



Solutions with Compact Time Spectrum to Nonlinear Klein–Gordon and Schrödinger Equations and the Titchmarsh Theorem for Partial Convolution

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Received: 20 October 2018 / Revised: 5 September 2019 / Accepted: 10 September 2019 /
Published online: 27 September 2019

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Abstract

We prove that finite energy solutions to the nonlinear Schrödinger equation and nonlinear Klein–Gordon equation which have the compact time spectrum have to be one-frequency solitary waves. The argument is based on the generalization of the Titchmarsh convolution theorem to partial convolutions.

Keywords Multifrequency solitary waves · Compact time spectrum · Nonlinear Klein–Gordon equation · Nonlinear Schrödinger equation · Soliton resolution conjecture · Titchmarsh convolution theorem

1 Introduction

Let us consider nonlinear Schrödinger and nonlinear Klein–Gordon equations,

$$\begin{aligned} i\partial_t u &= -\Delta u + \alpha(|u|^2)u, & -\partial_t^2 u &= -\Delta u + m^2 u + \alpha(|u|^2)u, & u(x, t) &\in \mathbb{C}, \\ x &\in \mathbb{R}^n, & n &\in \mathbb{N}, \end{aligned} \quad (1.1)$$

where $m > 0$ and the nonlinearity is represented by a function $\alpha \in C^1(\mathbb{R})$, $\alpha(0) = 0$. These $U(1)$ -invariant equations are well-known to admit solitary wave solutions of the form

$$u(x, t) = \phi(x)e^{-i\omega t}, \quad \omega \in \mathbb{R}, \quad (1.2)$$

To Rafail Kalmanovich Gordin on the occasion of his 70th birthday—with love and admiration.

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with $\phi(x)$ decaying at infinity (Strauss 1977; Berestycki and Lions 1983). Do these equations admit multifrequency solitary wave solutions of the form $\sum_{j=1}^N \phi_j(x)e^{-i\omega_j t}$? Indeed, such solutions have been found in similar systems; see below for more details. More generally, we would like to know whether besides one-frequency solitary waves there are finite energy solutions with compact time-spectrum, defined as follows.

Definition 1 Let $u \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}, \mathbb{C})$, and let $\tilde{u}(x, \omega) = \int_{\mathbb{R}} e^{i\omega t} u(x, t) dt$ be its partial Fourier transform in time. We say that the time spectrum of u is compact if there is a finite interval $I \subset \mathbb{R}$ such that

$$\text{supp } \tilde{u} \subset \mathbb{R}^n \times I.$$

In the present article, in Sect. 3, we will prove that in the nonlinear Schrödinger or Klein–Gordon equations under certain assumptions on the nonlinearity (polynomial or some algebraic functions), there are no finite energy solutions with compact time spectrum except the one-frequency solitary waves of the form (1.2). See Theorem 6 below for the precise formulation. The approach is based on the form of the Titchmarsh convolution theorem reformulated for partial convolutions; see Sect. 2 and in particular Theorem 2.

Soliton Resolution Conjecture This conjecture states that the long-time asymptotics of any finite energy solution to a nonlinear dispersive system with $U(1)$ -symmetry is given by a superposition of outgoing solitary waves and an outgoing dispersive wave; see Komech (2003, 2016), Soffer (2006), Tao (2007) and Komech and Komech (2007). For the recent results for the Schrödinger and Klein–Gordon equations with the critical power nonlinearity, see Duyckaerts et al. (2016) and the references therein. Let us also mention the probabilistic approach (Chatterjee 2014; Bonanno 2015). One strategy to attack this problem was proposed in Komech (2003): one notices that any solution converges to *radiationless solution*, the one that does not lose the energy any more. Then one needs to complete the following two steps:

1. Prove that any radiationless solution has a compact time spectrum;
2. Prove that any solution with compact time spectrum has a time spectrum consisting of a single point, and hence is a solitary wave:

$$u(x, t) = \phi(x)e^{-i\omega t}, \quad \omega \in \mathbb{R}, \quad \phi \in H^1(\mathbb{R}^n, \mathbb{C}). \tag{1.3}$$

Above, $H^1(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n); \|u\|_{H^1}^2 := \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 < \infty\}$ is the standard Sobolev space of order one.

Thus, any finite energy solution converges to a radiationless solution, which in turn is a solitary wave. Both steps of the program were accomplished for several models without translation invariance, namely, for the Klein–Gordon equation interacting with one oscillator (Komech 2003; Komech and Komech 2007),

$$-\partial_t^2 u = -\partial_x^2 u + m^2 u + \delta(x)\alpha(|u|^2)u, \quad u(x, t) \in \mathbb{C}, \quad x \in \mathbb{R},$$

where $m > 0$ and $\alpha(\tau)$ is a polynomial, for several nonlinear oscillators (Komech and Komech 2010b), for the Klein–Gordon and Dirac equations with the mean-field

self-interaction (Komech and Komech 2009, 2010a) (in any spatial dimension), for the Klein–Gordon with the mean-field self-interaction at several points (Comech 2012), and also for the Klein–Gordon equation in the discrete time-space coupled to a nonlinear oscillator (Comech 2013). In other words, in the models mentioned above, *the weak global attractor is formed by solitary waves*: any finite energy solution converges to the solitary manifold,

$$\mathbf{S} = \{\phi_\omega(x)e^{-i\omega t}; \omega \in \mathbb{R}\}.$$

The convergence is in the weak topology, in weighted spaces such as $H_{-s}^1(\mathbb{R}^n) = \{u \in H_{\text{loc}}^1(\mathbb{R}^n); \langle x \rangle^{-s} u \in H^1(\mathbb{R}^n)\}$, with $s > 0$, where $\langle x \rangle$ is [the operator of multiplication by] the function $(1 + x^2)^{1/2}$, $x \in \mathbb{R}^n$; in this sense, we are talking about the *weak attractor*. The weight makes sure that we forget about the excess energy, which is being carried away by the dispersive waves. One then says that the convergence to the attractor is caused by *friction by dispersion*; this is the substitute for the dissipation which is absent in a hamiltonian system.

In the present article we prove that, under certain assumptions on the nonlinearity, any solution with a compact time spectrum is a single-frequency solitary wave.

Multifrequency Solitary Waves If a particular model admits multifrequency solutions, defined as exact localized solutions with several frequencies, then they also belong to the attractor. One can show that multifrequency solitary waves exist in the Klein–Gordon equation with the mean-field self-interaction Komech and Komech (2009) and with several nonlinear oscillators Komech and Komech (2010b). Bi-frequency solitary waves exist in systems of nonlinear Schrödinger equations (Barashenkov et al. 2012) (the vector case may admit solutions with several harmonics when the nonlinearity does not produce higher harmonics due to cancellations, which are absent in the scalar case). In a similar fashion, bi-frequency solitary waves exist in the Soler model and Dirac–Klein–Gordon model with Yukawa self-interaction (Boussaïd and Comech 2018):

$$\psi(x, t) = \phi(x)e^{-i\omega t} + \chi(x)e^{i\omega t}, \quad \text{for particular } \phi, \chi \in H^1(\mathbb{R}^n, \mathbb{C}^N).$$

Sometimes one may place some restriction on the parameters of the problem [such as the spacings between the nonlinear oscillators in Komech and Komech (2010b)] to ensure that multifrequency solutions would be absent.

In Comech (2013), based on the Titchmarsh theorem for distributions on the circle (Komech and Komech 2013), it was shown that the global attractor of the Klein–Gordon equation in discrete time-space coupled with a nonlinear oscillator, besides usual one-frequency solitary waves $\phi e^{-i\omega T}$, could also contain two- and four-frequency solutions:

$$\phi e^{-i\omega T} + \chi e^{-i(\omega+\pi)T}, \quad \phi_1 e^{-i\omega_1 T} + \phi_2 e^{-i\omega_2 T} + \chi_1 e^{-i(\omega_1+\pi)T} + \chi_2 e^{-i(\omega_2+\pi)T},$$

where $T \in \mathbb{Z}$ is the discrete time and $\phi, \chi, \dots \in l^2(\mathbb{Z}^n)$ are particular functions of the discrete spatial variable $X \in \mathbb{Z}^n$, and indeed examples of such solutions were given.

According to Theorem 6 (see below), the nonlinear Schrödinger and Klein–Gordon equations with a certain class of nonlinearities do not admit multifrequency solitary wave solutions.

Breathers Let us contrast our results to the existence of breathers, which are exact periodic solutions in the context of completely integrable systems. For example, the completely integrable sine–Gordon equation

$$-\partial_t^2 u = -\partial_x^2 u + \sin u, \quad u(x, t) \in \mathbb{R}, \quad x \in \mathbb{R}, \quad (1.4)$$

admits solutions of the following form Ablowitz et al. (1973):

$$u(x, t) = 4 \arctan \left(\frac{\sqrt{1 - \omega^2} \cos(\omega t)}{\omega \cosh(\sqrt{1 - \omega^2} x)} \right), \quad \omega \in (-1, 1),$$

which are exponentially localized in space and are periodic in time. Note that the time spectrum of this solution is unbounded, and moreover the nonlinearity in (1.4) is not of algebraic type; thus, this solution does not contradict our statement on the absence of nontrivial compact spectrum solutions (other than one-frequency solitary waves) to the nonlinear Klein–Gordon equation with certain algebraic nonlinearities.

Similarly, the cubic nonlinear Schrödinger equation

$$i\partial_t u = -\partial_x^2 u - 2|u|^2 u, \quad u(x, t) \in \mathbb{C}, \quad x \in \mathbb{R},$$

admits exact solutions Akhmediev et al. (1987) such as the following one:

$$u(x, t) = \frac{\cos x + i\sqrt{2} \sinh t}{\sqrt{2} \cosh t - \cos x} e^{it}.$$

We notice that the frequency spectrum of this solution is not compact; moreover, this solution has an infinite L^2 -norm and energy. For more examples of such solutions, see Akhmediev et al. (1987).

