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RESEARCH CONTRIBUTION

On Osculating Framing of Real Algebraic Links

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Abstract
For a real algebraic link in $\mathbb{RP}^3$, we prove that its encomplexed writhe (an invariant introduced by Viro) is maximal for a given degree and genus if and only if its self-linking number with respect to the framing by the osculating planes is maximal for a given degree.

1 Introduction and Statement of the Main Result

By *real algebraic curve* in $\mathbb{RP}^3$ we mean a complex curve in $\mathbb{CP}^3$ invariant under complex conjugation. We use the same notation for a real curve and the set of its complex points and, if it is denoted by A, then $\mathbb{R}A$ stands for the set of real points which is called a *real algebraic link* if it is non-empty and A is smooth. A real algebraic link is called *maximally writhed* or MW_λ*-link* if $|w_\lambda(L)|$ [a variation of Viro's invariant (Viro 2002)] attains the maximal possible value $(d-1)(d-2)/2-g$ where d and g is the degree and genus of A respectively. We refer to Mikhalkin and Orevkov (2019) for a precise definition of w_λ.

In Mikhalkin and Orevkov (2019, Thm. 2) we proved that several topological and geometric invariants are maximized on MW_λ-links. In this paper we add one more item to this collection: we show that the self-linking number of L with respect to the osculating framing attains its maximal value (for links of a given degree) if and only if L is an MW_λ-link. The proof is very similar to that of the main theorem of Mikhalkin and Orevkov (2019). Let us give precise definitions and statements.

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Let L be an oriented link in a rational homology 3-sphere. A *framing* of L is a continuous 1-dimensional subbundle of the normal bundle of L or, equivalently, a continuous field (defined on L) of 2-dimensional planes tangent to L. Given a framed oriented link L, its *self-linking number* is defined as follows. Let F be an embedded annulus or Möbius band with core L, tangent to the framing. Then the self-linking number is $\frac{1}{2}\operatorname{lk}(L, \partial F)$ where the boundary ∂F of F is oriented so that $[\partial F] = 2[L]$ in $H_1(F)$.

For an oriented link L in $\mathbb{RP}^3$, the *osculating framing* is the framing defined by the field of osculating planes. We denote the self-linking number of L with respect to this framing by $\operatorname{osc}(L)$. If L is a non-oriented link and O an orientation of L, we use the notation $\operatorname{osc}(L, O)$ which is self-explained.

Recall that a smooth irreducible real algebraic curve A is called an *M-curve* if $\mathbb{R}A$ has $g + 1$ connected components where g is the genus of A. In this case $\mathbb{R}A$ divides A into two halves. The boundary orientation on $\mathbb{R}A$ induced by any of these halves is called a *complex orientation*. The main result of the paper is the following.

Theorem 1 *Let $L = \mathbb{R}A$ be an irreducible real algebraic link of degree $d \geq 3$ and O be an orientation of L. Then:*

(a) $|\operatorname{osc}(L, O)| \leq d(d-2)/2$.
(b) $|\operatorname{osc}(L, O)| = d(d-2)/2$ *if and only if L is an MW_λ-link* [by Mikhalkin and Orevkov (2019, Thm. 2), in this case A is an M-curve of genus at most $d-3$] *and O is its complex orientation.*

Remark In the space of real algebraic links of a given degree and genus we can distinguish three kinds of "walls". The walls of the first kind correspond to curves with a double point with real local branches. When crossing such walls, both invariants $w_\lambda(L)$ and $\operatorname{osc}(L)$ are changed by ± 2. The walls of the second kind correspond to curves with a real double point with complex conjugate local branches. When crossing such walls, $w_\lambda(L)$ does change but $\operatorname{osc}(L)$ does not. The third kind of wall corresponds to curves which have a local branch parametrized by $t \mapsto (t, t^3 + o(t^3), t^4 + o(t^4))$ in some affine chart. When crossing such a wall, $w_\lambda(L)$ does not change but $\operatorname{osc}(L)$ does. So, in general, the invariants $w_\lambda(L)$ and $\operatorname{osc}(L)$ are more or less independent. Nevertheless, Theorem 1 implies that the chamber where they have maximal value is bounded only by the walls of the first kind—common for the both invariants.

