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RESEARCH CONTRIBUTION

Faithful tropicalizations of elliptic curves using minimal models and inflection points

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Abstract

We give an elementary proof of the fact that any elliptic curve E over an algebraically closed non-archimedean field K with residue characteristic $\neq 2, 3$ and with $v(j(E)) < 0$ admits a tropicalization that contains a cycle of length $-v(j(E))$. We first define an adapted form of minimal models over non-discrete valuation rings and we recover several well-known theorems from the discrete case. Using these, we create an explicit family of marked elliptic curves (E, P), where E has multiplicative reduction and P is an inflection point that reduces to the singular point on the reduction of E. We then follow the strategy as in Baker et al. (Algebraic Geom 3(1):63–105, 2016) and construct an embedding such that its tropicalization contains a cycle of length $-v(j(E))$. We call this a numerically faithful tropicalization. A key difference between this approach and the approach in Baker et al. (2016) is that we do not require any of the analytic theory on Berkovich spaces such as the *Poincaré–Lelong formula* or (Baker et al. 2016) to establish the numerical faithfulness of this tropicalization.

Keywords Faithful tropicalizations · Tropical geometry · Elliptic curves · Minimal models

Mathematics Subject Classification 14T05 · 11G07 · 14M25 · 12J25

1 Introduction

In this paper, we study the tropicalization of one-parameter families of algebraic curves. The tropicalization process takes such a one-parameter family and assigns a piecewise-linear limit to it. Over the complex numbers, this limit has been known for some time as the *logarithmic limit set* or the *nonarchimedean amoeba* assigned

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to the family, see Bergman (1971) and Einsiedler et al. (2006, Definition 1.1.1.) This process has proven to be interesting, because the piecewise limit can retain much of the geometry of each of its members if the family is chosen carefully enough. In this paper, the families we will be aiming for are those that give rise to *faithful tropicalizations*. In the special case of a family of elliptic curves, we will show that we can modify the family such that the new family defines a faithful tropicalization.

Algebraically, the notion of a one-parameter family of complex curves can be viewed as a curve defined over the field of Puiseux series K. These Puiseux series are power series over $\mathbb{C}$ in one variable t of the form

$$f(t) = c_1 t^{a_1} + c_2 t^{a_2} + \dots \tag{1}$$

where the c_i are nonzero complex numbers for all i, and $a_1 < a_2 < \dots$ are rational numbers that have a common denominator. For instance, the formal power series

$$\frac{1}{(1 - t^{1/2})} = 1 + t^{1/2} + t + t^{3/2} + t^2 + \dots \tag{2}$$

is contained in K. An example of an algebraic curve over this field is then given by the equation

$$y^2 = x^3 + (x - t)^2. \tag{3}$$

We view this curve over K as giving a family of curves by evaluating t at complex numbers $t_0 \in \mathbb{C}$. The corresponding picture over $\mathbb{R}$ can be found in Fig. 1 for some small values of t.

We are now interested in the degeneration of this family to $t = 0$, where the curve becomes a singular curve. To study this process, we consider the valuation map $v : K^* \to \mathbb{Q}$, which sends a power series f to the lowest power of t in its power series expansion. For instance, we have $v(t^{-1/2}) = -1/2$ and $v(t^2 + t^3) = 2$. If we now apply this map to points on an algebraic curve with values in K^*, then we obtain our first notion of a *tropicalization*, see Sect. 2 for the general construction. The reader can find an example of a tropicalization in Fig. 2. In this case, we have tropicalized a family of *elliptic curves*, which are complex curves whose topological

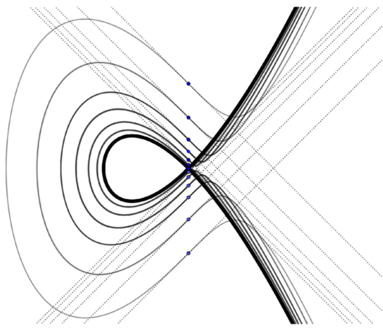
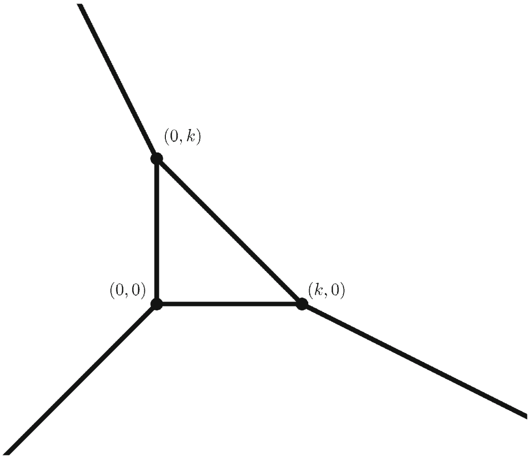


Fig. 1 A family of elliptic curves degenerating to a nodal curve. Every member in the family has two blue *inflection points* for which the corresponding dotted tangent lines intersect the curve only at that point. As the curves get closer to the nodal curve, the tangent lines converge to the two distinct tangent directions of the singular curve

Fig. 2 The tropicalizations obtained in Theorem 1.1 from the elliptic curves in Eq. 4



genus is one. As the reader can see from Fig. 2, this tropicalization contains a subgraph of Betti number one. This does not happen for all families of elliptic curves: it can only happen for families that satisfy the technical condition $v(j(E)) < 0$, where $j(E)$ is the j-invariant associated to the family of elliptic curves. Our goal in this paper is to modify a given family of elliptic curves such that its tropicalization contains a cycle of length $-v(j(E))$. We call this a *numerically faithful tropicalization*. We now state our main result:

Theorem 1.1 *Let E be an elliptic curve over K with $v(j(E)) < 0$, where $j(E)$ is the j-invariant of E. Then there exists an embedding $E \to \mathbb{P}^2$ such that its tropicalization contains a cycle of length $-v(j(E))$.*

The field K in the theorem can be more general than the field of Puiseux series, but the reader can assume throughout this paper that it is the field of Puiseux series as above. To be precise: we assume that our field K is an algebraically closed non-archimedean field with valuation ring R, maximal ideal $\mathfrak{m}$, residue field k and valuation $v : K^* \to \mathbb{R}$. We assume furthermore throughout this paper that $\text{char}(k) \neq 2, 3$.

Several proofs of Theorem 1.1 have already been given, for instance Chan and Sturmfels (2013, Proposition 2.1) and Baker et al. (2016, Theorem 6.2). Our proof will follow the strategy of Baker et al. (2016, Theorem 6.2), but we will not use the analytic slope formula, nor any of the results on faithful tropicalizations. The idea is to use reduction theory for elliptic curves in Weierstrass form $y^2 = x^3 + Ax + B$ to explicitly give a pair $(f, g) \in (K(E))^2$ such that the corresponding tropicalization has the desired cycle. To find these elements, we will use inflection points on families of elliptic curves, see Fig. 1 for an explicit example over the real numbers. Under an appropriate choice of f and g, the affine equation that cuts out the image of E in $\mathbb{P}^2$ is then given by

$$f^2 g + 2a' fg - fg^2 - 2a'b = 0 \tag{4}$$

for a' and b satisfying $v(a') = 0$ and $v(b) > 0$. The tropicalization of this curve then contains a triangle with sides of length $v(b)$ as in Fig. 2 and we show that the valuation of the j-invariant of E is $-3v(b)$.

Since reduction theory is usually only given in the discrete case (see Silverman 2009, Chapter VII), we give a more or less full treatment for the non-discrete case. We define minimal models, reduction types and we show that any elliptic curve has multiplicative reduction if and only if $v(j(E)) < 0$. These tools don't seem to be as well-known in tropical geometry as in arithmetic algebraic geometry due to the advent of Tate uniformizations, Berkovich spaces and (formal) semistable models. We chose to include most of the classical results on reduction theory (albeit in an altered form) in this text. We also included an introduction to tropical geometry in Sect. 2, giving most of the definitions, the fundamental theorem of tropical geometry and the structure theorem.

We now give a quick review of the proofs of Theorem 1.1 given in Chan and Sturmfels (2013) and Baker et al. (2016). In Chan and Sturmfels (2013), one starts with a two parameter family of elliptic curves whose Newton complexes are unimodular triangulations, the so-called elliptic curves in symmetric honeycomb form. More explicitly, these are given by

$$a \cdot (x^3 + y^3 + z^3) + b \cdot (x^2y + x^2z + y^2x + y^2z + z^2y + z^2x) + xyz, \tag{5}$$

where $a, b \in K$ satisfy

$$v(a) > 2 \cdot v(b) > 0. \tag{6}$$

For any $a, j \in K$ with $v(j) < 0$ and $v(a) + v(j) > 0$, they then show that there are exactly 6 values of b such that the j-invariant of the genus 1 curve $E_{a,b}$ in Eq. 5 is equal to j. Since unimodular triangulations automatically induce faithful tropicalizations by Baker et al. (2016, Corollary 5.28), the theorem is then proved. One can also skip this theorem on faithful tropicalizations using the results in Katz et al. (2009). There it is shown computationally that any curve in symmetric honeycomb form automatically has a cycle of length $-v(j(E_{a,b}))$. This then also directly gives the theorem.

In the proof of Baker et al. (2016, Theorem 6.2), they start with the *Tate uniformization theorem*, which can be stated as follows. Let E be an elliptic curve over a nonarchimedean, complete, algebraically closed field K with $v(j(E)) < 0$. Then there exists an isomorphism

$$E(K) \simeq K^*/\langle q \rangle \tag{7}$$

for some $q \in K$ with $v(q) = -v(j(E))$. They then set out to find functions f and g whose associated piecewise linear functions $-\log|f|$ and $-\log|g|$ induce an isometric embedding $\Sigma \to \mathbb{R}^2$ of the minimal skeleton Σ (in the associated Berkovich space of E) into $\mathbb{R}^2$. To ensure that this is an isometric embedding, they need that at least one of the piecewise linear functions has slope equal to 1 on every edge. They then create these functions f and g using the image of the three-torsion point $P = q^{1/3}$ in $E(K)$.

Our approach mostly follows the strategy outlined in the previous paragraph. There are three differences however. First of all, we do not use the Tate uniformization theorem to obtain the desired three-torsion point. Instead, we will use the geometric interpretation of three-torsion points being *inflection points* (see Lemma 3.12). To find the correct analogue of $q^{1/3}$, we work out a reduction theory for elliptic curves over K and we use some classical results from arithmetic geometry to find the desired inflection point P. In doing this, we obtain an *explicit family* of elliptic curves E with $v(j(E)) < 0$ and a marked inflection point P reducing to the singular point, see Lemmas 4.1 and 4.2. Moreover, other ingredients in the proof of Baker et al. (2016, Theorem 6.2) are also made explicit, as we give a pair of functions $(f, g) \in (K(E))^2$ which induce a closed embedding, and we explicitly calculate its image. The corresponding tropicalization then contains a triangle and we show that its length is equal to $-v(j(E))$. This then also highlights the second key difference: we do not use any analytic material such as the *Poincaré–Lelong formula* (see Baker et al. 2014, Theorem 5.15) or the criterion for faithful tropicalizations (see Baker et al. 2016, Corollary 5.28) to abstractly conclude that our tropicalizations contain a cycle of the right length. We also note that we don't assume that our nonarchimedean field is complete. This allows us to directly use our results on the field of Puiseux series $\mathcal{P}$ for instance.

