



Invariant Spanning Trees for Quadratic Rational Maps

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Abstract

We study Thurston equivalence classes of quadratic post-critically finite branched coverings. For these maps, we introduce and study invariant spanning trees. We give a computational procedure for searching for invariant spanning trees. This procedure uses bisets over the fundamental group of a punctured sphere. We also introduce a new combinatorial invariant of Thurston classes—the ivy graph.

Keywords Complex dynamics · Invariant tree · Iterated monodromy group

Mathematics Subject Classification Primary 37F20; Secondary 37F10

1 Introduction

Rational maps acting on the Riemann sphere are among central objects in complex dynamics. *Thurston's characterization theorem* allows to study these algebraic objects by topological tools. It views rational maps within a much wider class of topological branched coverings. (Branched self-coverings of the sphere whose critical points have finite orbits are called *Thurston maps*.) There is a natural equivalence relation on Thurston maps such that different rational functions are almost never equivalent. (All exceptions are known and well-understood.) Thurston's theorem provides a topological criterion for a Thurston map being equivalent to a rational function. Thus, classification of Thurston maps up to equivalence is an important problem. This funda-

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mental problem has applications beyond complex dynamics, e.g., in group theory; it is a focus of recent developments, see e.g. (Bartholdi and Nekrashevych 2006; Bartholdi and Dudko 2017; Cordwell et al. 2015; Kelsey and Lodge 2018; Hlushchanka 2017). We approach the problem via analogs of Hubbard trees for quadratic rational maps: *invariant spanning trees*.

We will write \mathbb{S}^2 for the oriented topological 2-sphere. By a *graph* in the sphere, we here mean a 1-dimensional cell complex embedded into \mathbb{S}^2 . By *vertices* and *edges*, we mean 0-cells and 1-cells, respectively. For a graph G , we will write $V(G)$ for the set of vertices of G and $E(G)$ for the set of edges of G . A *tree* is a simply connected graph. A vertex x of a tree T is called a *branch point* if $T - \{x\}$ has more than 2 components. Suppose that $P \subset \mathbb{S}^2$ is some finite subset. A tree T in \mathbb{S}^2 such that $P \subset V(T)$ is called a *spanning tree* for P if $V(T) - P$ consists of branch points.

Let $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be an orientation preserving branched covering of degree 2. The map f has two critical points $c_1(f)$ and $c_2(f)$. Let $v_1(f)$ and $v_2(f)$ be the corresponding critical values. The *post-critical set* of f is defined as the smallest closed f -stable set including $\{v_1(f), v_2(f)\}$. The post-critical set of f will be denoted by $P(f)$. If $P(f)$ is finite, then f is said to be *post-critically finite*. Recall that a *Thurston map* is a post-critically finite orientation preserving branched covering. In this paper, we will only consider degree two Thurston maps.

Definition 1.1 (*Invariant spanning tree*) Let $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be a Thurston map. A spanning tree T for $P(f)$ is called an *invariant spanning tree* for f if:

1. we have $f(T) \subset T$;
2. vertices of T map to vertices of T .

This notion is close to what is called “invariant trees” in Hlushchanka (2017). Note that the restriction of f to an edge of T is injective unless the edge contains a critical point of f . Consideration of invariant spanning trees is justified by the following examples (we describe those of them, which are quadratic maps, in more detail later, see Sect. 2):

1. *Hubbard trees* (Douady and Hubbard 1984, 1985; Bielefeld et al. 1992; Poirier 1993) can be connected to infinity to form invariant spanning trees.
2. Invariant spanning trees can be constructed for *formal matings* by joining the two Hubbard trees. (However, the tree structure sometimes does not survive in the corresponding topological matings).
3. Classical *captures* in the sense of Wittner (1988), Rees (1992) often come with invariant spanning trees. In fact, the original approach of Wittner used invariant trees.
4. Sufficiently high iterates of expanding Thurston maps possess invariant spanning trees by Hlushchanka (2017). This result has also been extended in Hlushchanka (2017) to rational maps with Sierpinski carpet Julia set.
5. Extended Newton graphs constructed in \mathbb{C} for post-critically finite Newton maps are often invariant trees. Invariant spanning trees can be obtained from them by erasing some of the vertices.
6. With each *critically fixed* rational map f , a certain bipartite graph is associated in Cordwell et al. (2015). Every edge of this graph is invariant, thus every spanning

tree of this graph is an invariant tree for f . Again, an invariant spanning tree can be obtained by erasing some of the vertices.

7. Some invariant spiders in the sense of Hubbard and Schleicher (1994) are invariant spanning trees (possibly after removal of the critical leg in case of a strictly preperiodic critical point). Note that invariant spiders may fail to be trees.

Section 2 deals with examples of invariant spanning trees. However, we restrict our attention to the case of degree 2 Thurston maps.

Let x be a vertex of a tree $T \subset \mathbb{S}^2$, and e be an edge of T . If x is in the closure of e , then we say that x is *incident* to e . We also say that e is incident to x . The following result shows how to recover the Thurston equivalence class of f from an invariant spanning tree of f . Recall that a *ribbon graph* (also known as a fat graph, or a cyclic graph) is an abstract graph in which the edges incident to each particular vertex are cyclically ordered. By MacLane (1941), ribbon trees are the same as isomorphism classes of embedded trees in \mathbb{S}^2 . Here an isomorphism of embedded trees is an orientation preserving self-homeomorphism of \mathbb{S}^2 that takes one tree to another. Recall that the orientation of \mathbb{S}^2 is assumed to be fixed. For a spanning tree T for $P(f)$, we write $C(T)$ for the set of critical points of f in T .

Theorem A *Suppose that $f, g : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ are two Thurston maps of degree 2. Let T_f and T_g be invariant spanning trees for f and g , respectively. Suppose that there is a cellular homeomorphism $\tau : T_f \rightarrow T_g$ with the following properties:*

1. *The map τ is an isomorphism of ribbon graphs.*
2. *We have $\tau \circ f = g \circ \tau$ on $V(T_f) \cup C(T_f)$.*
3. *The critical values of f are mapped to critical values of g by τ .*

Then f and g are Thurston equivalent.

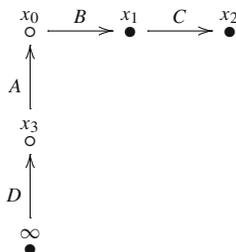
In other words, to know the Thurston equivalence class of f , it suffices to know the following data:

1. the ribbon graph structure of T_f ;
2. the restriction of the map f to the set $V(T_f) \cup C(T_f)$.

These data are discrete and can be encoded symbolically.

Note that higher degree analogs of Theorem A will require an additional structure on an invariant spanning tree (something like a structure of an angled tree), as is shown by the following example.

Example 1.2 This is a slightly modified example of Milnor (2009, Figure 21). Consider the following invariant tree T of a cubic polynomial f .



Here we have $x_0 \mapsto x_1 \mapsto x_2 \mapsto x_3 \hookrightarrow$. The points x_0 and x_1 are simple critical points. The corresponding critical values (marked as solid) are x_1 and x_2 . The edges, oriented as shown in the picture, are mapped as follows:

$$A \mapsto A \cup B, \quad B \mapsto C \mapsto C^{-1} \cup B^{-1} \cup A^{-1}, \quad D \mapsto D.$$

The inverses here indicate the change of orientation. The part of the tree T that is obtained by removing the edge D and the vertex ∞ is the Hubbard tree for f . This Hubbard tree is shown as an *angled tree*. For example, the angle between B and A^{-1} (computed counterclockwise) is $\frac{3}{4}$. We measure angles so that the full turn corresponds to angle 1. In order to recover the Thurston class of f by its Hubbard tree, we need to know the angles. For example, if we change the angle between B and A^{-1} to $\frac{1}{4}$, then we obtain a different polynomial and a different Thurston class. However, there is still a correspondence τ between the two trees that satisfies the assumptions of Theorem A.

An important algebraic invariant of a Thurston map is its *biset* over the fundamental group of $\mathbb{S}^2 - P(f)$. A biset is a convenient algebraic structure that carries a complete information about the Thurston class of f . The (perhaps better known) *iterated monodromy group* of f can be immediately recovered from the biset. A formal definition of a biset will be given in Sect. 5. For now, we just emphasize that bisets admit compact symbolic descriptions somewhat similar to presentations of groups by generators and relations or presentation of linear maps by matrices.

Theorem B *Suppose that f is a Thurston map of degree 2, and T is an invariant spanning tree for f . There is an explicit presentation of the biset of f based only on the following data*

1. *the ribbon graph structure on T ,*
2. *the restriction of f to $V(T) \cup C(T)$.*

In Sect. 5.5, we make the statement of Theorem B more precise. We provide an automaton representing the biset of f in Theorem 5.6. The construction is algorithmic. Theorem B solves, in a particular case, the problem of combinatorial encoding of Thurston maps by means of invariant graphs. Different contexts of this problem are addressed in Cannon et al. (2001), Bonk and Meyer (2017), Lodge et al. (2015) for specific families of rational maps. A relationship between the properties of invariant trees and the properties of the iterated monodromy group has been studied in Hlushchanka (2017).

1.1 Dynamical Tree Pairs

It is not always easy to find an invariant spanning tree for a Thurston map f . However, for any spanning tree T for $P(f)$, it is easy to find another spanning tree T^* that maps onto T . The tree T^* is not uniquely defined; there are several ways of choosing suitable subtrees in the graph $f^{-1}(T)$. Suppose that we want to find an invariant spanning tree for f . It is natural to look at an iterative process, a single step of which is the transition from T to T^* . Such an iterative process will be described below under the name of *ivy iteration*.