2 A Variant of Klein's Formula for the Number of Real Inflection Points

Let $C \in \mathbb{P}^2$ be a nodal real irreducible algebraic curve. It may have three types of nodes: real nodes with real local branches of C, real nodes with imaginary local branches of C, or non-real nodes (coming in conjugate pairs). Denote the number of nodes of each type with h, e, and i respectively.

A *real flex* is a local real branch of C with the order of tangency ω to its tangent line greater than 1 (i.e. the local intersection number is $\omega + 1 \geq 3$). The multiplicity of a real flex is $\omega - 1$. In an affine chart of $\mathbb{P}^2$ a flex corresponds to a critical point of the Gauss map. It is easy to see that the multiplicity of a flex equals the multiplicity of the corresponding critical point. Thus a multiple flex can be thought of as $\omega - 1$ ordinary flexes collected at the same point. We denote with F the number of flexes counted with multiplicities.

A *solitary real bitangent* is a real line $L \subset \mathbb{P}^2$ which is tangent to C at a non-real point (and thus also at the complex conjugate point). The *multiplicity* of L is the sum of the orders ω over all local branches of $C \backslash \mathbb{RP}^2$ tangent to L. We denote with B the number of solitary real bitangents counted with multiplicities. Clearly, B is an even number.

Lemma 2.1 (Klein's formula Klein (1876) for nodal curves). *For a nodal real irreducible curve of degree d in $\mathbb{P}^2$ we have*

$$F + B = d(d-2) - 2h - 2i.$$

Proof As in Viro (1988), we use additivity of the Euler characteristic χ to derive Klein's formula. Let $\nu : \tilde{C} \to C$ be the normalization. The space of all real lines in $\mathbb{P}^2$ is homeomorphic to $\mathbb{RP}^2$, and thus has the Euler characteristic 1. For a real line L the set $\nu^{-1}(L)$ consists of d distinct points unless L is tangent to C. Each tangency decreases the size of this set by ω.

Consider the space $X = \{(p, L) \mid p \in C,\ L \ni p\}$, where $L \subset \mathbb{RP}^2$ is a real line. From the observation above we deduce

$$\chi(X) + B + F + \chi(\mathbb{R}\tilde{C}) = d.$$

Note that $\chi(\mathbb{R}\tilde{C}) = 0$ and $\chi(X) = \chi(\nu^{-1}(C \backslash \mathbb{R}C)) = \chi(\tilde{C}) - 2e$, as each point of $\mathbb{R}C$ lifts to a circle in X while $\chi(S^1) = 0$. The lemma now follows from the adjunction formula $\chi(\tilde{C}) = 3d - d^2 + 2e + 2h + 2i$. □

Remark 2.2 Lemma 2.1 can be also obtained as an almost immediate consequence from Schuh's generalization (Schuh 1903–1904) of another Klein's formula

$$d - \sum_{x \in C \cap \mathbb{RP}^2} (m(x) - r(x)) = d^\vee - \sum_{x \in C^\vee \cap \mathbb{RP}^{2\vee}} (m^\vee(x) - r^\vee(x))$$

[see Viro (1988, Thm. 6.D) for a proof via Euler characteristics] combined with the class formula $d^\vee = d(d-1) - 2e - 2h - 2i$. Here $C^\vee$ is the dual curve, $d^\vee$ is its degree, $m(x)$ and $r(x)$ (resp. $m^\vee(x)$ and $r^\vee(x)$) are the multiplicity and the number of real local branches of C (resp. of $C^\vee$) at x.

3 Proof of the Main Theorem

Let $L = \mathbb{R}A$ be a smooth irreducible real algebraic link of degree d endowed with an orientation O. Let $\mathcal{U}$ be the set of points p in $\mathbb{RP}^3 \backslash L$ such that the projection of L from p is a nodal curve.

Fix a point $p \in \mathcal{U}$. Let $C_p = \pi_p(A)$ where $\pi_p : \mathbb{P}^3\setminus\{p\} \to \mathbb{P}^2$ is the linear projection from p. Consider the field of tangent planes to L passing through p, (so-called blackboard framing). Let $b_p(L)$ be the self-linking number with respect to it. We have

$$b_p(L) = \sum_q s(q), \quad \text{thus} \quad |b_p(L)| \le h(C_p) \tag{1}$$

where q runs over the hyperbolic (i. e., with real local branches) double points of C_p, $h(C_p)$ is the number of them, and $s(q)$ is the sign of the crossing at q in the sense of knot diagrams. The difference $|\operatorname{osc}(L) - b_p(L)|$ is bounded by one half of the number of those points where the osculating plane passes through p (each such point contributes $\pm 1/2$ or 0 to $\operatorname{osc}(L)$). This is the number of real flexes of C_p which we denote by $f(C_p)$. We have $f(C_p) \le d(d-2) - 2h(C_p)$ by Lemma 2.1. Thus

$$|\operatorname{osc}(L)| \le |\operatorname{osc}(L) - b_p(L)| + |b_p(L)| \le \tfrac{1}{2} f(C_p) + h(C_p) \le \tfrac{1}{2} d(d-2) \tag{2}$$

which is Part (a) of Theorem 1.