Convergence of Small Initial Data to One-Frequency Solitary Waves Let us mention the results on convergence of small solutions to (one-frequency) solitary waves, particularly in the context of the nonlinear Schrödinger equation: in other words, the attractor of small solutions is formed by small amplitude solitary waves. See in particular Tsai and Yau (2002), Soffer and Weinstein (2004), Cuccagna and Maeda (2015), Cuccagna et al. (2016) and Cuccagna and Tarulli (2016).

2 Titchmarsh Theorem for Partial Convolution

The original formulation of the Titchmarsh (1926) convolution theorem is as follows:

If $\phi(t)$ and $\psi(t)$ are integrable functions, such that $\int_0^x \phi(t)\psi(x-t) dt = 0$ almost everywhere in the interval $0 < x < \kappa$, then $\phi(t) = 0$ almost everywhere in $(0, \lambda)$, and $\psi(t) = 0$ almost everywhere in $(0, \mu)$, where $\lambda + \mu \geq \kappa$.

Above, λ and μ are some particular values ≥ 0 . An equivalent reformulation is that $\inf \text{supp } \phi * \psi = \inf \text{supp } \phi + \inf \text{supp } \psi$, for any $\phi, \psi \in \mathcal{E}'(\mathbb{R})$, where $\mathcal{E}'(\mathbb{R})$ is the space of distributions with compact support (dual to the space $\mathcal{E}(\mathbb{R})$ which is $C^\infty(\mathbb{R})$ with the topology defined by the seminorms $\sup_{\omega \in K} |f^{(k)}(\omega)|$, with $k \in \mathbb{N}_0$ and K a compact subset of \mathbb{R}). A higher dimensional generalization can be stated in terms of the convex hulls of the supports of distributions (Lions 1951):

Theorem 1 [Titchmarsh Convolution Theorem (Lions 1951)] For $f, g \in \mathcal{E}'(\mathbb{R}^n)$,

$$\text{conv supp } f * g = \text{conv supp } f + \text{conv supp } g. \quad (2.1)$$

Above, conv denotes the convex hull of a set.

We need a version of this theorem for a partial convolution with respect to only a subset of variables.

2.1 Maximal Lower Semicontinuous Function and Minimal Upper Semicontinuous Function

Lemma 1 Let $n \geq 1$. For any function $\mu : \mathbb{R}^n \rightarrow \mathbb{R}$ there is a maximal lower semicontinuous function on \mathbb{R}^n which does not exceed μ ; we will denote this function by $\mu^L(x)$. Similarly, there is a minimal upper semicontinuous function on \mathbb{R}^n which is not exceeded by μ ; we will denote this function by $\mu^U(x)$. For any $\mu, \nu : \mathbb{R}^n \rightarrow \mathbb{R}$, one has

$$\mu^L \leq \mu \leq \mu^U, \quad (2.2)$$

$$(\mu + \nu)^L \geq \mu^L + \nu^L, \quad (\mu + \nu)^U \leq \mu^U + \nu^U. \quad (2.3)$$

Proof The function is lower semicontinuous if and only if its *epigraph* (the set of points lying on or above its graph), $\text{epi } \mu = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}; y \geq \mu(x)\}$, is closed, or, equivalently, if and only if its *strict epigraph*,

$$\text{hyp}_S(\mu) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}; y < \mu(x)\},$$

is open. For a function $\mu : \mathbb{R}^n \rightarrow \mathbb{R}$ let us consider the complement to its epigraph, the strict *hypograph*,

$$\text{hyp}_S(\mu) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}; y < \mu(x)\}.$$

Let $\mu_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$, $\alpha \in I$, be a subset of the set of lower semicontinuous functions. Then

$$\text{hyp}_S \left(\sup_{\alpha \in I} \mu_\alpha \right) = \bigcup_{\alpha \in I} \text{hyp}_S(\mu_\alpha)$$

is open (as a union of any collection of open sets), hence $\mu^L := \sup_{\alpha \in I} \mu_\alpha$ is lower semicontinuous.

The inequalities (2.2) and (2.3) readily follow from the definition of μ^L and μ^U . □

Remark 1 The example of upper semicontinuous functions

$$\mu = \begin{cases} 1, & x \leq 0 \\ 0, & x > 0 \end{cases}, \quad \nu = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}, \quad \mu + \nu = \begin{cases} 1, & x \neq 0 \\ 2, & x = 0 \end{cases},$$

with $\mu^L = \begin{cases} 1, & x < 0 \\ 0, & x \geq 0 \end{cases}$, $\nu^L = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$, $(\mu + \nu)^L \equiv 1$, shows that the strict inequalities in (2.3) are possible (for \mathbb{R} -valued functions we say that $f < g$ if there is at least one point x in their domains such that $f(x) < g(x)$).

We recall that the space of distributions $\mathcal{D}'(\mathbb{R}^n)$ is defined as the dual to $\mathcal{D}(\mathbb{R}^n) = C^\infty_{\text{comp}}(\mathbb{R}^n)$, while $\mathcal{E}'(\mathbb{R}^n)$ is the space of distributions with compact support (the dual to $C^\infty(\mathbb{R}^n)$).

Definition 2 Let $f \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$. We define the functions A_f and B_f by

$$A_f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}, \quad x \mapsto \inf \{ \omega \in \mathbb{R}; (x, \omega) \in \text{supp } f \};$$

$$B_f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}, \quad x \mapsto \sup \{ \omega \in \mathbb{R}; (x, \omega) \in \text{supp } f \}.$$

It follows that A_f is lower semicontinuous, while B_f is upper semicontinuous:

$$A_f = A_f^L, \quad B_f = B_f^U.$$

Definition 3 Let $f \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$. We define $\Sigma[f]$ to be the projection of $\text{supp } f \subset \mathbb{R}^n \times \mathbb{R}$ onto the first factor:

$$\Sigma[f] = \{x \in \mathbb{R}^n; (\{x\} \times \mathbb{R}) \cap \text{supp } f \neq \emptyset\} \subset \mathbb{R}^n.$$

Thus, one has

$$x \notin \Sigma[f] \Leftrightarrow A_f(x) = +\infty \Leftrightarrow B_f(x) = -\infty.$$

Lemma 2 Let $f \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$. If there is a finite interval $I \subset \mathbb{R}$ such that $\text{supp } f \subset \mathbb{R}^n \times I$, then the set $\Sigma[f] \subset \mathbb{R}^n$ is closed.

Remark 2 $\Sigma[f]$ is not necessarily closed for $f \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$.

Lemma 3 For any distribution $f \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$, one has

$$A_f(x) \leq A_f^U(x) \leq B_f(x), \quad A_f(x) \leq B_f^L(x) \leq B_f(x), \quad \forall x \in \Sigma[f]; \tag{2.4}$$

$$\left(A_f^U\right)^L \geq A_f, \quad \left(B_f^L\right)^U \leq B_f. \tag{2.5}$$

Proof Note that B_f is upper semicontinuous with $B_f(x) \geq A_f(x)$ for all $x \in \Sigma[f]$, while A_f^U is the *smallest* upper semicontinuous function which is not smaller than A_f (cf. Lemma 1); thus, $A_f \leq A_f^U \leq B_f$. The second relation in (2.4) is proved similarly.

For the relations (2.5), one can see that for any lower semicontinuous function $a : \mathbb{R}^n \rightarrow \mathbb{R}$ one has $(a^U)^L \geq a$ (since a is a lower semicontinuous function which is not larger than a^U), and similarly for any upper semicontinuous function $b : \mathbb{R}^n \rightarrow \mathbb{R}$, one has $(b^L)^U \leq b$. □

Remark 3 In (2.4), A_f^U is not necessarily smaller than B_f^L ; it suffices to consider the example $f(x, \omega) = \theta(-x)\delta_{-1}(\omega) + \theta(x)\delta_1(\omega)$, with $x, \omega \in \mathbb{R}$. Also, the inequalities in (2.5) are not necessarily strict, as the example $f(x, \omega) = \delta(\omega) + \delta(x)\mathbb{1}_{[-1,1]}(\omega)$ shows (in detail, $B_f(0) = 1, B_f(x) = 0$ for $x \neq 0; A_f(0) = -1, B_f(x) = 0$ for $x \neq 0; B_f^L \equiv 0 \equiv A_f^U, (B_f^L)^U \equiv 0 \equiv (A_f^U)^L$).