We now proceed with a more formal exposition. Let $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be a Thurston map of degree two. Consider two spanning trees T^* and T for $P(f)$ such that $f(T^*) \subset T$. Moreover, we assume that

1. the vertices of T^* are mapped to vertices of T under f ;
2. all critical values of f are vertices of T ;

If these assumptions are fulfilled, then (T^*, T) is called a *dynamical tree pair* for f . Clearly, an invariant spanning tree T for f gives rise to a dynamical tree pair (T, T) . Thus, dynamical tree pairs generalize invariant spanning trees. Observe also that the restriction of f to every edge of T^* is injective unless the edge contains a critical point of f .

A spanning tree T for $P(f)$ gives rise to a distinguished generating set \mathcal{E}_T of the fundamental group $\pi_1(\mathbb{S}^2 - P(f), y)$ with $y \in \mathbb{S}^2 - T$. Namely, \mathcal{E}_T consists of the identity element and the homotopy classes of smooth loops based at y intersecting T only once and transversely (we will make this more precise later).

In Sect. 5, we state a theorem (Theorem 5.6) generalizing Theorem B. It follows from Theorem 5.6 that the biset of f is determined by a dynamical tree pair (T^*, T) . More precisely, the biset can be explicitly presented knowing the following discrete data:

1. the ribbon graph structures on T^*, T ;
2. the map $f : V(T^*) \cup C(T^*) \rightarrow V(T)$;
3. how elements of \mathcal{E}_{T^*} are expressed through elements of \mathcal{E}_T (or how both $\mathcal{E}_{T^*}, \mathcal{E}_T$ are expressed through some other generating set of $\pi_1(\mathbb{S}^2 - P(f), y)$).

Vice versa, given T and a presentation of the biset of f in the basis associated with T , these data can be recovered.

1.2 The Ivy Iteration

The principal objective of this paper is to introduce a computational procedure for finding invariant (or, more generally, periodic) spanning trees of degree 2 Thurston maps.

Let $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be a degree 2 Thurston map. Ivy iteration operates on isotopy classes (rel. $P(f)$) of spanning trees for $P(f)$, which we call *ivy objects*. Let $\text{Ivy}(f)$ denote the set of all ivy objects for f . Let T be a spanning tree, and $[T]$ be the corresponding ivy object. A symbolic presentation of the biset of f plus a symbolic encoding of the ribbon tree structure on T give rise to several choices of a spanning tree T^* such that (T^*, T) is a dynamical tree pair for f . Roughly speaking, several choices for T^* are related with different ways of choosing a spanning subtree in $f^{-1}(T)$. Consider the *pullback relation* $[T] \dashrightarrow [T^*]$ on $\text{Ivy}(f)$.¹ It equips $\text{Ivy}(f)$ with a structure of an abstract directed graph. A subset $C \subset \text{Ivy}(f)$ is said to be *pullback invariant* if $[T] \in C$ and $[T] \dashrightarrow [T^*]$ imply $[T^*] \in C$.

With the help of a computer, we found finite pullback invariant subsets in $\text{Ivy}(f)$ for several simplest quadratic Thurston maps f . Within these pullback invariant subsets,

¹ On a somewhat similar note, the pullback relation on isotopy classes of simple closed curves in $\mathbb{S}^2 - P(f)$ is discussed in Pilgrim (2003), and Koch et al. (2016).

we found all invariant ivy objects. Invariant ivy objects, obviously, correspond to invariant (up to homotopy) spanning trees for f . Having a picture for a pullback invariant subset, we can also see many periodic ivy objects of various periods. Some of these examples will be described in Sect. 2. Note that, if we found a spanning tree for $P(f)$ that is f -invariant up to homotopy, then this tree is a genuine invariant spanning tree for some map homotopic to f . This is good enough since we are interested in classification of Thurston maps up to Thurston equivalence (in particular, homotopic maps are in the same class).

How the Ivy Iteration Compares with Known Combinatorial Algorithms

The ivy iteration can be considered in the following general context. The biset associated with f is a way of compactly representing the “combinatorics” of f . Unfortunately, the same biset may have very different presentations in different bases. (A linear algebra analog is that the same linear map has different matrices in different bases.) Usually, a combinatorial description of f yields a presentation of its biset. However, given different combinatorial descriptions of a Thurston map, the problem is whether they describe the same thing. This problem translates into comparison of bisets: do different presentations correspond to the same biset?

Up to date, there are several algorithmic approaches to the comparison of bisets. We restrict our attention to bisets associated with rational, i.e., unobstructed, Thurston maps. A natural idea is to look for the “best” presentation of a biset. (This idea is somewhat similar to finding a Jordan normal form—or some other normal form—of a linear map.) This general idea works well for polynomials. In fact, the *combinatorial spider algorithm* of Nekrashevych (2009) aims at solving the problem. Given a presentation of a biset by a *twisted kneading automaton*, the algorithm searches for the best presentation, which is associated with a *kneading automaton*. The algorithm of Nekrashevych (2009) is a combinatorial implementation of the *spider algorithm* originally developed by Hubbard and Schleicher (1994) for quadratic polynomials. A version of the combinatorial spider algorithm was used in Bartholdi and Nekrashevych (2006) to identify twisted rabbits with the rabbit, co-rabbit, or the airplane polynomials, as well as to classify the twists of $z^2 + i$. In fact, the authors consider not only bisets over the fundamental group but also bisets over the pure mapping class group, which turns out to be useful for distinguishing twisted rabbits. In Bartholdi and Nekrashevych (2006), Bartholdi and Nekrashevych develop another approach to the same problem based on inspecting the correspondence on the moduli space associated with the Thurston pullback map. This second approach relies on the fact that, in examples under consideration, there are few (namely, four) points in $P(f)$ (if $P(f)$ consists of only four points, then the moduli space has complex dimension one).

The second approach of Bartholdi and Nekrashevych (2006) has been further developed in Kelsey and Lodge (2018), where all non-Euclidean Thurston maps with 4 or fewer post-critical points are classified. The authors also provide an algorithm for identifying the twists of all such maps. An extension of these results to Thurston maps with bigger post-critical sets is currently unavailable, not only because there are too many objects to classify but also because the technique is not easy to adapt. Thus, to the best of our knowledge, purely combinatorial tools available up to date for com-

paring bisets are restricted either to specific types of Thurston maps (say, topological polynomials, expanding maps, etc.) or to maps with few post-critical points.

The process of finding invariant trees described in Hlushchanka (2017) is in principle algorithmic. However, it applies under additional assumptions on the dynamics of the map (sphere or Sierpinski carpet Julia set) and produces an invariant tree only for a sufficiently high iterate of the map.

On the other hand, there are also “floating-point” algorithms, see e.g. (Bartholdi and Dudko 2017, Section V.2). Most of these algorithms aim at turning the Thurston iteration into an efficient computation. For example, given two bisets without obstructions, one can compare them as follows (see Corollary V.9 of Bartholdi and Dudko 2017). For each of the two bisets, compute the coefficients of the corresponding rational map using a version of Thurston’s algorithm. Then the two rational maps can be compared by comparing the corresponding coefficients. This approach works as an efficient computation but fails to provide good combinatorial tags to rational maps.

The ivy iteration may be regarded as an attempt to generalize the spider algorithm. In fact, for quadratic polynomials, the spider algorithm (applied to a not necessarily invariant spider of an actual polynomial) is the same as the ivy iteration, except that, the issue of arbitrary choices is resolved by specifying a kneading sequence. A spider is a tree of a very specific shape; it is a star. The ivy iteration may be in principle applied to arbitrary spanning trees, and to non-polynomial Thurston maps. In this sense it is more general. However, it is not as good as the spider algorithm because it is not an algorithm at all. An important ingredient (an analog of a kneading sequence that would allow to make specific choices) is missing so far. On the positive side, the ivy iteration can be implemented as a purely combinatorial procedure. At any step of the iteration, we obtain a presentation of the biset associated with a Thurston map. If the iteration converges, then we obtain a good presentation, and the hope is that there are only few good ones. Moreover, the result is a nice visual tag associated with a rational map. To the best of our knowledge, the ivy iteration does not coincide with other known computational procedures, although it is conceptually unsophisticated and is based on the same general idea: that of taking pullbacks.

Terminological Conventions

In this paper, we talk about graphs in the sphere as well as abstract graphs. The former notion belongs to topology, and the latter—to combinatorics. We try to clearly distinguish these notions. A graph (without a specification) usually means a graph in the sphere. When referring to an abstract graph, we always say “abstract”. We sometimes consider oriented edges of graphs in the sphere. These are edges, for which some orientation (= direction) is specified. On the other hand, we talk of directed edges in abstract directed graphs. In this sense, a directed edge is a fundamental notion, which can be defined as an ordered pair of vertices. It is not “an edge equipped with a direction”. Thus our terminological discrepancy between “oriented edges” and “directed edges” is intentional.

2 Examples of Invariant Spanning Trees

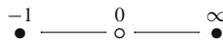
In this section, we describe some examples of invariant spanning trees. We confine ourselves with degree two rational maps.

2.1 Quadratic Polynomials

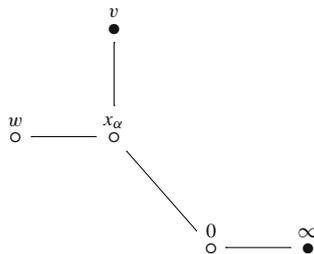
Let $p(z) = z^2 + c$ be a post-critically finite quadratic polynomial. We will write $J(p)$ for the Julia set of p and $K(p)$ for the filled Julia set of p . Set X to be the *forward p -orbit* of c , i.e., the set $\{p^{on}(c) \mid n \geq 0\}$. The landing point of the dynamical external ray $R_p(0)$ of p with argument 0 is denoted by x_β . This point is usually called the β -fixed point. We will use the terminology of Milnor (2009), Poirier (1993, 2010), in particular, the notion of a regulated hull. Define T as the union of $\{\infty\} \cup R_p(0)$ and the regulated hull of $X \cup \{x_\beta\}$. Then T is an invariant spanning tree for p .

