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RESEARCH CONTRIBUTION

On Generic Semi-simple Decomposition of Dimension Vector for an Arbitrary Quiver

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Abstract
Generic (canonical) decomposition of dimension vector for a quiver was introduced by Victor Kac as characterizing the generic module indecomposable summands dimensions, hence, the generic orbit. Derksen and Weyman proposed an elegant algorithm to compute that decomposition, extensively using Schofield's results. We consider generic *semi-simple* decomposition, which corresponds to generic *closed* orbit and provide a simple and fast algorithm to compute this decomposition. Generic semi-simple decomposition has two useful application. First, it reduces the computation of generic decomposition to the case of quiver without oriented cycles in a geometric way. Second, it provides a nice novel presentation of the algebra of invariants of quiver representations as a tensor product of similar algebras for the summands.

Keywords Quiver · Orthogonal sequence · Generic decomposition

Mathematics Subject Classification 14L30 · 16G20

1 Introduction

Let Q be a quiver, that is an oriented graph with the sets Q_0 of vertices and Q_1 of arrows together with two maps, $t, h : Q_1 \to Q_0$ such that $t\varphi$ and $h\varphi$ are the tail and the head of an arrow $\varphi \in Q_1$. We consider representations of Q over an algebraically closed field $\mathbf{k}$ of characteristic 0 and let $\dim V \in \mathbf{Z}_+^{|Q_0|}$ denote the dimension vector of a representation, V. Let $R(Q,\alpha)$ denote the affine space of all representations of a fixed dimension vector α. The affine space $R(Q,\alpha)$ is naturally acted upon by the group $GL(\alpha) = \prod_{i\in Q_0} GL_{\alpha(i)}$ and the orbits of this action are the isomorphism classes of

Dedicated to my teacher in Mathematics Rafail Kalmanovich Gordin on his 70th anniversary.

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representations. In spirit of this correspondence the concept of *generic orbit* natural for algebraic group theory can be translated into quiver representation language. Namely, Victor Kac introduced in Kac (1982) the concept of *canonical decomposition* of dimension vector $\alpha = \sum_{i=1}^{k} \beta_i$ such that there is an open dense subset $R^0 \subset R(Q, \alpha)$ such that the summands of the decomposition into indecomposables for every $V \in R^0$ have the dimensions $\beta_1, \ldots, \beta_k$ up to permutation. Moreover, the summands of generic decomposition are *Schur* roots, which means that the generic representation of this dimension is Schur, having no endomorphisms besides scalar operators.

From our perspective and also following other authors we believe that the term *generic* is more suitable for this decomposition, because it reflects the feature and also because the term *canonical decomposition* was used before Kac for a different concept in quiver representations. Kac himself tried to approach the algorithm of finding generic decomposition using his observation that the Euler–Ringel form takes non-negative values on different summands β_i, β_j of this decomposition, $\langle \beta_i, \beta_j \rangle \geq 0$ and conjectured that a decomposition into a sum of Schur roots with the above positivity property is the generic one. For quivers without oriented cycles Schofield gave in Schofield (1992) a counter-example to the above conjecture and proposed an algorithm for the generic decomposition, but his algorithm was not efficient numerically because needed to have similar decompositions for all smaller dimension vectors, as the input. Derksen and Weyman proposed in Derksen and Weyman (2002) an elegant algorithm for quivers without oriented cycles, which incrementally updates a decomposition trying to get the positivity of Euler form on different summands, and stops when reaches this positivity. As the starting point Derksen–Weyman algorithm uses any decomposition of dimension vector into a sum of Schur roots, which constitute an *orthogonal sequence*, and such an initial decomposition can easily be provided for quivers without oriented cycles. Actually, the summands of this decomposition are simple roots corresponding to the quiver vertices. For quivers with oriented cycles this decomposition is not an orthogonal sequence anymore. However, in Derksen and Weyman (2002) generic decomposition for arbitrary quiver was reduced to that for an extended quiver without oriented cycles having twice more vertices and additionally $|Q_0|$ more arrows compared with Q. This trick is easily applied but leads to a boost of complexity. It would be natural to get the generic decomposition for arbitrary quivers in terms of equivariant geometry of $R(Q, \alpha)$ itself and using the geometrical approach to Invariant theory in the spirit of Popov and Vinberg (1994). In this paper we complete this task. We introduce the following definition:

Definition 1.1 A decomposition $\alpha = \sum_{i=1}^{k} \beta_i$ is called *semi-simple* if each summand $\beta_i, i = 1, \ldots, k$ is a dimension vector of a simple representation.