For $f, g \in C_{\text{comp}}^\infty(\mathbb{R}^n \times \mathbb{R})$, we define the partial convolution

$$\begin{aligned} *_{\omega} : C_{\text{comp}}^\infty(\mathbb{R}^n \times \mathbb{R}) \times C_{\text{comp}}^\infty(\mathbb{R}^n \times \mathbb{R}) &\rightarrow C_{\text{comp}}^\infty(\mathbb{R}^n \times \mathbb{R}), \\ (f *_{\omega} g)(x, \omega) &= \int_{\mathbb{R}} f(x, \omega - \tau)g(x, \tau) d\tau, \quad (x, \omega) \in \mathbb{R}^n \times \mathbb{R}. \end{aligned} \tag{2.6}$$

This operation can be continuously extended to $f, g \in \mathcal{E}'(\mathbb{R}, L^2(\mathbb{R}^n))$:

$$*_{\omega} : \mathcal{E}'(\mathbb{R}, L^2(\mathbb{R}^n)) \times \mathcal{E}'(\mathbb{R}, L^2(\mathbb{R}^n)) \rightarrow \mathcal{E}'(\mathbb{R}, L^1(\mathbb{R}^n)).$$

Indeed, let $f, g, \varphi \in C_{\text{comp}}^\infty(\mathbb{R}^n \times \mathbb{R})$. Then

$$\begin{aligned} \langle f *_{\omega} g, \varphi \rangle &= \left\langle \int_{\mathbb{R}} f(x, \omega - \tau)g(x, \tau) d\tau, \varphi(x, \omega) \right\rangle \\ &= \int_{\mathbb{R}} \left(g(x, \tau) \int_{\mathbb{R}^n \times \mathbb{R}} f(x, \omega - \tau)\varphi(x, \omega) dx d\omega \right) d\tau, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ refers to the pairing of $L^2(\mathbb{R}^n \times \mathbb{R})$ -functions. The integral $\int_{\mathbb{R}^n \times \mathbb{R}} f(x, \omega - \tau)\varphi(x, \omega) dx d\omega$ makes sense for $f \in \mathcal{E}'(\mathbb{R}, L^2(\mathbb{R}^n))$ and $\varphi \in \mathcal{E}(\mathbb{R}, L^\infty(\mathbb{R}^n))$, defining an element from $\mathcal{E}'(\mathbb{R}, L^2(\mathbb{R}^n))$, which could then be coupled with $g \in \mathcal{E}'(\mathbb{R}, L^2(\mathbb{R}^n))$. Let us mention that for (complex) Banach spaces A, B and the space of bounded linear maps $\mathcal{B}(A, B)$, the space of $\mathcal{B}(A, B)$ -valued distributions $\mathcal{D}'(\mathbb{R}, \mathcal{B}(A, B))$ is defined as the space of bounded linear maps from $\mathcal{D}(\mathbb{R}, A)$ (A -valued test functions) to B , and similarly for $\mathcal{B}(A, B)$ -valued tempered distributions \mathcal{E}' ; for the general theory of Banach-space-valued distributions, see Zemanian (1972, Chapter 3).

For any $f, g \in \mathcal{E}'(\mathbb{R}, L^2(\mathbb{R}^n))$ there are immediate relations

$$A_{f \ast_w g} \geq A_f + A_g, \quad B_{f \ast_w g} \leq B_f + B_g. \tag{2.7}$$

We will show that the relations (2.7) are equalities, in the appropriate sense.

Theorem 2 (Titchmarsh theorem for partial convolution) *Let $f, g \in \mathcal{E}'(\mathbb{R}, L^2_{loc}(\mathbb{R}^n))$. Then*

$$A_{f \ast_w g} = (A_f^U + A_g)^L = (A_f + A_g^U)^L, \quad B_{f \ast_w g} = (B_f^L + B_g)^U = (B_f + B_g^L)^U.$$

Remark 4 Let us prove a similar statement for elements from the space $C(\mathbb{R}^n, \mathcal{E}'(\mathbb{R}))$, defined as the space of functions $F : \mathbb{R}^n \rightarrow \mathcal{E}'(\mathbb{R})$ which satisfy $\lim_{x \rightarrow x_0} F(x) = F(x_0)$ for any $x_0 \in \mathbb{R}^n$, with the convergence in the topology of $\mathcal{E}'(\mathbb{R})$. For $f, g \in C(\mathbb{R}^n, \mathcal{E}'(\mathbb{R}))$, since f and g depend continuously on x , the Titchmarsh convolution theorem can be applied pointwise in x , yielding

$$\inf \text{supp}(f \ast_w g)(x, \cdot) = \inf \text{supp } f(x, \cdot) + \inf \text{supp } g(x, \cdot), \quad \forall x \in \mathbb{R}^n, \tag{2.8}$$

and similarly for sup. Let $f \in C(\mathbb{R}^n, \mathcal{E}'(\mathbb{R}))$ and let $\rho \in \mathcal{D}(\mathbb{R})$. If $\mathcal{O} \subset \mathbb{R}^n$ is an open set such that $\langle \rho, f(x, \cdot) \rangle = 0$ for all $x \in \mathcal{O}$, then, by continuity of f in x , one also has $\langle \rho, f(x, \cdot) \rangle = 0$ for all x from the closure of \mathcal{O} . Therefore, given an open set $\Omega \subset \mathbb{R}$, if $\Omega \cap \text{supp } f(x, \cdot) = \emptyset$ for $x \in \mathcal{O} \subset \mathbb{R}^n$, then $\Omega \cap \text{supp } f(x, \cdot) = \emptyset$ for x from the closure of \mathcal{O} ; it then follows that

$$A_f^U(x) = \inf \text{supp } f(x, \cdot), \quad B_f^L(x) = \sup \text{supp } f(x, \cdot) \quad \text{for any } x \text{ such that } f(x, \cdot) \neq 0.$$

Applying the above to each of the terms in (2.8) (and similarly for sup) leads to the relations

$$A_{f \ast_w g}^U(x) = A_f^U(x) + A_g^U(x), \quad B_{f \ast_w g}^L(x) = B_f^L(x) + B_g^L(x)$$

which are similar to the relations stated in Theorem 2.

2.2 Convex Hulls and Partial Convolution Theorem in Higher Dimensions

Let us give a higher dimensional version of the partial convolution theorem in terms of convex hulls, following Lions (1951). Let $n, m \geq 1$. For any set-valued map $M : \mathbb{R}^n \rightarrow \{\text{closed subsets of } \mathbb{R}^m\}$ there is a maximal inner semicontinuous set-valued map $\mathbb{R}^n \rightarrow \{\text{closed subsets of } \mathbb{R}^m\}$ which does not exceed M ; we denote this map by

$$M^L(x) = \lim_{\epsilon \rightarrow 0^+} \bigcap_{y \in \mathbb{B}_\epsilon(x)} M(y), \quad \forall x \in \mathbb{R}^n.$$

Note that for each $x \in \mathbb{R}^n$, the set $M^L(x) \subset \mathbb{R}^m$ is closed (as an intersection of an arbitrary number of closed sets). Similarly, there is a minimal outer semicontinuous set-valued map $\mathbb{R}^n \rightarrow \{\text{closed subsets of } \mathbb{R}^m\}$ which is not exceeded by M ; we denote this map by

$$M^U(x) = \lim_{\epsilon \rightarrow 0^+} \bigcup_{y \in \mathbb{B}_\epsilon(x)} M(y), \quad \forall x \in \mathbb{R}^n.$$

(Note that for each $x \in \mathbb{R}^n$, the set $M^U(x) \subset \mathbb{R}^m$ is closed: if $\omega_j \in M^U(x)$ converges to some $\omega_* \in \mathbb{R}^m$ as $j \rightarrow \infty$, then there are sequences $\omega_{j,N} \in M(y_N)$ with $|x - y_N| < 1/N$, $N \in \mathbb{N}$ such that, for each $j \in \mathbb{N}$, $\omega_{j,N} \rightarrow \omega_j$ as $N \rightarrow \infty$, but then one can choose a diagonal subsequence ω_{j_r, N_r} converging to ω_* . Thus, $\omega_* \in M^U(x)$, so $M^U(x)$ is closed.) Thus,

$$M^L(x) \subset M(x) \subset M^U(x), \quad \forall x \in \mathbb{R}^n.$$

The following lemma is an immediate generalization of Lemma 1.

Lemma 4 For any $M, N : \mathbb{R}^n \rightarrow \{\text{closed subsets of } \mathbb{R}^m\}$, one has

$$(M + N)^L(x) \supset M^L(x) + N^L(x), \quad (M + N)^U(x) \subset M^U(x) + N^U(x), \quad \forall x \in \mathbb{R}^n. \tag{2.9}$$

Above, the sum of two subsets $A, B \subset \mathbb{R}^m$ is defined by $A + B = \{a + b \in \mathbb{R}^m; a \in A, b \in B\} \subset \mathbb{R}^m$.

We recall that, given a set $S \subset \mathbb{R}^n$, then $\text{conv } S$ denotes its convex hull. For a set $S \subset \mathbb{R}^n \times \mathbb{R}^m$, with $m, n \in \mathbb{N}$, let us define $\text{conv}_\omega S$ as a map from \mathbb{R}^n to convex subsets of \mathbb{R}^m by

$$\text{conv}_\omega S : x \mapsto \text{conv} (S \cap (\{x\} \times \mathbb{R}^m)) \subset \mathbb{R}^m.$$

If S is closed, this map is outer semicontinuous.

For a closed subset $S \subset \mathbb{R}^n \times \mathbb{R}^m$, we define

$$(\text{conv}_\omega S)^L : \mathbb{R}^n \rightarrow \{\text{closed subsets of } \mathbb{R}^n \times \mathbb{R}^m\}$$

as the largest inner semicontinuous map from \mathbb{R}^n to closed convex subsets of \mathbb{R}^m which satisfies

$$(\text{conv}_\omega S)^L(x) \subset (\text{conv}_\omega S)(x), \quad \forall x \in \mathbb{R}^n.$$

Remark 5 For $f \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$, there is an obvious relation

$$\begin{aligned} (\text{conv}_\omega \text{supp } f)^L(x) &= [A_f^U(x), B_f^L(x)] \subset (\text{conv}_\omega \text{supp } f)(x) \\ &= [A_f(x), B_f(x)], \quad \forall x \in \Sigma[f]. \end{aligned}$$

Theorem 3 (Titchmarsh theorem for partial convolution: convex hulls) *Let $f, g \in \mathcal{E}'(\mathbb{R}^m, L^2_{loc}(\mathbb{R}^n))$. Then*

$$\begin{aligned} \text{conv}_\omega \text{ supp } f *_\omega g &= \left((\text{conv}_\omega \text{ supp } f)^L + \text{conv}_\omega \text{ supp } g \right)^U \\ &= \left(\text{conv}_\omega \text{ supp } f + (\text{conv}_\omega \text{ supp } g)^L \right)^U. \end{aligned}$$

The proof of Theorem 3 follows the same lines as that of Theorem 2 [using the language of Lions (1951)].