In our Main Theorem, we show that for any elliptic curve E/K with $v(j(E)) < 0$, we can find a tropicalization such that it contains a cycle of length $-v(j(E))$. We call this a numerically faithful tropicalization, see Sect. 2.1. This is not the same as a faithful tropicalization however. The difference is subtle and in Sect. 2.2 we explore this difference. Simply put, by expanding some of the edges and contracting others, we can obtain a cycle that has the right length, but which does not faithfully represent the minimal skeleton of the associated Berkovich space since it is not injective on the contracted parts. In Sect. 2.2, we abstractly prove the existence of a numerically faithful tropicalization that is not faithful, see Example 2.26.

Over the last couple of years, various new results on faithful tropicalizations have been published. For instance, in Baker and Rabinoff (2014) they proved using tropical Jacobians that for any curve C, there exists a rational map $C \to \mathbb{P}^3$ such that its tropicalization is an isometry onto its image. For genus two and Mumford curves, there are other more specific results in this direction, see Wagner (2015) or Jell (2018). On the other hand, one can also consider faithful tropicalizations of more general algebraic varieties. For results in these directions, we would like to direct the reader to Cueto et al. (2014) for faithful tropicalizations of the Grassmannian of planes and Kutler (2017) for faithful tropicalizations of hypertoric varieties.

The paper is structured as follows. We start by summarizing some well-known results in tropical geometry in Sect. 2. We cover the tropical semiring, tropical varieties, the fundamental theorem (Theorem 2.8), the structure theorem (Theorem 2.12) and we give the definition of a *numerically faithful tropicalization*. We then compare this definition to that of a faithful tropicalization in Sect. 2.2. Here we give two examples that are numerically faithful but not faithful. In Sect. 3, we introduce a reduction theory for elliptic curves over K as in Silverman (2009, Chapter VII) and in Sect. 4 we prove Theorem 1.1.

We tried to keep the text as elementary as possible, giving examples of the notions introduced wherever possible. As such, we believe that this paper serves as a didactic tool in understanding some of the more abstract material in Baker et al. (2016) and Baker et al. (2014) in concrete terms. To further aid the reader in this, we will point out any similarities and differences between our approach and the one in Baker et al. (2016) as we come across them.

We will be using most of the conventions regarding algebraic geometry as introduced in Silverman (2009, Chapters II and III). For tropical geometry, we will mostly be using Maclagan and Sturmfels (2015, Chapter 3). We also refer the reader to those two books for more background information regarding these topics.

2 Tropicalizations

In this section we discuss the notion of a tropicalization of a closed variety X/K inside $(K^*)^n$, where K is a nonarchimedean field as in Sect. 1. We will be mostly interested in the case where X is induced by an algebraic curve such as an elliptic curve over K. We introduce the tropical semiring and we recall the fundamental theorem of tropical geometry, which we will use in our main theorem as an easy tool to calculate tropicalizations. We refer the reader to Maclagan and Sturmfels (2015) for more background information regarding this topic. In Sect. 2.1, we study the underlying combinatorial structure of the tropicalization of a variety and we discuss the tropical structure theorem: Theorem 2.12. This in turn allows us to define the lattice length on the tropicalization of a curve, see Definition 2.15. After this we define a numerically faithful tropicalization for the special case of elliptic curves in Definition 2.20. In Sect. 2.2 we then compare this definition to the definition of a faithful tropicalization given in Baker et al. (2016, Sect. 5.15) and we give two examples of a numerically faithful tropicalization that is not faithful.

Consider the extended real line $\mathcal{R} := \mathbb{R} \cup \{\infty\}$ with its natural total order. We turn this into a semiring by defining the following two operations on $\mathcal{R}$:

$$a \oplus b = \min\{a, b\},$$
$$a \odot b = a + b.$$

Here we set $\infty \odot b = b \odot \infty = \infty$ for any $b \in \mathcal{R}$. We note that these operations mimic the following two identities of the valuation function $v : K \to \mathcal{R}$:

$$v(a + b) \geq \min\{v(a), v(b)\},$$
$$v(a \cdot b) = v(a) + v(b).$$

A multivariate tropical polynomial in n variables $\mathbf{x} = (x_1, x_2, ..., x_n)$, written as $f(\mathbf{x}) = \bigoplus_{i \in I} a_i \odot \mathbf{x}^i$ for $a_i \in \mathbb{R}$, is then the function $\mathcal{R}^n \to \mathcal{R}$ given by

$$\mathbf{x} \mapsto \bigoplus_{i \in I} a_i \odot \mathbf{x}^i. \tag{8}$$

Here I is a finite subset of $\mathbb{N}^n$, similar to the case of multivariate polynomials $K[x_1, x_2, ..., x_n]$. Any monomial of the form $a_i \odot \mathbf{x}^i$ is referred to as a term of f.

Example 2.1 Consider the tropical polynomial $f = (1 \odot x \odot y) \oplus (2 \odot 2x) = \min\{1 + x + y, 2 + 2x\}$. This defines the following piecewise linear function on $\mathbb{R}^2$:

$$f(x, y) = \begin{cases} 2 + 2x & \text{for } y \leq 1 + x, \\ 1 + x + y & \text{for } y > 1 + x. \end{cases} \tag{9}$$

Using a slight alteration of the definition of a tropical polynomial, we obtain the notion of a tropical Laurent polynomial. The definition imitates that of the ring $K[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$. A tropical Laurent polynomial is by definition a piecewise linear function as in Eq. 8, where we now allow the index set I to be a subset of $\mathbb{Z}^n$. For instance, for tropical Laurent polynomials in one variable x, we have that x^{-i} (written tropically) is equal to the function $-i \cdot x$. The multivariate case is similar.

We now define the tropicalization of an algebraic set $V(I) \subset (K^*)^n$ corresponding to an ideal $I \subset K[x_1, x_1^{-1}, ..., x_n, x_n^{-1}] =: K[\mathbf{x}, \mathbf{x}^{-1}]$. To that end, we first define the tropicalization of a multivariate polynomial $f \in K[\mathbf{x}, \mathbf{x}^{-1}]$. Let $f = \sum_{i \in I} a_i \mathbf{x}^i$ be a multivariate Laurent polynomial. We define the tropicalization of f to be the tropical Laurent polynomial given by

$$\mathrm{trop}(f) = \bigoplus_{i \in I} v(a_i) \odot \mathbf{x}^i, \tag{10}$$

where the product $\mathbf{x}^i$ is now a tropical product.

Example 2.2 Let $f = \varpi^2 + \varpi x^2 + y^3 + x^{-1}$, where $\varpi \in K$ satisfies $v(\varpi) = 1$. Then $\mathrm{trop}(f) = \min\{2, 1 + 2x, 3y, -x\}$.

For any multivariate Laurent polynomial f, we now introduce the notions of a tropical hypersurface and a tropical variety.

Definition 2.3 Let f be a multivariate Laurent polynomial with monomial terms h_i. We define the **tropical hypersurface** corresponding to a f to be the set of points $\mathbf{x} \in \mathbb{R}^n$ such that $\mathrm{trop}(f)(\mathbf{x}) = \mathrm{trop}(h_i)(\mathbf{x}) = \mathrm{trop}(h_j)(\mathbf{x})$ for at least two different terms of f. It is denoted by $\mathrm{trop}(V(f))$. A **tropical pre-variety** is then an intersection of these tropical hypersurfaces and a **tropical variety** is a subset of $\mathbb{R}^n$ of the form

$$\mathrm{trop}(V(I)) = \bigcap_{f \in I} \mathrm{trop}(V(f)), \tag{11}$$

where I is an ideal in the Laurent polynomial ring $K[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$. We refer to this subset $\mathrm{trop}(V(I))$ as the *tropicalization* of I.

Example 2.4 Let f be the tropical polynomial from Example 2.1. Then the tropical hypersurface corresponding to f is given by $\mathrm{trop}(V(f)) = \{(x, y) \in \mathbb{R}^2 : y = 1+x\}$.

Example 2.5 Not every tropical pre-variety is a tropical variety, as we will see in Example 2.11. The idea is to take a maximal ideal I inside $K[x, x^{-1}, y, y^{-1}]$ with two generators f_1 and f_2 such that the intersection of the two corresponding tropical hypersurfaces is a half-ray. This intersection is one-dimensional, whereas the zero set of the ideal I is zero-dimensional. We will see that this causes any ideal J with $\text{trop}(V(J)) = \text{trop}(V(f_1)) \cap \text{trop}(V(f_2))$ to be zero-dimensional, which contradicts the fundamental theorem of tropical geometry: Theorem 2.8. The details are given in Example 2.11.

We now relate the construction of tropical varieties to another, perhaps more natural, construction. Consider the "naive" tropicalization map

$$\text{val} : (K^*)^n \to \mathbb{R}^n \tag{12}$$

$$(x_1, x_2, ..., x_n) \mapsto (v(x_1), v(x_2), ..., v(x_n)). \tag{13}$$

Definition 2.6 Let I be an ideal of the Laurent polynomial ring $K[\mathbf{x}, \mathbf{x}^{-1}]$. The *naive tropicalization* of an algebraic set $V(I)$ is defined to be the closure of the image of $V(I)$ under $\text{val}(\cdot)$. We denote it by $\text{val}(V(I))$. It is also referred to as the **non-archimedean amoeba** associated to I, see Einsiedler et al. (2006, Definition 1.1.1.).

Example 2.7 Let C be the plane curve defined by $f := \varpi^{-3}x_1 + \varpi^{-2}x_2 - 1 = 0$. We consider three cases.

- Suppose that $v(x_1) > 3$. Then we must have $v(x_2) = 2$ by considering the valuations of both sides of the equation $\varpi^{-3}x_1 = 1 - \varpi^{-2}x_2$.
- Suppose that $v(x_2) > 2$. Then similarly $v(x_1) = 3$.
- Suppose now that $v(x_1) \leq 3$ and $v(x_2) \leq 2$. Then the valuations of $\varpi^{-3}x_1$ and $\varpi^{-2}x_2$ have to be equal in order to obtain $v(1) = 0$. In other words, $-3 + v(x_1) = -2 + v(x_2)$, which gives $v(x_1) = 1 + v(x_2)$.

We thus see that the nonarchimedean amoeba consists of three linear pieces, as depicted in Fig. 3.

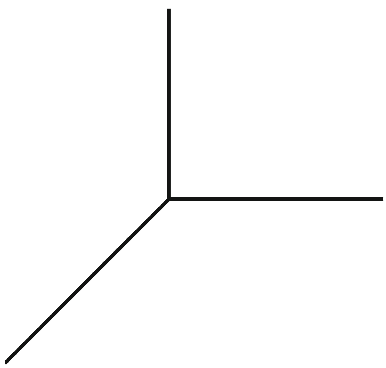


Fig. 3 The tropicalization obtained in Example 2.7

The good news now is that the naive tropicalization of $V(I)$ coincides with the tropicalization defined in Definition 2.3. This is also known as the fundamental theorem of tropical geometry.

Theorem 2.8 (Fundamental theorem of tropical geometry) *Let* $\mathrm{trop}(V(I))$ *be the tropical variety defined in Definition* 2.3 *and let* $\mathrm{val}(V(I))$ *be the naive tropicalization defined in Definition* 2.6*. Then*

$$\mathrm{trop}(V(I)) = \mathrm{val}(V(I)). \tag{14}$$

Proof See (Maclagan and Sturmfels 2015, Theorem 3.2.3). □

Remark 2.9 Since the algebraic variety $V(I)$ only depends on the radical $\sqrt{I}$ of I in the sense that $V(I) = V(\sqrt{I})$, we conclude by Theorem 2.8 that $\mathrm{trop}(V(I)) = \mathrm{trop}(V(\sqrt{I}))$.

