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RESEARCH CONTRIBUTION

Laughlin states and gauge theory

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Abstract
Genus one Laughlin wavefunctions, describing the gas of interacting electrons on a two dimensional torus in the presence of a strong magnetic field, analytically continued in the filling fraction, are related to the partition functions of half-BPS surface defects in four dimensional $\mathcal{N} = 2$ supersymmetric gauge theory.

Keywords Many-body systems · Instantons · Electrons · Fractional Hall effect · Laughlin states

1 Introduction

The relation between the low-energy physics of gauge theories and many-body systems has been explored for quite some time. For example, in Gorsky and Nekrasov (1994a, b, 1995a), Nekrasov (1996) and Fock and Rosly (1999) certain many-body quantum mechanical systems were identified as subsectors of gauge theories in two, three, and four dimensions. In Nekrasov and Shatashvili (2009a, b, 2010) the Bethe/gauge correspondence has been proposed, building on the prior work above and Moore et al. (2000), Gerasimov and Shatashvili (2008, 2007). This correspondence identifies the supersymmetric vacua of gauge theories with $\mathcal{N} = (2, 2)$ $d = 2$ Poincare supersymmetry (see Witten 1993 for the introduction into the subject) with the stationary states of some quantum integrable system.

We point out a curious relation between Laughlin wavefunctions describing the many-electron states in fractional quantum Hall effect on a torus, and the partition functions of certain surface defects in supersymmetric gauge theories in four dimen-

To Raphail Kalmanovich Gordin on his 70th anniversary.

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sions. To use this relation in the analysis of the physics of strongly correlated electron systems is a challenge left for future work.

2 Laughlin states

2.1 Electrons in magnetic field

Consider N non-relativistic electrons, moving on a two-dimensional Riemann surface Σ with the metric g, subject to a magnetic field $B = dA$, A being a connection on some principal $U(1)$-bundle P over Σ. Let $\mathbf{x}_1, \dots, \mathbf{x}_N \in \Sigma$ denote the positions of electrons. The Hamiltonian governing their dynamics is given by a formula, somewhat similar to the Eq. (3.1):

$$\hat{H} = -\frac{\hbar^2}{2m^*} \sum_{i=1}^{n} \nabla^2 + \sum_{i \neq j} \mathcal{U}\left(\mathbf{x}_i, \mathbf{x}_j\right), \tag{2.1}$$

where $\nabla = d + A$, $\nabla^2 = \nabla \star \nabla$. The Hamiltonian $\hat{H}$ acts on the N-electron states, the wavefunctions $\Psi \in \mathcal{H} = \Lambda^N H$. The space H of the single particle states is the space of L^2-sections ψ of a complex line bundle $L \to \Sigma$, associated to P.

2.1.1 Kaluza–Klein picture

The bundle P is easy to describe in cases where Σ is a sphere or a torus. For $\Sigma = \mathbb{S}^2$ one can take $P = \mathbb{S}^3 \approx SU(2)$. For $\Sigma = \mathbb{T}^2$, P is the quotient of a three dimensional Euclidean space $\mathbb{R}^3$ by the action of the Heisenberg group Γ which is the central extension of the lattice $\mathbb{Z}^2$ by $\mathbb{Z}$, corresponding to the 2-cocycle $c : \mathbb{Z}^2 \to \mathbb{Z}$ given by

$$c(n_1, m_1; n_2, m_2) = n_1 m_2. \tag{2.2}$$

Explicitly, as a set $\Gamma = \mathbb{Z}^3$ with the non-abelian multiplication

$$(n_1, m_1, l_1) \cdot (n_2, m_2, l_2) = (n_1 + n_2,\ m_1 + m_2,\ l_1 + l_2 + c(n_1, m_1; n_2, m_2)) . \tag{2.3}$$

The homomorphisms $\mathbb{Z} \to \Gamma$ and $\Gamma \to \mathbb{Z}^2$ are given by $l \mapsto (0, 0, l)$ and $(m, n, l) \mapsto (m, n)$, respectively. So, Γ is a non-abelian group built out of two abelian ones. The group Γ acts on $\mathbb{R}^3$ as follows:

$$(n, m, l) \cdot (x, y, w) = (x + n, y + m, w + ny + l). \tag{2.4}$$

We can now define a charge q particle on Σ. We first study the quantization of a particle on P. The Hilbert space $\mathcal{H}_P$ is the space of L^2-functions on P. Since P has the $U(1)$-symmetry, the Hilbert space decomposes into the representations of $U(1)$:

$$\mathcal{H}_P = \bigoplus_{q\in\mathbb{Z}} \mathcal{H}_P^q \tag{2.5}$$

where $\mathcal{H}_P^q$ consists of functions ψ on P, such that $g = e^{2\pi \mathrm{i}t} \in U(1)$ acts via: $g^*\psi = e^{2\pi \mathrm{i}qt}\psi$. By definition, $\mathcal{H}$ - the space of states of the charge q particle on Σ is the subspace $\mathcal{H}_P^q$. Explicitly: $\psi \in \mathcal{H}_P^q$ iff

$$\psi(x+n, y+m, w+l+ny) = \psi(x, y, w)\,, \qquad \psi(x, y, w) = e^{2\pi \mathrm{i}qw}\chi(x, y) \tag{2.6}$$

We should stress that the charge q here is really the first Chern class of the line bundle L over Σ whose sections are the wavefunctions of the particle on Σ subject to magnetic field. In physical terms q is the total flux of the magnetic field through Σ.

The kinetic energy of the charged particle moving in Σ is simply the kinetic energy of an ordinary particle moving in P, i.e. the Laplacian corresponding to the $U(1)$-invariant metric on P:

$$ds_P^2 = V^{-1}(dw - xdy)^2 + \frac{A}{\tau_2}\left((dy + \tau_1 dx)^2 + \tau_2^2 dx^2\right), \tag{2.7}$$

where $\tau = \tau_1 + \mathrm{i}\tau_2$ with $\tau_2 > 0$ defines the complex structure on Σ, and $V = V(x, y)$ is some positive function on Σ:[1]

$$\hat{h} = -\frac{1}{A\tau_2}\left(\left(\tau_2\partial_y + 2\pi \mathrm{i}q\tau_2 x\right)^2 + \left(\partial_x - \tau_1\partial_y - 2\pi \mathrm{i}q\tau_1 x\right)^2\right) + 4\pi^2 q^2 V(x, y). \tag{2.8}$$