2.3 Proof of Partial Convolution Theorem for $f *_\omega f$ for $f \in \mathcal{D}(\mathbb{R}, L^2_{loc}(\mathbb{R}^n))$

Following (Hörmander 1983, Proof of Theorem 4.3.3), we first prove the theorem for $f, g \in \mathcal{D}(\mathbb{R}, L^2_{loc}(\mathbb{R}^n))$. To consider the case $f = g$, we need the two lemmata, which are the immediate adaptations of Hörmander (1983, Lemmata 4.3.4, 4.3.5).

Lemma 5 *For $f \in \mathcal{D}(\mathbb{R}, L^2_{comp}(\mathbb{R}^n))$, one has*

$$\|f *_\omega f^\sharp\|_{L^2(\mathbb{R}^n \times \mathbb{R})} = \|f *_\omega f\|_{L^2(\mathbb{R}^n \times \mathbb{R})},$$

where

$$f^\sharp(x, \omega) = \overline{f(x, -\omega)}. \tag{2.10}$$

Lemma 6 *For any finite open interval $\Omega \subset \mathbb{R}$, there is $C < \infty$ such that*

$$\|f\|_{L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))} \leq C \left\| \partial_\omega^2 f \right\|_{L^2(\mathbb{R}^n \times \Omega)}, \quad \forall f \in \mathcal{D}(\Omega, L^2_{comp}(\mathbb{R}^n)).$$

Now we can give the proof for the case $f = g \in \mathcal{D}(\mathbb{R}, L^2_{loc}(\mathbb{R}^n))$.

Lemma 7 *Let $f \in \mathcal{D}(\mathbb{R}, L^2_{loc}(\mathbb{R}^n))$. There are the relations*

$$A_{f *_\omega f} = 2A_f, \quad B_{f *_\omega f} = 2B_f.$$

Proof For any open set $\mathcal{O} \subset \mathbb{R}^n$, one has:

$$\begin{aligned} \|f\|_{L^2(\mathcal{O} \times \Omega)}^2 &= \|f *_\omega f^\sharp(\cdot, 0)\|_{L^2(\mathcal{O})} = \|f *_\omega f^\sharp\|_{L^2(\mathcal{O}, L^\infty(\Omega))} \\ &\leq C \left\| \partial_\omega^2 (f *_\omega f^\sharp) \right\|_{L^2(\mathcal{O} \times \Omega)} = C \left\| \partial_\omega f *_\omega \partial_\omega f^\sharp \right\|_{L^2(\mathcal{O} \times \Omega)} \\ &= C \left\| \partial_\omega f *_\omega \partial_\omega f \right\|_{L^2(\mathcal{O} \times \Omega)}; \end{aligned}$$

in the second line, we applied Lemma 6 and then Lemma 5. Applying the above inequality to $f_\xi(x, \omega) = f(x, \omega)e^{\omega\xi}$, we arrive at the inequality

$$\|f_\xi\|_{L^2(\mathcal{O}\times\Omega)}^2 \leq C \left\| \partial_\omega^2(f_\xi *_\omega f_\xi) \right\|_{L^2(\mathcal{O}\times\Omega)}. \tag{2.11}$$

This inequality is satisfied for arbitrarily large $|\xi|$, while $f_\xi *_\omega f_\xi(x, \omega) = e^{\omega\xi}(f *_\omega f)(x, \omega)$ for a given function f ; hence, twice the support of the integrand in the left-hand side of (2.11) is contained in $\cup_{x \in \mathcal{O}} \text{conv}(\{x\} \times \Omega \cap \text{supp } f *_\omega f)$. Sending $\mathcal{O} \rightarrow \{x\}$, we conclude that $2B_f(x) \leq B_{f *_\omega f}(x)$, for all $x \in \mathbb{R}^n$. We conclude that

$$2B_f(x) \leq B_{f *_\omega f}(x), \quad \forall x \in \mathbb{R}^n. \tag{2.12}$$

Due to an immediate inequality $2B_f(x) \geq B_{f *_\omega f}(x)$ which follows from the definition (2.6), one has $2B_f(x) = B_{f *_\omega f}(x)$. Similarly, $2A_f(x) = A_{f *_\omega f}(x)$. \square

2.4 Proof of Partial Convolution Theorem for $f, g \in \mathcal{D}(\mathbb{R}, L^2_{\text{loc}}(\mathbb{R}^n))$

Lemma 8 *Let $f, g \in \mathcal{D}(\mathbb{R}, L^2_{\text{loc}}(\mathbb{R}^n))$. Then, for any polynomials $\alpha(\omega)$ and $\beta(\omega)$,*

$$A_{(\alpha f) *_\omega (\beta g)} \geq A_{f *_\omega g}, \quad B_{(\alpha f) *_\omega (\beta g)} \leq B_{f *_\omega g}.$$

Proof We closely follow the argument from Hörmander (1983, Proof of Theorem 4.3.3). It suffices to prove the second inequality, and only for the polynomials $\alpha(\omega) = \omega, \beta(\omega) = 1$. Denote

$$f_n(x, \omega) = \omega^n f(x, \omega), \quad g_n(x, \omega) = \omega^n g(x, \omega), \quad B_{mn}(x) := B_{f_m *_\omega g_n}(x). \tag{2.13}$$

Let us assume that, contrary to the statement of the Lemma, there is $x \in \mathbb{R}^n$ such that

$$B_{f_1 *_\omega g}(x) > B_{f *_\omega g}(x); \tag{2.14}$$

from now on, all the quantities are evaluated at this particular value of x . The inequality (2.14) can be rewritten as

$$B_{10} - B_{00} > 0. \tag{2.15}$$

Due to the relation $\omega(f *_\omega g)(\omega) = (f_1 *_\omega g)(\omega) + (f *_\omega g_1)(\omega)$, we have:

$$B_{f_1 *_\omega g + f *_\omega g_1} = B_{\omega(f *_\omega g)(\omega)} \leq B_{f *_\omega g} = B_{00}. \tag{2.16}$$

It follows that

$$B_{f_1 *_\omega g *_\omega f_1 *_\omega g + f_1 *_\omega g + f *_\omega g_1} \leq B_{f_1 *_\omega g} + B_{f_1 *_\omega g + f *_\omega g_1} \leq B_{10} + B_{00}.$$

If we had $B_{f_1 \ast g \ast f_1 \ast g} \neq B_{f_1 \ast g \ast f \ast g_1}$, then both these quantities would be smaller than or equal to $B_{10} + B_{00}$. By Lemma 7 and (2.15), this would lead to $B_{f_1 \ast g} \leq (B_{10} + B_{00})/2 < B_{10}$, contradicting (2.13). Thus, $B_{f_1 \ast g \ast f_1 \ast g} = B_{f_1 \ast g \ast f \ast g_1}$, leading to

$$B_{f_1 \ast g \ast f_1 \ast g} = B_{f_1 \ast g \ast f \ast g_1} \leq B_{f \ast g} + B_{f_1 \ast g_1}. \tag{2.17}$$

By Lemma 7, $B_{f_1 \ast g \ast f_1 \ast g} = 2B_{f_1 \ast g}$; then (2.17) could be rewritten as

$$2B_{f_1 \ast g} \leq B_{f \ast g} + B_{f_1 \ast g_1}. \tag{2.18}$$

This gives

$$B_{11} - B_{10} \geq B_{10} - B_{00} > 0. \tag{2.19}$$

In the last inequality, we took into account (2.15). The inequalities (2.19) imply that

$$B_{f_1 \ast g_1} > B_{f_1 \ast g}. \tag{2.20}$$

Just as we derived (2.18) from (2.14), we could use (2.20) to derive

$$2B_{f_1 \ast g_1} \leq B_{f_1 \ast g} + B_{f_2 \ast g_1}. \tag{2.21}$$

The inequality (2.21) could be written as $B_{21} - B_{11} \geq B_{11} - B_{10}$, and, together with (2.19), this yields

$$B_{21} - B_{11} \geq B_{11} - B_{10} \geq B_{10} - B_{00} > 0.$$

Proceeding by induction, we prove that

$$B_{32} - B_{22} \geq B_{22} - B_{21} \geq B_{21} - B_{11} \geq B_{11} - B_{10} \geq B_{10} - B_{00} > 0,$$

hence

$$B_{nn} \geq B_{00} + 2n(B_{10} - B_{00}). \tag{2.22}$$

At the same time, since $B_{f_n} \leq B_f, B_{g_n} \leq B_g$, we know that $B_{f_n \ast g_n} \leq B_{f_n} + B_{g_n} \leq B_f + B_g$. This would be in contradiction to (2.22). Hence, (2.14) is not true. This finishes the proof of the lemma. \square

Proof of Theorem 2 for $f, g \in \mathcal{D}(\mathbb{R}, L^2_{loc}(\mathbb{R}^n))$ Now we complete the proof of the Titchmarsh theorem for $f \ast g$. For our convenience, we assume that $\text{supp } f \subset \mathbb{R}^n \times [1, +\infty)$ and $\text{supp } g \subset \mathbb{R}^n \times [1, +\infty)$.