Semi-simple decompositions stand in a clear relationship with semi-simple representations. Indeed, the dimensions of simple summands of a semi-simple representations constitute a semi-simple decomposition of dimension vector. For each α, there is a generic semi-simple decomposition of α, as follows:

Theorem 1.2 *There is a generic semi-simple decomposition of α such that the semi-simple representations having simple summands corresponding to this decomposition constitute an open dense subset in the closure of the set of all semi-simple representations of dimension α.*

Theorem 1.3 *A semi-simple decomposition $\alpha = \sum_{i=1}^{k} m_i \beta_i$ with $\beta_i \neq \beta_j$ for $i \neq j$ is generic if and only if after some reordering $(\beta_1, \ldots, \beta_k)$ is an orthogonal sequence and $m_i = 1$ for $\langle \beta_i, \beta_i \rangle < 0$.*

By Theorem 1.3 the generic semi-simple decomposition can serve as initial point for the Derksen–Weyman algorithm. Therefore, to have a complete algorithm of generic decomposition for any quiver we only need a one for generic semi-simple decomposition and we do it in this note. Moreover, this decomposition and the Luna-Richardson Theorem Luna and Richardson (1979) allow for a nice description of invariants:

Theorem 1.4 *Let $\alpha = \sum_{i=1}^{k} m_i \beta_i$ be the generic semi-simple decomposition such that $\beta_i \neq \beta_j$ for $i \neq j$. Choose a vector subspace $L = \bigoplus_{i=1}^{k} R(Q, m_i \beta_i) \subseteq R(Q, \alpha)$. Then restricting $GL(\alpha)$-invariants to L gives rise to an algebra isomorphism:*

$$\mathrm{Res}|_L : \mathbf{k}[R(Q,\alpha)]^{GL(\alpha)} \xrightarrow{\sim} \bigotimes_{1 \leq i \leq k : \langle \beta_i, \beta_i \rangle \leq 0} \mathbf{k}[R(Q, m_i \beta_i)]^{GL(m_i \beta_i)}. \qquad (1)$$

Moreover, if $\langle \beta_i, \beta_i \rangle < 0$, then $m_i = 1$. If $\langle \beta_i, \beta_i \rangle = 0$, then $\mathbf{k}[R(Q, m_i \beta_i)]^{GL(m_i \beta_i)}$ is a polynomial algebra in m_i variables.

The rest of the paper is organized as follows. In Sect. 2 we prove Theorems 1.2, 1.3, and 1.4 in Sect. 3 we introduce our Algorithm 3.4 for generic semi-simple decomposition and prove it.

2 Semi-simple Decompositions

A representation V of quiver Q is an assignment of a (finite dimensional) vector space $V(i)$ over $\mathbf{k}$ for every vertex, $i \in Q_0$, and of a linear map $V(\varphi) : V(t\varphi) \to V(h\varphi)$, for every arrow $\varphi \in Q_1$. A morphism of representations $f : V \to W$ is a linear map $f(i) : V(i) \to W(i)$ defined for every $i \in Q_0$ such that for every $\varphi \in Q_1$ holds a commutative property: $f(h\varphi)V(\varphi) = W(\varphi) f(t\varphi)$. A direct sum of two representations is defined in the natural way and a representation is called indecomposable if not isomorphic to a non-trivial direct sum of representations. Krull–Remak–Schmidt theorem states that every representation V has a unique decomposition $V = \sum_{i=1}^{k} R_i$ into indecomposable summands up to permutation of summands. By dimension vector of a representation V, we mean the vector $(\dim V(i), i \in Q_0) \in \mathbf{Z}_+^n$, where $n = |Q_0|$ is the number of vertices in Q. For such a vector, $\alpha \in \mathbf{Z}_+^n$, let $R(Q, \alpha)$ denote the affine space of all representations supported by fixed vector spaces $V(i) = \mathbf{k}^{\alpha(i)}$, $i \in Q_0$.

Recall that the Euler form $\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha(i)\beta(i) - \sum_{\varphi \in Q_1} \alpha(t\varphi)\beta(h\varphi)$ was introduced by Ringel and the important equality holds

$$\langle \dim V, \dim W \rangle = \dim \mathrm{Hom}(V, W) - \dim \mathrm{Ext}^1(V, W), \qquad (2)$$

where $\mathrm{Ext}^1(V, W)$ denotes the $\mathbf{k}$-module of extensions of W by V modulo isomorphism (a reader can find the notion and statements e.g. in Assem et al. 2006, A.5).

In particular, $\mathrm{Ext}^1(V, W)$ vanishes if and only if any extension splits. The main idea of Derksen and Weyman (2002) is an extensive usage of the concept of *orthogonal sequence*:

Definition 2.1 Dimension vectors $\beta, \gamma \in \mathbf{Z}^n$ are called orthogonal, $\beta \perp \gamma$, if for generic representations $V \in R(Q, \beta)$ and $W \in R(Q, \gamma)$ holds $\mathrm{Hom}(V, W) = 0$, $\mathrm{Ext}^1(V, W) = 0$, hence, $\langle \beta, \gamma \rangle = 0$ by formula (2). A sequence $\beta_1, \beta_2, \ldots, \beta_k$ is called orthogonal if for any $i < j$ holds $\beta_i \perp \beta_j$.