Now, let us choose $q > 0$ for definiteness, and assume, for now, that $V(x, y)$ is a constant. Then up to a positive shift the operator $\hat{h}$ is equal to:

$$\hat{h} \sim \frac{1}{A\tau_2}\bar{D}^\dagger\bar{D} = \frac{1}{A\tau_2}\left(-\partial_x + \bar{\tau}\partial_y + 2\pi \mathrm{i}q\bar{\tau}x\right)\left(\partial_x - \tau\partial_y - 2\pi \mathrm{i}q\tau x\right), \tag{2.9}$$

where

$$\bar{D} = \partial_x - \tau\partial_y - 2\pi \mathrm{i}q\tau x. \tag{2.10}$$

The ground states $\psi \in H_{\mathrm{gnd}}$ are annihilated by $\bar{D}$:

$$\bar{D}\psi = 0 \Leftrightarrow \chi(x, y) = \gamma(z)e^{\pi \mathrm{i}q\tau x^2}, \tag{2.11}$$

where $z = y + \tau x$, and (2.6) implies:

$$\gamma(z + m + n\tau)e^{2\pi \mathrm{i}qnz + \pi \mathrm{i}qn^2\tau} = \gamma(z). \tag{2.12}$$

[1] One may think of q^2V as representing the impurities.

For $q = 1$ the only solution to (2.12), up to a constant multiple, is the theta function (Mumford 1984) (a section of a line bundle L over E_τ, with $c_1(L) = 1$):

$$\gamma(z) = a\,\theta(z|\tau) := a \sum_{n\in\mathbb{Z}} e^{2\pi \mathrm{i} n z + \pi \mathrm{i} n^2 \tau}. \tag{2.13}$$

For $q > 1$, the space of solutions to (2.12) has complex dimension q. One way to parametrize the solutions is the product ansatz:

$$\gamma(z) = a \prod_{i=1}^{q} \theta(z - w_i|\tau), \tag{2.14}$$

with $w_1, \ldots, w_q$ constrained by:

$$w_1 + \cdots + w_q = 0. \tag{2.15}$$

Another option, related to the so-called real polarization in geometric quantization, is to use the periodicity in m shifts in (2.12) to write χ as:

$$\gamma(z) = \sum_{r\in\mathbb{Z}} \hat{\gamma}_r\, |r\rangle, \tag{2.16}$$

where $\hat{\gamma}_r$ obey:

$$\hat{\gamma}_{r+nq} = \hat{\gamma}_r, \tag{2.17}$$

and the orthonormal basis on the space of ground states (the lowest Landau level) is given by

$$|r\rangle \sim \sqrt{2q\tau_2} \sum_{n\equiv r(q)} e^{\pi \mathrm{i}\tau q (x+n/q)^2} e^{2\pi \mathrm{i} n y}, \tag{2.18}$$

with $r = 0, 1, \ldots, q-1$.

2.1.2 Excitations

The Hamiltonian (2.9) is easy to diagonalize exactly: $\hat{h}\chi = \epsilon_k \chi$, where

$$\chi(x, y) = \sum_{r\in\mathbb{Z}} \chi_r\,(x + r/q)\; e^{2\pi \mathrm{i} r y} e^{\pi \mathrm{i}\tau_1 q (x+r/q)^2}, \tag{2.19}$$

where (2.6) translates to the periodicity:

$$\chi_{r+q}(x) = \chi_r(x), \tag{2.20}$$

with the eigenproblem for $\hat{h}$ becoming that of the usual harmonic oscillator on $\mathbb{R}$

$$\hat{h}\chi_r = \frac{1}{A\tau_2}\left(-\partial_x^2 + \Omega^2 x^2 - \Omega\right)\chi_r\,, \qquad \Omega = 2\pi\tau_2 q, \tag{2.21}$$

so that the spectrum is equidistant with the gap $4\pi q/A$,

$$\epsilon_k = \frac{2\Omega}{A\tau_2}k = \frac{4\pi q}{A}k\,, \qquad k = 0, 1, 2, \ldots, \tag{2.22}$$

and each energy level degenerate with the degeneracy q.

2.1.3 Topology and quantization

The spectrum of the single-particle Hamiltonian in constant magnetic field is not universal. For example, on a round two-sphere $\Sigma = \mathbb{S}^2$ of area A the spectrum would be:

$$\epsilon_k = \frac{2\pi}{A}k(k + 2q + 2)\,, \qquad k = 0, 1, 2, \ldots \tag{2.23}$$

with the degeneracy $q + k + 1$. The general statement is that the space H_{gnd} of ground states is the space of holomorphic sections $H^0(\Sigma, L)$ of a line bundle L over Σ with the first Chern class $c_1(L) = q$. The Riemann–Roch formula plus Kodaira theorem imply that for $q > g - 1$ the degeneracy is equal to $q + 1 - g$. The space H_{gnd} is the Hilbert space of the geometric quantization of Σ with the Planck constant $\sim 1/q$. In this sense the symmetry of the first Landau level is the quantized algebra of functions on Σ. For $\Sigma = \mathbb{S}^2$ this algebra is a quotient of $U(\mathfrak{sl}_2)$ by the Casimir relation $X^2 + Y^2 + Z^2 = q(q + 2)$, also known as the "fuzzy sphere". For $\Sigma = \mathbb{T}^2$ this is the so-called "fuzzy torus" algebra, generated by $U_{0,1}$ and $U_{1,0}$ subject to the relation

$$U_{0,1}U_{1,0} = e^{\frac{2\pi \mathrm{i}}{q}} U_{1,0}U_{0,1}. \tag{2.24}$$

The operators

$$U_{a,b} = \Pi_{\text{gnd}} e^{2\pi \mathrm{i}(ax+by)} \Pi_{\text{gnd}}, \tag{2.25}$$

with Π_{gnd} the orthogonal projection onto H_{gnd}, act on H_{gnd} as:

$$U_{a,b}\,|r\,\rangle = e^{-\frac{\pi \mathrm{i} a(b+2r)}{q} - \frac{\pi|a-b\tau|^2}{2q\tau_2}}\,|r + b \bmod q\,\rangle. \tag{2.26}$$

Note that $U_{a,b}$ differs from $U_{1,0}^a U_{0,1}^b$ by an (a, b)-dependent scalar multiplier, while $Z_{a,b} = U_{a,b}^q = (-1)^{ab} e^{-\frac{\pi|a-b\tau|^2}{2\tau_2}}$ are in the center of the algebra.

2.1.4 Fermi statistics

The wavefunction $\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ is antisymmetric in the electron positions

$$\Psi(\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(N)}) = (-1)^{\sigma} \Psi(\mathbf{x}_1, \ldots, \mathbf{x}_N), \tag{2.27}$$

for any permutation $\sigma \in \mathcal{S}(N)$.