Fix $x \in \mathbb{R}^n$. Let $\epsilon \in (0, 1)$ be arbitrarily small. Due to lower semicontinuity of $A_{f \ast g}$, for any $\omega_0 \in (A_{f \ast g}(x) - \epsilon, A_{f \ast g}(x))$, there is a nonempty open neighborhood $\mathcal{O} \subset \mathbb{B}_\epsilon(x)$, $\mathcal{O} \ni x$, such that $\omega_0 < A_{f \ast g}(y)$ for all $y \in \mathcal{O}$. This implies that

$$\int_{\mathcal{O}} \rho(y) \int_0^\omega f(y, \omega - \tau)g(y, \tau) d\tau dy = 0, \quad \forall \omega \in (0, \omega_0), \quad \forall \rho \in C^\infty_{\text{comp}}(\mathcal{O}). \tag{2.23}$$

By Lemma 8, the relation (2.23) leads to

$$\int_{\mathcal{O}} \rho(y) \int_0^\omega f(y, \omega - \tau)g(y, \tau)\tau^N d\tau dy = 0, \quad N \in \mathbb{N}, \quad \forall \omega \in (0, \omega_0), \quad \forall \rho \in C^\infty_{\text{comp}}(\mathcal{O}).$$

It follows that

$$f(y, \omega - \tau)g(y, \tau) = 0, \quad \forall y \in \mathcal{O}, \quad \forall \omega \in (0, \omega_0). \tag{2.24}$$

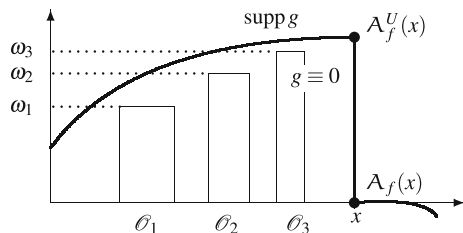
Since we consider the case $f \in \mathcal{D}(\mathbb{R}, L^2_{\text{loc}}(\mathbb{R}^n)) \subset L^2_{\text{loc}}(\mathbb{R}^n \times \mathbb{R})$, for a given open neighborhood $\mathcal{O} \ni x$ there is an open neighborhood $\mathcal{O}_1 \subset \mathcal{O}$, an open interval $\Omega_1 \subset (A_f(x) - \epsilon, A_f(x) + \epsilon)$, and $\delta > 0$ such that $|f| \geq \delta$ almost everywhere on $\mathcal{O}_1 \times \Omega_1$. (If not, then one would conclude that $f = 0$ almost everywhere in $\mathcal{O} \times (A_f(x) - \epsilon, A_f(x) + \epsilon)$, contradicting the definition of $A_f(x)$.) It follows from (2.24) that $g(y, \omega - \tau) = 0$ for all $y \in \mathcal{O}_1$, $\omega \in (0, \omega_0)$, $\tau \in \Omega_1$. Therefore,

$$g(y, \omega) \equiv 0 \quad \text{almost everywhere in the rectangle } (y, \omega) \in \mathcal{O}_1 \times (0, \omega_1),$$

where $\omega_1 := \omega_0 - A_f(x) - \epsilon$. Choosing $\epsilon = 2^{-j}$, $j \in \mathbb{N}$, in the above construction, we obtain a sequence $(\omega_j)_{j \in \mathbb{N}}$ which converges to $A_{f \ast g}(x) - A_f(x)$ and $\mathcal{O}_j \subset \mathbb{B}_{\epsilon_j}(x)$ such that $(\mathcal{O}_j \times (0, \omega_j)) \cap \text{supp } g = \emptyset$.

It follows that $A^U_g(x) \geq A_{f \ast g}(x) - A_f(x)$, and similarly $B^L_g(x) \leq B_{f \ast g}(x) - B_f(x)$. Since $x \in \mathbb{R}^n$ was arbitrary, this finishes the proof (Fig. 1). \square

Fig. 1 A_f and A^U_f



2.5 Proof of Partial Convolution Theorem for $f, g \in \mathcal{E}'(\mathbb{R}, L^2_{loc}(\mathbb{R}^n))$

Lemma 9 *Let $f, g \in \mathcal{E}'(\mathbb{R}, L^2_{loc}(\mathbb{R}^n))$. Let $\varphi \in \mathcal{D}(\mathbb{R})$, $\int_{\mathbb{R}} \varphi(\omega) d\omega = 1$. Then $\text{supp } f *_\omega \varphi \rightarrow \text{supp } f$ as $\text{supp } \varphi \rightarrow \{0\}$, and moreover, for each $x \in \mathbb{R}^n$,*

$$\begin{aligned} A_{f *_\omega \varphi}(x) &\rightarrow A_f(x), & A^U_{f *_\omega \varphi}(x) &\rightarrow A^U_f(x) \text{ as } \text{supp } \varphi \rightarrow \{0\}, \\ B_{f *_\omega \varphi}(x) &\rightarrow B_f(x), & B^L_{f *_\omega \varphi}(x) &\rightarrow B^L_f(x) \text{ as } \text{supp } \varphi \rightarrow \{0\}. \end{aligned}$$

Proof If $(x, \omega) \in \text{supp } f$, then there is an arbitrarily small open neighborhood $\mathcal{O} \times \Omega$ of (x, ω) and the functions $\psi \in L^2(\mathcal{O})$ and $\theta \in \mathcal{D}(\Omega)$ such that $\langle f, \psi \otimes \theta \rangle \neq 0$. One has $\theta *_\omega \varphi = \theta * \varphi \xrightarrow{\mathcal{D}} \theta$ as $\text{supp } \varphi \rightarrow \{0\}$ [see (Hörmander 1983, Theorem 1.3.2)]; then

$$0 \neq \langle f, \psi \otimes \theta \rangle = \lim_{\text{supp } \varphi \rightarrow \{0\}} \langle f, \psi \otimes (\varphi * \theta) \rangle.$$

Therefore, one has $\langle f *_\omega \varphi, \psi \otimes \theta \rangle = \langle f, \psi \otimes (\varphi * \theta) \rangle \neq 0$ for $\text{supp } \varphi$ small enough. For such φ , one has

$$\text{dist}(\text{supp } f, \text{supp } f *_\omega \varphi) \leq \text{diam}(\mathcal{O}) + \text{diam}(\Omega) + \text{diam}(\text{supp } \varphi).$$

Since \mathcal{O} and Ω are arbitrarily small, the conclusion follows. □

Proof of Theorem 2 We follow the proof of (Hörmander 1983, Theorem 4.3.3). Let $0 \leq \varphi \in \mathcal{D}(\mathbb{R})$ be such that $\int_{\mathbb{R}} \varphi(\omega) d\omega = 1$; we apply the version of Theorem 2 for $f, g \in \mathcal{D}(\mathbb{R}, L^2_{loc}(\mathbb{R}^n))$ (which we proved in Sect. 2.4) to $f *_\omega \varphi$ and $g *_\omega \varphi$ to conclude that

$$B_{f *_\omega \varphi} + B^L_{g *_\omega \varphi} \leq B_{(f *_\omega \varphi) *_\omega (g *_\omega \varphi)} = B_{(f *_\omega g) *_\omega (\varphi *_\omega \varphi)}.$$

Considering the limit $\text{supp } \varphi \rightarrow \{0\}$ and applying Lemma 9, we arrive at

$$B_f(x) + B^L_g(x) \leq B_{f *_\omega g}(x), \quad x \in \mathbb{R}^n. \tag{2.25}$$

Lemma 10 *Let $f \in \mathcal{E}'(\mathbb{R}, L^2_{loc}(\mathbb{R}^n))$. Then $(A^U_f)^L = A_f$ and $(B^L_f)^U = B_f$.*

Proof It is enough to prove the second statement. Let us first prove it for f measurable. For $x \in \mathbb{R}^n \setminus \Sigma[f]$, since $\Sigma[f]$ is closed (see Lemma 2), there is an open neighborhood $\mathcal{O} \subset \mathbb{R}^n$, $\mathcal{O} \ni x$, such that $\mathcal{O} \cap \Sigma[f] = \emptyset$, hence

$$B_f|_{\mathcal{O}} \equiv -\infty, \quad B^U_f|_{\mathcal{O}} \equiv -\infty, \quad (B^U_f)^L|_{\mathcal{O}} \equiv -\infty.$$

Now let us consider $x \in \Sigma[f] \subset \mathbb{R}^n$. For any $\varepsilon > 0$, there is $\delta > 0$, $\mathcal{O} \subset \mathbb{B}_\varepsilon(x)$, and $\Omega \subset (B_f(x) - \varepsilon, B_f(x) + \varepsilon)$ such that $|f| \geq \delta$ for almost all $(x, \omega) \in \mathcal{O} \times \Omega$

(otherwise, f vanishes almost everywhere in an open neighborhood of $(x, B_f(x))$, hence $(x, B_f(x)) \notin \text{supp } f$, which is in contradiction to the definition of B_f).

For $f \in \mathcal{E}'(\mathbb{R}, L^2_{\text{loc}}(\mathbb{R}^n))$, we fix $\varphi \in C^\infty_{\text{comp}}(\mathbb{R})$, $\varphi \geq 0$, $0 \in \text{supp } \varphi$, $\int_{\mathbb{R}} \varphi(\omega) d\omega = 1$, and consider $f *_\omega \varphi$. Since $f *_\omega \varphi \in \mathcal{D}(\mathbb{R}, L^2_{\text{loc}}(\mathbb{R}^n))$ is measurable, the first part of the proof applies, showing that

$$(B_{f *_\omega \varphi}^L)^U(x) = B_{f *_\omega \varphi}(x), \quad \forall x \in \mathbb{R}^n. \tag{2.26}$$

It remains to notice that $B_f + \text{diam}(\text{supp } \varphi) \geq B_{f *_\omega \varphi} \geq B_f$, for all $x \in \mathbb{R}^n$, with the last inequality due to (2.25), and to send $\text{supp } \varphi \rightarrow \{0\}$; then (2.26) turns into $(B_f^L)^U = B_f$, for all $x \in \mathbb{R}^n$. \square

Lemma 11 *Assume that $f, g \in \mathcal{E}'(\mathbb{R}, L^2_{\text{loc}}(\mathbb{R}^n))$. Then*

$$(A_f + A_g^U)^L = A_{f *_\omega g} = (A_f + A_g)^U, \quad (B_f + B_g^L)^U = B_{f *_\omega g} = (B_f + B_g)^L.$$

Proof It is enough to prove the second statement. From (2.25), we conclude that $(B_f + B_g^L)^U \leq B_{f *_\omega g}$, while

$$B_{f *_\omega g} \leq B_f + B_g \Rightarrow B_{f *_\omega g}^L \leq (B_f + B_g)^L \Rightarrow (B_{f *_\omega g}^L)^U \leq ((B_f + B_g)^L)^U.$$