Recall that the regulated hull of X is called the *Hubbard tree* of p . Thus T is strictly bigger than the Hubbard tree of p . It is important that the tree T contains both critical values of p . In our terminology, the Hubbard tree itself is not a spanning tree for p . To specify a graph structure on T , we need to define vertices. By definition, the vertices of T are post-critical points and branch points of T . On the other hand, x_β is never a vertex of T .

As an example, an invariant spanning tree T for the basilica polynomial $p(z) = z^2 - 1$ looks as follows:



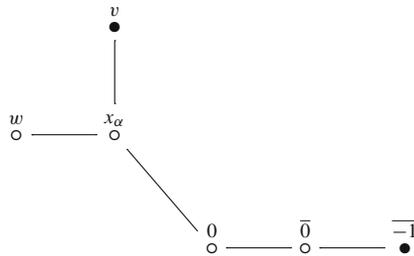
Critical values are shown as solid, and other vertices of T as circles. For the rabbit polynomial $z^2 + c$, where $c \approx -0.122561 + 0.744862i$, the tree T looks as follows:



Here x_α is the α -fixed point, i.e., the landing point of external rays with arguments $\frac{1}{7}, \frac{2}{7}$ and $\frac{4}{7}$. The edge 0∞ contains the β -fixed point x_β and the points $-x_\alpha, -w$. The latter three points are not in $V(T)$ since they are neither post-critical nor branch points.

2.2 Matings

Let p and q be two post-critically finite quadratic polynomials. Consider a compactification \mathbb{C} of \mathbb{C} obtained by adding a circle at infinity. More precisely, the circle at infinity is parameterized by the arguments of external rays. For an angle $\theta \in \mathbb{R}/\mathbb{Z}$, we let $R_p(\theta)$ be the corresponding external ray in the dynamical plane of p . We will write $E_p(\theta)$ for the corresponding point in the circle at infinity. Let p and q act on different copies of \mathbb{C} , say, p acts on \mathbb{C}_p and q on \mathbb{C}_q . Then $E_p(\theta)$ and $E_q(\theta)$ will refer to points in \mathbb{C}_p and \mathbb{C}_q , respectively. Consider the disjoint union $Y = \mathbb{C}_p \sqcup \mathbb{C}_q$. Let \sim be an equivalence relation on Y defined as follows. We have $x \sim y$ and $x \neq y$ if and only if one of the two points, say, x , has the form $E_p(\theta)$, and the other point y has the form $E_q(-\theta)$. The quotient space $\mathbb{S}^2_{p \sqcup q} = Y / \sim$ is called the *formal mating space* of p and q . It is easy to see that $\mathbb{S}^2_{p \sqcup q}$ is homeomorphic to \mathbb{S}^2 . The map $F : Y \rightarrow Y$ defined as p on \mathbb{C}_p and q on \mathbb{C}_q descends to the quotient space. Thus we have a naturally defined map $f : \mathbb{S}^2_{p \sqcup q} \rightarrow \mathbb{S}^2_{p \sqcup q}$. We write $f = p \sqcup q$ and call f the *formal mating* of p and q . To construct an invariant spanning tree for f , it suffices to construct invariant spanning trees for p and q as above, and then take the union of the two trees. Below, the thus constructed invariant spanning tree is shown for $p \sqcup q$, where p is the rabbit polynomial, and q is the basilica polynomial.



Here, $\bar{0}$ and $\bar{-1}$ refer to the points 0 and -1 in the dynamical plane \mathbb{C}_q (more precisely, in the image of this plane in the space $\mathbb{S}^2_{p \sqcup q}$). The point $\infty = \overline{\infty}$ (more precisely, the image of $E_p(0)$ and $E_q(0)$ in $\mathbb{S}^2_{p \sqcup q}$) is not a critical value anymore. Moreover, this point is not a vertex of the invariant spanning tree shown above. It belongs to the edge connecting 0 with $\bar{0}$.

2.3 Captures

The following definition of a capture is equivalent to the one from Rees (1992). However, we phrase the definition somewhat differently. Introduce a smooth structure on \mathbb{S}^2 . We also fix a smooth spherical metric on \mathbb{S}^2 . Given a vector v_x at some point $x \in \mathbb{S}^2$ and $\varepsilon > 0$, there is a vector field $D(v_x, \varepsilon)$ such that

1. outside of the ε -neighborhood of x with respect to the spherical metric, $D(v_x, \varepsilon) = 0$;
2. at point x , the vector $D(v_x, \varepsilon)_x$ coincides with v_x .

We may consistently choose vector fields $D(v_x, \varepsilon)$ for all x, v_x and ε so that they depend continuously (or even smoothly) on all parameters. Consider a smooth path $\beta : [0, 1] \rightarrow \mathbb{S}^2$ and choose a small $\varepsilon > 0$. Define the map $\sigma_\beta : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ as the time $[0, 1]$ flow of the non-autonomous vector field $D(\dot{\beta}(t), \varepsilon)$. Here $\dot{\beta}(t)$ is the velocity vector of β at the point $\beta(t)$. The map σ_β is a self-homeomorphism of \mathbb{S}^2 with the following properties:

1. we have $\sigma_\beta(\beta(0)) = \beta(1)$;
2. the map σ_β is the identity outside of the ε -neighborhood $U_\varepsilon(\beta)$ of $\beta[0, 1]$;
3. the map σ_β is homotopic to the identity modulo $\mathbb{S}^2 - U_\varepsilon(\beta)$.

The homeomorphism σ_β depends on β, ε and on a particular choice of $D(v_x, \varepsilon)$. However, if the path β is fixed, then any two such homeomorphisms σ_β and $\tilde{\sigma}_\beta$ are homotopic relative to $\mathbb{S}^2 - U_\varepsilon(\beta)$.

We can consider a composition $\sigma_\beta \circ p$, where p is a post-critically finite quadratic polynomial, and the choice of β depends on p . Set $\beta(0) = \infty$, and place $\beta(1)$ at some strictly preperiodic point that is not postcritical. If $U_\varepsilon(\beta)$ does not contain finite post-critical points of the map p and iterated images of $\beta(1)$, then all such maps $\sigma_\beta \circ p$ with fixed β are equivalent. In other words, the Thurston equivalence class of $f = \sigma_\beta \circ p$ depends only on β and p . The post-critical set of f is the union of $P(p)$ and the forward orbit of $\beta(1)$, including $\beta(1)$. Note that $\beta(1)$ is a critical value of f , the image of the critical point ∞ . In fact, the homotopy class of f does not change if we deform β within the same homotopy class relative to $P(f)$. When talking about $\sigma_\beta \circ p$, we will always assume that the set $\beta[0, 1]$ is disjoint from $P(p)$ and from the forward orbit of $\beta(1)$. The path β is called a *capture path* for p .

Definition 2.1 The map $\sigma_\beta \circ p$ defined as above is called the (generalized) *capture* of p associated with β . The capture $\sigma_\beta \circ p$ is said to be *simple* if there is only one $t_0 \in [0, 1]$ with $\beta(t_0) \in J(p)$. In the latter case, the corresponding capture path is called a *simple capture path*.

Suppose that $\beta(1)$ is eventually mapped to a periodic critical point of p , i.e., to 0 if $p(z) = z^2 + c$. Then a simple capture path $\beta : [0, 1] \rightarrow \mathbb{S}^2$ looks as follows. There is a parameter $t_0 \in (0, 1)$ such that $\beta[0, t_0]$ is in the basin of infinity, $\beta(t_0, 1]$ is in the Fatou component eventually mapped to a super-attracting periodic basin, and $\beta(t_0)$ is a point of the Julia set. We may arrange $\beta|_{[0, t_0]}$ to go along an external ray, and $\beta|_{(t_0, 1]}$ to go along an internal ray. If $\beta(1) \in J(p)$, then $\beta[0, 1]$ can be chosen as the union of an external ray and its landing point. Different simple capture paths lead to at most two different Thurston equivalence classes of captures provided that p and $\beta(1)$ are fixed, cf. (Rees 2010, Section 2.8).

Generalized captures were first defined by Rees (1992). Simple captures go back to Wittner (1988). Both Wittner and Rees used the word “capture” to mean simple capture. We, on the contrary, use the word “capture” to mean a generalized capture. It is worth noting that the original approach of Wittner also used invariant trees. The study of captures is motivated by the following theorem of Rees:

Theorem 2.2 (Polynomial-and-Path Theorem, Section 1.8 of Rees 1992) *Suppose that R is a rational function of degree two with a periodic critical point c_1 . Suppose also*

that the other critical point c_2 of R is not periodic but is eventually mapped to c_1 . Then R is equivalent to some capture $\sigma_\beta \circ p$. Moreover, the quadratic polynomial p has a periodic critical point of the same period as c_1 .

Suppose that β is a simple capture path for p , and $f = \sigma_\beta \circ p$ is the corresponding capture. Let T be the minimal subtree of the extended Hubbard tree of p that includes $P(f)$. Then T satisfies the property $p(T) \subset T$. Note that it may happen that $f(T) \not\subset T$, so that T is not an invariant spanning tree for f . For example, let p be the airplane polynomial. Choose $\beta(1)$ to be an iterated p -preimage of 0 on an edge of the Hubbard tree of p . Then T coincides with the Hubbard tree set-theoretically but has more vertices. Some edge e of T maps under p so that the $\beta(1) \in p(e)$ but $\beta(1)$ is not an endpoint of $p(e)$. The latter is a consequence of the fact that there are no vertices of T mapping to $\beta(1)$. The homeomorphism σ_β displaces $p(e)$ so that $\sigma_\beta(p(e))$ no longer contains $\beta(1)$. Thus T is not forward invariant under $f = \sigma_\beta \circ p$.

It may seem plausible that T can be deformed slightly into a genuine invariant spanning tree. Unfortunately, this is not always true. It is known that different simple captures (even those for which $\beta(1)$ is the same) may yield different Thurston equivalence classes, see e.g. (Rees 2010, Section 2.8). If T were deformable into an invariant spanning tree, then, by Theorem A, all simple captures with given $\beta(1)$ would be Thurston equivalent, a contradiction.