Example 2.10 Let $f = \varpi^{-3}x_1 + \varpi^{-2}x_2 - 1$ as in Example 2.7. Its tropical polynomial is then given by

$$\mathrm{trop}(f) = \min\{x_1 - 3, x_2 - 2, 0\}. \tag{15}$$

We then easily see that the points (x_1, x_2) where the minimum is attained at least twice is exactly equal to the non-archimedean amoeba calculated in Example 2.7.

Example 2.11 We now continue the train of thought in Example 2.5 and give an example of a tropical pre-variety S that is not a tropical variety. Our proof that S is not a tropical variety will use Theorem 2.8 together with some commutative algebra.

Consider the ideal $I = (1 + x + y, x + 2y)$ in $K[x, x^{-1}, y, y^{-1}]$. One then easily checks that $V(I) = \{(-1, 2)\}$, so Theorem 2.8 tells us that $\mathrm{trop}(V(I))$ consists of a single point. We now consider the tropicalizations of the hypersurfaces defined by $f_1 = 1 + x + y$ and $f_2 = x + 2y$. The tropicalization of $V(1 + x + y)$ is then as in Examples 2.7 and 2.10 and the tropicalization of $V(x + 2y)$ is a single line segment. They intersect in the half-ray $S = \{(w_0, w_1) : w_0 = w_1 < 0\} \subset \mathbb{R}^2$, see Fig. 4. We claim that this half-ray is not of the form $\mathrm{trop}(V(J))$ for any ideal J in $K[x, x^{-1}, y, y^{-1}]$.

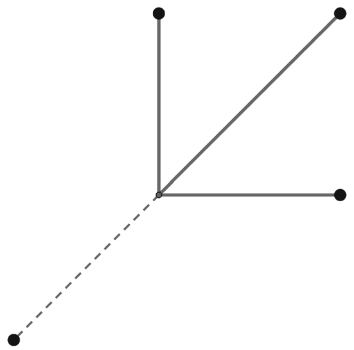


Fig. 4 The dotted line gives a tropical pre-variety which is not a tropical variety, see Example 2.11 for the proof

Suppose for a contradiction that we can write $S = \text{trop}(V(J))$ for some ideal J. By Remark 2.9, we can and do assume that J is radical. By (Eisenbud 1995, Corollary 2.12) we find that J is the intersection of all prime ideals that contain J and thus it is the intersection of the finitely many minimal prime ideals that contain it (see Eisenbud 1995, Chapter 3 for the notion of a minimal prime ideal). We will show that these prime ideals all have height 2, which shows that they are maximal since the dimension of $K[x, x^{-1}, y, y^{-1}]$ is two.

Note that none of these prime ideals can have height zero, since $J \neq 0$. Suppose on the other hand that the height of such a $\mathfrak{p}_i$ is one. Then $\mathfrak{p}_i$ is principal, since $K[x, x^{-1}, y, y^{-1}]$ is a unique factorization domain (being a nonzero localization of a unique factorization domain). We can thus write $\mathfrak{p}_i = (f_i)$ for an irreducible element $f_i \in K[x, x^{-1}, y, y^{-1}]$. Since $\mathfrak{p}_i = (f_i) \supset J$, we obtain $V(f_i) \subset V(J)$ and thus $\text{trop}(V(f_i)) \subset \text{trop}(V(J)) = S$. But this can't happen: $\text{trop}(V(f_i))$ will contain points with positive valuation. Indeed, for almost all x_0 (i.e. a Zariski dense set in K^1),[1] we have that $f_i(x_0, y)$ is a nonconstant polynomial. Since K is algebraically closed, there exists a y_0 such that $f_i(x_0, y_0) = 0$ and for any such y_0, we then have that $P = (x_0, y_0) \in V(f_i) \subset V(J)$. If we now chose x_0 such that $v(x_0) > 0$, this will contradict $\text{trop}(V(f_i)) \subset \text{trop}(V(J))$. Indeed, Theorem 2.8 gives us $\text{val}(P) \in \text{val}(V(f_i)) = \text{trop}(V(f_i)) \subset S$, which is the desired contradiction. We thus conclude that the minimal primes of J are maximal ideals. That is, J defines a zero-dimensional variety. But then $S = \text{trop}(V(J)) = \text{val}(V(J))$ must also consist of finitely many points, another contradiction. We conclude that S is not a tropical variety.

We now investigate some of the combinatorics underlying a tropical variety. To that end, we first recall some polyhedral geometry. A good reference for the material here is Ziegler (1995). We will mostly follow Maclagan and Sturmfels (2015, Sect. version of this theorem can also2.3) for the tropical side of things.

A *polyhedron* P is a subset of $\mathbb{R}^n$ that can be written as

$$P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}, \tag{16}$$

where A is a $d \times n$-matrix and $\mathbf{b} \in \mathbb{R}^d$. A *face* of a polyhedron P is then a subset of P that can be written as

$$\text{face}_{\mathbf{w}}(P) = \{\mathbf{x} \in P : \mathbf{w}(\mathbf{x}) \leq \mathbf{w}(\mathbf{y}) \text{ for all } \mathbf{y} \in P\}, \tag{17}$$

where $\mathbf{w}$ is some element of the dual space $(\mathbb{R}^n)^\vee := \text{Hom}(\mathbb{R}^n, \mathbb{R})$. A *facet* of a polyhedron is a face that is not contained in a larger proper face. A *polyhedral complex* is a finite collection Σ of polyhedra in $\mathbb{R}^n$ such that the following two hold:

1. If $P \in \Sigma$, then any face of P is also in Σ.
2. If P and Q are in Σ, then $P \cap Q$ is either empty or a face of both P and Q.

[1] To be explicit, we can write $f = \sum c_i(x) y^i$ where the $c_i(x)$ are elements of $K[x, x^{-1}]$ and consider a nonzero coefficient $c_i(x)$ for $i \neq 0$. The set $U = K \backslash V(c_i(x))$ inside K with its Zariski topology then has the required properties.

The elements of a polyhedral complex are referred to as *cells* and cells of a polyhedral complex that are not faces of any larger cell are the *facets* of the complex. The underlying points of a polyhedral complex Σ, or the *support* of Σ, is defined by

$$\operatorname{supp}(\Sigma) = |\Sigma| := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in P \text{ for some } P \in \Sigma\}. \tag{18}$$

For a polyhedron P inside a polyhedral complex Σ, the *affine span* of P is the smallest affine linear space that contains it. The *dimension* of the polyhedron P is then the dimension of this affine linear space. We say that a polyhedral complex Σ is pure of dimension n if every facet of Σ has dimension n.

We now consider Γ-rational polyhedra and polyhedral complexes, where Γ is a subgroup of the additive group of $\mathbb{R}$. For us, this Γ will always be the value group of the discretely valued field K. A *Γ-rational polyhedron* is a polyhedron that is defined by a matrix A with integer entries and a vector $\mathbf{b} \in (\Gamma)^d$ as in Eq. 16. Similarly, a Γ-rational polyhedral complex is a polyhedral complex consisting of Γ-rational polyhedra and its support is just its support as a polyhedral complex.

Theorem 2.12 (Structure theorem for tropical varieties) *Let I be an ideal of $K[\mathbf{x}, \mathbf{x}^{-1}]$ defining the variety $X := V(I)$. Let* $\operatorname{trop}(X)$ *be its tropicalization, as defined in Definition 2.3. Then* $\operatorname{trop}(X)$ *is the support of a Γ-rational polyhedral complex Σ. If X is an irreducible variety of dimension n, then the Γ-rational polyhedral complex Σ obtained above is pure of dimension n.*

Proof See Maclagan and Sturmfels (2015, Theorem 3.2.3.(2), and Lemma 3.2.10). An early version of this theorem can also be found in Bieri and Groves (1984, Theorem A). □

Example 2.13 Let us show that the tropicalization in Examples 2.7 and 2.10 are $\mathbb{Q}$-rational polyhedral complexes by explicitly giving the matrices A and the vectors $\mathbf{b}$ as in Eq. 16. We will use x_1 and x_2 as coordinates in $\mathbb{R}^2$.

We have three linear pieces $S_i \subset \mathbb{R}^2$. S_1 is defined by the equations $x_1 = 3$ and $x_2 \geq 2$, S_2 is defined by $x_2 = 2$ and $x_1 \geq 3$ and S_3 is defined by $x_2 = x_1 - 1$ and $x_2 \leq 2$ and $x_1 \leq 3$. The corresponding matrices A_i and vectors $\mathbf{b}_i$ are then given by:

$$A_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{b}_1 = \begin{pmatrix} 3 \\ -3 \\ -2 \end{pmatrix}, \tag{19}$$

$$A_2 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}, \tag{20}$$

$$A_3 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}. \tag{21}$$

Note that this polyhedral complex is pure of dimension one, which makes sense in view of Theorem 2.12 since $V(I)$ defines an irreducible curve.

2.1 Lattice lengths and numerically faithful tropicalizations

In this section we study the case of tropical curves in more detail. Using the structure theorem from the last section, we define the lattice length for any segment of a tropical curve. We then specialize to elliptic curves and define the notion of a *numerically faithful tropicalization* using the j-invariant of the elliptic curve. In the next section, we will relate this to the notion of a *faithful tropicalization*, as introduced and studied in Bieri and Groves (2016, Sect. 5.15).

We start with the definition of a tropical curve associated to an algebraic curve.

Definition 2.14 *(Tropical curves)* Let $X \subset (K^*)^n$ be an irreducible curve. We define the tropical curve associated to X to be its tropicalization $\mathrm{trop}(X)$ as in Definition 2.3.

By Theorem 2.12, we have that $\mathrm{trop}(X)$ is the support of a Γ-rational polyhedral complex, pure of dimension one. If we have a bounded edge e in $\mathrm{trop}(X)$, then it belongs to some polyhedral cone P defined by a matrix A with values in $\mathbb{Z}$ and a vector $\mathbf{b}$ with values in $(\Gamma)^n$. The directional vector $\mathbf{v}$ corresponding to e then lies in the polyhedral cone P' defined by $A\mathbf{x} \leq 0$.

We now recall the *fundamental theorem of polyhedral geometry*, also known as the Minkowski-Weyl theorem. Let L be a subfield of the real numbers (such as $\mathbb{Q}$). The fundamental theorem then says that a subset of L^n is defined by linear inequalities $A\mathbf{x} \leq 0$ (where A is a matrix with entries in L) if and only if it is equal to the cone generated by a finite set of vectors in L^n. Here the cone generated by a finite set of vectors $Y = \{y_1, \dots, y_k\} \subset L^n$ is given by

$$\mathrm{cone}(Y) = \{\lambda_1 y_1 + \cdots + \lambda_k y_k : \lambda_i \geq 0\}. \tag{22}$$

We direct the reader to Ziegler (1995, Theorem 1.2) for a proof of the fundamental theorem. We note that it is usually only stated and proved for the field of real numbers $\mathbb{R}$, but the result holds for any subfield L of $\mathbb{R}$. Indeed, the Fourier–Motzkin method described in Ziegler (1995, Sect. 1.2) only uses the ordering on L (and not any completeness properties), so the same proof works over L.

We now apply this theorem with $L = \mathbb{Q}$ to the polyhedral cone P' defined by $A\mathbf{x} \leq 0$. Using the fact that P' is one-dimensional, we see that P' is equal to the cone generated by a single vector $\mathbf{v}$ in $\mathbb{Q}^n$. There is then a unique primitive $\mathbb{Z}$-valued vector $\mathbf{w} \in P'$ and we can use this vector $\mathbf{w}$ to define the *lattice length* of e.