2.2 Interactions

With the inclusion of the electron-electron interactions the Hamiltonian has the form:

$$-\frac{\hbar^2}{2m^*}\sum_{i=1}^{N} \hat{h}_i + \sum_{i \neq j} \mathcal{U}\left(z_i - z_j, \bar{z}_i - \bar{z}_j\right), \tag{2.28}$$

where m^* is the effective mass of the electrons, and $\mathcal{U}$ is the two-body electron potential, which we shall assume isotropic at short distances and double-periodic globally:

$$\mathcal{U}(z, \bar{z}) = \int_0^{\infty} d\mathsf{a}\, u_{\mathsf{a}} \sum_{m,n \in \mathbb{Z}} e^{-\frac{\pi A}{\mathsf{a}\tau_2}|z+m+n\tau|^2}, \tag{2.29}$$

with some formfactor function u_{a}. In the limit $\Delta = \frac{2\pi\hbar^2 q}{m^* A} \gg \mathcal{U}$ the dynamics of dominated by the kinetic term forcing the wavefunction to obey, in the first approximation:

$$D_i \Psi = 0, \qquad i = 1, \ldots, N, \tag{2.30}$$

2.2.1 Degeneracy lift, QHE states

To solve the Schrödinger equation in the next order approximation, we need to project the perturbation $\sum_{i \neq j} \mathcal{U}_{ij}$ onto $\Lambda^N H_{\mathrm{gnd}}$. We shall call the elements of $\Lambda^N H_{\mathrm{gnd}}$ the QHE states. As we explained above, these can be identified with the completely antisymmetric functions χ of $(z_1, \ldots, z_N) \in \mathbb{C}^N$, obeying

$$\chi(z_1 + m_1 + n_1\tau, \ldots, z_N + m_N + n_N\tau) = \chi(z_1, \ldots, z_N)\, e^{-\pi \mathrm{i} q \sum_{\alpha=1}^{N}(2n_\alpha z_\alpha + n_i^2 \tau)}, \tag{2.31}$$

for any $\vec{m}, \vec{n} \in \mathbb{Z}^N$.

We have: $\mathcal{U}_{\mathrm{gnd}} \equiv \sum_{i \neq j} \Pi^N_{\mathrm{gnd}} \mathcal{U}_{ij} \Pi^N_{\mathrm{gnd}}$ so that, when acting on a two-particle space of states $H^i_{\mathrm{gnd}} \otimes H^j_{\mathrm{gnd}}$ it has the form:

$$\mathcal{U}_{\mathrm{gnd}} |r_i, r_j\rangle = \sum_{s,r=0}^{q-1} \hat{u}_{r,s}\, e^{\frac{2\pi \mathrm{i} s(r_j - r_i - r)}{q}} |r_i + r, r_j - r\rangle, \tag{2.32}$$

where

$$\hat{u}_{r,s} = \sum_{m,n \in \mathbb{Z}} \int d\mathsf{a}\, u_{\mathsf{a}}\, e^{-\left(\frac{\mathsf{a}}{A} + \frac{1}{q}\right)\frac{\pi}{\tau_2}|r\tau - s + q(m\tau - n)|^2}. \tag{2.33}$$

Thus, $\mathcal{U}_{\text{gnd}}$ is a long-range spin chain Hamiltonian, with q-valued spins located at the sites $i = 1, \ldots, N$. Here we shall present a very naive argument leading to the ansatz of Laughlin (1983). Neglecting the periodicity of x and y, the Eq. (2.25) suggest to think of the projected coordinates $\hat{x} = \Pi_{\text{gnd}} x \Pi_{\text{gnd}}$, $\hat{y} = \Pi_{\text{gnd}} y \Pi_{\text{gnd}}$, as of the operators obeying

$$[\hat{x}, \hat{y}] = \frac{\mathrm{i}}{2\pi q}. \tag{2.34}$$

Then $\frac{|\hat{x}\tau + \hat{y}|^2}{\tau_2}$ becomes, in the holomorphic polarization, the dilatation operator $\frac{1}{\pi q} z \frac{\partial}{\partial z}$. Thus, the eigenfunctions of $\mathcal{U}_{\text{gnd}}$ are of the form $\prod_{i \neq j} (z_i - z_j)^{\nu}$ with some ν (which should be an odd integer in order for the wavefunction to be antisymmetric). Globally, one completes this to:

$$\chi(z_1, \ldots, z_N) = \left(\prod_{i<j} \vartheta_{11}(z_i - z_j\,;\,\tau) \right)^{\nu} \times \chi_{\nu}\left(\sum_{i=1}^{N} z_i\,;\,\tau \right), \tag{2.35}$$

where

$$\vartheta_{11}(z;\tau) = \mathfrak{q}^{\frac{1}{8}} e^{\pi \mathrm{i}(z + \frac{1}{2})} \theta\left(z + \tfrac{1}{2} + \tfrac{\tau}{2} \,|\, \tau\right), \tag{2.36}$$

and $\chi_{\nu}(z;\tau)$ is the section of L^{ν} over E_{τ}, e.g.

$$\chi_{\nu}(\xi;\tau) = a \prod_{I=1}^{\nu} \vartheta_{11}(\xi - w_I;\ \tau), \tag{2.37}$$

and w_I, obeying $\sum_I w_I = 0$ are the remaining parameters of the state, i.e. the space of Laughlin states on $\mathbb{T}^2$ is ν-dimensional. The (2.35) is a section of $p_1^* L^q \otimes \ldots p_N^* L^q$ bundle over $E_{\tau}^{\times N}$ where

$$q = \nu N. \tag{2.38}$$

The parameter $1/\nu = N/q$ is the filling fraction. In the physical system ν needs not be an odd integer, this is why the wavefunctions (2.35) are not the end of the story.

3 From many-body systems to gauge theories

In the trigonometric limit $\mathfrak{q} \to 0$ the function (2.35) approaches the ground state wavefunction of the Calogero-Moser-Sutherland model, which is the system of N

identical particles $x_1, x_2, \ldots, x_N$ in one dimension (a line $\mathbb{R}^1$ or a circle $\mathbb{S}^1$) which interact via pair-wise potential:

$$\hat{H} = -\frac{1}{2}\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \nu(\nu-1)\sum_{i<j} U(x_i - x_j), \tag{3.1}$$

where the potential can be rational or trigonometric:

$$U(x) = \frac{1}{x^2} + \frac{\omega^2}{2}x^2, \quad \text{or} \quad U(x) = \frac{1}{4R^2 \sin^2\left(\frac{x}{2R}\right)}. \tag{3.2}$$

The naive expectation is to identify (2.35) with the groundstate wavefunction of the elliptic Calogero-Moser system Olshanetsky and Perelomov (1983, 1981),[2]

$$U(x) = \frac{1}{4\pi^2 R^2}\wp\left(\frac{x}{2\pi R}; \tau\right). \tag{3.3}$$

Unfortunately, they are not known explicitly, although the Bethe/gauge correspondence gives an ansatz (Nekrasov and Shatashvili 2009a, b, 2010) for general ν, and Felder and Varchenko (1997) for integer ν. In what follows we recall the precise relation between supersymmetric gauge theories and the elliptic Calogero-Moser system. We then modify the gauge theoretic setup so as to produce precisely the Laughlin type function (2.35).