By Lemma 10, $(B_{f *_\omega g}^L)^U = B_{f *_\omega g}$; therefore, we conclude from the above relations that

$$(B_f + B_g^L)^U \leq B_{f *_\omega g} \leq ((B_f + B_g)^L)^U. \tag{2.27}$$

On the other hand, let us pick $x \in \mathbb{R}^n$; there is a sequence $x_j \rightarrow x$ such that $B_g(x_j) \rightarrow B_g^L(x)$. Then $\limsup B_f(x_j) \leq B_f(x)$, hence we conclude that

$$\begin{aligned} B_f(x) + B_g^L(x) &= B_f(x) + \lim_{j \rightarrow \infty} B_g(x_j) \geq \limsup_{j \rightarrow \infty} (B_f(x_j) + B_g(x_j)) \\ &\geq \liminf_{j \rightarrow \infty} (B_f(x_j) + B_g(x_j)) \geq (B_f + B_g)^L(x), \end{aligned}$$

and then we conclude that

$$(B_f + B_g^L)^U(x) \geq ((B_f + B_g)^L)^U(x), \quad \forall x \in \mathbb{R}^n. \tag{2.28}$$

Combining (2.27) and (2.28), we arrive at $B_f(x) + B_g^L(x) = B_{f *_\omega g}(x) = (B_f + B_g)^L(x)$, for all $x \in \mathbb{R}^n$. \square

This completes the proof of Theorem 2. \square

3 Compact Spectrum Solutions to the Nonlinear Klein–Gordon Equation

Let us first recall the unique continuation property (**UCP**) for the Laplace operator.

Theorem 4 (Unique continuation property for the Laplace operator) *Let $n \geq 1$. Assume that $u \in H^1(\mathbb{R}^n)$ satisfies the relation*

$$|\Delta u| \leq |Vu| \tag{3.1}$$

almost everywhere in a connected open domain $\mathcal{O} \subset \mathbb{R}^n$, with $V \in L^p_{\text{loc}}(\mathbb{R}^n)$, with $p \geq n/2$, $n \geq 2$, and with $p = 1$ for $n = 1$. If u vanishes almost everywhere in an open subset $\mathcal{O}_0 \subset \mathcal{O}$, then it vanishes almost everywhere in \mathcal{O} .

Wolff (1992, Theorem 3) proved the unique continuation in \mathbb{R}^n , $n \geq 3$, with $V \in L^p_{\text{loc}}(\mathbb{R}^n)$, where $p \geq n/2$ if $n \geq 5$, $p > 2$ if $n = 4$, and $p \geq 2$ if $n = 3$. The optimal unique continuation results for (3.1) were obtained in Koch and Tataru (2001); in particular, it follows that the strong unique continuation property holds for $V \in L^{n/2}_{\text{loc}}(\mathbb{R}^n)$ for any $n \geq 2$ [the sufficient conditions on V in Koch and Tataru (2001, Theorem 1.1) are slightly weaker]. We also mention that the sufficient condition $V \in L^1_{\text{loc}}(\mathbb{R})$ in the one-dimensional case is a consequence of the two-dimensional result when considering functions with a trivial dependence on x_2 . Or, arguing directly, one could assume that $I \subset \mathbb{R}$ is an open interval and $u \in H^1(\mathbb{R})$ vanishes in an open neighborhood $\mathcal{O}_0 \subset I$ and satisfies $|u''| \leq |Vu|$ almost everywhere in I , with some $V \in L^1_{\text{loc}}(\mathbb{R})$. Let us show that $u|_I \equiv 0$. Without loss of generality, we may assume that I is bounded. Given $x_0 \in \mathcal{O}_0$, then, for any $x \in I$, one has $|u'(x)| \leq \int_{x_0}^x |V(y)u(y)| dy \leq \sup_{y \in [x_0, x]} |u(y)| \|V\|_{L^1(I)}$, hence

$$|u(x)| \leq \int_{x_0}^x |u'(y)| dy \leq \sup_{y \in [x_0, x]} |u(y)| |x - x_0| \|V\|_{L^1(I)},$$

showing that $u(x) = 0$ as long as $x \in I$ is close enough to x_0 so that $|x - x_0| < 1/\|V\|_{L^1(I)}$. It follows that $u \equiv 0$ in I .

Remark 6 For $n \leq 2$, the Sobolev embedding gives $u \in H^1(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ for any $2 \leq q < \infty$ (including $q = \infty$ when $n = 1$), hence $V(x) := \alpha(|u(x)|^2)$ with $\alpha(\tau)$ from (3.3) satisfies $V \in L^p_{\text{loc}}(\mathbb{R}^n)$ for any $1 < p < \infty$. Therefore, for $n \leq 2$, the unique continuation takes place for any $\kappa > 0$.

For $n \geq 3$, by the Sobolev embeddings, $u \in H^1(\mathbb{R}^n) \subset L^{2^*}(\mathbb{R}^n)$, with $2^* = 2n/(n - 2)$. Then $V(x) := \alpha(|u(x)|^2)$ satisfies

$$V \in L^p_{\text{loc}}(\mathbb{R}^n), \quad \text{with } p = \frac{2^*}{2\kappa} = \frac{n}{(n - 2)\kappa}.$$

For the unique continuation to take place, we need the relation $p = \frac{n}{(n-2)\kappa} \geq \frac{n}{2}$, so for $n \geq 3$ we need $\kappa \leq 2/(n - 2)$.

Now we recall the local well-posedness results for the Klein–Gordon equation.

Theorem 5 [NLKG global well-posedness (Kato 1986, Proposition 2.1)] *Let $n \in \mathbb{N}$, $m > 0$. Let $f \in C^1(\mathbb{C}, \mathbb{C})$ with $f(0) = 0$ and $f(e^{is}u) = e^{is}f(u)$, $\forall u \in \mathbb{C}$, $\forall s \in \mathbb{R}$; $F(u) = \int_0^{|u|} f(v) dv$, $u \in \mathbb{C}$. Assume that there are $c_0 > 0$ and $c_1 > 0$ such that*

$$F(u) \geq -c_0|u|, \quad |f'(u)| \leq c_1(1 + |u|^{p-1}), \quad \forall u \in \mathbb{C},$$

with some $p \in (1, +\infty)$ if $n \leq 2$; $p \in (1, 1 + 4/(n - 2))$ if $n \geq 3$. Then there is a unique, strongly continuous solution $u \in C(\mathbb{R}, H^1(\mathbb{R}^n, \mathbb{C}))$, $\forall t \in \mathbb{R}$, to the Cauchy problem

$$-\partial_t^2 u = -\Delta u + m^2 u + f(u), \quad u(x, t) \in \mathbb{C}, \quad x \in \mathbb{R}^n; \quad (u, \dot{u})|_{t=0} \in H^1(\mathbb{R}^n, \mathbb{C}) \times L^2(\mathbb{R}^n, \mathbb{C}). \tag{3.2}$$

Its energy is conserved: $E(u(t)) = E(u(0))$ for all $t \in \mathbb{R}$.

Assumption 1 $f(u) = \alpha(|u|^2)u$, with $\alpha \in C(\overline{\mathbb{R}_+})$, $\alpha(0) = 0$, and there is $C < \infty$ such that

$$|\alpha(\tau)| \leq C\langle \tau \rangle^\kappa, \quad \forall \tau \geq 0, \quad \text{with } \kappa \text{ satisfying } \begin{cases} \kappa > 0, & n \leq 2; \\ 0 < \kappa \leq 2/(n - 2), & n \geq 3. \end{cases} \tag{3.3}$$

We note that the restriction on κ is such that the unique continuation property from Theorem 4 applies to $V(x) = \alpha(|u|^2)$ with $u \in H^1(\mathbb{R}^n)$ (see Remark 6). We also note that the well-posedness result from Theorem 5 applies if e.g. $\alpha(\tau) = C\tau^\kappa$, $\tau \geq 0$ (or if $\alpha(\tau)$ is a polynomial of degree $\kappa \in \mathbb{N}$, with $\kappa > 0$ for $n \leq 2$, $0 < \kappa < 2/(n - 2)$ if $n \geq 3$ (cf. (3.3)).

We will be able to consider not only polynomial nonlinearities, but also certain algebraic nonlinearities.

Assumption 2 Assume that $\alpha \in C(\overline{\mathbb{R}_+})$ is a non-constant algebraic function, so that there is $J \in \mathbb{N}$ and polynomials $M_j(\tau)$, $0 \leq j \leq J$, with $M_J(\tau) \not\equiv 0$, such that $w(\tau) := \tau\alpha(\tau)$ satisfies the relation $\mathcal{M}(\tau, w(\tau)) = 0$, $\forall \tau \geq 0$, where

$$\begin{aligned} \mathcal{M}(\tau, w(\tau)) &:= \tau^J \sum_{j=0}^J P_j(\tau)\alpha(\tau)^j = \sum_{j=0}^J M_j(\tau)(\tau\alpha(\tau))^j \\ &= \sum_{j=0}^J M_j(\tau)w(\tau)^j, \quad \forall \tau \geq 0. \end{aligned} \tag{3.4}$$

Moreover, assume that

$$\deg M_0 > \deg M_j + j, \quad \forall j, \quad 1 \leq j \leq J.$$

If $n \geq 3$, additionally assume that

$$\deg M_j + (\kappa + 1)j \leq n/(n - 2), \quad \forall j, \quad 0 \leq j \leq J. \tag{3.5}$$

Example 1 Assume that α is a polynomial: $\alpha(\tau) = \sum_{j=0}^{\kappa} \alpha_j \tau^j$, with $\kappa \in \mathbb{N}$, $\kappa \leq 2/(n - 2)$ if $3 \leq n \leq 4$, and $\alpha_{\kappa} \neq 0$. Let $M_0(\tau) = -\tau\alpha(\tau)$ and $M_1(\tau) = 1$, so that $\deg M_0 = \kappa + 1$ and $\deg M_1 = 0$. Then

$$\mathcal{M}(\tau, \tau\alpha(\tau)) = M_0(\tau) + M_1(\tau)\tau\alpha(\tau) = -\tau\alpha(\tau) \cdot 1 + 1 \cdot \tau\alpha(\tau) = 0, \quad \forall \tau \in \mathbb{R}.$$

One can see that Assumption 2 is satisfied (including the requirement (3.5) when $n \geq 3$).