Definition 2.15 *(Lattice length for tropical curves)* Let $e = PQ$ be a finite edge in the tropicalization $\mathrm{trop}(X)$ of a curve $X \subset (K^*)^n$ with directional vector $\mathbf{v}$. Let $\mathbf{w}$ be the unique primitive $\mathbb{Z}$-valued vector in the direction of $\mathbf{v}$ as above. Writing $\mathbf{v} = \lambda \cdot \mathbf{w}$ for some $\lambda \in \mathbb{R}_{\geq 0}$, we then define $\ell(e) = \lambda$. This is referred to as the **lattice length** of e.

Example 2.16 **(Main Example)** Consider the polynomial $f = x^2y + xy + xy^2 + \varpi^k$ with $v(\varpi) = 1$ and k some positive integer.

The tropical polynomial associated to f is

$$\mathrm{trop}(f) = \min\{2x + y, x + y, x + 2y, k\}. \tag{23}$$

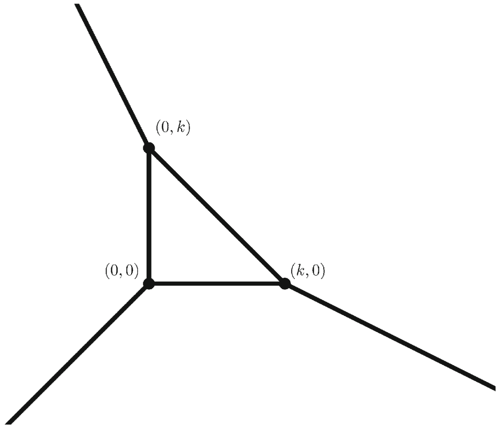


Fig. 5 The tropicalizations of the curves in Example 2.16

The tropicalization trop($V(f)$) then contains a triangle with vertices $(0, 0)$, $(0, k)$ and $(k, 0)$, see Fig. 5. The *ordinary length* of the edge between $(0, k)$ and $(k, 0)$ is $\sqrt{2} \cdot k$. The *lattice length* of this edge is just k, since the primitive directional vector in the direction of e is $\pm(1, -1)$.

We now define the notion of a cycle.

Definition 2.17 A *cycle* inside the tropicalization trop(C) of an irreducible curve $C \subset (K^*)^n$ is a finite set of bounded edges inside trop(C) that form a leafless, connected subgraph of Betti number one. Here a leaf is an edge connected to a vertex of valence one. The *length* of the cycle is the sum of the lattice lengths (as defined in Definition 2.15) of the edges in the subgraph.

Example 2.18 Let C be the curve defined by $f = x^2y + xy + xy^2 + \varpi^k = 0$, as in Example 2.16, which defines an elliptic curve $\overline{C}$. In Sect. 4, we will see a Weierstrass form of this equation. By the calculations in Example 2.16, the corresponding tropicalization contains a triangle with edges of length k. This thus defines a cycle inside trop(C) of total length $3k$.

Example 2.19 Consider an elliptic curve given by a Weierstrass equation

$$y^2 = x^3 + Ax + B. \tag{24}$$

We denote the corresponding curve in $(K^*)^2$ by C. There are then two options for the tropicalization of $C \subset (K^*)^2$. For $v(A)/2 < v(B)/3$, we have that trop(C) consists of five line segments as in Fig. 6. Explicitly, they are given (from left to right) by:

- The line S_1 given by: $2y = 3x$ for $x \leq v(A)/2$, $y \leq 3/4 \cdot v(A)$,
- The line S_2 given by: $x = v(A)/2$ for $y > 3/4 \cdot v(A)$,
- The line S_3 given by: $2y = v(A) + x$ for $v(A)/2 < x < v(B)/3$,
- The line S_4 given by: $x = v(B)/3$ for $y > v(B)/2$,
- The line S_5 given by: $y = v(B)/2$ for $x > v(B)/3$.

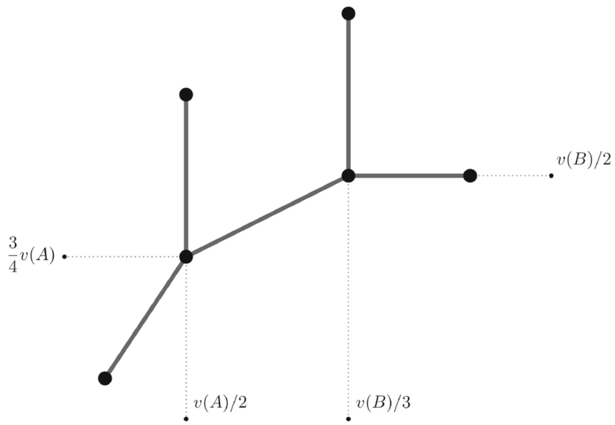


Fig. 6 The tropicalizations in Example 2.19 of elliptic curves in Weierstrass form with $v(A)/2 < v(B)/3$. The dotted lines highlight the x and y-coordinates

For $v(A)/2 \geq v(B)/3$, the two vertical line segments S_2 and S_4 become one segment and S_3 reduces to a single point, as the reader can easily check. In this case, $\text{trop}(C)$ consists of three line segments as in Fig. 3. In particular, we now see that the tropicalization of an elliptic curve in Weierstrass form never gives rise to a cycle.

We now specialize to elliptic curves. Recall from Silverman (2009, Chapter III, Proposition 1.4(b)) that to any elliptic curve E/K, we can associate a number $j(E) \in K$ such that $j(E) = j(E')$ for two elliptic curves E, E' if and only if E and E' are isomorphic. This number is known as the *j-invariant* of the elliptic curve. This j-invariant thus determines the isomorphism class of an elliptic curve over an algebraically closed field. If K is not algebraically closed, then the statement becomes a little bit different, see Silverman (2009, Chapter X, Proposition 5.4). If an elliptic curve E is given in Weierstrass form $y^2 = x^3 + Ax + B$, then the j-invariant can be given explicitly by:

$$j(E) = -1728\frac{(4A)^3}{\Delta}, \tag{25}$$

where $\Delta = -16(4A^3 + 27B^2)$ is the *discriminant* of the elliptic curve.

Suppose now that we are given an irreducible curve $C \subset (K^*)^n$. Then C can be compactified to give a smooth curve $\overline{C}$ over K, see Hartshorne (1977, Chapter 1, Sect. 6). If $\overline{C}$ has genus one, then we say that C has genus one. Choosing a rational point on $\overline{C}$ (since K is algebraically closed, there is no problem in finding such a point), we find that $\overline{C}$ is an elliptic curve and thus has a j-invariant $j(\overline{C})$. We can then associate a j-invariant $j(C)$ to such a curve C by setting $j(C) = j(\overline{C})$.

We now come to the key definition of a numerically faithful tropicalization.

Definition 2.20 *(Numerically faithful tropicalizations)* Let $C \subset (K^*)^n$ be an irreducible curve of genus one and let $\text{trop}(C)$ be its tropicalization, as in Definition 2.3.

Suppose that $v(j(C)) < 0$. We say that trop(C) is a *numerically faithful tropicalization* if there is a cycle of of length $-v(j(C))$ in trop(C), where $j(C)$ is the j-invariant associated to C. Here the length is as defined in Definition 2.15.

Example 2.21 Let C again be an elliptic curve in Weierstrass form, as in Example 2.19. We saw in that Example that the tropicalization never contains a cycle. We thus see that this tropicalization is never numerically faithful.

Example 2.22 Let f again be as in Examples 2.16 and 2.18. We will give an explicit formula for the j-invariant of the corresponding elliptic curve in our proof of Theorem 1.1, see Eq. 93. A calculation then shows that $-v(j(C)) = 3k$, showing that it is a numerically faithful tropicalization.

2.2 Faithful tropicalizations

We now come to the notion of a *faithful tropicalization*, as defined in Baker et al. (2016, Sect. 5.15). To fully define a faithful tropicalization would take us too far into the field of Berkovich spaces and nonarchimedean geometry, so we will first give a summary of the necessary concepts together with references where the reader can learn more about them. After this, we will state the definition and work out some of the relevant notions in an example. We will then point out some key differences between a *faithful* tropicalization and a *numerically faithful* tropicalization. The upcoming material is not strictly necessary to understand the final result of this paper and is somewhat more technical, so the reader can skim through it on a first read-through. We will assume that K is complete throughout this section.

Let us start with a summary of some of the concepts necessary to define a faithful tropicalization:

- To any connected curve X over a nonarchimedean field K, one can associate its so-called *Berkovich analytification* X^{an}, as in Berkovich (2012). It is a path-connected Hausdorff space that contains the rational points $X(K)$ as a dense open subset and locally looks like a *metric tree*. Some good references to learn more about Berkovich spaces are: Baker and Rumely (2010), Jonsson (2016) and Berkovich (2012).
- Let $M \simeq \mathbb{Z}^n$ be a lattice, $N = \text{Hom}(M, \mathbb{Z})$ its dual lattice and $T = \text{Spec}(K[M])$ the corresponding torus, written as the spectrum of the group ring over K corresponding to M. We write $N_{\mathbb{R}} := N \otimes \mathbb{R} \simeq \mathbb{R}^n$ for the real vector space corresponding to N. Consider a toric variety Y_{Δ}, defined by a polyhedral fan Δ in $N_{\mathbb{R}}$. For every polyhedral cone σ in Δ there is a tropicalization map from Y_{σ}^{an} to the space of additive semigroup homomorphisms $\text{Hom}(\sigma^{\vee} \cap M, \mathbb{R} \cup \{\infty\})$ and we write $N_{\mathbb{R}}(\sigma)$ for the image of Y_{σ}^{an} under this map. For an inclusion of cones $\tau \preceq \sigma$, this yields $N_{\mathbb{R}}(\tau) \subset N_{\mathbb{R}}(\sigma)$ and we write $N_{\mathbb{R}}(\Delta)$ for the space obtained by gluing along these inclusions induced by cones in Δ (see Baker et al. 2016, Definition 4.3 for the details). This then induces a *generalized tropicalization map:*

$$\text{trop} : Y_{\Delta}^{\text{an}} \to N_{\mathbb{R}}(\Delta). \tag{26}$$

In terms of the tropicalizations studied in the beginning of Sect. 2, these concepts translate as follows. The toric variety used there is $(K^*)^n$, given in scheme-theoretic form by $Y_\Delta = \mathrm{Spec}(K[\mathbf{x}, \mathbf{x}^{-1}])$, and $N_\mathbb{R}(\Delta)$ is just $\mathbb{R}^n$. The K-rational points of Y_Δ (which can be identified with $(K^*)^n$) naturally embed into Y_Δ^{an} and the generalized tropicalization map in Eq. 26 coincides on these K-rational points with the tropicalization map $\mathrm{val}(\cdot)$ given in Eq. 12. For more background information regarding toric varieties and generalized tropicalization maps, we refer the reader to Maclagan and Sturmfels (2015, Chapter 6) and Baker et al. (2016, Sects. 4 and 5). Other good references for toric varieties are Fulton (1993) and Cox et al. (2011).

– One can also use this generalized tropicalization map to define a tropicalization map for closed subschemes of a toric variety Y_Δ. In the context of curves and their compactifications, this tropicalization map works as follows. Let X be a smooth and connected curve over K and let $\overline{X}$ be its smooth compactification as in Hartshorne (1977, Chapter 1, Sect. 6). Let $\overline{X} \to Y_\Delta$ be a closed immersion of $\overline{X}$ into a toric variety Y_Δ with polyhedral fan Δ such that $\overline{X}$ meets the dense torus T of Y_Δ. This then induces a tropicalization map

$$\mathrm{trop} : \overline{X}^{\mathrm{an}} \to N_\mathbb{R}(\Delta) \tag{27}$$

by taking the composition of the embedding and the generalized tropicalization map in Eq. 26, see Baker et al. (2016, Definition 4.3). We will use this map to define faithful tropicalizations of the proper curve $\overline{X}$.