4 Supersymmetric instanton count

Consider the $\mathcal{N} = 2^*$ supersymmetric gauge theory in four dimensions, i.e. softly broken maximally supersymmetric Yang–Mills theory, subject to the Ω-deformation (Nekrasov 2003), with parameters $\varepsilon_1, \varepsilon_2$, and the noncommutative deformation, as in Nekrasov and Schwarz (1998) and Seiberg and Witten (1999). The theory depends on the microscopic gauge coupling and the theta angle, which are conveniently packaged into the elliptic curve modulus τ and the nome $\mathfrak{q}$:

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi \mathrm{i}}{g^2}, \quad \mathfrak{q} = e^{2\pi \mathrm{i}\tau} \tag{4.1}$$

and the mass $\mathtt{m}$ of the adjoint hypermultiplet.

4.1 Partition functions

It turns out (Nekrasov 2004) that the partition function of supersymmetric gauge theory with eight supercharges with supersymmetric boundary conditions at infinity can be

[2] $\wp(x;\tau) = \frac{1}{x^2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(x+m+n\tau)^2} - \frac{1}{(m+n\tau)^2}$.

computed exactly. For the theory with gauge group $U(N)$ it can be represented as a sum (a big advantage over the original path integral!) over N-tuples $\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(N)})$ of partitions (Young diagrams):

$$\begin{aligned} & Z(\mathbf{a}, \mathtt{m}, \varepsilon_1, \varepsilon_2, \mathfrak{q}) \\ & \quad = Z^{\text{tree}}(\mathbf{a}, \mathtt{m}, \varepsilon_1, \varepsilon_2, \mathfrak{q}) \times Z^{1-\text{loop}}(\mathbf{a}, \mathtt{m}, \varepsilon_1, \varepsilon_2, \mathfrak{q}) \times Z^{\text{inst}}(\mathbf{a}, \mathtt{m}, \varepsilon_1, \varepsilon_2, \mathfrak{q}), \end{aligned} \tag{4.2}$$

where $\mathbf{a} = (a_1, \ldots, a_N) \in \mathbb{C}^N$ are the so-called Coulomb parameters, $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$ are the so-called Ω-deformation parameters, $\mathfrak{q}$ is as in (4.1)

$$Z^{\text{tree}}(\mathbf{a}, \mathtt{m}, \varepsilon_1, \varepsilon_2, \mathfrak{q}) = \exp \frac{2\pi \mathrm{i} \tau}{2\varepsilon_1 \varepsilon_2} \sum_{\alpha=1}^{N} a_\alpha^2, \tag{4.3}$$

$Z^{1-\text{loop}}$ is given by an explicit formula involving the product of ratios

$$\frac{\Gamma_2(a_\alpha - a_\beta; \varepsilon_1, \varepsilon_2)}{\Gamma_2(a_\alpha - a_\beta + \mathtt{m}; \varepsilon_1, \varepsilon_2)}, \tag{4.4}$$

of Barnes double Gamma-functions over the roots of $SU(N)$, and:

$$Z^{\text{inst}} = \sum_{\Lambda} \mathfrak{q}^{|\Lambda|} \, \mathsf{E}\left[(1 - e^{\mathtt{m}}) \left(\mathsf{N} K^* + \mathsf{N}^* K q_1 q_2 - (1 - q_1)(1 - q_2) K K^*\right)\right], \tag{4.5}$$

where

$$\begin{aligned} \mathsf{N} &= \sum_{\alpha=1}^{N} \mathsf{N}_\alpha, \qquad K = \sum_{\alpha=1}^{N} K_\alpha \\ \mathsf{N}_\alpha e^{a_\alpha}, &\qquad K_\alpha = e^{a_\alpha} \sum_{i=1}^{\infty} q_1^{i-1} \sum_{j=1}^{\lambda_i^{(\alpha)}} q_2^{j-1}, \\ |\Lambda| &= \sum_{\alpha=1}^{N} |\lambda^{(\alpha)}| \equiv \sum_{\alpha=1}^{N} \sum_{i} \lambda_i^{(\alpha)}, \end{aligned} \tag{4.6}$$

λ_i is the length of the i'th row of Young diagram of λ, the operation E is the plethystic exponent, so that

$$\mathsf{E}\left[\sum_i e^{\xi_i} - \sum_J e^{\eta_J}\right] = \frac{\prod_J \eta_J}{\prod_i \xi_i}, \tag{4.7}$$

and the conjugation * acts on the virtual characters as

$$\chi^* \left(e^{a_\alpha}, q_1, q_2\right) = \chi \left(e^{-a_\alpha}, q_1^{-1}, q_2^{-1}\right). \tag{4.8}$$

The partition function (4.2) has been studied from many points of view. Its asymptotics, as $\varepsilon_{1,2} \to 0$, capture an important piece of physics: namely, the limit

$$\mathcal{F}(\mathbf{a}, \mathtt{m}, \mathfrak{q}) = \mathrm{Lim}_{\varepsilon_1, \varepsilon_2 \to 0}\, \varepsilon_1 \varepsilon_2 \log Z(\mathbf{a}, \mathtt{m}, \varepsilon_1, \varepsilon_2, \mathfrak{q}), \tag{4.9}$$

is the prepotential of the low-energy effective action (Seiberg and Witten 1994a, b):

$$S_{\mathrm{eff}} = \frac{1}{2}\int_{\mathbb{R}^4} \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} F_i^- \wedge F_j^- - \frac{\partial^2 \bar{\mathcal{F}}}{\partial \bar{a}_i \partial \bar{a}_j} F_i^+ \wedge F_j^+ + \int_{\mathbb{R}^4} \mathrm{Im} \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} da_i \wedge \star d\bar{a}_j + \dots \tag{4.10}$$

where ... stand for fermionic terms and the terms with higher derivatives, $a_i = a_i(x)$ are the complex valued scalar fields of the low-energy theory, and $F_i = dA_i$ are the field strengths of the $U(1)$ gauge fields $A_i, i = 1, \dots, N$. It was argued in Donagi and Witten (1996) [and proven in Nekrasov and Okounkov (2006)] that $\mathcal{F}$ can be recovered from the classical symplectic geometry of an algebraic integrable system (the earlier indications for the connections between the special geometry of $\mathcal{N} = 2$ theories and integrable systems see Gorsky et al. 1995; Martinec and Warner 1996), which was previously (Gorsky and Nekrasov 1994a, b, 1995a; Nekrasov 1996) identified with the classical elliptic Calogero-Moser system: namely, the action variables of that system

$$a_i = \oint_{C_i} \mathbf{p} d\mathbf{z} \qquad a_D^i = \oint_{C_i^\vee} \mathbf{p} d\mathbf{z}, \tag{4.11}$$