Now suppose that $|\operatorname{osc}(L)| = d(d-2)/2$. Then for any choice of $p \in \mathcal{U}$ we have the equality sign everywhere in (2), in particular, we have the equality sign in (1), i.e., all crossings are of the same sign, say, positive:

$$s(q) = +1 \quad \text{for any hyperbolic crossing } q \text{ of } C_p. \tag{3}$$

By Lemma 2.1, the equality sign in the last inequality of (2) implies that all flexes of C_p are ordinary for any choice of $p \in \mathcal{U}$. This implies that L has non-zero torsion at each point. Indeed, otherwise there exists a real plane P which has tangency with L of order greater than 3. It is easy to check that $\mathcal{U}$ has non-empty intersection with any plane, thus we can choose a point $p \in \mathcal{U} \cap P$, and then C_p would have a k-flex with $k > 3$. Moreover, the positivity of all crossings for any generic projection implies that the torsion is everywhere positive [cf. the proof of Mikhalkin and Orevkov (2018, Prop. 1)].

Similarly to Mikhalkin and Orevkov (2018, Lem. 7) and Mikhalkin and Orevkov (2019, Lem. 4.10), we derive from these conditions that the real tangent surface TL (the union of all real lines in $\mathbb{RP}^3$ tangent to L) is a union of (non-smooth) embedded tori. Indeed, suppose that two tangent lines cross. Let P be the plane passing through them (any plane passing through them if they coincide) and let ℓ be the line passing through the two tangency points. Let p be a generic real point on ℓ. Then C_p has two real local branches at the same point such that each of them is either singular or tangent to the line $\pi_p(P)$. Since L has non-zero torsion, all singular branches of C_p are ordinary cusps. Then we can find a generic point close to p such that the projection from it does not satisfy (3).

Let $K_1, \dots, K_n$ be the connected components of L, and let TK_i be the connected component of TL that contains K_i (the union of real lines tangent to K_i). The same arguments as in Mikhalkin and Orevkov (2019, Lemma 4.12) show that, for some positive integers $a_1, \dots, a_n$, there exist real lines ℓ_i, ℓ_i', $i = 1, \dots, n$, such that (for

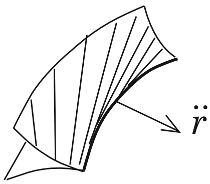

Fig. 1 A cuspidal edge of the tangent surface

suitable choice of the orientations) the linking numbers of their real loci $l_i = \mathbb{R}\ell_i$ and $l'_i = \mathbb{R}\ell'_i$ with the components of L are:

$$2\,\mathrm{lk}(l_i, K_i) = a_i + 2, \quad 2\,\mathrm{lk}(l'_i, K_i) = a_i. \tag{4}$$

Moreover, each TK_i splits $\mathbb{RP}^3$ into two solid tori U_i and V_i such that $l_i \subset U_i$, $l'_i \subset V_i$, the homology classes $[l_i]_U$ and $[l'_i]_V$ generate $H_1(U_i)$ and $H_1(V_i)$ respectively, and we have $[K_i]_U = a_i[l_i]_U$ and $[K_i]_V = (a_i + 2)[l'_i]_V$. It follows that

$$2\,\mathrm{osc}(K_i) = a_i(a_i + 2) \tag{5}$$

(the linking number of K_i with its small shift disjoint from TL). Indeed, if K_i is parametrized by $t \mapsto r(t)$ and the torsion is non-zero, then TK_i has a cuspidal edge along K_i and a small shift of K_i in the direction of the vector field $\ddot{r}$ is disjoint from TK_i (see Fig. 1). A priori this argument proves (5) up to sign only. However the positivity of the torsion implies that $\mathrm{osc}(K_i)$ is positive.