– Let $\overline{\mathcal{X}}$ be a semistable R-model for $\overline{X}$. That is, it is an integral scheme $\overline{\mathcal{X}}$ with a flat proper morphism $\overline{\mathcal{X}} \to \mathrm{Spec}(R)$ and an isomorphism $\overline{\mathcal{X}}_\eta \simeq \overline{X}$ such that the special fiber $\overline{\mathcal{X}}_s$ is a semistable curve in the sense of Liu (2006, Chapter 10, Definition 3.1).

To any semistable R-model $\overline{\mathcal{X}}$ of $\overline{X}$, one can associate a *skeleton* as follows.[2] The generic points of the special fiber $\overline{\mathcal{X}}_s$ give rise to type-2 points of the Berkovich analytification as in Theorem (Baker et al. 2014, Theorem 4.6(1)) using the reduction map. This set of type-2 points is known as a semistable vertex set and we denote it by V. The complement $X^{\mathrm{an}}\backslash V$ then decomposes as a disjoint union of open balls and generalized open annuli. We define the skeleton $\Sigma(\overline{X}, V)$ to be the union of the skeleta of these generalized open annuli together with the set V, see Baker et al. (2014, Definition 3.3).

– There is a metric on the set $\mathbf{H}_0(\overline{X}^{\mathrm{an}})$ of skeletal points, which is the set of points of type 2 and 3 inside $\overline{X}^{\mathrm{an}}$. This metric is known as the *skeletal metric*. Explicit local formulas for this metric are given in Baker et al. (2014, Sect. 5.3). A *finite subgraph* of $\overline{X}^{\mathrm{an}}$ is then an isometric embedding $\Gamma \to \mathbf{H}_0(\overline{X}^{\mathrm{an}})$ of a finite connected metric graph Γ into this set of skeletal points.

[2] In Baker et al. (2014) they use formal semistable R-models to define the skeleton, but (Amini et al. 2015, Lemma 5.1) tells us that the category of semistable models is equivalent to the category of formal semistable R-models by using the "completion functor", so there is no harm in using this category.

To familiarize ourselves with some of the concepts given here, let us give an example of a skeleton using an explicit semistable model. We will work out the semistable model and the skeleton for the curve defined in Example 2.16.

Example 2.23 Let $f = x^2y + xy + xy^2 + \varpi^k$ be the polynomial from Examples 2.16 and 2.18 defining a curve inside $(K^*)^2$. We consider the (affine) model

$$\mathcal{C} = \mathrm{Spec}(R[x, y]/(f)) \tag{28}$$

over the valuation ring R of K. The special fiber is then given by the spectrum of the tensor product $R[x, y]/(f) \otimes_R k = k[x, y]/(\overline{f})$, where k is the residue field and $\overline{f}$ is the reduction of f to the polynomial ring $k[x, y]$. We then have $\overline{f} = x^2 + xy + xy^2$. This polynomial is reducible: $\overline{f} = xy(x + y + 1)$. We now see that the special fiber contains three irreducible components given by the factors x, y and $x + y + 1$. The corresponding prime ideals in $\mathcal{C}$ are given by $\mathfrak{p}_1 = (x, \varpi)$, $\mathfrak{p}_2 = (y, \varpi)$ and $\mathfrak{p}_3 = (x + y + 1, \varpi)$. The corresponding components $\Gamma_i = V(\mathfrak{p}_i)$ intersect each other transversally in a cyclic fashion: the component Γ_1 intersects Γ_2 in the maximal ideal $\mathfrak{n}_{1,2} = (x, y, \varpi)$, Γ_2 intersects Γ_3 in $\mathfrak{n}_{2,3} = (y, \varpi, x + 1)$ and Γ_3 intersects Γ_2 in $(x, y + 1, \varpi)$. The incidence graph (see Liu 2006, Chapter 10, Definition 3.17) thus consists of a cyclic graph with three vertices and three edges, which already shows a strong connection with the tropicalization of C calculated in Example 2.18.

Using some birational geometry, this $\mathcal{C}$ can be considered as an open affine subset of a semistable model $\overline{\mathcal{C}}$ for the compactified $\overline{C}$, where the generic points of the nontrivial components of $\overline{\mathcal{C}}$ are all in this open affine $\mathcal{C}$. By Theorem (Baker et al. 2014, Theorem 4.7), this gives a so-called *semistable vertex set* V of $\overline{C}$ and the skeleton $\Sigma(\overline{C}, V)$ is just the incidence graph of $\mathcal{C}$ (see Baker et al. 2014, Sect. 4.9). A quick calculation shows that the lengths of the ordinary double points are k [with respect to ϖ, see Liu (2006, Chapter 10, Definition 3.23) for the definition], which again shows the strong connection with the tropicalization in Example 2.18.

Definition 2.24 *(Faithful tropicalizations)* (See Baker et al. 2016, Sect. 5.15) Let X be a smooth, connected curve over a complete, algebraically closed field K with smooth compactification $\overline{X}$ and consider a closed immersion $\overline{X} \to Y_\Delta$ of $\overline{X}$ into a toric variety Y_Δ with dense torus T. Assume that $T \cap \overline{X} \neq \emptyset$ and consider the tropicalization map given in Eq. 27.

- We say that a finite subgraph Γ of the Berkovich analytification $\overline{X}^{\mathrm{an}}$ is *faithfully tropicalized* by the tropicalization map $\mathrm{trop} : \overline{X}^{\mathrm{an}} \to N_{\mathbb{R}}(\Delta)$ if $\mathrm{trop}(\cdot)$ maps Γ homeomorphically and isometrically onto its image.
- We say that $\mathrm{trop}(\cdot)$ is a faithful tropicalization of $\overline{X}$ if a skeleton of $\overline{X}$ is faithfully tropicalized.

Example 2.25 It can be shown that any polynomial $f \in K[x, y]$ whose *Newton complex* (see Rabinoff (2012) and Maclagan and Sturmfels (2015), Proposition 3.1.6) is a unimodular triangulation gives rise to a faithful tropicalization. This is (Baker et al. 2016, Corollary 5.28(2)).

A quick calculation shows that the Newton complex defined by $f = x^2y + xy + xy^2 + \varpi^k$ is as in Fig. 7, which is a unimodular triangulation. This implies that the

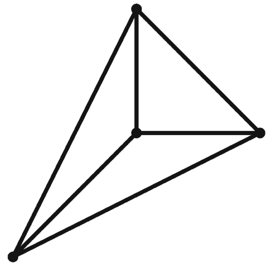


Fig. 7 The Newton complex in Example 2.25

corresponding tropicalization trop(C) is faithful and we thus conclude that $v(j(C)) = -3k$ from Example 2.18. In Theorem 1.1, we will show that $v(j(C)) = -3k$ without this result on faithful tropicalizations.

We now compare the notions of a numerically faithful tropicalization and a faithful tropicalization. First, a faithful tropicalization is automatically numerically faithful (if we adopt the more general form of tropicalizations using toric varieties as explained earlier). Indeed, if $v(j(E)) < 0$, then the Berkovich space contains a cycle with length $-v(j(E))$ and this subgraph is mapped isometrically (for the lattice length on trop(C)) onto its image in trop(C), so trop(C) contains a cycle of length $-v(j(E))$. See the discussion in Baker et al. (2016, Sect. 6) for more details. It is not true however that every numerically faithful tropicalization is faithful, as the following example shows.

Example 2.26 We will use the slope formula from Baker et al. (2014, Theorem 5.15) to calculate the abstract tropicalization as in Baker et al. (2016, Theorem 6.2). Throughout this example we will be making heavy use of the results in Baker et al. (2016) and Baker et al. (2014), so we refer the reader to those papers for more information on these concepts. Consider an elliptic curve E with multiplicative reduction, that is: $v(j(E)) < 0$. Then E can be analytically uniformized:

$$E^{\mathrm{an}} \simeq \mathbb{G}_m^{\mathrm{an}}/q^{\mathbb{Z}} \tag{29}$$

for some $q \in K^*$ with $v(q) = -v(j(E))$. This is also known as the Tate uniformization.

We now choose a fourth root $q^{1/4}$ of q and we consider the four-torsion point $P \in E(K)[4]$ corresponding to it under the Tate uniformization above. We write $P_i = i \cdot P$ for the multiples of P under the group law on $E(K)$. In particular we have that $P_0 = P_4$ is the identity element. The minimal skeleton Σ of E is then a circle of length $-v(j(E))$ and there is a natural retraction map $E^{\mathrm{an}} \to \Sigma$ onto this skeleton. If we consider the natural map $K^* \to E^{\mathrm{an}}$ arising from the Tate uniformization, then the composite map $K^* \to \Sigma$ is just given by $z \mapsto [\mathrm{val}(z)]$, see the proof of Baker et al. (2016, Theorem 6.2). We denote the type-2 points the points P_i retract to by Γ_i. See Fig. 8 for a pictorial description of this retraction.

Now consider the following divisors:

$$D_1 = 6P_2 - 4P_1 - 2P_3,$$
$$D_2 = 2P_3 - P_2 - P_0.$$

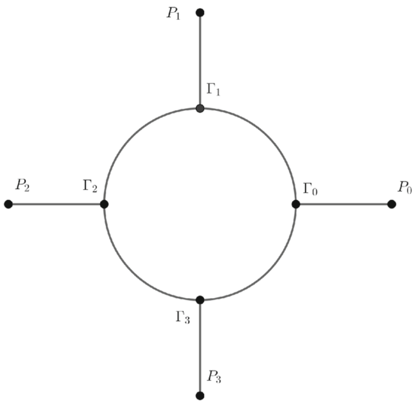


Fig. 8 The Berkovich minimal skeleton in Example 2.26, together with a set of four-torsion points that retract nontrivially onto the skeleton

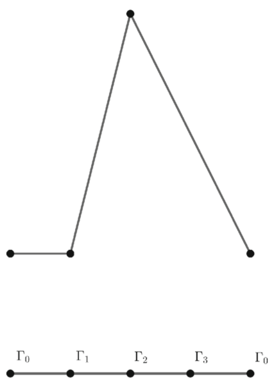


Fig. 9 The piecewise linear function F_1 constructed in Example 2.26

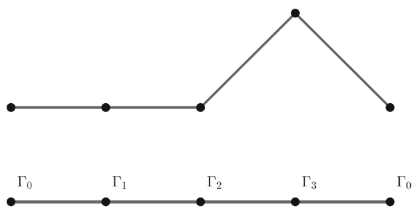


Fig. 10 The piecewise linear function F_2 constructed in Example 2.26

Since they have degree zero and add up to zero under the group law, we find that they are principal by Silverman (2009, Chapter III, Corollary 3.5). We can thus write $\text{div}(f_i) = D_i$ for f_i in the function field of E. We calculate the corresponding piecewise-linear functions on the skeleton Σ using the slope formula. The result is in Figs. 9 and 10. We thus see that both $F_i = -\log|f_i|$ have slope zero on $\Gamma_0\Gamma_1$ and slope divisible by 2 on $\Gamma_1\Gamma_2$. The slope of F_2 on $\Gamma_2\Gamma_3$ and $\Gamma_3\Gamma_0$ is ± 1, whereas the slope of F_1 is -2 on these edges. The slope of F_1 on $\Gamma_1\Gamma_2$ is 4.