In the following lemma, by a *support* of a homeomorphism $\sigma : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ we mean the closure of the set of points $x \in \mathbb{S}^2$ with $\sigma(x) \neq x$.

Lemma 2.3 *Let p, β and T be as above. Assume that the support of σ_β is a sufficiently narrow neighborhood of $\beta[0, 1]$, i.e., a subset of the ε -neighborhood of $\beta[0, 1]$ for sufficiently small $\varepsilon > 0$. Then T is an invariant spanning tree for $f = \sigma_\beta \circ p$ whenever $\beta[0, 1] \cap p(T) = \emptyset$.*

Recall our assumption that the capture path β is simple.

Proof Suppose that $\beta[0, 1] \cap p(T) = \emptyset$. Then the support of σ_β can be made disjoint from $p(T)$. It follows that $\sigma_\beta = id$ on $p(T)$, therefore, $f(T) = \sigma_\beta(p(T)) = p(T) \subset T$. □

3 Proof of Theorem A

Let $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be a Thurston map of degree two. It will be convenient to mark the critical points of f , i.e., to distinguish between $c_1(f)$ and $c_2(f)$.

Definition 3.1 (*Marked Thurston maps*) A (critically) marked Thurston map of degree two is an ordered triple (f, c_1, c_2) , where f is a Thurston map of degree two, and $\{c_1, c_2\}$ is the set of all critical points of f . Thus, if $c_1 \neq c_2$, then (f, c_1, c_2) and (f, c_2, c_1) are different marked Thurston maps. To lighten the notation, we will sometimes write f for a Thurston map (f, c_1, c_2) . In this case, we will write $c_1(f), c_2(f)$ to emphasize the dependence on f .

We now recall the definition of Thurston equivalence.

Definition 3.2 (*Thurston equivalence*) Let f and g be two Thurston maps. They are said to be *Thurston equivalent* if there are two orientation preserving homeomorphisms $\phi, \psi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ with the following properties:

1. We have $\phi = \psi$ on $P(f)$, and $\phi(P(f)) = P(g)$.
2. The maps ϕ and ψ are isotopic modulo $P(f)$.
3. We have $\psi \circ f = g \circ \phi$.

If f and g are marked Thurston maps of degree two, then we additionally require that $\phi(v_i(f)) = v_i(g)$ for $i = 1, 2$.

For example, two topologically conjugate Thurston maps are Thurston equivalent. The following is another particular case of Thurston equivalence.

Lemma 3.3 Let $f_t, t \in [0, 1]$ be a continuous family of Thurston maps with $P(f_t) = P(f_0)$. Then all f_t are Thurston equivalent.

Thurston maps f_0 and f_1 from Lemma 3.3 are said to be *homotopic*. This lemma is known but we will sketch a proof for completeness.

Sketch of a proof By the covering homotopy theorem, there is a homotopy ϕ_t with $\phi_0 = id$ and $f_t \circ \phi_t = f_0$. It is easy to see that ϕ_t are orientation preserving homeomorphisms. Setting $g = f_t, f = f_0, \phi = \phi_t, \psi = id$, we see that the requirements of Definition 3.2 are fulfilled. \square

3.1 Cyclic Sets and Pseudoaccesses

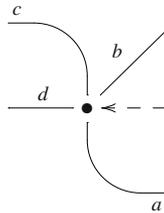
Recall Theorem A. We are given two quadratic Thurston maps f, g with invariant spanning trees T_f, T_g . There is an isomorphism $\tau : T_f \rightarrow T_g$ of ribbon graphs that conjugates f with g on $V(T_f) \cup C(T_f)$ and maps critical values to critical values. We want to prove that τ extends to a Thurston equivalence between f and g .

Modify the trees T_f, T_g by adding to their vertices the critical points of f belonging to T_f, T_g , respectively. We will write \bar{T}_f, \bar{T}_g for the modified trees. These are also ribbon graphs. Note that τ takes the vertices of \bar{T}_f to the vertices of \bar{T}_g . Moreover, τ induces an isomorphism of ribbon graphs.

The proof will consist of two steps. The first step is to define a ribbon graph isomorphism between $f^{-1}(T_f)$ and $g^{-1}(T_g)$. In other words, if just T_f is given (in which the critical values are marked), then $f^{-1}(T_f)$ can be recovered as a ribbon graph, even without knowing f . In order to recover $f^{-1}(T_f)$, a classical construction of the Riemann surface for f^{-1} helps. (This construction is essentially the same as for the Riemann surface of $z \mapsto \sqrt{z}$.) We make a cut between two critical values of f , and then glue two copies of the slitted sphere along the slits. If the cut is disjoint from T_f (except the endpoints), then it suffices to see how copies of T_f in the two slitted spheres are glued together. To translate this process to combinatorics, we need some terminology related to cyclic sets and pseudoaccesses. The next definition follows the terminology of Poirier (1993).

Definition 3.4 (*Pseudoaccess*) Let A be a *cyclic set*, i.e., a set with a distinguished cyclic order of elements. A *pseudoaccess* of A is an (ordered) pair (a, b) of elements

of A such that b is the immediate successor of a in the cyclic order. The terminology is motivated by the following picture. Suppose that A consists of Jordan arcs in the plane that share an endpoint and are otherwise disjoint (the cyclic order on A follows the counterclockwise direction around the endpoint). A Jordan arc disjoint from all elements of A except for the same endpoint defines a pseudoaccess of A . This is illustrated by the figure below, in which the pseudoaccess (a, b) is represented by the dashed segment.



The cyclic set A here is represented by the four arcs a, b, c, d in this cyclic order.

3.2 A Homeomorphism Between $f^{-1}(T_f)$ and $g^{-1}(T_g)$

Consider a Thurston map f of degree 2 with an invariant spanning tree T_f , and a Thurston map g of degree 2 with an invariant spanning tree T_g . We will work under the assumptions of Theorem A. In particular, we consider a homeomorphism $\tau : T_f \rightarrow T_g$ with the properties listed there. The first step in the proof of Theorem A is to extend τ to $G_f = f^{-1}(T_f)$.

Note that we view G_f not only as a subset of \mathbb{S}^2 but also as a graph. Vertices of G_f are defined as preimages of vertices of T_f . Edges of G_f are defined as components of $f^{-1}(T_f - V(T_f))$.

Definition 3.5 (*Pseudoaccesses of graphs*) Let G be a graph in the sphere. A *pseudoaccess* of G at a vertex a is defined as a pseudoaccess of $E(G, a)$. Here $E(G, a)$ is the cyclic set of all edges of G incident to a . Recall that the cyclic order of edges incident to a is induced by the orientation of \mathbb{S}^2 .

Consider a Jordan arc C_f that connects $v_1(f)$ with $v_2(f)$ and is otherwise disjoint from T_f , see Fig. 1, top left. Then C_f defines two pseudoaccesses of T_f , one at each of the critical values. Since τ preserves the cyclic order of edges at every vertex, it defines a correspondence between pseudoaccesses of T_f and pseudoaccesses of T_g . The two pseudoaccesses of T_f defined by C_f give rise to two distinguished pseudoaccesses of T_g . Since τ maps the critical values of f to the critical values of g , the two distinguished pseudoaccesses of T_g are at $v_1(g)$ and $v_2(g)$. Clearly, there exists a Jordan arc C_g connecting $v_1(g)$ with $v_2(g)$, otherwise disjoint from T_g and defining the two distinguished pseudoaccesses of T_g .

The set $U_f = \mathbb{S}^2 - C_f$ is a disk. The restriction of f to $f^{-1}(U_f)$ is an unbranched covering since both critical values of f are in C_f . Therefore, $f^{-1}(U_f)$ is a disjoint union of two open disks U_f^0 and U_f^1 . These disks are shown as hemispheres in Fig. 1,

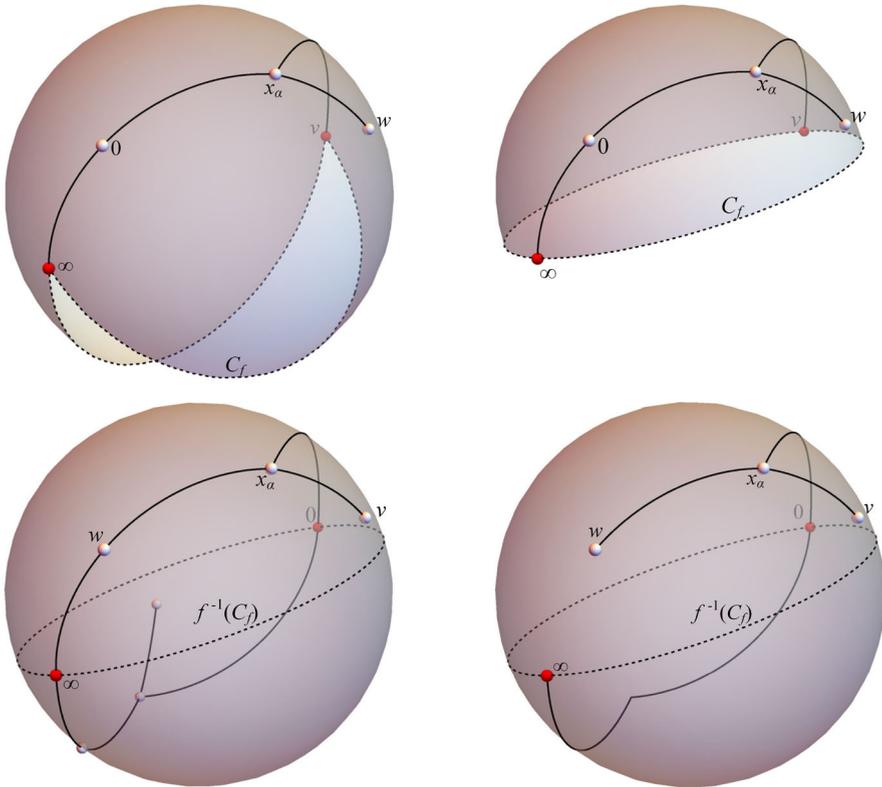


Fig. 1 Recovering $G_f = f^{-1}(T_f)$ from T_f . Top left: we start with T_f and make a cut along a simple curve C_f connecting the critical values outside of T_f . As T_f , we took an invariant spanning tree for the rabbit polynomial p ; we set $v = p(0)$ and $w = p(v)$. Top right: widening the cut, we obtain a hemisphere U_f^0 with a copy T_f^0 of T_f . Vertices of T_f^0 are labeled as the corresponding vertices of T_f . Bottom left: attach the opposite hemisphere U_f^1 with another copy T_f^1 of T_f . Then G_f is the union of T_f^0 and T_f^1 . This construction shows that G_f is uniquely defined as a ribbon graph once T_f is given and the critical values are distinguished among the vertices of T_f . Vertices of G_f are now labeled as they appear in G_f (new labels are preimages of the former labels). Bottom right: we now removed the edges of G_f that do not appear in T_f . What remains is a tree that identifies with T_f (one should rotate the sphere and deform the trees to attain the coincidence)

