$ and has kernel $\mathbb{C} \cdot q_2$ for $a = 2$. Thus, Q is reducible or non-reduced if and only if $y^2 - w_2 + \lambda q_2$ is a product of linear forms, for some $\lambda \in \mathbb{C}$. But this is only possible if $w_2 - \lambda q_2$ is a square of a linear form. For fixed q_2 such w_2 form a set of codimension $\binom{M+2}{2} - (M+1) - 1$ in $\mathcal{P}_{2,M+1}$, so the codimension of this component of the first set in $\mathcal{F}^o$ is

$$2(M+1) + \binom{M+2}{2} - (M+1) - 1 = \frac{(M+4)(M+1)}{2} - 1. \tag{11}$$

Now assume that Q is an irreducible and reduced complete intersection of dimension $M-1$ and $\mathrm{codim}_Q \mathrm{Sing}(Q) \leqslant 8$. A point $p \in Q$ is a singularity of Q if and only if the tangent space to Q_2 in p is contained in the tangent space to $W_2 = \{y^2 - w_2 = 0\}$ in p, or vice versa. In both cases there exists a $\lambda = (\lambda_1 : \lambda_2) \in \mathbb{P}^1$ such that $W(\lambda) = \lambda_1 q_2 + \lambda_2 (y^2 - w_2)$ has a singularity in p. Thus

$$\mathrm{Sing}(Q) \subset \bigcup_{\lambda \in \mathbb{P}^1} \mathrm{Sing}(W(\lambda)),$$

and since $\dim \mathrm{Sing}(Q) \geqslant M - 9$ we have $\max_{\lambda \in \mathbb{P}^1} \dim \mathrm{Sing}(W(\lambda)) \geqslant M - 10$. Since $W(\lambda)$ is the vanishing locus of a quadric in $M + 2$ variables, $\dim \mathrm{Sing}(W(\lambda)) = M + 1 - \mathrm{rk}(W(\lambda))$, and this implies $\min_{\lambda \in \mathbb{P}^1} \mathrm{rk}(W(\lambda)) \leqslant 11$.

We distinguish two cases: If $\mathrm{rk}(q_2) \leq 11$ the inequality above is satisfied for $\lambda = (1 : 0)$. The codimension of this component of the second set in $\mathcal{F}^o$ where q_2 satisfies this condition is $\geqslant$ to

$$2(M+1)+\frac{(M-9)(M-10)}{2}=\frac{(M-9)(M-6)}{2}+20. \tag{12}$$

If $\mathrm{rk}(q_2)>11$ we must find a $\mu\in\mathbb{C}$ such that $\mathrm{rk}(y^2-w_2+\mu q_2)\leqslant 11$. This is the case if and only if $\mathrm{rk}(w_2-\mu q_2)\leqslant 10$, so for fixed q_2 the quadratic polynomial w_2 lies in the cone in $\mathcal{P}_{2,M+1}$ spanned by the vertex q_2 and all the quadratic polynomials of rank $\leqslant 10$ in $M+1$ variables. This cone has codimension $\frac{(M-8)(M-9)}{2}-1$ in $\mathcal{P}_{2,M+1}$, so the Zariski closure of the set of all pairs (g,h) in $\mathcal{F}^o$ where q_2 and w_2 satisfy the above conditions has codimension $\geqslant$ to

$$2(M+1)+\frac{(M-8)(M-9)}{2}-1=\frac{(M-8)(M-5)}{2}+17 \tag{13}$$

3.3 Proof of Theorem 2

Using the estimates (2)–(13) Proposition 3.1 tells us that we will obtain a lower bound for the codimension of the regular locus $\mathcal{F}_{\mathrm{reg}}$ in $\mathcal{F}$ by subtracting M from the minimum of

$$\binom{M+1}{2},\ \binom{M+2}{2},\ \frac{(M-4)(M-3)}{2}+5,$$

subtracting $M-1$ from the minimum of

$$\binom{M+1}{2},\ \binom{M+2}{2},\ \binom{M}{2},\ \frac{(M-4)(M-3)}{2}+4,$$
$$\frac{(M-5)(M-4)}{2}+6,\ \binom{M+2}{2},\ \frac{(M+4)(M+1)}{2}-1,$$
$$\frac{(M-9)(M-6)}{2}+20,\ \frac{(M-8)(M-5)}{2}+17$$

and taking the smaller of the two numbers. For each $M\geqslant 10$ an elementary calculation yields the lower bound $\xi(M)$ as defined in Sect. 1.

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