If we now consider the embedding given by (f_1, f_2), it might not be a closed embedding. There are two ways out of this problem. Consider the union Z of the supports of

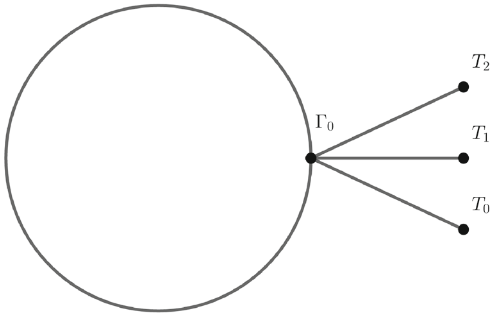


Fig. 11 The minimal skeleton in Example 2.26 with three-torsion points retracting to the same point on Σ

the f_i inside E. Let $U = E\backslash Z$. We can then define a generalized tropicalization map on U. To that end, we first define a map[3] $U \to (K^*)^2$ using these f_i and we then define an induced tropicalization map from U^{an} to $\mathbb{R}^2$, see Baker and Rabinoff (2014, Sect. 8). The tropicalization induced by (f_1, f_2) in this sense then has the following properties. It collapses $\Gamma_0\Gamma_1$ to a point since both F_i have slope zero on that segment. It expands $\Gamma_1\Gamma_2$ by a factor 2, since the greatest common divisor of the slopes of the F_i is 2, see (Baker et al. 2016, Remark 5.6). On the remaining two edges the greatest common divisor is just one, so it is isometric there. This implies that the tropicalization contains a cycle of length $0 + 2 \cdot (v(q)/4) + 1 \cdot (v(q)/4) + 1 \cdot (v(q)/4) = v(q) = -v(j(E))$, so it is numerically faithful. It is not faithful however, since one of the segments of the minimal skeleton is contracted and another is expanded.

We can also modify the embedding to make it a closed embedding. The following idea can also be found in an earlier version of Baker et al. (2016). We first construct a closed embedding with a "trivial tropicalization" on the minimal skeleton. That is, we construct two functions whose piecewise linear functions are trivial on Σ, but still define a closed embedding. After that, we combine the two to make a closed embedding with the desired tropicalization.

We can in fact use the argument in Baker et al. (2016, Theorem 6.2), but with different points.[4] Choose a primitive third root of unity $\zeta_3 \in K^*$ and consider the point T in $E(K)[3]$ that corresponds to it under the Tate uniformization. We will denote the multiples of T by T_i again, where $T_1 = T$ and $T_0 = T_3$ is the identity. For a pictorial description, see Fig. 11.

The divisors $D_3 = 2T_1 - T_2 - T_0$ and $D_4 = 2T_2 - T_1 - T_0$ are principal and we can write $\mathrm{div}(f_3) = D_3$ and $\mathrm{div}(f_4) = D_4$. The functions f_3 and f_4 are then global generating sections of the sheaf $\mathcal{O}_E(D)$ (see Liu 2006, Sect. 7.1.2), where $D = T_0 + T_1 + T_2$. Furthermore, the corresponding map $E \to \mathbb{P}^2$ is a closed embedding since $\deg(D) > 2$, see Liu (2006, Chapter 7, Proposition 4.4(b)). Note that the piecewise linear functions of f_3 and f_4 are trivial on the skeleton Σ. Indeed, the T_i all retract

[3] To do this, it is enough to define a ring map $K[x, x^{-1}, y, y^{-1}] \to \mathcal{O}_E(U)$, see Liu (2006, Chapter 2, Proposition 3.25). This is given by mapping $x \mapsto f_1$ and $y \mapsto f_2$. It is well-defined because the f_i are invertible on U.

[4] The old version of Baker et al. (2016) uses a mix of a two-torsion point and a three-torsion point. In our argument, we only use three-torsion points.

to the same point on Σ. We thus see that the corresponding tropicalization does not contain a cycle.

We now have three morphisms: $E \to \mathbb{P}^2$ arising from the pair (f_3, f_4), and two morphisms $E \to \mathbb{P}^1$ arising from f_1 and f_2. Using the universal property of the product, we obtain a morphism $E \to \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$, which is a closed embedding because it is a closed embedding on the first factor. The embedding now tropicalizes as follows. First, note that the tropical functions $-\log|f_3|$ and $-\log|f_4|$ are constant on Σ, so their coordinates don't change. The tropicalization on the remaining coordinates is as before, so we conclude that there is an induced cycle of length $-v(j(E))$ and thus the tropicalization is numerically faithful. It is not faithful however, since one of the segments is contracted to a point and another is expanded.

3 Minimal models over non-Noetherian valuation rings

In this section, we give a reduction theory for elliptic curves over K, similar to the one studied in Silverman (2009, Chapter VII). For simplicity, we will assume that $\mathrm{char}(k) \neq 2, 3$. In the general case, one can still write down minimal models using a variant of Tate's algorithm, see Silverman (1994, Chapter IV, Algorithm 9.4). In the discretely valued case, the most convenient way to study the reduction type of the elliptic curve E is through its Néron model $\mathcal{E}$. Since we are in the non-discrete case however, we cannot use this machinery. Furthermore, there is no direct generalization of Néron models to the non-discrete case available at the present, so we will study the reduction of E/K using the theory of minimal models, which does generalize to the non-discrete case.

Let E/K be an elliptic curve, as defined in Silverman (2009, Chapter III). Using the Riemann-Roch theorem, one can show that every such elliptic curve can be described by a Weierstrass equation:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \tag{30}$$

see Silverman (2009, Chapter III, Proposition 3.1). By applying an appropriate scaling transformation, one can then assume that $v(a_i) \geq 0$ for every i. We will call such Weierstrass equations *integral Weierstrass models* or integral Weierstrass equations. For any such Weierstrass model, one obtains a reduced Weierstrass equation over k by reducing the coefficients a_i mod $\mathfrak{m}$. This reduced Weierstrass equation is not canonical in any sense however: two K-isomorphic models can lead to non-isomorphic reduced curves, as the following example shows.

Example 3.1 Consider the integral Weierstrass equation

$$W : y^2 = x^3 + \varpi^4x + \varpi^6. \tag{31}$$

The reduced curve in this case is given by the equation

$$y^2 = x^3, \tag{32}$$

which defines a singular curve. We now consider an isomorphic curve whose reduction is nonsingular. Dividing by ϖ^6 on both sides of Eq. 31 and taking $y' = \frac{y}{\varpi^3}$ and $x' = \frac{x}{\varpi^2}$, we obtain

$$y'^2 = x'^3 + x' + 1. \tag{33}$$

Reducing the coefficients mod $\mathfrak{m}$ then yields a nonsingular curve, which consequently is not isomorphic to the reduced curve in Eq. 32.

We are thus led to impose an additional condition on the models over R to ensure some kind of canonicity. The notion we will be using is that of a *minimal model*, as in Silverman (2009, Chapter VII).

Definition 3.2 Let E/K be an elliptic curve with integral Weierstrass equation

$$W : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{34}$$

and discriminant $\Delta \in K$ (see Silverman 2009, Chapter III, Page 42 for an explicit formula). Then W is said to be *minimal* if $v(\Delta)$ is minimal among all integral Weierstrass equations for E. We refer to such a model as a *minimal Weierstrass model* and we denote it by W/R.

Example 3.3 Let W be the Weierstrass equation given by $y^2 = x^3 + \varpi^4 x + \varpi^6$ as in Example 31. Then the discriminant of W is given by $\Delta_W = -496 \cdot \varpi^{12}$ and $v(\Delta_W) = 12$. The Weierstrass equation W' in Eq. 33 is also a Weierstrass equation for the same elliptic curve, but it has $v(\Delta_{W'}) = 0$, so W is not minimal. Note that W' is automatically minimal, since the valuation of the discriminant cannot become any smaller for integral Weierstrass equations.

Using our assumption on the residue field, we now give the following convenient criterion for an integral Weierstrass model W to be minimal. First recall that for any field of characteristic not equal to 2 or 3, any minimal Weierstrass model is isomorphic to one of the form

$$y^2 = x^3 - 27c_4x - 54c_6. \tag{35}$$

Indeed, the transformations

$$y \longmapsto \frac{1}{2}(y - a_1x - a_3) \tag{36}$$

and

$$(x, y) \longmapsto \left(\frac{x - 3b_2}{36}, \frac{y}{108}\right) \tag{37}$$

on Silverman (2009, Pages 42 and 43) are invertible over R and the valuations of their discriminant are the same by the tables on Silverman (2009, Page 45).

Lemma 3.4 *Let W/R be a Weierstrass equation for an elliptic curve E/K. Then W/R is a minimal model if and only if*

$$\min\{v(c_4), v(c_6)\} = 0. \tag{38}$$

Proof Suppose that either $v(c_4) = 0$ or $v(c_6) = 0$ and suppose for a contradiction that there exists an integral Weierstrass equation W'/R with $v(\Delta') < v(\Delta)$. We then have

$$u^4 \cdot c_4' = c_4 \tag{39}$$

$$u^6 \cdot c_6' = c_6. \tag{40}$$

for a standard transformation relating W' and W as in Silverman (2009, Page 44) [every isomorphism is of this form by Silverman (2009), Chapter III, Proposition 3.1]. But $v(u) > 0$ (since $v(\Delta') < v(\Delta)$), so either $v(c_4') < 0$ or $v(c_6') < 0$, a contradiction. Note that this proof doesn't use the assumption on the characteristic.

Suppose now that W/R is a minimal model, which we can assume to be of the form

$$y^2 = x^3 - 27c_4x - 54c_6 \tag{41}$$

by our assumption on the residue characteristic. Suppose that $v(c_4), v(c_6) > 0$ and consider

$$m := \min\{v(c_4)/4, v(c_6)/6\}. \tag{42}$$

Let u be any element with valuation m (which exists because K is algebraically closed) and consider the transformation

$$x = u^2 \cdot x',$$
$$y = u^3 \cdot y'.$$

By Eq. 39, we see that $v(c_4'), v(c_6') \geq 0$ (in fact, one of them has to be zero) and by

$$u^{12}\Delta' = \Delta, \tag{43}$$

we see that $v(\Delta') < v(\Delta)$, a contradiction. This proves the lemma. □

Remark 3.5 Note that the proof of Lemma 3.4 gives an explicit way of determining a minimal Weierstrass model: we take any integral equation and determine the c_i. By applying the transformation in the proof, we then immediately obtain a minimal model.

Remark 3.6 We note that there exist minimal Weierstrass models over valued fields with residue characteristic 2 and 3 with $v(c_4) > 0$ and $v(c_6) > 0$. As an example, let $K = \mathbb{Q}_2$ be the field of 2-adic numbers and consider the elliptic curve given by

$$y^2 + y = x^3. \tag{44}$$

This curve has good reduction, so $v(\Delta) = 0$. We then have $c_4 = 0$ and $c_6 = -216$, so both of the invariants have strictly positive valuation.

Proposition 3.7 *Let E/K be an elliptic curve. Then the following hold.*

- *E has a minimal Weierstrass model W/R.*
- *A minimal Weierstrass model is unique up to a change of coordinates*

$$x = u^2x' + r \tag{45}$$
$$y = u^3y' + u^2sx' + t, \tag{46}$$

with $u \in R^*$ and $r, s, t \in R$.
- *Let W/R be an integral Weierstrass equation. Then any change of coordinates*

$$x = u^2x' + r \tag{47}$$
$$y = u^3y' + u^2sx' + t \tag{48}$$

that turns W/R into a minimal Weierstrass model W'/R satisfies $u, r, s, t, \in R$.