Recall that Kac proved in Kac (1980) that the dimensions of indecomposable representations form a root system corresponding to the quadratic Tits form $q_Q(\alpha) = \langle \alpha, \alpha \rangle$. Simple roots are the vectors $\varepsilon_i(j) = \delta^i_j$, $i \in Q_0$. *Real* roots are those roots α such that $\langle \alpha, \alpha \rangle = 1$ and for real root $R(Q, \alpha)$ contains precisely one isomorphism class of indecomposable representations. In particular, real Schur roots have this isomorphism class open dense in $R(Q, \alpha)$. The roots, which are not real are called *imaginary* and then either $\langle \alpha, \alpha \rangle = 0$ (such roots are called *isotropic*) or $\langle \alpha, \alpha \rangle < 0$.

Throughout the algorithm of Derksen and Weyman (2002) different decompositions $\alpha = \sum_{i=1}^k m_i \beta_i$ are considered such that:

(i) each β_i is a Schur root
(ii) $\beta_i \perp \beta_j$ holds for $i < j$
(iii) $m_i > 0$ for each i and $m_i = 1$ if $\langle \beta_i, \beta_i \rangle < 0$.

Notice that the conditions (*i–iii*) "almost imply" that the summands are distinct. Indeed, assume $\beta_i = \beta_j, i < j$. Then by (ii) β_i is an isotropic root, so we might reduce the decomposition by replacing the term $m_i\beta_i + m_j\beta_j$ with $(m_i + m_j)\beta_i$ and this new decomposition fullfils (*i–iii*) because the original one does. The algorithm starts with an initial decomposition built out of simple roots corresponding to the vertices of Q, that is $\alpha = \sum_{i \in Q_0} \alpha(i)\varepsilon_i$. The condition ($i$) obviously holds and the condition (ii) holds for the ordering of vertices such that all arrows go from smaller indices to bigger ones. Indeed, such an order exists if and only if the quiver has no oriented cycles. Moreover, the condition (iii) holds because the quiver has no loops. From this initial decomposition the algorithm derives new ones with summands of bigger total dimension and keeps the above conditions (*i–iii*) hold true. In particular, the summands of the generic decomposition constitute an orthogonal sequence of Schur roots and the multiplicities of non-isotropic imaginary roots are equal to 1.

Passing to the general case of a quiver with oriented cycles and loops, we can not use the above decomposition of a dimension vector into simple roots as initial, because the condition (ii) fails for oriented cycles and the condition (iii) for loops. The following fact is well-known:

Theorem 2.2 *A representation V of Q of dimension α has a closed $GL(\alpha)$-orbit if and only if it is semi-simple.*

This fact is so obvious for experts that it is not clear, which reference to provide, so we will sketch the proof. The main observation is that for any exact sequence $0 \to R_1 \to V \to R_2 \to 0$ of quiver representation morphisms there is a 1-dimensional torus $T \subseteq GL(\alpha)$ such that $\lim_{t \to 0} tV = R_1 + R_2$. Indeed, define the action of t on

$V(i) = R_1(i) \oplus R_2(i)$ as multiplication by t and t^{-1} on $R_1(i)$ and $R_2(i)$, respectively. From this observation one can easily deduce that the sum of Jordan–Hölder factors of V belongs to the orbit closure of V, so the part "only if" of the Theorem. Conversely, let V be a semi-simple representation and the closure of $GL(\alpha)V$ contains a unique closed orbit $GL(\alpha)W$, $W \in R(Q, \alpha)$. By (Popov and Vinberg 1994, Theorem 6.9), there is a 1-dimensional torus $T \subseteq GL(\alpha)$ such that $W = \lim_{t\to 0} tV$. Let V_m, $V_{>m} \subseteq \oplus_{i\in Q_0} V(i)$ be the eigenspace of T with the minimal weight and the direct sum of other eigenspaces, respectively. We therefore have a vector space decomposition $W(i) = V(i) = V_m(i) \oplus V_{>m}(i)$ for any $i \in Q_0$. Then we claim that V_m is a submodule in V. Indeed, for any $\varphi \in Q_1$ the image $V(\varphi)V_m(t\varphi)$ must belong to $V_m(h\varphi)$ otherwise the limit $\lim_{t\to 0} tV$ does not exist. Also V_m is a submodule in each tV, so in W by continuity. By the "only if" part, W is also semi-simple, so we have semi-simple modules V_+, W_+ such that holds $V = V_m \oplus V_+$, $W = V_m \oplus W_+$ and W_+ belongs to the closure of the V_+-orbit, hence, we are done by induction on the number of irreducible summands. □