(there are twice as many action variables compared to the real Liouville theory) are related to each other via:

$$a_D^i = \frac{\partial \mathcal{F}}{\partial a_i} \tag{4.12}$$

where C_i and $C_i^\vee$ form the symplectic basis of 1-cycles in the homology of the level set of the integrals of motion, of which the first two are the momentum and the Hamiltonian (we choose the units in which $R = 1/2\pi$):

$$P = \sum_{i=1}^{N} p_i\,, \qquad H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \mathtt{m}^2 \sum_{i<j} \wp(z_i - z_j; \tau), \tag{4.13}$$

The limit $\varepsilon_2 \to 0$, with $\varepsilon_1 = \hbar$ fixed (Nekrasov and Shatashvili 2009a, b, 2010) contains the information about the spectra of the quantized integrals of motion. In order to see the wavefunctions, one introduces the surface defects.

4.1.1 Surface defects

The surface operator $\mathcal{O}_t[\Sigma]$ associated with a codimension two surface $\Sigma \subset \mathbb{R}^4$ is the prescription to do the path integral over the gauge fields A with a singularity near Σ, such that for each $x \in \Sigma$ the holonomy $g_x = P\exp\oint_{C_x} A$ around the small loop C_x around Σ has a fixed conjugacy class $[g_x] = t \in G/Ad(G)$. The operators defined in this way were discussed in Kronheimer and Mrowka (1993, 1995), Losev et al. (1996), Nekrasov (2003), Gukov and Witten (2006, 2010), Gaiotto (2012a, b), Alday et al. (2010), Gaiotto and Kim (2016) and Gukov (2016). A very useful approach was proposed in Finkelberg and Rybnikov (2010), and studied in Kanno and Tachikawa (2011), Alday and Tachikawa (2010), Nekrasov (2016a, b, 2017, 2018, 2019). The defects depend on the complex couplings $\mathfrak{q}_0, \ldots, \mathfrak{q}_{N-1}$, which we view as an N-periodic function $\mathbb{Z} \to \mathbb{C}^{\times}$: $\mathfrak{q}_{\omega+N} = \mathfrak{q}_{\omega}$. These parameters are the Kähler moduli of the sigma model on defect's worldsheet. The relation of these parameters to the coordinates of Calogero particles $z_1, \ldots, z_N$ (which we also extend to $\mathbb{Z} \to \mathbb{C}$ function with N-quasiperiodicity $z_{\omega+N} = z_{\omega} + \tau$) and τ are the following:

$$\mathfrak{q}_{\omega} = e^{2\pi \mathrm{i}(z_{\omega+1} - z_{\omega})}, \qquad \omega = 1, \ldots, N-1, \qquad \mathfrak{q}_0 = e^{2\pi \mathrm{i}(z_1 - z_N + \tau)}, \tag{4.14}$$

Thus, $\mathfrak{q} = \mathfrak{q}_0 \mathfrak{q}_1 \ldots \mathfrak{q}_{N-1}$. The surface defects are defined with the help of the orbifold projection, associated with the action of a cyclic group $\mathbb{Z}_N$ (one may consider other cyclic groups, e.g. $\mathbb{Z}_p$ but in our story we shall only use $p = N$) on $\mathbb{C}^2$ via $(z_1, z_2) \mapsto (z_1, e^{\frac{2\pi \mathrm{i}}{N}} z_2)$, accompanied by the action on the gauge bundle. On the fixed set $z_2 = 0$ the gauge bundle N splits as a sum $\mathsf{N} = \bigoplus_{\omega=0}^{N-1} \mathcal{N}_{\omega} \otimes \mathcal{R}_{\omega}$ over the irreducible representations $\mathcal{R}_{\omega}$ of $\mathbb{Z}_N$. The irreps of $\mathbb{Z}_N$ obey

$$\mathcal{R}_{\omega'} \otimes \mathcal{R}_{\omega''} = \mathcal{R}_{\omega'+\omega'' \bmod N}, \tag{4.15}$$

In our case all $\mathcal{N}_{\omega}$ are one-dimensional. The color decomposition assigns to each ω a color $c(\omega) = 1, \ldots, N$, so that the corresponding characters decompose as (note the change $q_2 \to q_2^{\frac{1}{N}}$):

$$\mathsf{N} = \sum_{\omega=0}^{N-1} \mathsf{N}_{c(\omega)} \otimes \mathcal{R}_{\omega},$$
$$\mathsf{K} = \sum_{\omega=0}^{N-1} e^{a_{c(\omega)}} \sum_{i=1}^{\infty} q_1^{i-1} \sum_{j=1}^{\lambda_i^{(c(\omega))}} q_2^{\frac{j-1}{N}} \mathcal{R}_{\omega+j-1 \bmod N} = \sum_{\omega=0}^{N-1} \mathsf{K}_{\omega} \otimes \mathcal{R}_{\omega}. \tag{4.16}$$