If L is connected (i. e., $n = 1$), it remains to note that then the condition $2\,\mathrm{osc}(K_1) = d(d-2)$ implies $(a_1+2)a_1 = d(d-2)$, hence $a_1 = d-2$. Thus L satisfies Condition (v) of Mikhalkin and Orevkov (2019, Thm. 1) which completes the proof that L is an MW-knot.

If L is not necessarily connected, we argue as follows. By Murasugi's result Murasugi (1991, Prop. 7.5) (see also Mikhalkin and Orevkov 2019, Prop. 1.2), the number of crossings of any projection of K_i is at least $(a_i+2)(a_i-1)/2$. Hence, for $h = h(C_p)$, we have

$$2h \geq \sum_{i=1}^{n} (a_i + 2)(a_i - 1) + \sum_{i \neq j} |\,\mathrm{lk}(K_i, K_j)|. \tag{6}$$

On the other hand, if we choose p on a line passing through a pair of complex conjugate points of A, then C_p has at least one elliptic double point (i. e., a real double point with complex conjugate local branches), whence by the genus formula we obtain

$$h \leq (d-1)(d-2)/2 - g - 1 \leq (d-1)(d-2)/2 - n \tag{7}$$

(the second inequality in (7) is the Harnack's bound). Hence

$$\begin{aligned} d(d-2) = 2\,\mathrm{osc}(L) &= 2\sum_{i=1}^{n}\mathrm{osc}(K_i) + \sum_{i\neq j}\mathrm{lk}(K_i, K_j) \\ &\le \sum_{i=1}^{n} a_i(a_i+2) + 2h - \sum_{i=1}^{n}(a_i+2)(a_i-1) \qquad \text{by (5) and (6)} \\ &= 2h + 2n + \sum_{i=1}^{n} a_i \le (d-1)(d-2) + \sum_{i=1}^{n} a_i. \quad \text{by (7)} \end{aligned}$$

Thus $\sum a_i \ge d-2$ and we conclude that L is an MW_λ-link. This fact follows from Mikhalkin and Orevkov (2019, Prop. 1.1) (which implies that $\mathrm{ps}(L) = \sum a_i$) combined with Mikhalkin and Orevkov (2019, Thm. 2) (which claims, in particular, that L is an MW_λ-link as soon as $\mathrm{ps}(L) \ge d-2$). Here we denote with $\mathrm{ps}(L)$ the *plane section number* of L. It is a topological invariant of a link in $\mathbb{RP}^3$ defined in Mikhalkin and Orevkov (2019) as the minimal number of intersection points with a generic plane where the minimum is taken over the isotopy class of the link.

Let us show that O is a complex orientation of L. It is easy to see that the plane section number is at most $d-2$ for any algebraic link of degree d. Indeed, it is enough to consider a small shift of a non-osculating tangent plane in a suitable direction. Thus the inequality in $\mathrm{ps}(L) = \sum a_i \ge d-2$ is in fact an equality. It follows that the equality is attained in all the inequalities used in the proof, in particular, we have $|\mathrm{lk}(K_i, K_j)| = \mathrm{lk}(K_i, K_j)$ for $i \neq j$. Since all components of an MW_λ-link endowed with a complex orientation are positively linked (see Mikhalkin and Orevkov 2019), we are done. This completes the proof of the "only if" part of (b).

To prove the "if" part of (b), we notice that by Mikhalkin and Orevkov (2019, Thm. 3 and §4.4), any MW_λ-link L of degree d and genus g is a union of $g+1$ knots $K_0 \cup \cdots \cup K_g$ and $\mathrm{lk}(K_i, K_j) = a_i a_j$, $i \neq j$, for some positive integers $a_0, \ldots, a_g$ with $a_0 + \cdots + a_g = d-2$. Furthermore, the torsion of L is everywhere positive and each knot K_i is arranged on its tangent surface TK_i as described above, thus (5) holds for each i, and we obtain

$$\begin{aligned} 2\,\mathrm{osc}(L) &= \sum_{i=0}^{g}\mathrm{osc}(K_i) + \sum_{i\neq j}\mathrm{lk}(K_i, K_j) = \sum_{i=0}^{g} a_i(a_i+2) + \sum_{i\neq j} a_i a_j \\ &= \Big(\sum a_i\Big)^2 + 2\sum a_i = (d-2)^2 + 2(d-2) = d(d-2). \end{aligned}$$

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