Proof The first part follows from Lemma 3.4 and Remark 3.5. The second and third part follow in exactly the same way as in Silverman (2009, Chapter VII, Proposition 1.3). Note that the proofs of these parts do not use the discreteness of the valuation, nor the completeness of K. We leave it to the reader to fill in the details. □

Let W/R be a minimal Weierstrass model given by

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \tag{49}$$

where $a_i \in R$. Using the natural reduction map $R \to R/\mathfrak{m} = k$, we can then consider the reduced Weierstrass equation

$$y^2 + \overline{a}_1xy + \overline{a}_3y = x^3 + \overline{a}_2x^2 + \overline{a}_4x + \overline{a}_6. \tag{50}$$

By Proposition 3.7, any two minimal Weierstrass models W and W' are related by a coordinate change as in Eq. 45 with $u \in R^*$ and $r, s, t, \in R$. Reducing this coordinate change $\bmod \mathfrak{m}$, we obtain a standard coordinate change over the residue field. We thus see that any two minimal Weierstrass models give rise to isomorphic reduced curves. The reduced equation is thus independent of the minimal Weierstrass model, up to standard coordinate changes over the residue field. Note that the notion of minimality is crucial here, as the reduced curve of a non-minimal Weierstrass equation can be non-isomorphic to the reduced curve of a minimal Weierstrass equation, see Example 3.1. The reduced curve associated to any minimal Weierstrass model W/R will be denoted by $\overline{E}/k$ or $\overline{E}$.

We now give a reduction map $E(K) \to \overline{E}(k)$ in terms of projective coordinates. Write P as $P = [x_0, y_0, z_0]$. By scaling these coordinates, we can find an equivalent triple such that at least one of x_0, y_0, z_0 is a unit. The reduced point

$$\overline{P} = [\overline{x}_0, \overline{y}_0, \overline{z}_0] \tag{51}$$

then lies in $\overline{E}(k)$. This gives us the reduction map

$$E(K) \to \overline{E}(k), \qquad P \mapsto \overline{P}. \tag{52}$$

We now recall some facts regarding the reduced curve $\overline{E}/k$. This curve is singular if and only if $\overline{\Delta} = 0$, where Δ is the discriminant associated to the Weierstrass equation. Furthermore, any Weierstrass equation can have only one singularity, which is either a cusp or a node. If it has a singularity, then we can put a group structure on the smooth points by Silverman (2009, Chapter III, Proposition 2.5). There are three possible singularities and for each one we have a different group structure. We can in fact characterize the type of singularity we get by reducing the invariants c_i, as the following proposition shows.

Proposition 3.8 *Let W/R be a minimal Weierstrass model for an elliptic curve E/K with reduced curve $\overline{E}$. Let $\overline{E}_{\text{ns}}$ be the set of nonsingular points. Then the following hold:*

- *$\overline{E}$ is an elliptic curve if and only if $v(\Delta) = 0$. We have $\overline{E} = \overline{E}_{\text{ns}}$. In this case, the elliptic curve E is said to have good reduction.*
- *$\overline{E}$ has a cusp if and only if $v(\Delta) > 0$ and $\overline{c}_4 = 0$. We have $\overline{E}_{ns} \simeq k^+$, the additive group of the residue field k. In this case, the elliptic curve is said to have additive reduction.*
- *$\overline{E}$ has a node if and only if $v(\Delta) > 0$ and $\overline{c}_4 \neq 0$. We have $\overline{E}_{ns} \simeq k^*$, the multiplicative group of the residue field k. In this case, the elliptic curve is said to have multiplicative reduction.*

Proof The proof of Silverman (2009, Chapter VII, Proposition 5.1) still works in the non-discrete case, as one can easily check. We leave the details to the reader. □

Example 3.9 Let E be the elliptic curve defined by the Weierstrass minimal model

$$y^2 = x^3 + x^2 + \varpi^2, \tag{53}$$

where $v(\varpi) > 0$. The reduced curve is then given by

$$y^2 = x^3 + x^2, \tag{54}$$

which has the singularity (0, 0). Note that this singularity is a node, so E has multiplicative reduction.

We now relate the reduction type of an elliptic curve to the valuation of the j-invariant. To that end, we will need the following formula:

$$j = \frac{c_4^3}{\Delta} = 1728 \frac{c_4^3}{c_4^3 - c_6^2}, \tag{55}$$

see Silverman (2009, Chapter III, Page 42).

Proposition 3.10 *Let E/K be an elliptic curve with minimal model W/R. Then:*

- *E has good reduction if and only if $v(j) \geq 0$,*
- *E has multiplicative reduction if and only if $v(j) < 0$.*

In particular, we see that E cannot have additive reduction.

Proof Suppose that E has good reduction. Then $v(\Delta(E)) = 0$ and consequently $v(j) = 3v(c_4) - v(\Delta) \geq 0$, as desired. Suppose that $v(j) \geq 0$ and let W/R be a minimal model of the form

$$y^2 = x^3 - 27c_4x - 54c_6. \tag{56}$$

Suppose that $v(\Delta) > 0$. Then we must have $3v(c_4) = 2v(c_6)$. But by Lemma 3.4, we see that either $v(c_4) = 0$ or $v(c_6) = 0$, so $v(c_4) = 0 = v(c_6)$. But then $v(j) = 3v(c_4) - v(\Delta) < 0$, a contradiction. We conclude that $v(\Delta) = 0$ and thus E has good reduction.

Suppose now that $v(j) < 0$. Then we must have $v(\Delta) > 0$. Suppose that E has additive reduction. Then $v(c_4) > 0$ by Proposition 3.8 and consequently $v(c_6) = 0$ by Lemma 3.4. But then $v(\Delta) = 0$, a contradiction. We conclude that E has multiplicative reduction, as desired. Suppose that E has multiplicative reduction. By what was proved earlier, we cannot have $v(j) \geq 0$, so we must have $v(j) < 0$. This concludes the proof. □

Consider the following subset of $E(K)$:

$$E_0(K) = \{P \in E(K) : \overline{P} \in \overline{E}_{\text{ns}}(k)\}. \tag{57}$$

By Silverman (2009, Chapter VII, Proposition 2.1) (note that the proof only uses the fact that R is Henselian), we find that $E_0(K)$ is a subgroup of $E(K)$ and we have an exact sequence

$$0 \to E_1(K) \to E_0(K) \to \overline{E}_{\text{ns}}(k) \to 0. \tag{58}$$

Here $E_1(K)$ is the kernel of the reduction map, i.e.

$$E_1(K) = \{P \in E(K) : \overline{P} = \overline{\mathcal{O}}\}, \tag{59}$$

where $\mathcal{O}$ is the point at infinity. Note that the projective point $[0, 1, 0]$ is always nonsingular for any Weierstrass equation (see Silverman 2009, Chapter III, Proposition 1.2), so we have $E_1(K) \subseteq E_0(K)$. Since $E_0(K)$ is a subgroup, we can consider the quotient

$$E/E_0(K) := E(K)/E_0(K). \tag{60}$$

A point $P \in E(K)$ then gives rise to a nontrivial point in $E/E_0(K)$ if and only if P reduces to a singular point.

Example 3.11 Let E again be the elliptic curve defined by the Weierstrass minimal model

$$y^2 = x^3 + x^2 + \varpi^2, \tag{61}$$

where $v(\varpi) > 0$. We saw in Example 3.9 that E has multiplicative reduction. We now give an example of a nontrivial point in $E/E_0(K)$, $E_0(K)$ and $E_1(K)$ respectively. Consider the point

$$P_1 = (0, \varpi) \in E(K). \tag{62}$$

Since P_1 reduces to the singular point, we see that $P_1 \notin E_0(K)$. Let α be a square root of $2 + \varpi^2$ in K and consider the point $P_2 = (1, \alpha)$. Its reduction is then given by $\overline{P}_2 = (\overline{1}, \overline{\alpha}) \in \overline{E}_{\text{ns}}$, so we find that $P_2 \in E_0(K)$. Lastly, let β be a square root of $1 + \varpi^2 + \varpi^8$ and consider the projective point $P_3 = [\varpi, \beta, \varpi^3]$. In terms of x and y coordinates, this is given by

$$P_3 = \left(\frac{1}{\varpi^2}, \frac{\beta}{\varpi^3}\right). \tag{63}$$

Since $\beta \notin \mathfrak{m}$, we find that $P_3 \in E_1(K)$.

We now have two subgroups of E at our disposal: E_1 and E_0. We are interested in the torsion structure of these groups. That is, for any abelian group G and integer $n > 1$, we consider the subgroup

$$G[n] = \{g \in G : n \cdot g = e\}, \tag{64}$$

where e is the identity of G. For elliptic curves, we will denote the n-torsion subgroup of the K-valued points by $E[n](K)$. For K algebraically closed and n coprime to $\text{char}(K)$, we then have

$$E[n](K) = (\mathbb{Z}/n\mathbb{Z})^2 \tag{65}$$

see Silverman (2009, Chapter III, Corollary 6.4). We are especially interested in the case $n = 3$. In this case, the torsion points have a very geometric flavor to them, as the following well-known lemma shows:

Lemma 3.12 (Inflection points) *Let E/K be an elliptic curve with a point $P \in E(K)$. Then $P \in E[3](K)$ if and only if P is an inflection point. That is, the tangent line at P only intersects E at P.*

Proof Suppose that the tangent line is given by $H(X, Y, Z) = \alpha X + \beta Y + \gamma Z = 0$ and that it only intersects E at P. By Bezout's theorem applied to $E \subset \mathbb{P}^2$, it intersects E triply. The divisor of H/Z is then $3(P) - 3(\mathcal{O})$ and thus P is a point of order three, as desired. Conversely, let P be a point of order three and let $H(X, Y, Z)$ be the tangent line at P. Then $\text{div}(H/Z) = 2(P) + (Q) - 3(\mathcal{O})$ for some $Q \in E(K)$ and consequently the degree zero divisor $(Q) - (\mathcal{O})$ is the inverse of $2(P) - 2(\mathcal{O})$ in $\mathrm{Pic}^0(E)$. But this inverse is exactly $(P) - (\mathcal{O})$, so we find $P = Q$, as desired. □

Example 3.13 Let E be the elliptic curve defined by the affine (minimal) Weierstrass equation

$$y^2 = x^3 + (x - \varpi)^2 \tag{66}$$

This contains the rational point $P = (0, \varpi)$ and we claim that this is an inflection point. Indeed, rewriting the equation yields

$$(y - (x - \varpi))(y + (x - \varpi)) = x^3. \tag{67}$$

We then have that

$$\mathrm{div}(y + (x - \varpi)) = 3(P) - 3\mathcal{O}, \tag{68}$$

so P is a three-torsion point because the group structure on E is induced by that on $\mathrm{Pic}^0(E)$, see Silverman(2009, Chapter III, Proposition 3.4(d)). By Lemma 3.12, we then see that P is indeed an inflection point. The real picture for this family can be found in Fig. 12. Here we evaluated ϖ at real numbers close to zero. A similar family will be used in Sect. 4 for the proof of Theorem 1.1.

We will use this characterization of inflections points in Lemma 3.12 in the proof of Lemma 4.1 to give an explicit family of elliptic curves with a marked inflection point on each member. This explicit form then allows us to find the desired tropicalization we are after.