The fractional instanton charges k_{ω} are defined by:

$$k_{\omega} = \sum_{\omega'=0}^{N-1} \sum_{j=1}^{N} \delta_{\omega, \omega'+j-1 \bmod N} \left(\lambda_j^{(c(\omega'))}\right)^t, \tag{4.17}$$

where λ^t is the partition whose Young diagram is transposed Young diagram of λ (rows and columns are exchanged). The surface defect partition function $Z_c^{U(N)}$ is given by the formula analogous to (4.2) with $\tilde{a}_{\alpha} = a_{c(\alpha-1)} + \varepsilon_2 \frac{\alpha-1}{N}$,

$$Z_c^{\text{tree}} = \prod_{\alpha=1}^{N} \mathfrak{q}^{\frac{\tilde{a}_\alpha^2}{2\varepsilon_1\varepsilon_2}} e^{\frac{2\pi \mathrm{i} z_\alpha \tilde{a}_\alpha}{\varepsilon_1}}, \tag{4.18}$$

$$Z_c^{1-\text{loop}} = Z^{1-\text{loop}}(\tilde{a}_1, \ldots, \tilde{a}_N, \mathtt{m}, \varepsilon_1, \varepsilon_2) \times \prod_{\alpha=1}^{N} \prod_{\beta=1}^{N+1-\alpha} \frac{\Gamma\left(\frac{\tilde{a}_\beta - \tilde{a}_\alpha}{\varepsilon_1}\right)}{\Gamma\left(\frac{\tilde{a}_\beta - \tilde{a}_\alpha + \mathtt{m}}{\varepsilon_1}\right)}, \tag{4.19}$$

$$Z_c^{\text{inst}} = \sum_{\Lambda} \prod_{\omega=0}^{N-1} \mathfrak{q}_\omega^{k_\omega(\Lambda)} \mathsf{E}\left[e^{\tilde{a}_\omega} K_\omega^* + e^{-\tilde{a}_\omega} K_{\omega-1} q_1 - (1-q_1)(K_\omega - K_{\omega-1}) K_\omega^*\right], \tag{4.20}$$

with the understanding $K_{-1} = q_2 K_{N-1}$. This is an explicit albeit complicated power series in $\mathfrak{q}_\omega$'s with coefficients which are rational functions of $\tilde{a}$'s, ε's and $\mathtt{m}$, with rational poles. There are $N!$ such functions, which differ by the choices of the one-to-one coloring functions $c : \{0, \ldots, N-1\} \to \{1, \ldots, N\}$ (more general defects correspond to the coloring functions which are not one-to-one).

One of the consequences of Nekrasov (2016a, b, 2017, 2018, 2019) is the Knizhnik–Zamolodchikov–Bernard equation obeyed by all $Z_c^{U(N)}$:

$$\left[N\varepsilon_1\varepsilon_2 \mathfrak{q} \frac{\partial}{\partial \mathfrak{q}} + \frac{\varepsilon_1^2}{2} \sum_{\alpha=1}^{N} \frac{\partial^2}{\partial z_\alpha^2} + \mathtt{m}(\mathtt{m}+\varepsilon_1) \sum_{\alpha<\beta} \wp(z_\alpha - z_\beta; \tau) \right] Z_c^{U(N)} = 0, \tag{4.21}$$

which in the limit $\varepsilon_2 \to 0$ becomes the eigenvalue problem for the elliptic Calogero-Moser Hamiltonian (3.3), (4.13):

$$Z_c^{U(N)}(\tilde{\mathbf{a}}, \varepsilon_1, \varepsilon_2, \mathtt{m}; \mathbf{z}, \mathfrak{q}) / Z(\tilde{\mathbf{a}}, \varepsilon_1, \varepsilon_2, \mathtt{m}; \mathfrak{q}) \longrightarrow_{\varepsilon_2 \to 0} \Psi_{\tilde{\mathbf{a}}}(\mathbf{z}). \tag{4.22}$$

Finally, we can define in an analogous fashion the surface defect in the $U(1)^N$ theory (recall Nekrasov and Schwarz 1998 that even the $U(1)$ theory on a noncommutative space has instantons)

$$Z^{U(1)^N} = \prod_{\omega=1}^{N} Z_\omega^{U(1)}, \tag{4.23}$$

where $Z_\omega^{U(1)}$ is given by the $N = 1$ version of (4.20) with the coloring function $c(\omega) = 1$, i.e. it is a sum over the set of partitions (recall the periodicity $\mathfrak{q}_{\omega+N} = \mathfrak{q}_\omega$):

$$\begin{aligned} Z_\omega^{U(1)} = \sum_{\lambda} \prod_{(i,j)\in\lambda} & \mathfrak{q}_{\omega+j-1} \\ & \times \left(\frac{\mathtt{m} + \varepsilon_1(a_{i,j}+1) - \varepsilon_2 l_{i,j}}{\varepsilon_1(a_{i,j}+1) - \varepsilon_2 l_{i,j}} \right)^{\delta_{l_{i,j} \bmod N}} \\ & \times \left(\frac{\mathtt{m} - \varepsilon_1 a_{i,j} + \varepsilon_2(l_{i,j}+1)}{-\varepsilon_1 a_{i,j} + \varepsilon_2(l_{i,j}+1)} \right)^{\delta_{l_{i,j}+1 \bmod N}}, \end{aligned} \tag{4.24}$$

where $a_{i,j} = \lambda_i - j$ and $l_{i,j} = \lambda^t_j - i$ are the arm-length and the leg-length of the box (i, j) in the Young diagram of λ, cf. (Nakajima 1999). The results of Nekrasov (2016a, b, 2017, 2018, 2019) imply the following identity

$$Z^{U(1)^N} = \left(\mathfrak{q}^{-\frac{N^2-1}{24}} \phi(\mathfrak{q})^{\frac{\mathtt{m}+\varepsilon_2}{\varepsilon_2}} \prod_{\alpha=1}^{N} e^{2\pi \mathrm{i}(\alpha - \frac{N+1}{2}) z_\alpha} \Theta_{A_{N-1}}(\mathbf{z};\tau) \right)^{-\frac{\mathtt{m}+\varepsilon_1}{\varepsilon_1}}, \quad (4.25)$$

where $\phi(\mathfrak{q}) = \prod_n (1 - \mathfrak{q}^n)$, and the rank $N-1$ theta function is given by

$$\Theta_{A_{N-1}}(\mathbf{z};\tau) = \eta(\tau)^{N-1} \prod_{\alpha>\beta} \frac{\vartheta_{11}\left(z_\alpha - z_\beta;\, \tau\right)}{\eta(\tau)}, \quad (4.26)$$

It obeys the *heat equation* cf. (Kac and Peterson 1984)

$$4\pi \mathrm{i} N \frac{\partial}{\partial \tau} \Theta_{A_{N-1}}(\mathbf{z};\tau) = \sum_{\alpha=1}^{N} \frac{\partial^2}{\partial z_\alpha^2} \Theta_{A_{N-1}}(\mathbf{z};\tau). \quad (4.27)$$

Thus, the essential building block of Laughlin wavefunction (2.35) is the partition function of the surface defect in the $U(1)^N$ gauge theory, with the filling fraction $1/\nu$ being essentially the ratio $\varepsilon_1/\mathtt{m}$. In string/M-theory realizations, using (Douglas 1997, 1998; Douglas and Moore 1996; Witten 1997) the $U(N)$ surface defects appear naturally, as opposed to the ones for $U(1)^N$. We therefore suggest to look for the many-body wavefunctions describing the fractional Hall effect states among the surface defects of the $U(N)$ gauge theory. Let us conclude with a few remarks aiming to clarify some of the confusions.[3]