Example 2 Assume that $\alpha(\tau) = A(\tau)^{1/N}$, with $N \in \mathbb{N}$, $N \geq 2$, and with $A(\tau) = \sum_{j=0}^a A_j \tau^j$ a polynomial of degree $a = \deg A \geq 1$; if N is even, we additionally assume that $A(\tau) \geq 0$ for $\tau \geq 0$. Let $M_0(\tau) = -\tau^N A(\tau)$, and $M_N(\tau) = 1$; $\deg M_0 = a + N$ and $\deg M_N = 0$. Then

$$\begin{aligned} \mathcal{M}(\tau, \tau\alpha(\tau)) &= M_0(\tau) + M_N(\tau)(\tau\alpha(\tau))^N \\ &= -\tau^N A(\tau) \cdot 1 + 1 \cdot (\tau\alpha(\tau))^N = 0, \quad \forall \tau \geq 0. \end{aligned}$$

Assumption 2 is satisfied if $n \leq 2$. If $n \geq 3$, we additionally need $0 < \kappa = a/N \leq 2/(n - 2)$ and (3.5) to be satisfied; there are nontrivial examples only when $a = 1$, $N = 2$.

Example 3 Consider $\alpha(\tau) = A(\tau)/B(\tau)$, with A, B polynomials of degrees $a = \deg A \geq 0$ and $b = \deg B \geq 1$; $B(\tau) \neq 0$ for $\tau \geq 0$. Let $M_0(\tau) = -\tau A(\tau)$, $M_1(\tau) = B(\tau)$. Then

$$\mathcal{M}(\tau, \tau\alpha(\tau)) = -\tau A(\tau) \cdot 1 + B(\tau) \cdot \tau\alpha(\tau) = 0, \quad \forall \tau \geq 0.$$

For Assumption 2 to be satisfied, we need $\deg M_0 = a + 1 > \deg M_1 + 1 = b + 1$, so for $n \leq 2$ one only needs $a > b$. If $n \geq 3$, one additionally needs $0 < \kappa = a - b \leq 2/(n - 2)$ and (3.5); since $a > b \geq 1$, there are nontrivial examples only when $n = 3$, $a = 2$, $b = 1$.

Now we can formulate and prove our main result: under rather generic assumptions the only type of solutions with compact time spectrum is the one-frequency solitary waves.

Theorem 6 *Let $n \in \mathbb{N}$, $m > 0$. Let $f(u) = \alpha(|u|^2)u$ be such that both Assumption 1 and Assumption 2 are satisfied. Assume that $u \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n))$ is a solution to the nonlinear Schrödinger or Klein–Gordon equation (1.1). If there is a finite interval $I \subset \mathbb{R}$ such that $\text{supp } \tilde{u} \subset \mathbb{R}^n \times I$, with $\tilde{u}(x, \omega)$ the Fourier transform of u with respect to time, then*

$$u(x, t) = \phi_0(x)e^{-i\omega_0 t}, \quad \text{with some } \phi_0 \in H^1(\mathbb{R}^n, \mathbb{C}) \text{ and } \omega_0 \in \mathbb{R}.$$

Note that, in particular, the above theorem applies to finite energy solutions to the nonlinear Klein–Gordon equation from Theorem 5.

Proof The proof for the nonlinear Schrödinger equation and the nonlinear Klein–Gordon equation is the same; for definiteness, we consider the latter case. Assume that $u \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n, \mathbb{C}))$ is a solution to (3.2) with compact time spectrum, so that the Fourier transform of u in time,

$$\tilde{u}(x, \omega) = \int_{\mathbb{R}} u(x, t)e^{i\omega t} dt, \quad \tilde{u} \in \mathcal{E}'(\mathbb{R}, H^1(\mathbb{R}^n, \mathbb{C})),$$

satisfies $\text{supp } \tilde{u} \subset \mathbb{R}^n \times [a, b]$, with some $a, b \in \mathbb{R}, a < b$. We denote

$$\begin{aligned} \Sigma &:= \Sigma[\tilde{u}] = \{x \in \mathbb{R}^n; (\{x\} \times \mathbb{R}) \cap \text{supp } u \neq \emptyset\} \\ &= \{x \in \mathbb{R}^n; (\{x\} \times \mathbb{R}) \cap \text{supp } \tilde{u} \neq \emptyset\} \end{aligned} \quad (3.6)$$

to be the projection of the support of u onto \mathbb{R}^n . Then, since $\text{supp } \tilde{u} \subset \mathbb{R}^n \times [a, b]$,

$$B_{\tilde{u}}|_{\Sigma} \geq a \quad \Rightarrow \quad B_{\tilde{u}}^L|_{\Sigma \setminus \emptyset \Sigma} \geq a; \quad A_{\tilde{u}}|_{\Sigma} \leq b \quad \Rightarrow \quad A_{\tilde{u}}^U|_{\Sigma \setminus \emptyset \Sigma} \leq b.$$

Lemma 12 $\alpha(|u(x, t)|^2)$ and $|u(x, t)|$ do not depend on time, and moreover

$$B_{\tilde{u}}^L = A_{\tilde{u}}, \quad B_{\tilde{u}} = A_{\tilde{u}}^U, \quad \forall x \in \Sigma.$$

Proof The Sobolev embedding leads to

$$u \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n, \mathbb{C})) \subset L^\infty(\mathbb{R}, L_{\text{loc}}^q(\mathbb{R}^n, \mathbb{C})), \quad \begin{cases} 1 \leq q < \infty, & n \leq 2; \\ 1 \leq q \leq 2n/(n-2), & n \geq 3. \end{cases} \quad (3.7)$$

The inclusion (3.7) together with (3.3) lead to

$$v \in L^\infty\left(\mathbb{R}, L_{\text{loc}}^{q/(2\kappa)}(\mathbb{R}^n, \mathbb{R})\right), \quad v(x, t) := \alpha(|u(x, t)|^2). \quad (3.8)$$

By (3.2) and (3.8),

$$\left(\partial_t^2 - \Delta + m^2\right)u = -\alpha(|u|^2)u \in L^\infty\left(\mathbb{R}, L_{\text{loc}}^{q/(2\kappa+1)}(\mathbb{R}^n, \mathbb{C})\right). \quad (3.9)$$

Applying the Fourier transform to (3.9) and denoting by $\tilde{v}(x, \omega)$ the Fourier transform of $v(x, t) := \alpha(|u(x, t)|^2)$ in time, one has

$$(m^2 - \omega^2 - \Delta)\tilde{u} = -\tilde{v} *_\omega \tilde{u}. \quad (3.10)$$

- Let us consider the case when $\alpha(\tau)$ is a polynomial of degree $\kappa = \deg \alpha \geq 1$, with either $n \leq 2, \kappa \in \mathbb{N}$; or $n = 3, \kappa = 1, 2$; or $n = 4, \kappa = 1$. Applying Theorem 2 to the right-hand side of the above relation, we arrive at

$$B_{(m^2 - \omega^2 - \Delta)\tilde{u}}(x) = B_{\tilde{v} \ast_{\omega} \tilde{u}}(x) \geq B_{\tilde{v}}^L(x) + B_{\tilde{u}}(x), \quad \forall x \in \Sigma.$$

Due to the inclusion $\text{supp } \Delta \tilde{u} \subset \text{supp } \tilde{u}$, the above yields $B_{\tilde{u}} \geq B_{\tilde{v}}^L + B_{\tilde{u}}$ for all $x \in \Sigma$, hence $B_{\tilde{v}}^L \leq 0$ and therefore

$$B_{\tilde{v}}(x) \leq 0, \quad \forall x \in \Sigma. \tag{3.11}$$

Similarly, $A_{\tilde{v}} \geq 0$, for all $x \in \Sigma$; thus, $\text{supp } \tilde{v} \subset \mathbb{R}^n \times \{0\}$, and we conclude that

$$v(x, t) = \sum_{j \in \mathbb{N}_0} t^j v_j(x). \tag{3.12}$$

Above, in agreement with the general theory of distributions (Hörmander 1983), the summation in $j \in \mathbb{N}_0$ is locally finite (there are finitely many terms for $x \in K$ for each compact subset $K \subset \mathbb{R}^n$) (cf. Hörmander 1983, Theorem 2.3.5). The terms with derivatives of $\delta(\omega)$ do not appear since this would lead to $v(x, t)$ growing in time, contradicting (3.8). This implies that in (3.12) the only nonzero term is the one with $j = 0$. Thus, $V(x) := v(x, t) = \alpha(|u(x, t)|^2)$ does not depend on time. Since $\alpha(\tau)$ is a nonconstant algebraic function, $|u(x, t)|^2$ also does not depend on time:

$$\text{supp } \widetilde{|u|^2} \subset \mathbb{R}^n \times \{0\}. \tag{3.13}$$

Using the above relation and applying Theorem 2 to $\widetilde{|u|^2} = \tilde{u}^\# \ast_{\omega} \tilde{u}$, where $\tilde{u}^\# = \widetilde{\tilde{u}}$ (see (2.10)), we conclude that

$$0 = B_{\widetilde{|u|^2}}(x) \geq B_{\tilde{u}}^L(x) + B_{\tilde{u}^\#}(x) = B_{\tilde{u}}^L(x) - A_{\tilde{u}}(x), \quad \forall x \in \Sigma.$$