We now consider the problem of determining how the n-torsion of an elliptic curve is distributed over E_1, E_0 and the quotient E/E_0. To that end, we first consider the n-torsion of E_1. By Silverman (2009, Chapter VII, Proposition 2.2), there is an isomorphism

$$\hat{E}(\mathfrak{m}) \to E_1(K), \tag{69}$$

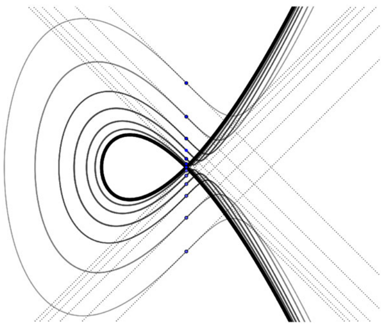


Fig. 12 The family of elliptic curves in Example 3.13. The blue points indicate inflection points on the members in this family. The tangent lines at these points intersect the curve only at that inflection point

where $\hat{E}$ is the formal group associated to E (see Silverman 2009, Chapter IV) and $\mathfrak{m}$ the maximal ideal of R. Since the multiplication by n map is invertible on $\hat{E}$ for any n coprime to the residue characteristic, we obtain the following

Proposition 3.14 *Let E/K be an elliptic curve and let n be an integer that is coprime to the characteristic of the residue field k. Then $E_1(K)[n] = (0)$.*

Proof By Silverman (2009, Chapter IV, Proposition 2.3.b), the multiplication by n-map $\hat{E} \to \hat{E}$ is an isomorphism. This directly implies that $\hat{E}(\mathfrak{m})[n] = (0)$, as desired. □

Example 3.15 Consider again the elliptic curve defined by

$$y^2 = x^3 + x^2 + \varpi^2. \tag{70}$$

We saw in Example 3.11 that $P_3 = (\frac{1}{\varpi^2}, \frac{\beta}{\varpi^3}) \in E_1(K)$, where $\beta^2 = 1 + \varpi^2 + \varpi^8$. If we assume that $\mathrm{char}(k) = 0$, then using Proposition 3.14, we see that P_3 cannot be a torsion point, so P_3 has infinite order. This trick can be used more generally to create points of infinite order on families of elliptic curves.

Lemma 3.16 *Suppose that E has multiplicative reduction with singular point $x \in \overline{E}(k)$ and let n be coprime to the characteristic of the residue field. Then there exists a point $P \in E(K)$ of order n such that $\overline{P} = x$.*

Proof Suppose that every point P of order n of E reduces to a nonsingular point. Then $E[n](K) \subset E_0(K)$. By Proposition 3.14 and the exact sequence from Eq. 58, we see that $E[n](K)$ injects into $\overline{E}_{\mathrm{ns}}(k)$ under the reduction map. But this is impossible: $\overline{E}_{\mathrm{sm}}[n] \simeq k^*[n] \simeq \mathbb{Z}/n\mathbb{Z}$ has order n, whereas $E[n](K) \simeq (\mathbb{Z}/n\mathbb{Z})^2$ has order n^2. We conclude that there exists a P of order n reducing to the singular point, as desired. □

Example 3.17 Let E be the curve in Example 3.13. The inflection point $P = (0, \varpi)$ then reduces to the singular point $(0, 0)$ on the reduction $y^2 = x^3 + x^2$. We thus see that P defines a nontrivial element in $E/E^0(K)[3]$, as guaranteed by Lemma 3.16.

Remark 3.18 Let us assume that K is complete. Using the analytic uniformization theorem for elliptic curves with split multiplicative reduction (see Eq. 29), it is now much easier to obtain the point P in this Lemma. Indeed, for any such elliptic curve E/K with $v(j(E)) < 0$, one considers the analytic isomorphism

$$E(K) \to K^*/\langle q \rangle, \tag{71}$$

where $q \in K$ is such that $-v(j(E)) = v(q) > 0$. To find the point P as in Lemma 3.16, one simply takes $P = q^{1/n}$. We invite the reader to compare this to the material in Sect. 2.2 and the construction in Baker et al. (2016, Theorem 6.2).

4 Creating numerically faithful tropicalizations using minimal models

In this section, we will show that any elliptic curve E/K with $v(j(E)) < 0$ admits a numerically faithful tropicalization as in Definition 2.20. Using the criterion in Baker et al. (2016, Corollary 5.28(2)) and Example 2.25, we then also see that this particular embedding defines a faithful tropicalization. To find this numerically faithful tropicalization, we use a three torsion point P that reduces to the singular point of the reduced curve corresponding to a Weierstrass minimal model, which exists by Lemma 3.16. We then construct two principal divisors $\text{div}(f), \text{div}(g) \in \text{Prin}(E)$ using this torsion point. These two principal divisors then give rise to a closed embedding

$$E \to \mathbb{P}^2, \tag{72}$$

whose tropicalization is easily shown to contain a cycle of length $-v(j(E))$, which concludes the proof. The idea of the proof is mostly based on the construction in Baker et al. (2016, Theorem 6.2). In that proof, they abstractly show using the Poincaré–Lelong formula that the corresponding piecewise linear functions $-\log|f|$ and $-\log|g|$ separate the points of the Berkovich minimal skeleton of E.

Let E be an elliptic curve with $v(j(E)) < 0$. By Proposition 3.10, this implies that E has multiplicative reduction. By Lemma 3.16, there exists a $P \in E[3](K)$ such that P reduces to the singular point of a minimal Weierstrass model. In other words, the class of P in $E/E_0(K)$ is nontrivial. Let W be a minimal Weierstrass model for E of the form

$$y^2 = x^3 + b_2x^2 + b_4x + b_6. \tag{73}$$

Since P reduces to the singular point on $\overline{E}$, we see in particular that $v(x(P)) \geq 0$. The transformation $x \mapsto x - x(P)$ then transforms W into another *integral* Weierstrass model, which is again minimal by Proposition 3.7. We will again denote this minimal Weierstrass model by W. In this new model, we have $x(P) = 0$.

Lemma 4.1 *Let E, W and P be as above. Then $\Delta(b_2x^2+b_4x+b_6) = b_4^2-4b_2b_6 = 0$. In other words, we can write*

$$y^2 = x^3 + a(x-b)^2 \tag{74}$$

for a and b in R.

Proof Let $P = (0, y(P))$ be the 3-torsion point reducing to the singular point on $\overline{E}$ as above. By Lemma 3.12, it is an inflection point: its tangent line intersects E only at P. The tangent line at P is given by the equation

$$\frac{\partial f}{\partial x}(P) \cdot x + \frac{\partial f}{\partial y}(P)(y - y(P)) = 0, \tag{75}$$

where $f = y^2 - (x^3 + b_2x^2 + b_4x + b_6)$. We thus obtain

$$-b_4x + 2y(P)(y - y(P)) = 0. \tag{76}$$

In terms of y, we obtain

$$y = \frac{2b_6 + b_4 x}{2y(P)}. \tag{77}$$

Note here that $y(P) \neq 0$, since otherwise P would be a 2-torsion point.

Squaring the last expression and equating it to $x^3 + b_2x^2 + b_4x + b_6$, we obtain the cubic equation

$$x^3 + (b_2 - \frac{b_4^2}{4b_6})x^2 + h(x) = 0, \tag{78}$$

where $h(x)$ is some linear polynomial. Since $x = 0$ is a triple root of this equation, we must have $b_2 - \frac{b_4^2}{4b_6} = 0$, or in other words:

$$b_4^2 = 4b_2b_6. \tag{79}$$

This means that the discriminant of the quadratic form $b_2x^2 + b_4x + b_6$ is zero and we can thus write it as $a(x - b)^2$ for some a and b in R. This concludes the proof of the lemma. □

Lemma 4.2 *Let E, W and P be as above. Then $v(a) = 0$ and $v(b) > 0$.*

Proof If $v(a) > 0$, then E has additive reduction since the reduced curve is given in this case by $y^2 = x^3$. This contradicts our assumption that E has multiplicative reduction and we thus see that $v(a) = 0$. For the second part, we will use our assumption that P reduces to the singular point of $\overline{E}$. Let a' be a square root of a. We then have

$$P = (0, a'b) \text{ or } P = (0, -a'b). \tag{80}$$

Without loss of generality, we can assume that $P = (0, a'b)$. Since P reduces to the singular point on $\overline{E}$, we must have

$$\left(\frac{\partial f}{\partial x}(\overline{P}), \frac{\partial f}{\partial y}(\overline{P})\right) = (0, 0), \tag{81}$$

where $f = y^2 - x^3 - \overline{a}(x - \overline{b})^2 \in k[x, y]$. But then $\overline{a'b} = 0$ and consequently $\overline{b} = \overline{0}$, since $\overline{a'} \neq \overline{0}$ by $v(a) = 0$. This proves that $v(b) > 0$, as desired. □

Let W be given by

$$y^2 = x^3 + a(x - b)^2. \tag{82}$$

We can rewrite this as

$$(y - a'(x - b))(y + a'(x - b)) = x^3. \tag{83}$$

We then quickly see that $\mathrm{div}(y - a'(x - b)) = 3P' - 3\mathcal{O}$ and $\mathrm{div}(y + a'(x - b)) = 3P - 3\mathcal{O}$, where $P' = (0, -a'b)$. Another calculation then shows that

$$\mathrm{div}(x) = P + P' - 2\mathcal{O}. \tag{84}$$

We now explicitly give the two principal divisors f and g that will give the desired embedding into $\mathbb{P}^2$. We take

$$f = \frac{x^2}{y - a'(x - b)} \tag{85}$$

$$g = \frac{x^2}{y + a'(x - b)}. \tag{86}$$

Using the above identities for the divisors of x and $y \pm a'(x - b)$, we then obtain

$$\mathrm{div}(f) = 2P - P' - \mathcal{O}$$
$$\mathrm{div}(g) = 2P' - P - \mathcal{O}.$$

We now explicitly calculate the closed embedding induced by (f, g).

Lemma 4.3 *Let E be as above and let f and g be as in Eqs. 85 and 86. Then the image in $\mathbb{P}^2$ of the embedding induced by f, g is cut out by the affine equation*

$$f^2 g + 2a' fg - fg^2 - 2a'b = 0. \tag{87}$$

Proof First note that $fg = x$, by virtue of

$$(y - a'(x - b))(y + a'(x - b)) = x^3. \tag{88}$$

We now express y in terms of f and g. Note that $(y - a'(x - b)) \cdot f = x^2$, so

$$y = \frac{x^2}{f} + a'(x - b) = fg^2 + a'(fg - b). \tag{89}$$

Plugging in x and y in $y + a'(x - b) = \dfrac{x^2}{g} = f^2 g$, we obtain

$$y + a'(x - b) = fg^2 + a'(fg - b) + a'(fg - b) = f^2 g \tag{90}$$

and thus

$$fg^2 + 2a' fg - 2a'b - f^2 g = 0, \tag{91}$$

as desired. □

We are now ready to prove the main theorem of this paper.

Theorem 4.4 (Main Theorem) *Let E be an elliptic curve over K with $v(j(E)) < 0$. Then there exists an embedding $\phi : E \to \mathbb{P}^2$ such that its tropicalization contains a cycle of length $-v(j(E))$.*

Proof We consider the embedding from Lemma 4.3. By Lemma 4.2, we have that $v(a) = 0$ and $v(b) > 0$. We then see that the tropicalization is given by the tropical polynomial

$$\operatorname{trop}(fg^2 + 2a'fg - 2a'b - f^2g) = \min\{f + 2g, f + g, v(b), 2f + g\}. \quad (92)$$

This tropicalization contains a cycle of length $3v(b)$, see Example 2.16. We now calculate the j-invariant of E using the formulas on Silverman (2009, Page 45). This gives

$$j(E) = \frac{-256 \cdot a(a + 6b)^3}{4ab^3 + 27b^4} \quad (93)$$

Since $v(a) = 0$ is strictly less than $v(6 \cdot b) = v(b)$, we have that $v(a + 6b) = 0$. Therefore, the numerator of $j(E)$ has valuation zero. Similarly, we find that the valuation of the denominator is $3v(b)$. This is the length of the cycle in the tropicalization of ϕ, so we see that this embedding is numerically faithful. □

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