The surface defect partition functions are defined for by the choice of the coloring function c. In addition, the expansion (4.20) is valid in the chamber $|\mathfrak{q}_\omega| \ll 1$ where the two dimensional theory on the surface of the defect is weakly coupled. When two electrons approach each other, e.g. $z_\omega \to z_{\omega+1}$, one of the couplings $\mathfrak{q}_\omega \to 1$ approaches a strong coupling point, beyond which the two dimensional theory can be continued (this is an analogue of the flop transition in Gromov-Witten theory), using the Eq. (4.21). Indeed, the leading singularity $\nu(\nu-1)/(z_\omega - z_{\omega+1})^2$ implies the solutions behave as

$$a_c^+ (z_\omega - z_{\omega+1})^\nu (1 + \cdots) + a_c^- (z_\omega - z_{\omega+1})^{1-\nu} (1 + \cdots). \quad (4.28)$$

From this the analytic continuation to a different chamber follows.

To ensure the antisymmetry of the wavefunction we must, therefore, take a linear combination of surface defects corresponding to all $N!$ one-to-one coloring functions. The matching conditions will impose constraints on the quasi-momenta $a_1, \ldots, a_N$, similar to the quantization conditions in those in the last reference in Nekrasov and Shatashvili (2009a, b, 2010). We expect that for the rational value of ν the filling

[3] We thank the anonymous referee for pointing them out.

fraction to be again given by $1/\nu$. Roughly speaking, the additional multi-valuedness of the analytically continued wavefunction for non-integer ν can be compensated by going to a finite cover of the torus, where the effective flux is an integer multiple of N.

We note the symmetry $\nu \leftrightarrow 1-\nu$ is part of the flavor symmetry of the four dimensional gauge theory (it exchanges the two chiral multiplets comprising the hyper-multiplet). In the fractional quantum Hall effect it is the symmetry $1/\nu \leftrightarrow 1-1/\nu$ which is sought for (it is the electron-hole duality) Son (2018). In the trigonometric case this duality follows from the arm-leg duality of Jack polynomials. We have some preliminary results on the elliptic analogue of this duality, which uses the qq-character (Nekrasov 2016a, b, 2017, 2018, 2019) representation of the $\varepsilon_2 \to 0$ limit of the surface defect partition function, which we hope to present in a subsequent publication.

We should point out that an alternative approach to maintain the particle-hole duality is to use the elliptic super-Calogero-Moser system, describing two species of particles (in the trigonometric limit this model has been studied in, e.g. Sergeev and Veselov 2004; Atai and Langmann 2017). The wavefunctions of that system can also be found using surface defects in supersymmetric field theories (Nekrasov 2016a, b, 2017, 2018, 2019).

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References

Alday, L.F., Gaiotto, D., Gukov, S., Tachikawa, Y., Verlinde, H.: Loop and Surface Operators in N = 2 Gauge Theory and Liouville Modular Geometry. arXiv:0909.0945 [hep-th]

Alday, L.F., Tachikawa, Y.: Affine $SL(2)$ conformal blocks from 4d gauge theories. Lett. Math. Phys. **94**, 87–114 (2010). arXiv:1005.4469 [hep-th]

Atai, F., Langmann, E.: Deformed Calogero-Sutherland model and fractional quantum Hall effect. J. Math. Phys. **58**, 11902 (2017)

Donagi, R., Witten, E.: Supersymmetric Yang–Mills theory and integrable systems. Nucl. Phys. B **460**, 299 (1996). https://doi.org/10.1016/0550-3213(95)00609-5. arXiv:hep-th/9510101

Douglas, M.R., Moore, G.W.: D-branes, quivers, and ALE instantons. arXiv:hep-th/9603167

Douglas, M.R.: Branes within branes, in Cargese, Strings, Branes and Dualities (1997) pp. 267–275 arXiv:hep-th/9512077

Douglas, M.R.: Gauge fields and D-branes. J. Geom. Phys. **28**, 255 (1998). arXiv:hep-th/9604198

Felder, G., Varchenko, A.: Three formulas for Eigen functions of integrable Schrodinger operators. arXiv:hep-th/9511120

Finkelberg, M., Rybnikov, L.: Quantization of Drinfeld Zastava in type A. arXiv:1009.0676 [math.AG]

Fock, V., Rosly, A.: Poisson structure on moduli of flat connections on Riemann surfaces and r-matrix. Am. Math. Soc. Transl. **191**, 67 (1999). arXiv:math/9802054 [math-qa]

Gaiotto, D., Kim, H.-C.: Surface defects and instanton partition functions. arXiv:1412.2781 [hep-th]

Gaiotto, D.: N = 2 dualities. JHEP **1208**, 034 (2012a). arXiv:0904.2715 [hep-th]

Gaiotto, D.: Surface operators in $\mathcal{N}=2$ 4d Gauge theories. JHEP **1211**, 090 (2012b). arXiv:0911.1316 [hep-th]

Gerasimov, A.A., Shatashvili, S.L.: Two-dimensional gauge theories and quantum integrable systems. In: Proceedings of Symposia in Pure Mathematics, May 25–29 (2007), University of Augsburg, Germany. arXiv:0711.1472 [hep-th]