bottom left. There is an ambiguity in labeling U_f^0 and U_f^1 . One of the two disks has to be labeled U_f^0 , and the other U_f^1 . However, which disk gets which label is up to us. Similarly, $g^{-1}(U_g)$ is a disjoint union of two disks U_g^0 and U_g^1 . Again, the labeling of these disks should be specified somehow.

The common boundary of the disks U_f^0, U_f^1 is the Jordan curve $f^{-1}(C_f)$. Consider the closure T_f^i (in \mathbb{S}^2) of the f -pullback of $T_f - C_f$ in U_f^i , where $i = 0, 1$. Clearly, T_f^i is a tree isomorphic to T_f . Moreover, T_f^i and T_f have isomorphic ribbon graph structures. We may view T_f^0 and T_f^1 as two copies of T_f . Observe that these two copies are glued at the critical points of f to form the graph $G_f = f^{-1}(T_f)$. Observe also

that the critical points of f are the vertices of T_f^i that correspond, under the natural isomorphism between T_f^i and T_f , to the critical values $v_1(f)$ and $v_2(f)$. Thus there is an abstract description of the ribbon graph G_f . It involves making two copies of T_f and gluing them at the vertices corresponding to $v_1(f)$, $v_2(f)$. See again Fig. 1. Note that the representation of G_f as a union $T_f^0 \cup T_f^1$ depends on the choice of C_f . More precisely, it depends on the choice of the two pseudoaccesses of T_f . A similar representation can be obtained for $G_g = g^{-1}(G_g)$.

Lemma 3.6 *Either $\tau(T_f \cap T_f^i) \subset T_g \cap T_g^i$ for every $i = 0, 1$ or $\tau(T_f \cap T_f^i) \subset T_g \cap T_g^{1-i}$ for every $i = 0, 1$.*

Lemma 3.6 is a manifestation of the fact that the construction shown in Fig. 1 is essentially unique. Define the label $\ell(e)$ of an edge $e \in E(\overline{T}_f)$ so that $e \subset T_f^{\ell(e)}$. Thus the label of an edge can take values 0 or 1. Labels are defined on edges of \overline{T}_f and on edges of $f^{-1}(T_f)$ but may not be well-defined on edges of T_f . (Recall that \overline{T}_f was defined above as a subdivision of T_f , in which critical points of f in T_f become vertices.) Lemma 3.6 asserts that τ either preserves all labels or reverses all labels. We will choose the labeling of U_g^0, U_g^1 so that all labels are preserved by τ .

Proof of Lemma 3.6 We have to show that, if $\tau(e_r) \subset T_g^{\ell(e_r)}$ for some $e_r \in E(\overline{T}_f)$, then the same holds for every edge e of \overline{T}_f . In other words, τ preserves all labels. To this end, we compare every edge e with e_r . The latter will be called the *reference edge*. Consider critical points of f in \overline{T}_f . Note, however, that T_f does not have to contain all critical points of f . Suppose that some critical point c (which is necessarily $c_1(f)$ or $c_2(f)$) lies in \overline{T}_f . Let $v = f(c)$ be the corresponding critical value. The curve $f^{-1}(C_f)$ defines two pseudoaccesses of \overline{T}_f at c , not necessarily different. We will call these distinguished pseudoaccesses *critical pseudoaccesses*. Clearly, critical pseudoaccesses depend only on the ribbon graph structure of \overline{T}_f and on the choice of C_f , more precisely, on the two pseudoaccesses defined by C_f . The latter two pseudoaccesses will be referred to as *post-critical pseudoaccesses*. The two critical pseudoaccesses of \overline{T}_f at c may separate some pairs of edges incident to c .

The values of the function ℓ can be computed step by step, starting at e_r and passing from edges to adjacent edges. Suppose that $\ell(e)$ is known, and e' shares a vertex a with e . If a is not critical, then $\ell(e') = \ell(e)$. If a is critical, then $\ell(e') \neq \ell(e)$ if and only if e and e' are separated by the critical pseudoaccesses at a . The just described computational description of ℓ follows from the observation that edges $e, e' \in E(T_f, a)$ are separated by the critical pseudoaccesses at a if and only if they are separated by $f^{-1}(C_f)$, i.e., lie in different components of $f^{-1}(U_f)$.

We now need to prove that $\ell(\tau(e)) = \ell(e)$. To this end, it is enough to observe that τ maps $C(T_f)$ to $C(T_g)$ and that τ maps critical pseudoaccesses of \overline{T}_f to critical pseudoaccesses of \overline{T}_g . Indeed, suppose that c is a critical point in \overline{T}_f . Then $v = f(c)$ is a critical value, and $\tau(v)$ is also a critical value by property (3) of τ listed in the statement of Theorem A. On the other hand, by property (2) of τ , we have $\tau(v) = \tau \circ f(c) = g(\tau(c))$. Since $g(\tau(c))$ is a critical value, and g has degree two, $\tau(c)$ is a critical point. It is also clear that critical pseudoaccesses at c map under τ to critical pseudoaccesses at $\tau(c)$. We conclude that τ preserves the labels, as desired. \square

Recall that $G_f = f^{-1}(T_f)$ and $G_g = g^{-1}(T_g)$. Recall also that $V(G_f) \supset V(\overline{T}_f) \supset V(T_f)$, and similarly for g .

Proposition 3.7 *There is a ribbon graph isomorphism $\tau_* : G_f \rightarrow G_g$ with the following properties:*

1. We have $\tau_* = \tau$ on $V(\overline{T}_f)$.
2. We have $\tau_* \circ f = g \circ \tau_*$ on all vertices of G_f .

Proof Recall that, by Lemma 3.6, the map $\tau : T_f \rightarrow T_g$ preserves labels. This map lifts to T_f^i , where $i = 0, 1$, by the homeomorphisms $f : T_f^i \rightarrow T_f$ and $g : T_g^i \rightarrow T_g$. In other words, we can define a map $\tau^i : T_f^i \rightarrow T_g^i$ by the formula $\tau^i = g_i^{-1} \circ \tau \circ f$, where g_i^{-1} is the inverse of $g : T_g^i \rightarrow T_g$. We set τ_* to be the map from G_f to G_g , whose restriction to T_f^i is τ^i . Then we need to prove that properties (1)–(2) hold for τ_* .

Let us first prove that $\tau_* = \tau$ on $V(\overline{T}_f)$. On $V(\overline{T}_f) \cap T_f^i$, the map τ satisfies the property $\tau \circ f = g \circ \tau$ by the assumptions of Theorem A. By Lemma 3.6, under τ the set $T_f \cap T_f^i$ maps to $T_g \cap T_g^i$ (note that $\overline{T}_f = T_f$ as sets, and hence $\overline{T}_f \cap T_f^i = T_f \cap T_f^i$ set-theoretically). Therefore, we have $\tau = g_i^{-1} \circ \tau \circ f$ on $V(\overline{T}_f) \cap T_f^i$. It remains to note that the right hand side coincides with the definition of τ^i .

We now prove that the cyclic order of edges incident to a vertex $a_* \in V(G_f)$ is preserved by τ_* . Let f_i^{-1} be the inverse of $f : T_f^i \rightarrow T_f$. If $a_* = f_i^{-1}(a)$, where a is not a critical value, then the statement is obvious since both $f : U_f^i \rightarrow \mathbb{S}^2$ and $g : U_g^i \rightarrow \mathbb{S}^2$ preserve the orientation. Now, if a is a critical value, then the restriction of τ_* to the union of edges incident to a_* is glued from the two maps τ^0 and τ^1 . The cyclic order of edges of G_f at a_* is as follows. First come all edges of T_f^0 incident to a_* that are mapped by τ^0 in an order preserving fashion. Then come all edges of T_f^1 incident to a_* that are mapped by τ^1 in an order preserving fashion. It follows that τ_* preserves the cyclic order on edges of G_f incident to a_* .

It remains to prove that $\tau_* \circ f = g \circ \tau_*$ on all vertices of G_f . Indeed, let a_* be a vertex of G_f . Then $a_* = f_i^{-1}(a)$ for $i = 0$ or 1 . We have

$$\tau_* \circ f(a_*) = \tau(a) = g \circ \tau^i \circ f_i^{-1}(a) = g \circ \tau_*(a_*).$$

In the first equality, we used that $\tau^* = \tau$ on $V(T_f)$. In the second equality, we used the definition of τ^i .

3.3 An Extension of τ to the Sphere

We keep the notation of Theorem A. Consider the homeomorphism $\tau_* : G_f \rightarrow G_g$ constructed in Proposition 3.7. The restriction of τ_* to \overline{T}_f is in general different from τ . However, these two maps match on $V(\overline{T}_f) = V(T_f) \cup C(T_f)$. Moreover, τ_* restricted to T_f also satisfies assumptions (1)–(3) of Theorem A. Thus we may consider τ_* in place of τ .