As a Corollary of the above Theorem we get Theorem 1.2, because the existence of generic closed orbit with respect to the action of reductive groups is well-known, see e.g. (Popov and Vinberg 1994, 7.4,7.5). □

For a dimension vector α of a quiver Q we denote by $S(\alpha) \subseteq Q$ the support of α, $S(\alpha)_0 = \{a \in Q_0 : \alpha(a) > 0\}$, $S(\alpha)_1 = \{\varphi \in Q_1 : t\varphi, h\varphi \in S(\alpha)_0\}$.

Corollary 2.3 *Let V be a simple representation of a quiver Q, $\alpha = \dim V$.*

1. *If $\langle\alpha, \alpha\rangle = 1$, then $S(\alpha)$ has a single vertex and no edges (loops).*
2. *If $\langle\alpha, \alpha\rangle = 0$, then $S(\alpha)$ is an oriented cycle quiver $\widetilde{A_n}$ and $\alpha(i) \le 1, i \in Q_0$.*

Proof 1. Since V is simple, then the $GL(\alpha)$-orbit of V is closed by Theorem 2.2. Since α is a Schur root, this orbit is open in $R(Q, \alpha)$, so we have $R(Q, \alpha)$ is a point, which is equivalent to α being a simple root corresponding to a vertex without loops.
2. By (Le Bruyn and Procesi 1990, Theorem 4) $S(\alpha)$ is strongly connected and the inequalities $\langle\alpha, \varepsilon_i\rangle \le 0$ and $\langle\varepsilon_i, \alpha\rangle \le 0$ hold for every $i \in Q_0$. Taking into account the equality $\langle\alpha, \alpha\rangle = \sum_{i\in Q_0} \alpha(i)\langle\alpha, \varepsilon_i\rangle = 0$, we get $\langle\alpha, \varepsilon_i\rangle = 0$ for every $i \in Q_0$ and analogously $\langle\varepsilon_i, \alpha\rangle = 0$ for every $i \in Q_0$. In other words for every $j \in S(\alpha)_0$ the equalities hold:

$$\sum_{\varphi\in Q_1, h\varphi=j} \alpha(t\varphi) = \alpha(j) = \sum_{\psi\in Q_1, t\psi=j} \alpha(h\psi). \tag{3}$$

Then for every $\varphi \in S(\alpha)_1$ the inequality $\alpha(t\varphi) < \alpha(h\varphi)$ contradicts (3) for $j = t\varphi$ and analogously $\alpha(t\varphi) > \alpha(h\varphi)$ is a contradiction. Since $S(\alpha)$ is connected, $\alpha(i)$ is constant over $S(\alpha)_0$. Then (3) implies that every $i \in S(\alpha)_0$ is incident to exactly one $\varphi \in S(\alpha)_1$ with $t\varphi = i$ and exactly one $\psi \in S(\alpha)_1$ with $h\psi = i$. Therefore $S(\alpha)$ is the cyclic quiver $\widetilde{A_n}$ for some n, and for this case (Le Bruyn and Procesi 1990, Theorem 4) yields that $\alpha(i) = 1$ for every $i \in S(\alpha)_0$. □

Now we pass to the proof of Theorem 1.3 and recall the concept of étale slice introduced in Luna (1973). For any reductive group G acting on an affine variety Z

and for any $z \in Z$ having a closed G-orbit in Z there is a G_z-stable affine locally closed subvariety $S_z \subseteq Z$, $z \in S_z$ such that the natural map $\varphi_z : G *_{G_z} S_z \to Z$, $[g, s] \to gs$ is étale, hence, the action of G in the neighborhood of z can be studied via that of G_z on S_z. For the case of quiver representations it was remarked in Le Bruyn and Procesi (1990) that the slice at a semi-simple representation $V = \sum_{i=1}^{k} m_i S_i$ is Aut(V)-isomorphic to $\mathrm{Ext}^1(V, V)$. By formula (2) and since V is semi-simple, we have $\mathrm{Aut}(V) \cong \prod_{i=1}^{k} GL(\mathbf{k}^{m_i})$,

$$\mathrm{Ext}^1(V, V) = \oplus_{1 \le i, j \le k}\, e_{ij} \mathrm{Hom}_{\mathbf{k}}(\mathbf{k}^{m_i}, \mathbf{k}^{m_j}), \quad e_{ij} = \delta^{ij} - \langle \dim S_i, \dim S_j \rangle. \quad (4)$$

So following Le Bruyn and Procesi (1990) we introduce a quiver Q_V with k vertices and e_{ij} arrows from vertex i to vertex j, in particular, e_{ii} loops at vertex i. Moreover $\gamma = (m_i, i = 1, \ldots, k)$ is a dimension vector for Q_V so that we have an equivariant isomorphism: $(\mathrm{Aut}(V), \mathrm{Ext}^1(V, V)) \cong (GL(\gamma), R(Q_V, \gamma))$.