Gerasimov, A.A., Shatashvili, S.L.: Higgs bundles, Gauge theories and quantum groups. Commun. Math. Phys. **277**, 323 (2008). arXiv:hep-th/0609024
Gorsky, A., Nekrasov, N.: Elliptic Calogero-Moser system from two-dimensional current algebra. (1994b) arXiv:hep-th/9401021
Gorsky, A., Nekrasov, N.: Relativistic Calogero-Moser model as gauged WZW theory. Nucl. Phys. B **436**, 582 (1995a). https://doi.org/10.1016/0550-3213(94)00499-5. arXiv:hep-th/9401017
Gorsky, A., Nekrasov, N.: Hamiltonian systems of Calogero type and two-dimensional Yang–Mills theory. Nucl. Phys. B **414**, 213 (1994). https://doi.org/10.1016/0550-3213(94)90429-4. arXiv:hep-th/9304047
Gorsky, A., Krichever, I., Marshakov, A., Mironov, A., Morozov, A.: Integrability and Seiberg–Witten exact solution. Phys. Lett. B **355**, 466 (1995). https://doi.org/10.1016/0370-2693(95)00723-X. arXiv:hep-th/9505035
Gukov, S., Witten, E.: Gauge theory, ramification, and the geometric. Langlands Program. arXiv:hep-th/0612073
Gukov, S.: Surface operators. In: Teschner, J. (ed.) New Dualities of Supersymmetric Gauge Theories. (2016). arXiv:1412.7127 [hep-th]
Gukov, S., Witten, E.: Rigid surface operators. Adv. Theor. Math. Phys. **14**(1), 87 (2010). https://doi.org/10.4310/ATMP.2010.v14.n1.a3. arXiv:0804.1561 [hep-th]
Kac, V., Peterson, D.: Infinite-dimensional Lie algebras, theta-functions and modular forms. Adv. Math. **53**, 125–264 (1984)
Kanno, H., Tachikawa, Y.: Instanton counting with a surface operator and the chain-saw quiver. JHEP **1106**, 119 (2011). arXiv:1105.0357 [hep-th]
Kronheimer, P.B., Mrowka, T.S.: Gauge theory for embedded surfaces, I. Topology **32**(4), 773–826 (1993)
Kronheimer, P.B., Mrowka, T.S.: Gauge theory for embedded surfaces, II. Topology **34**(1), 37–97 (1995)
Laughlin, R.B.: Anomalous quantum Hall effect: an incompressible quantum fluid with fractionally charged excitations. Phys. Rev. Lett. **50**(18), 1395–1398 (1983)
Losev, A., Moore, G., Nekrasov, N., Shatashvili, S.L.: Four dimensional avatars of two dimensional rational conformal field theory. Nucl. Phys. Proc. Suppl. **46**, 130–145 (1996). arXiv:hep-th/9509151
Martinec, E.J., Warner, N.P.: Integrable systems and supersymmetric gauge theory. Nucl. Phys. B **459**, 97 (1996). https://doi.org/10.1016/0550-3213(95)00588-9. arXiv:hep-th/9509161
Moore, G.W., Nekrasov, N., Shatashvili, S.: Integrating over Higgs branches. Commun. Math. Phys. **209**, 97 (2000). arXiv:hep-th/9712241
Mumford, D.: Tata lectures on theta, Bïrhauser, (1984)
Nakajima, H.: Lectures on Hilbert Schemes of Points on Surfaces. American Mathematical Society, Providence (1999)
Nekrasov, N.: Holomorphic bundles and integrable systems. Commun. Math. Phys. **180**(3), 587–603 (1996). arXiv:hep-th/9503157
Nekrasov, N.A.: Seiberg–Witten prepotential from instanton counting. Adv. Theor. Math. Phys. **7**(5), 831 (2003). arXiv:hep-th/0206161
Nekrasov, N.: On surface operators in susy gauge theories and KZ-type equations. Lecture at the DARPA program on Langlands duality and physics, IAS, Princeton, March 6–10 (2004)
Nekrasov, N.: BPS/CFT correspondence: non-perturbative Dyson–Schwinger equations and qq-characters. JHEP **1603**, 181 (2016a). arxiv:1512.05388 [hep-th]
Nekrasov, N.: BPS/CFT correspondence II: instantons at crossroads, moduli and compactness theorem. (2016b). arXiv:1608.07272 [hep-th]
Nekrasov, N.: BPS/CFT correspondence V: BPZ and KZ equations from qq-characters (2017). arXiv:1711.11582 [hep-th]
Nekrasov, N.: BPS/CFT correspondence III: Gauge Origami Partition Function and qq-characters. (2018). arXiv:1701.00189 [hep-th]
Nekrasov, N.: BPS/CFT correspondence IV: sigma models and defects in gauge theory (2019). https://doi.org/10.1007/s11005-018-1115-7. arXiv:1711.11011 [hep-th]
Nekrasov, N., Okounkov, A.: Seiberg–Witten theory and random partitions. Prog. Math. **244**, 525 (2006). arXiv:hep-th/0306238
Nekrasov, N., Schwarz, A.: Instantons on the noncommutative $\mathbb{R}^4$ and (0, 2) superconformal theory. Commun. Math. Phys. **198**, 689 (1998). https://doi.org/10.1007/s002200050490. arXiv:hep-th/9802068
Nekrasov, N.A., Shatashvili, S.L.: Supersymmetric vacua and Bethe ansatz. Nucl. Phys. Proc. Suppl. **192–193**, 91 (2009). https://doi.org/10.1016/j.nuclphysbps.2009.07.047. arXiv:0901.4744 [hep-th]

Nekrasov, N.A., Shatashvili, S.L.: Quantum integrability and supersymmetric vacua. Prog. Theor. Phys. Suppl. **177**, 105 (2009). https://doi.org/10.1143/PTPS.177.105. arXiv:0901.4748 [hep-th]

Nekrasov, N.A., Shatashvili, S.L.: Quantization of Integrable Systems and Four Dimensional Gauge Theories (2010). arXiv:0908.4052 [hep-th]

Olshanetsky, M., Perelomov, A.: Classical integrable finite dimensional systems related to Lie algebras. Phys. Rept. **71**, 313 (1981). https://doi.org/10.1016/0370-1573(81)90023-5

Olshanetsky, M., Perelomov, A.: Quantum Integrable systems related to Lie algebras. Phys. Rept. **94**, 313 (1983). https://doi.org/10.1016/0370-1573(83)90018-2

Seiberg, N., Witten, E.: Electric-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang–Mills theory. Nucl. Phys. B **426**, 19 (1994a). Erratum: [Nucl. Phys. B **430**, 485 (1994)]. arXiv:hep-th/9407087

Seiberg, N., Witten, E.: Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD. Nucl. Phys. B **431**, 484 (1994b). arXiv:hep-th/9408099

Seiberg, N., Witten, E.: String theory and noncommutative geometry. JHEP **9909**, 032 (1999). arXiv:hep-th/9908142

Sergeev, A., Veselov, A.: Deformed quantum Calogero–Moser problems and Lie superalgebras. Commun. Math. Phys. **245**(2), 249–278 (2004)

Son, D.T.: The Dirac composite fermion of the fractional quantum Hall effect. Ann. Rev. Condens. Mater. Phys. **9**, 397 (2018). https://doi.org/10.1146/annurev-conmatphys-033117-054227. arXiv:1805.04472 [cond-mat.mes-hall]

Witten, E.: Phases of N = 2 theories in two dimensions. Nucl. Phys. B **403**, 159 (1993). arXiv:hep-th/9301042

Witten, E.: Solutions of four-dimensional field theories via M theory. Nucl. Phys. B **500**, 3 (1997). arXiv:hep-th/9703166