We will now extend τ_* to the entire sphere. Such an extension is possible due to the following result.

Theorem 3.8 (Corollary 6.6 of Bielefeld et al. 1992) *Let G and G' be two connected graphs embedded into \mathbb{S}^2 . Consider a homeomorphism $h : G \rightarrow G'$ that induces an isomorphism of ribbon graphs. Then there is an orientation preserving homeomorphism $h_* : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ whose restriction to G is h .*

Applying Theorem 3.8 to our specific situation, we obtain the following corollary.

Corollary 3.9 *Suppose that $\tau_* : G_f \rightarrow G_g$ satisfies the properties listed in Proposition 3.7. Then τ_* extends to an orientation preserving homeomorphism $\tau_* : \mathbb{S}^2 \rightarrow \mathbb{S}^2$.*

The homeomorphism τ_* maps complementary components of G_f to complementary components of G_g . The following notion helps to say which components are mapped to which components in combinatorial terms:

Definition 3.10 (*Boundary Circuits*) Let G be a graph in \mathbb{S}^2 . If an orientation of an edge $e \in E(G)$ is fixed, then e is called an *oriented edge* of G . The endpoints of e form an ordered pair (a, b) , where a is the *initial endpoint* and b is the *terminal endpoint* of e . We also say that e *originates* at a and *terminates* at b . The same edge equipped with different orientations gives rise to two different oriented edges. A *boundary circuit* of G (also known as a left-turn path in G) is a cyclically ordered sequence $[e_0, \dots, e_{n-1}]$ of oriented edges of G with the following property: if e_i terminates at a vertex a , then $e_{i+1 \pmod n}$ originates at a , and $e_{i+1 \pmod n}$ is the immediate *predecessor* of e_i in the cyclic order on $E(G, a)$. Clearly, any oriented edge belongs to some boundary circuit.

As above, let W be some complementary component of G . There is a boundary circuit $\Sigma_W = [e_0, \dots, e_{n-1}]$ associated with W . Informally, it is obtained by tracing the boundary of W counterclockwise. The correspondence $W \mapsto \Sigma_W$ between components of $\mathbb{S}^2 - G$ and boundary circuits of G is one-to-one. Observe that the same edge may enter Σ_W twice with different orientations. Observe also that the rotation from e_k to e_{k+1} around the terminal point of e_k is clockwise.

3.4 Homotopy

Theorem A will be deduced from Theorem 3.11 stated below. Theorem 3.11 is not new: Proposition 3.4.3 of Hlushchanka (2017) contains a more general fact; it is based in turn on a similar statement from Bonk and Meyer (2017). However, since notation and terminology in Bonk and Meyer (2017), Hlushchanka (2017) are somewhat different, we sketch a proof here. The proof will be based on a technical lemma from Bielefeld et al. (1992).

Theorem 3.11 *Suppose that two Thurston maps f and g of degree two share an invariant spanning tree T . Moreover, suppose that $f^{-1}(T) = g^{-1}(T) = G$, that $f = g$ on $V(G)$, and that the critical values of f coincide with the critical values of g . Then there is an orientation preserving homeomorphism ψ isotopic to the identity relative to $V(T)$ and such that $f = g \circ \psi$.*

Note that the equality $f^{-1}(T) = g^{-1}(T)$ means the equality of graphs rather than just sets. In particular, we assume that the two graphs have the same vertices. Note also that all critical points of f are among vertices of these graphs. Theorem 3.11 implies that f and g are Thurston equivalent. In fact, they are even homotopic.

Proof of Theorem A using Theorem 3.11 Let f, g and $\tau : T_f \rightarrow T_g$ be as in Theorem A. As before, set $G_f = f^{-1}(T_f)$ and $G_g = g^{-1}(T_g)$. By Proposition 3.7, there is a homeomorphism $\tau_* : G_f \rightarrow G_g$ that induces an isomorphism of ribbon trees and is such that

1. we have $\tau_* = \tau$ on $V(T_f)$;
2. we have $g \circ \tau_* = \tau_* \circ f$ on all vertices of G_f .

Replacing τ with τ_* if necessary, we may assume that τ satisfies these properties. In particular, τ maps G_f to G_g .

By Corollary 3.9, the map τ extends to an orientation preserving homeomorphism $\tau : \mathbb{S}^2 \rightarrow \mathbb{S}^2$. Set $g_* = \tau^{-1} \circ g \circ \tau$. Clearly, this is a Thurston map of degree two. Then T_f is an invariant spanning tree for g_* . Since τ maps the critical values of f to the critical values of g , the maps f and g_* share the critical values. Finally,

$$g_*^{-1}(T_f) = \tau^{-1} \circ g^{-1} \circ \tau(T_f) = \tau^{-1}(G_g) = G_f = f^{-1}(T_f).$$

Thus all assumptions of Theorem 3.11 hold for f and g_* . By Theorem 3.11, the map g_* is homotopic to f . Since g is topologically conjugate to g_* , we conclude that g is Thurston equivalent to f . □

Consider two graphs G and T in the sphere. Let $h : G \rightarrow T$ be a continuous map that is injective on the edges of G and is such that the forward and inverse images of the vertices are vertices. Such a map is called a *graph map* in Bielefeld et al. (1992). Suppose that a graph map h has an extension $\bar{h} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$. If \bar{h} is an orientation preserving branched covering injective on every complementary component of G , then \bar{h} is called a *regular extension* of h . This terminology also follows Bielefeld et al. (1992).

Theorem 3.12 (Corollary 6.3 of Bielefeld et al. 1992) *Consider two graph maps $h, h' : G \rightarrow T$ admitting regular extensions \bar{h}, \bar{h}' . Suppose that $h = h'$ on $V(G)$ and $h(e) = h'(e)$ for every $e \in E(G)$. Then there is a homeomorphism $\psi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that $\bar{h} = \bar{h}' \circ \psi$, and ψ is isotopic to the identity relative to $V(G)$.*

We are now ready to deduce Theorem 3.11 from Theorem 3.12.

Proof of Theorem 3.11 Apply Theorem 3.12 to $h = f : G \rightarrow T$ and $h' = g : G \rightarrow T$. These are clearly graph maps admitting regular extensions. All assumptions of Theorem 3.12 are satisfied. It follows that there exists a homeomorphism $\psi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that $f = g \circ \psi$ on \mathbb{S}^2 , and ψ is isotopic to the identity relative to $V(G)$. □

Thus we proved Theorem 3.11, and the latter implies Theorem A.

4 No Dynamics: Spanning Trees

In this section, we associate certain combinatorial objects with a spanning tree. Recall that, given a finite set P of (marked) points in \mathbb{S}^2 , a *spanning tree* for P is a tree $T \subset \mathbb{S}^2$ with the property that $V(T) = P \cup B(T)$, where $B(T)$ is the set of branch points of T . Thus the notion of a spanning tree is an non-dynamical notion. Suppose that the sphere \mathbb{S}^2 is glued of a polygon Δ by identifying some edges of it. Then the boundary of Δ becomes a spanning tree for the set P of all vertices of Δ . Alternatively, some vertices of Δ can be dropped from P if these give rise to branch points of the tree.

4.1 The Generating Set \mathcal{E}_T of $\pi_1(\mathbb{S}^2 - P)$

Let T be a spanning tree for a finite marked set P . Assume that the base point $y \in \mathbb{S}^2 - T$ is fixed once and for all. We now define a certain generating set $\mathcal{E} = \mathcal{E}_T$ of $\pi_1(\mathbb{S}^2 - P) = \pi_1(\mathbb{S}^2 - P, y)$. (This is the same generating set as in Hlushchanka (2017); Hlushchanka refers to its elements as *edge generators*).

Endow \mathbb{S}^2 with some smooth structure. (It will be clear however that our construction is independent of this structure). Consider an oriented smooth Jordan arc A . Let γ be a smooth path that crosses A only once and transversely. By a transverse intersection we mean that the tangent lines to A and γ at the intersection point are different, and that the intersection point is not an endpoint of A . We say that γ approaches A from the left if, at the intersection point, the velocity vectors to γ and to A (in this order) form a positively oriented basis in the tangent plane to the sphere. With every oriented edge e of T , we associate an element $g_e \in \pi$ as follows. The homotopy class g_e is represented by a smooth loop γ_e that crosses e just once and transversely, approaches it from the left, and has no other intersection points with T . (We assume of course that the loop γ_e is based at y). A smooth loop γ_e with the indicated properties is said to be *adapted* to T at e . Consider the subset $\mathcal{E} = \mathcal{E}_T \subset \pi_1(\mathbb{S}^2 - P)$ consisting of id , the neutral element, and elements g_e , where e ranges through all oriented edges of T . Note that the same edge equipped with different orientations gives rise to two different elements of \mathcal{E} . These elements are inverse to each other.

Thus \mathcal{E} is a generating set of $\pi_1(\mathbb{S}^2 - P)$ that is symmetric ($\mathcal{E}^{-1} = \mathcal{E}$) and such that $id \in \mathcal{E}$.

Lemma 4.1 *Different oriented edges of T give rise to different elements of \mathcal{E} .*

Proof Consider two different oriented edges e_1, e_2 of T with $g_{e_1} = g_{e_2}$. Let $\gamma_{e_i} : [0, 1] \rightarrow \mathbb{S}^2$ be smooth simple loops as above so that $g_{e_i} = [\gamma_{e_i}]$. We can also arrange that $\gamma_{e_1}(0, 1)$ is disjoint from $\gamma_{e_2}(0, 1)$. Let D be the croissant shaped region bounded by $\gamma_{e_1}[0, 1]$ and $\gamma_{e_2}[0, 1]$.