Proposition 2.4 *V is generic semi-simple if and only if $m_i = 1$ for those i such that $\langle \dim S_i, \dim S_i \rangle \le 0$ and Q_V has no oriented cycles besides loops.*

Proof By (Luna 1973, Corollaire 6), V is generic semi-simple if and only if the only regular invariants for the action of $GL(\gamma)$ on $R(Q_V, \gamma)/R(Q_V, \gamma)^{GL(\gamma)}$ are constant functions. The invariant subspace $R(Q_V, \gamma)^{GL(\gamma)}$ is clearly generated by loops at vertices with dimension equal to 1. On the other hand, any loop at a vertex with dimension bigger than 1 and every oriented cycle on more than one vertex yield a regular non-constant $GL(\gamma)$-invariant function. By formula (4), the number of loops in Q_V at a vertex i is equal to $1 - \langle \dim S_i, \dim S_i \rangle$ so the claim follows. □

Now we can prove Theorem 1.3. Assume that a decomposition $\alpha = \sum_{i=1}^{k} m_i \beta_i$ is semi-simple generic such that $\beta_i \ne \beta_j$ for $i \ne j$ and let $V = \sum_{i=1}^{k} \sum_{t=1}^{m_i} S_i^t$ be the sum of generic representations in every summand of this decomposition. If $\langle \beta_i, \beta_i \rangle = 1$, then all simple representations of dimension β_i are isomorphic to each other, so $\sum_{t=1}^{m_i} S_i^t = m_i S_i$. Otherwise, if $\langle \beta_i, \beta_i \rangle < 0$, then $m_i \ge 2$ implies that there are at least 2 arrows of both directions between any 2 vertices out of m_i in Q_V, hence, an oriented cycle in contradiction to 2.4. Only if β_i is an isotropic root, then the term $m_i \beta_i$ yields m_i different vertices in Q_V. Anyway, by Proposition 2.4 Q_V has no oriented cycles, hence, after some reordering of the summands holds $\langle \beta_i, \beta_j \rangle = 0$ for $i < j$. Since S_i and S_j are simple and not isomorphic, then $\mathrm{Hom}(S_i, S_j) = 0$ holds, hence we conclude $\beta_i \perp \beta_j$. Conversely, if $(\beta_1, \ldots, \beta_k)$ is an orthogonal sequence, $m_i = 1$ for $\langle \beta_i, \beta_i \rangle < 0$, and V is a generic semi-simple representation corresponding to the decomposition $\alpha = \sum_{i=1}^{k} m_i \beta_i$, then all arrows of Q_V go from vertices with bigger indices to those with smaller ones and loops only exist at vertices corresponding to isotropic summands, which guarantees that V is generic by Proposition 2.4. □

Remark 2.1 In Shmelkin (2007) we introduced the concept of *locally semi-simple* representation and decomposition for quivers. This concept is similar but more subtle than semi-simple representations because it reflects semi-invariants of quivers in similar way as semi-simple ones reflect the regular invariants: the latter have closed orbits with respect to the action of $GL(\alpha)$ whereas the former have closed orbits with respect to the commutator subgroup $SL(\alpha) = (GL(\alpha), GL(\alpha))$ of $GL(\alpha)$. We proved in Shmelkin (2007) the existence of generic locally semi-simple representation and decomposition and in Shmelkin (2009) proposed an algorithm for finding this decomposition. Moreover, Theorem 4.3 from Shmelkin (2009) is similar to 1.3.

Now we prove Theorem 1.4:

Proof By Proposition 2.4 a generic semi-simple representation V has the following decomposition:

$$V = \sum_{i:\langle\beta_i,\beta_i\rangle=1} m_i V_i + \sum_{i:\langle\beta_i,\beta_i\rangle=0} V_i^1 + \cdots + V_i^{m_i} + \sum_{i:\langle\beta_i,\beta_i\rangle<0} V_i. \qquad (5)$$