Thus, $B_{\tilde{u}}^L \leq A_{\tilde{u}}$ for all $x \in \Sigma$. On the other hand, by Lemma 3, $B_{\tilde{u}}^L \geq A_{\tilde{u}}$ for all $x \in \Sigma$. We conclude that

$$B_{\tilde{u}}^L = A_{\tilde{u}} \quad \text{and similarly} \quad B_{\tilde{u}} = A_{\tilde{u}}^U, \quad \forall x \in \Sigma. \tag{3.14}$$

- Let us consider the case when $\alpha(\tau)$ is an algebraic function satisfying Assumption 2. Multiplying (3.9) by \tilde{u} , we have:

$$\tilde{u} \left(m^2 + \partial_t^2 - \Delta \right) u = -|u|^2 \alpha(|u|^2) \in L^\infty \left(\mathbb{R}, L_{\text{loc}}^{q/(2\kappa+2)}(\mathbb{R}^n, \mathbb{C}) \right), \tag{3.15}$$

with $1 \leq q < \infty$ if $n \leq 2$ and $1 \leq q \leq 2n/(n - 2)$ if $n \geq 3$. Let \mathcal{M} be as in (3.4). Applying $\mathcal{M}(|u|^2, \cdot)$ to both sides of the relation (3.15) leads to

$$\begin{aligned}
 0 &= \mathcal{M}\left(|u|^2, |u|^2\alpha(|u|^2)\right) = \mathcal{M}\left(|u|^2, -\bar{u}(m^2 - \omega^2 - \Delta)u\right) \\
 &= \sum_{j=0}^J M_j(|u|^2)\left(-\bar{u}(m^2 + \partial_t^2 - \Delta)u\right)^j.
 \end{aligned}
 \tag{3.16}$$

We need to make sure that the right-hand side is a well-defined distribution. Taking into account (3.7) and (3.9), we conclude that all the terms in the right-hand side are in $\mathcal{E}'(\mathbb{R}, L^1_{loc}(\mathbb{R}^n))$ as long as in (3.7) one can take $q \geq 1$ such that

$$\frac{2 \deg M_j}{q} + \frac{2\kappa + 2}{q}j \leq 1, \quad \forall j, \quad 0 \leq j \leq J.$$

For $n \leq 2$, we can satisfy the above by taking $1 \leq q < \infty$ arbitrarily large; for $n \geq 3$, the above is satisfied with $q = 2n/(n - 2)$ due to the inequality (3.5) in Assumption 2.

We note that $\tilde{u}^\# \circledast \tilde{u} = \widetilde{|u|^2}$ and that $\text{supp}(\tilde{u}^\# \circledast (m^2 - \omega^2 - \Delta)\tilde{u}) \subset \text{supp}(\tilde{u}^\# \circledast \tilde{u})$, hence

$$B_{\tilde{u}^\# \circledast (m^2 - \omega^2 - \Delta)\tilde{u}} \leq B_{\tilde{u}^\# \circledast \tilde{u}}, \quad \forall x \in \Sigma.
 \tag{3.17}$$

Now we apply Theorem 2 to the Fourier transform (in time) of the relation (3.16) and use Assumption 2, arriving at

$$B_{\tilde{u}^\# \circledast (m^2 - \omega^2 - \Delta)\tilde{u}}^L \leq 0;$$

then $B_{\tilde{u}^\# \circledast (m^2 - \omega^2 - \Delta)\tilde{u}} \leq 0$, and similarly $A_{\tilde{u}^\# \circledast (m^2 - \omega^2 - \Delta)\tilde{u}} \geq 0$. It follows that

$$\text{supp} \tilde{u}^\# \circledast (m^2 - \omega^2 - \Delta)\tilde{u} \subset \mathbb{R}^n \times \{0\},$$

hence, by the argument after (3.12), $|u|^2\alpha(|u|^2)$ is time-independent, and so is $|u|^2$ (we note that $\tau\alpha(\tau)$ is a nonconstant function of τ : indeed, if we had $0 = \mathcal{M}(\tau, \tau\alpha(\tau)) = -C + \tau\alpha(\tau)$, then $M_0(\tau) = C$ and $M_1(\tau) = 1$, not satisfying Assumption 2). Therefore, we again arrive at (3.13) and then (3.14) follows. \square

By Lemma 12,

$$V(x) := v(x, t) = \alpha(|u(x, t)|^2) \quad \text{does not depend on time;} \quad \tilde{v}(x, \omega) = 2\pi \delta(\omega)V(x).
 \tag{3.18}$$

Due to (3.18), Eq. (3.2) takes the form

$$\Delta \tilde{u} = m^2 \tilde{u} - \omega^2 \tilde{u} + V(x)\tilde{u}.
 \tag{3.19}$$

By (3.7), $|u|^2 \in L^\infty(\mathbb{R}, L^{q/2}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}))$, $\widetilde{|u|^2} \in \mathcal{E}'(\mathbb{R}, L^{q/2}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}))$ (we took into account the assumption that the spectrum of u is compact), with any $q \geq 1$ if $n \leq 2$ and $1 \leq q \leq 2n/(n - 2)$ if $n \geq 3$. Then, according to the assumption (3.3),

$$V(x) = \alpha(|u(x, t)|^2) \text{ satisfies } V \in L^{q/(2\kappa)}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}), \tag{3.20}$$

with any $q \geq 2\kappa$ for $n \leq 2$ and $q = 2n/(n - 2)$ for $n \geq 3$. Due to the requirement (3.3) on κ , the function $V(x)$ satisfies conditions needed for the unique continuation property (see Theorem 4 and Remark 6).

Let us show that $\Sigma[\tilde{u}]$ defined in (3.6) has to be the whole space.

Lemma 13 *If u is not identically zero, then $\Sigma[\tilde{u}] = \mathbb{R}^n$.*

Proof Assume that, on the contrary, $\Sigma[\tilde{u}] \subsetneq \mathbb{R}^n$; since $\Sigma[\tilde{u}]$ is closed, there is a nonempty connected open subset $\mathcal{O} \subset \mathbb{R}^n$ such that $\mathcal{O} \cap \Sigma[\tilde{u}] = \emptyset$. Let $\Omega \subset \mathbb{R}$ be an open interval; since $\mathcal{O} \cap \Sigma[\tilde{u}] = \emptyset$, one has $(\mathcal{O} \times \Omega) \cap \text{supp } \tilde{u} = \emptyset$. Since V satisfies the assumptions of Theorem 4, we apply the unique continuation property to an L^2 -function \tilde{u} (valued in $\mathcal{D}'(\Omega)$) which solves (3.19), concluding that

$$(\mathbb{R}^n \times \Omega) \cap \text{supp } \tilde{u} = \emptyset.$$

When applying the unique continuation property to (3.19), we need to mention that the multiplication by ω^2 is a continuous automorphism in $\mathcal{D}(\Omega) = C^\infty_{\text{comp}}(\Omega)$ (in the Fréchet topology based on sup-norms in $C^k_{\text{comp}}(\Omega)$, $k \geq 0$), and hence also in $\mathcal{D}'(\Omega)$. □

Lemma 14 *There is $\omega_0 \in \mathbb{R}$ such that $\text{supp } \tilde{u} \subset \mathbb{R}^n \times \{\omega_0\}$.*

Proof Pick $x_1 \in \Sigma[\tilde{u}] = \mathbb{R}^n$. Denote $\omega_1 = B_{\tilde{u}}(x_1)$. We will show that for any open neighborhood $\Omega \subset \mathbb{R}$, $\inf \Omega > \omega_1$, one has $(\mathbb{R}^n \times \Omega) \cap \text{supp } \tilde{u} = \emptyset$.

Since $B_{\tilde{u}}$ is upper semicontinuous, for any $\varepsilon > 0$, which we choose to be $\varepsilon := \text{dist}(\omega_1, \Omega)/2 > 0$, there is an open neighborhood $\mathcal{O} \subset \mathbb{R}^n$, $\mathcal{O} \ni x_1$, such that $B_{\tilde{u}|_{\mathcal{O}}} < \omega_1 + \varepsilon$. Let $\varphi \in C^\infty_{\text{comp}}(\mathbb{R}, \mathbb{R})$, $\text{supp } \varphi \subset \Omega$. Using the unique continuation property exactly as in Lemma 13, we conclude that $(\mathcal{O} \times \Omega) \cap \text{supp } \tilde{u} = \emptyset$ implies that $(\mathbb{R}^n \times \Omega) \cap \text{supp } \tilde{u} = \emptyset$. Since the choice of $x_1 \in \mathbb{R}^n$ was arbitrary, we conclude that

$$\text{supp } \tilde{u} \subset \mathbb{R}^n \times (-\infty, \inf B_{\tilde{u}}].$$

Similarly one proves that

$$\text{supp } \tilde{u} \subset \mathbb{R}^n \times [\sup A_{\tilde{u}}, +\infty).$$

By Lemma 12, $B_{\tilde{u}}^L = A_{\tilde{u}}$; it follows that $\inf B_{\tilde{u}} = \sup A_{\tilde{u}} =: \omega_0$, and therefore $\text{supp } \tilde{u} \subset \mathbb{R}^n \times \{\omega_0\}$. □

By Lemma 14,

$$u(x, t) = e^{-i\omega_0 t} \sum_{j \in \mathbb{N}_0} \phi_j(x) t^j. \quad (3.21)$$

By the above arguments, the summation in (3.21) is locally finite for $x \in K$, for each compact subset $K \subset \mathbb{R}^n$. Since $u \in L^\infty(\mathbb{R}, H^1(\mathbb{R}^n, \mathbb{C}))$, we conclude that in (3.21) the terms with $j \geq 1$ are absent; thus, $u(x, t) = \phi_0(x)e^{-i\omega_0 t}$, with $\phi_0 \in H^1(\mathbb{R}^n, \mathbb{C})$. This concludes the proof of Theorem 6. \square

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