Since $g_{e_1} = g_{e_2}$, the loops γ_{e_i} are homotopic rel. P . Therefore, there are no points of P in D . We claim that there are also no branch points of T in D . Indeed, if x is such point, then there is a component of $T - \{x\}$ disjoint from both e_1 and e_2 . This component must end somewhere in D . On the other hand, by definition of a spanning tree, all endpoints of T are in P . A contradiction with the fact that $P \cap D = \emptyset$.

Since $e_1 \neq e_2$, there is at least one vertex x of T in D . However, this is impossible since all vertices of T are in $P \cup B(T)$.

Consider a set E of smooth loops based at y with the following properties. Firstly, we assume that every $\gamma \in E$ is adapted to T at some oriented edge e of T . Secondly, there is exactly one loop $\gamma_e \in E$ adapted to T at e , and, as we change the orientation of e , the corresponding loop also changes orientation but otherwise remains the same. Thirdly, we assume that the constant loop belongs to E , and that different loops from E are either disjoint (except the common basepoint y) or the same (up to the change of direction). If these assumptions are satisfied, then we say that E is an *adapted set of loops* for T . Clearly, any spanning tree admits an adapted set of loops. The set \mathcal{E}_T equals $[E]$, the set of classes in $\pi_1(\mathbb{S}^2 - P)$ of all elements from E .

The following lemma will help us translate the pullback operations on spanning trees into a combinatorial language. We will assume that the basepoint y is chosen outside of all spanning trees under consideration.

Lemma 4.2 *Let H be an inner automorphism of $\pi_1(\mathbb{S}^2 - P)$. Suppose that two spanning trees T and T' are such that $\mathcal{E}_T = H(\mathcal{E}_{T'})$ in $\pi_1(\mathbb{S}^2 - P)$. Then T and T' are isotopic rel. P .*

Proof Let h be an element of $\pi_1(\mathbb{S}^2 - P)$ such that H is the conjugation by h . We will write $\text{PMod}(\mathbb{S}^2, P)$ for the pure mapping class group of \mathbb{S}^2 with marked point set P . Consider the homomorphism $\mathcal{P}ush : \pi_1(\mathbb{S}^2 - P, y) \rightarrow \text{PMod}(\mathbb{S}^2, P \cup \{y\})$ from the Birman exact sequence (cf. Section 4.2.1 of Farb and Margalit 2012). It is easy to see that $\psi = \mathcal{P}ush(h)$ acts on $\pi_1(\mathbb{S}^2 - P, y)$ as H . Moreover, the Birman exact sequence implies that ψ is isotopic to the identity rel. P but not rel. $P \cup \{y\}$. Replacing T' with $\psi^{-1}(T')$, we can arrange that \mathcal{E}_T and $\mathcal{E}_{T'}$ coincide. Thus we will assume from now on that $\mathcal{E}_T = \mathcal{E}_{T'}$.

We may assume that both T and T' are composed of smooth arcs. Suppose that sets E, E' of smooth loops based at y are adapted to T, T' , respectively. The sets E, E' form embedded graphs Γ, Γ' , respectively, in \mathbb{S}^2 with the single vertex y . Every complementary component (“face”) of Γ contains a single vertex of T , and similarly for Γ' . There is a homeomorphism $\phi : \Gamma \rightarrow \Gamma'$ that is simultaneously a graph map. Moreover, for every edge of Γ , there is an isotopy transforming this edge to its ϕ -image. (Indeed, two loops are homotopic rel. P if and only if they are isotopic rel. P .) Then ϕ can be extended as an orientation preserving homeomorphism $\phi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ fixing P pointwise. This follows from Lemma 2.9 of Farb and Margalit (2012). Moreover, it follows from the same lemma that ϕ is isotopic to the identity rel. P . Applying ϕ^{-1} to T' and E' , we may now assume that $E = E'$. The corresponding edges of T and T' connect the same complementary components of Γ and cross the same edge of Γ . It follows that the corresponding edges of T and T' are homotopic rel. P , as desired. □

The converse of Lemma 4.2 is also true. We say that two subsets \mathcal{E} and \mathcal{E}' of a group π are *conjugate* if there is $u \in \pi$ such that \mathcal{E}' coincides with the set of all elements of the form uvu^{-1} , where v runs through \mathcal{E} .

Proposition 4.3 *Let T and T' be spanning trees for P . The trees T and T' are homotopic rel. P if and only if the corresponding generating sets \mathcal{E}_T and $\mathcal{E}_{T'}$ are conjugate.*

Proof of Proposition 4.3 To lighten the notation, we will write \mathcal{E} and \mathcal{E}' instead of \mathcal{E}_T and $\mathcal{E}_{T'}$. We silently assumed that both \mathcal{E} and \mathcal{E}' are subsets of the same group $\pi = \pi_1(\mathbb{S}^2 - P, y)$ corresponding to a certain basepoint y . Thus the basepoint is fixed.

Suppose first that \mathcal{E} and \mathcal{E}' are conjugate. Then T and T' are homotopic rel. P , by Lemma 4.2. Suppose now that T and T' are homotopic. We may assume that both T, T' are formed by smooth arcs. Let E be a set of smooth loops adapted to T .

Now consider a homotopy T_t of T (so that $T_0 = T, T_1 = T'$, and t runs through $[0, 1]$). We may assume that this homotopy is smooth. Then there is a homotopy $\phi_t : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ consisting of orientation preserving diffeomorphisms such that $\phi_0 = id$ and $\phi_t(T) = T_t$. Clearly, $E_t = \phi_t(E)$ is adapted to T_t . In particular, E_t represents the symmetric generating set $\mathcal{E}_t = \mathcal{E}_{T_t}$ in $\pi_1(\mathbb{S}^2 - P, \phi_t(y))$.

Recall that any homotopy class c of paths connecting two given points $y, y' \in \mathbb{S}^2 - P$ gives rise to an isomorphism $H_c : \pi_1(\mathbb{S}^2 - P, y) \rightarrow \pi_1(\mathbb{S}^2 - P, y')$. Two different isomorphisms of this type differ by an inner automorphism of the target group. All groups $\pi_1(\mathbb{S}^2 - P, \phi_t(y))$ can be identified along the path $t \mapsto \phi_t(y)$. In particular, $\pi_1(\mathbb{S}^2 - P, \phi_1(y))$ identifies with π .

Modifying the homotopy if necessary, we may arrange that $\phi_1(y) = y$. Thus, \mathcal{E}_1 and \mathcal{E}' lie in the same group, and, by definition of \mathcal{E}' , we must have $\mathcal{E}' = \mathcal{E}_1$. On the other hand, \mathcal{E}_1 identifies with $\mathcal{E}_0 = \mathcal{E}$ under the automorphism H_c , where c is the homotopy class of the loop $t \mapsto \phi_t(y)$. Since H_c is an inner automorphism, \mathcal{E} and \mathcal{E}' are conjugate. Thus the proposition is proved. □

4.2 Vertex Structures

Below, we will introduce some formal algebraic/combinatorial notions. The purpose of these is to translate topological objects, namely, spanning trees, into a symbolic language.

For any finite set \mathcal{E} , we write $FS(\mathcal{E})$ for the free semi-group generated by \mathcal{E} . The semi-group $FS(\mathcal{E})$ can also be thought of as the set of all finite words in the alphabet \mathcal{E} . The empty word is allowed as an element of $FS(\mathcal{E})$; it is the neutral element of the semi-group. For $g, h \in \mathcal{E}$, the product of g and h in $FS(\mathcal{E})$ will be written as $g \cdot h$.

Suppose now that π is a group and that $\mathcal{E} \subset \pi$. We also suppose that $id \in \mathcal{E}$. Here id means the identity element of π . It is not to be confused with the neutral element of $FS(\mathcal{E})$, which is not an element of \mathcal{E} or of π . We set \mathcal{E}^* to be the quotient of $FS(\mathcal{E})$ modulo the relations $id \cdot g = g \cdot id = g$ for all $g \in \mathcal{E}$. Now assume that $\mathcal{E} \ni id$ is symmetric, i.e., that $g \in \mathcal{E}$ implies $g^{-1} \in \mathcal{E}$. Here g^{-1} is the inverse of g in the group π . Then there is a natural map $\Pi : \mathcal{E}^* \rightarrow \pi$ that takes every word in the alphabet \mathcal{E} to the product of its symbols. (The latter product is with respect to the group operation in π .) We will refer to Π as the *evaluation map*. For example, an element $g_1 \cdot g_2 \in \mathcal{E}^*$ is mapped to $g_1 g_2 \in \pi$. Intuitively, an element $u \in \mathcal{E}^*$ is a way of writing the element $\Pi(u)$ of the subgroup of π generated by \mathcal{E} as a product of generators. Different ways of writing the same element may differ by a sequence of cancellations. However, we disregard all appearances of id . For example, $g \cdot h \cdot g \cdot g^{-1}$ is different from $g \cdot h$ as an element of \mathcal{E}^* . However, it is the same as $g \cdot id \cdot h \cdot g \cdot id \cdot g^{-1} \cdot id \cdot id$, for example.

A *vertex structure* on \mathcal{E} is a subset $\mathcal{V} \subset \mathcal{E}^*$ with the following property: for every $g \in \mathcal{E}$, there is a unique element of \mathcal{V} of the form $u_1 \cdot g \cdot u_2$ for some $u_1, u_2 \in \mathcal{E}^*$. Any vertex structure gives rise to an abstract directed graph $G(\mathcal{V})$ as follows. The vertices of $G(\mathcal{V})$ are identified with elements of \mathcal{V} . The oriented edges of $G(\mathcal{V})$ are labeled by elements of \mathcal{E} . Two vertices $v, w \in \mathcal{V}$ are connected with an oriented edge g (from v to w) if

$$v = v_1 \cdot g \cdot v_2, \quad w = w_1 \cdot g^{-1} \cdot w_2$$

for some elements v_1, v_2, w_1, w_2 of \mathcal{E}^* . Since \mathcal{E} is symmetric, the edges of $G(\mathcal{V})$ always come in pairs so that paired edges connect the same vertices but go in different directions. These pairs of edges correspond to pairs of the form $\{g, g^{-1}\}$ in \mathcal{E} . Thus $G(\mathcal{V})$ can also be regarded as an undirected graph, by identifying each pair of oppositely directed edges with an undirected edge. A vertex structure \mathcal{V} on \mathcal{E} is called a *tree structure* if $G(\mathcal{V})$ is a tree.