In other words, the imaginary Schur roots such that $\langle\beta_i, \beta_i\rangle < 0$ have multiplicity 1 in the decomposition. The real roots such that $\langle\beta_i, \beta_i\rangle = 1$ may have arbitrary multiplicity and the generic representation in dimension $m_i\beta_i$ is the sum of isomorphic indecomposable representations of dimension β_i. The isotropic roots such that $\langle\beta_i, \beta_i\rangle = 0$ can also have arbitrary multiplicity but a generic representation is a sum of non-isomorphic indecomposable representations of dimension β_i. Then the automorphism group has the following form:

$$\mathrm{Aut}(V) = \prod_{i:\langle\beta_i,\beta_i\rangle=1} GL(m_i) \times \prod_{i:\langle\beta_i,\beta_i\rangle=0} (\mathbf{k}^*)^{m_i} \times \prod_{i:\langle\beta_i,\beta_i\rangle<0} \mathbf{k}^*. \qquad (6)$$

Since V has a generic closed $GL(\alpha)$-orbit, Theorem of Luna and Richardson (1979) claims that restricting $GL(\alpha)$-invariant functions to invariant points, $R(Q,\alpha)^{\mathrm{Aut}(V)}$, gives rise to an isomorphism:

$$\mathbf{k}[R(Q,\alpha)]^{GL(\alpha)} \cong \mathbf{k}\left[R(Q,\alpha)^{\mathrm{Aut}(V)}\right]^{N_{GL(\alpha)}(\mathrm{Aut}(V))}. \qquad (7)$$

Recall that by L we denote a selected vector subspace $L = \bigoplus_{i=1}^k R(Q, m_i\beta_i) \subseteq R(Q,\alpha)$. Given the direct sum and product decompositions in (5) and (6), respectively, it is clear that the invariant subspace $R(Q,\alpha)^{\mathrm{Aut}(V)}$ belongs to L up to $GL(\alpha)$-conjugate. By Corollary 2.3.1 the direct summand $R(Q, m_i\beta_i)$ of L with $\langle\beta_i, \beta_i\rangle = 1$ is just zero as a vector space. For other direct summands of L the corresponding factor of $\mathrm{Aut}(V)$ is a m_i-dimensional torus, so that $R(Q, m_i\beta_i)^{\mathrm{Aut}(V)}$ is the whole of $R(Q,\beta_i)$ if $\langle\beta_i,\beta_i\rangle < 0$ and the direct sum of m_i copies of $R(Q,\beta_i)$ if $\langle\beta_i,\beta_i\rangle = 0$. The centralizer $Z_{GL(\alpha)}(\mathrm{Aut}(V))$ is in this case the direct product of m_i copies of $GL(\beta_i)$. The normalizer $N_{GL(\alpha)}(\mathrm{Aut}(V))$ is a finite extension of $Z_{GL(\alpha)}(\mathrm{Aut}(V))$ acting on the above components of $L^{\mathrm{Aut}(V)}$ by permutation of those direct summands $R(Q,\beta_i)$ of $L^{\mathrm{Aut}(V)}$ such that the corresponding subgroups $GL(\beta_i)$ are $GL(\alpha)$-conjugate. Because of condition $\beta_i \neq \beta_j$ for $i \neq j$ we get the presentation as follows:

$$N_{GL(\alpha)}(\mathrm{Aut}(V)) \cong Z_{GL(\alpha)}(\mathrm{Aut}(V)) \rtimes \prod_{i:\langle\beta_i,\beta_i\rangle=0} S_{m_i}, \qquad (8)$$

where S_{m_i} is the symmetric group of m_i symbols. Hence, formulae (8) and (7) imply the isomorphism (1). Also the fact that $\mathbf{k}[R(Q, m_i\beta_i)]^{GL(m_i\beta_i)}$ for $\langle\beta_i, \beta_i\rangle = 0$ is a polynomial algebra in m_i variables follows from Corollary 2.3.2. Indeed, by (7) and (8) the algebra of invariants for a cyclic quiver in dimension vector $(m, m, \ldots, m)$ is isomorphic to that of symmetric polynomes in m variables, which is known to be polynomial algebra in m variables. □

3 Algorithm for Generic Semi-simple Decomposition

Recall that a quiver Q is called *strongly connected* if for any $i, j \in Q_0$ there is an oriented path from i to j in Q. A subquiver $J \subseteq Q$ is called *full* if it contains every arrow in Q_1 with the tail and head in J_0. In general, every quiver Q can be decomposed uniquely into maximal strongly connected components Q^i, $i = 1, \ldots, m$ such that $Q_0 = Q_0^1 \sqcup \ldots \sqcup Q_0^m$ holds and every Q_0^i is a full strongly connected subquiver.

$$Q_0 = Q_0^1 \sqcup \ldots \sqcup Q_0^m,\ Q_1^i = \{\varphi \in Q_1 : t\varphi \in Q_0^i, h\varphi \in Q_0^i\}, i = 1, \ldots, m. \qquad (9)$$

Notice that if Q has no oriented cycles, then the strongly connected components of Q are just single vertex subquivers corresponding to all vertices. In general, the strongly connected components can be thought of as vertices of a quiver $SC(Q)$. For two components, Q^i and Q^j the direction of all arrows between vertices in Q_0^i and Q_0^j is the same by maximality condition, hence, all these arrows yield one arrow in $SC(Q)_1$. Remark that for Q connected $SC(Q)$ is a connected quiver without oriented cycles.

Proposition 3.1 *Let α^i be a restriction of dimension vector α to Q_0^i and let $\alpha^i = \sum_{t=1}^{k_i} m_t^i\beta_t^i$ be the generic semi-simple decomposition of α^i as dimension vector for Q^i, $i = 1, \ldots, m$. Then $\alpha = \sum_{i=1}^{m}\sum_{t=1}^{k_i} m_t^i\beta_t^i$ is the generic semi-simple decomposition of α.*

Proof By Theorem 1.3 the components $(\beta_1^i, \ldots, \beta_{k_i}^i)$ constitute an orthogonal sequence and multiplicities of non-isotropic imaginary roots are equal to 1. Then the union of these components over all vertices of $SC(Q)$ constitutes an orthogonal sequence, because $SC(Q)$ has no oriented cycles. Hence, the total decomposition is generic semi-simple by Theorem 1.3. □

The above proposition reduces the computation of generic semi-simple decomposition to the case of strongly connected quivers.

Proposition 3.2 *Let Q be a strongly connected quiver, α a dimension vector for Q, $m = \min\{\alpha(i), i \in Q_0\}$. For $p \in \mathbf{N}$ let γ_Q^p denote the dimension vector such that $\gamma_Q^p(i) = p$ for any $i \in Q_0$. Let ε_i be the simple root such that $\varepsilon_i(j) = \delta_j^i$, $i \in Q_0$. If $m > 0$, then holds:*

(i) If Q is an oriented cycle quiver $\widetilde{A_n}$, then γ_Q^1 is dimension of simple representation and the decomposition $\alpha = m\gamma_Q^1 + \sum_{i\in Q_0, \alpha(i)>m}(\alpha(i) - m)\varepsilon_i$ is generic semi-simple.

(ii) *If Q is not isomorphic to $\widetilde{A}_n$, then γ_Q^m is dimension of simple representation and the decomposition $\alpha = \gamma_Q^m + \sum_{i \in Q_0, \alpha(i) > m} (\alpha(i) - m)\varepsilon_i$ is semi-simple.*

Proof That γ_Q^1 and γ_Q^m are dimensions of simple representations in cases (i) and (ii) respectively is proven in Le Bruyn and Procesi (1990). Other components are simple as well. In case (i) the quiver Q_V corresponding to the given decomposition is a union of a full subquiver in $\widetilde{A}_n$ corresponding to those $i \in Q_0$ such that $\alpha(i) > m$ and m additional vertices corresponding to m simple components of dimension γ_Q^1. Since γ_Q^1 is isotropic, those vertices have no arrows between each other and also have no arrows to and from a vertex corresponding to $i \in Q_0$ because $\langle \gamma_Q^1, \varepsilon_i \rangle = \langle \varepsilon_i, \gamma_Q^1 \rangle = 0$. Because $\alpha(i) = m$ for at least one vertex $i \in Q_0$, the connected components of Q_V are equioriented Dynkin quivers A_k and single vertex single loop quivers. By Proposition 2.4 the decomposition is generic semi-simple. □

We generalize Proposition 2.4 as follows:

Proposition 3.3 *Let $V = \sum_{i=1}^k m_i S_i$ be a decomposition of a semi-simple representation into simple summands. Let A_V be the $k \times n$ matrix such that ith row contains the vector* $\dim S_i$, *$\gamma = (m_1, \ldots, m_k)$ a dimension vector for Q_V. If $\gamma = \sum_{j=1}^l p_j \rho_j$ is the generic semi-simple decomposition, then $\alpha = \gamma A_V = \sum_{j=1}^l p_j [\rho_j A_V]$ is the generic semi-simple decomposition.*

Proof From Luna (1973) follows that the generic closed orbit in the slice $(GL(\gamma), R(Q_V, \gamma))$ yields the generic closed orbit in the whole $R(Q, \alpha)$, which is equivalent to the statement. □

The strongly connected components of a quiver can be computed by Tarjan's algorithm Tarjan (1972), that we refer to as *Tarjan*. *Tarjan* has linear computational complexity $O(|Q_0| + |Q_1|)$. Based on *Tarjan* we propose our algorithm of generic semi-simple decomposition. The algorithm is recursive and uses notation introduced before and in Proposition 3.3:

Algorithm 3.4 *Generic semi-simple decomposition, $GSSD(Q, \alpha)$:*
Input: *Quiver Q, dimension vector α.* **Output:** $\alpha = \sum_{i=1}^k m_i \beta_i$
1: $Q = Q^\alpha : Q_0^\alpha = \{i \in Q_0 : \alpha_i > 0\}$, $Q_1^\alpha = \{\varphi \in Q_1 : \alpha(t\varphi) > 0, \alpha(h\varphi) > 0\}$.
2: Tarjan: $Q^1, \cdots, Q^l$ maximal strongly connected components of Q.
3: $\alpha = \alpha^1 + \cdots + \alpha^l$, α_i is dimension vector of Q^i
4: If $l > 1$
5:   return $GSSD(Q^1, \alpha_1) + \cdots + GSSD(Q^l, \alpha_l)$
6: else if $Q = \widetilde{A}_n$
7:   return $\alpha = m\gamma_Q^1 + \sum_{i \in Q_0, \alpha(i) > m} (\alpha(i) - m)\varepsilon_i$, $m = \min\{\alpha(i), i \in Q_0\}$.
8: else
9: $m = \min\{\alpha(i), i \in Q_0\}$,
10: V is semi-simple corresponding to $\alpha = \gamma_Q^m + \sum_{i \in Q_0, \alpha(i) > m} (\alpha(i) - m)\varepsilon_i$
11:   return $GSSD(Q_V, \gamma) A_V$

Proposition 3.5 *Algorithm 3.4 returns the generic semi-simple decomposition and finishes after finitely many steps*

Proof That Algorithm computes the generic semi-simple decomposition follows from Propositions 3.1, 3.2, and 3.3. Notice that recursive calls in the Algorithm apply to quivers with number of vertices strongly less than $|Q_0|$ in line *5* and less than or equal to $|Q_0|$ in line *11*. It is obvious for line *5* and for line *11* the vertices of Q_V are those of Q such that $\alpha(i) > m$ (so at most $|Q_0| - 1$ of such vertices) and an additional one corresponding to γ_Q^m. In this case, if $\alpha(i) = m$ for just 1 vertex, then the number of vertices for Q_V is equal to that for Q but the average total dimension of the components strictly increases, which inductively guarantees a finite number of steps in the Algorithm. □

Remark 3.1 Both our algorithm and the one from Derksen and Weyman (2002) reduce computation of generic decomposition to a quiver without oriented cycles. We believe that, although the reduction from Derksen and Weyman (2002) is quite straightforward, our method is computationally more efficient and it also clarifies the relations between generic and generic semi-simple decompositions in the way the generic and generic closed orbits are related in the framework of geometrical approach to Invariant Theory in the spirit of Popov and Vinberg (1994). To compare the algorithms complexity, we remark that the method from Derksen and Weyman (2002) increases the number of vertices in two times and starts with a decomposition consisting of $2|\alpha|$ summands (with multiplicities) of dimension 1, where $|\alpha| = \sum_{a\in Q_0} \alpha(a)$. In our method the Derksen–Weyman algorithm for orthogonal sequences typically starts from less than $|\alpha|$ summands of larger dimension because imaginary simple summands consolidate much of dimension. As for our Algorithm 3.4, it uses Tarjan's one with linear computational complexity, and very straightforward computations in lines *7*, *10*.

References

Assem, I., Skowronski, A., Simson, D.: Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory, London Mathematical Society Student Texts. Cambridge University Press, Cambridge (2006)

Derksen, H., Weyman, J.: On the canonical decomposition of quiver representation. Compositio Math. **133**, 245–265 (2002)

Kac, V.G.: Infinite root systems, representations of graphs and invariant theory. Invent. Math. **56**(1), 57–92 (1980)

Kac, V.G.: Infinite root systems, representations of graphs and invariant theory II. J. Algebra **78**(1), 141–162 (1982)

Luna, D.: Slices étales. Bull. Soc. Math. France **33**, 81–105 (1973)

Luna, D., Richardson, R.W.: A generalization of the Chevalley restriction theorem. Duke Math. J. **46**(3), 487–496 (1979)

Le Bruyn, L., Procesi, C.: Semisimple representations of quivers. Trans. AMS **317**, 585–598 (1990)

Popov, V.L., Vinberg, E.B.: Invariant Theory. In: Parshin, A.N., Shafarevich, I.R. (eds.) Algebraic Geometry IV. Encyclopaedia of Mathematical Sciences, vol. 55. Springer, Berlin (1994)

Schofield, A.: General representations of quivers. Proc. Lond. Math. Soc. **65**, 46–64 (1992)

Shmelkin, D.A.: Locally semi-simple representations of quivers. Transform. Groups **12**, 153–173.7 (2007)

Shmelkin, D.A.: Some algorithms for semi-invariants of quivers. J. Algebra **322**, 3997–4010 (2009)

Tarjan, R.: Depth-first search and linear graph algorithms. SIAM J. Comput. **1**, 146–160 (1972)