Observe that the graph $G(\mathcal{V})$ also carries a natural ribbon graph structure. Indeed, directed edges of $G(\mathcal{V})$ originating at a given vertex $v \in \mathcal{E}^*$ are linearly ordered. We refer to the linear order of symbols in words from \mathcal{E}^* . For example, consider a vertex represented by $a \cdot b \cdot c \in \mathcal{E}^*$ with $a, b, c \in \mathcal{E}$. Then we should think of a, b, c as appearing in this *clockwise* order around the given vertex. That is, the cyclic order of a, b, c at the given vertex is $[c, b, a]$.

4.3 Vertex Words

In this section, we explain how a spanning tree T for a finite marked set P defines a tree structure on $\mathcal{E} = \mathcal{E}_T$. To this end, we need to equip T with a bit of extra structure. Namely, we assume that some pseudoaccess is fixed at every vertex of T .

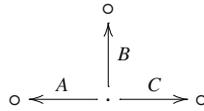
Recall that any oriented edge e of T gives rise to a group element (edge generator) $g_e \in \mathcal{E}$. Moreover, by Lemma 4.1, different edges correspond to different edge generators. Thus we may think of \mathcal{E} as a combinatorial analog for the set of oriented edges of T . We now define a combinatorial analog of a vertex.

Definition 4.4 (*Vertex word*) Let x be a vertex of T . Consider all edges e_0, \dots, e_{k-1} incident to x and oriented outwards. The linear order of these edges is well defined if we impose that

1. it follows the natural *clockwise* order around x ;
2. the chosen pseudoaccess at x coincides with (e_{k-1}, e_0) .

Then we define the *vertex word* of x as the product $g_{e_0} \cdots g_{e_{k-1}} \in \mathcal{E}^*$. For example, if $k = 3$, then $x = g_{e_0} \cdot g_{e_1} \cdot g_{e_2}$ (the product is in \mathcal{E}^* , not in $\pi_1(\mathbb{S}^2 - P)$!). Let \mathcal{V} be the set of all vertex words associated with the vertices of T . Then \mathcal{V} is clearly a tree structure on \mathcal{E} such that $G(\mathcal{V})$ is isomorphic to T as a ribbon graph.

The construction presented above may seem artificial. In order to shed some light on it, let us consider an example. The following is a spanning tree T for a set of three marked points:



(The marked points, shown as circles, are precisely the endpoints of the tree.) We write A, B, C for the oriented edges of T originating at the branch point. Set $a = g_A, b = g_B, c = g_C$. Then the generating set \mathcal{E}_T consists of 7 elements $id, a, a^{-1}, b, b^{-1}, c, c^{-1}$. The vertex word corresponding to the branch point of the tree is $a \cdot b \cdot c$. Note that this word is different from the neutral element of \mathcal{E}^* even through $\Pi(a \cdot b \cdot c) = abc = id$ in π . This example explains why we need to consider \mathcal{E}^* . The vertex structure associated with T is

$$\mathcal{V} = \{a \cdot b \cdot c, a^{-1}, b^{-1}, c^{-1}\}.$$

Clearly, the combinatorial structure of $G(\mathcal{V})$ represents that of T .

5 Dynamics: Computation of the Biset

In this section, we consider a Thurston map f of degree two with an invariant spanning tree T . We will find a presentation for the biset of f using only the combinatorics of the map $f : T \rightarrow T$. We start with recalling the terminology.

5.1 Bisets and Automata

A biset is a convenient algebraic invariant of a Thurston map, which fully encodes the Thurston equivalence class.

Fix some basepoint $y \in \mathbb{S}^2 - P(f)$. Define the set $\mathcal{X}_f(y)$ as the set of all homotopy classes of paths from y to $f^{-1}(y)$ in $\mathbb{S}^2 - P(f)$. To lighten the notation, we will write π_f for the fundamental group $\pi_1(\mathbb{S}^2 - P(f), y)$. There are natural left and right actions of π_f on $\mathcal{X}_f(y)$. For this reason, the set $\mathcal{X}_f(y)$ is referred to as a π_f -biset.

The left action of π_f on $\mathcal{X}_f(y)$ is the usual composition of paths. Let γ be a representative of an element $[\gamma] \in \pi_f$, and let α be a representative of an element $[\alpha] \in \mathcal{X}_f(y)$. Then $[\gamma][\alpha]$, the left action of the element $[\gamma] \in \pi_f$ on an element $[\alpha] \in \mathcal{X}_f(y)$, is defined as the element $[\gamma\alpha]$ of $\mathcal{X}_f(y)$ represented by the composition $\gamma\alpha$ of γ and α : we first traverse γ , and then α . According to our convention, paths are composed from left to right. The right action of π_f on $\mathcal{X}_f(y)$ is defined as follows. For $[\gamma] \in \pi_f$ and $[\alpha] \in \mathcal{X}_f(y)$ as above, let β be the composition of α and the pullback of γ originating at the terminal point of α . Then the element $[\alpha].[\gamma] \in \mathcal{X}_f(y)$, the right action of $[\gamma]$ on $[\alpha]$, is defined as $[\beta]$. We will refer to $\mathcal{X}_f(y)$ as *the biset of f* . Now

that we have a particular example at hand, we give a general algebraic definition of a biset.

Definition 5.1 (*Biset*) Let π be a group. A set \mathcal{X} is called a *biset over π* , or a π -*biset*, if commuting left and right actions of π on \mathcal{X} are given. The biset \mathcal{X} is said to be *left free* if there exists a subset $\mathcal{B} \subset \mathcal{X}$ such that every element $a \in \mathcal{X}$ can be uniquely represented as gb , where $g \in \pi$ and $b \in \mathcal{B}$. The subset \mathcal{B} is then called a *basis of \mathcal{X}* . Let π' be another group, and \mathcal{X}' be a π' -biset. A group isomorphism $\rho : \pi \rightarrow \pi'$ is said to *conjugate \mathcal{X} with \mathcal{X}'* if there is a bijection $\sigma : \mathcal{X} \rightarrow \mathcal{X}'$ with the property that $\sigma(g_1 a \cdot g_2) = \rho(g_1) \sigma(a) \rho(g_2)$ for all $g_1, g_2 \in \pi$ and $a \in \mathcal{X}$. If ρ and σ with these properties exist, then \mathcal{X} and \mathcal{X}' are said to be *conjugate*. If moreover $\pi = \pi'$ and $\rho = id$, we say that \mathcal{X} and \mathcal{X}' are *isomorphic*. For more details on these formal notions, we refer the reader to Nekrashevych (2005), Bartholdi and Dudko (2017) (note that bisets are called *bimodules* in Nekrashevych 2005, see Chapter 2).

Clearly, the biset of a Thurston map is well defined up to conjugation. Recall the following theorem of Nekrashevich (Theorem 6.5.2 of Nekrashevych 2005, see also Kameyama 2001; Pilgrim 2003), which says that, reversely, the conjugacy class of the biset determines the Thurston equivalence class of the map:

Theorem 5.2 Let f_1 and f_2 be Thurston maps, and \mathcal{X}_{f_i} be the corresponding π_{f_i} -bisets, $i = 1, 2$. Here π_{f_i} is the fundamental group of $\mathbb{S}^2 - P(f_i)$.

1. The maps f_1 and f_2 are Thurston equivalent if and only if there exists an orientation preserving homeomorphism $h : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that $h(P(f_1)) = P(f_2)$ and the induced isomorphism $h_* : \pi_{f_1} \rightarrow \pi_{f_2}$ conjugates \mathcal{X}_{f_1} with \mathcal{X}_{f_2} .
2. Suppose that $P(f_1) = P(f_2) = P$ and the base points chosen for $\mathcal{X}_{f_1}, \mathcal{X}_{f_2}$ coincide. The maps f_1 and f_2 are homotopic rel. P if and only if \mathcal{X}_{f_1} and \mathcal{X}_{f_2} are isomorphic.

Let us go back to a degree 2 Thurston map f . A basis of $\mathcal{X}_f(y)$ consists of two elements. These are homotopy classes of two paths connecting y with its preimages y_0, y_1 . Thus, to choose a basis of $\mathcal{X}_f(y)$ is the same as to choose two paths α_0, α_1 , up to homotopy rel. $P(f)$, so that α_ε connects y with y_ε , for $\varepsilon = 0, 1$. Once some basis of $\mathcal{X}_f(y)$ is chosen, we can associate an automaton with $\mathcal{X}_f(y)$.

Definition 5.3 (*Automaton*) Let A and S be some sets. In practically important cases both A and S are finite. The set A is called an *alphabet*, and its elements are called *symbols*. The set S is called the *set of states*, and its elements are called *states*. An *automaton* can be defined as a map $\Sigma : A \times S \rightarrow S \times A$, or rather as a triple (A, S, Σ) . Let $FS(A)$ be the set of finite words in the alphabet A , including the empty word. This is a free semi-group generated by A , thus the notation. If we fix some initial state $s_0 \in S$, then we obtain a self-map of $FS(A)$ as follows. Imagine that a machine reads a word $w \in FS(A)$ symbol by symbol, *right to left*. Suppose, at some point, it reads a symbol $a \in A$ and its state is s . Set $(t, b) = \Sigma(a, s)$. Then the machine writes b in place of a , changes the state to t , and moves one step left. In other words, an automaton (A, S, Σ) gives rise to a right action of S on $FS(A)$. If Σ is fixed, then it is common to write $\Sigma(a, s)$ simply as as .

Consider an abstract left free π -biset \mathcal{X} . Assume that some basis \mathcal{B} of \mathcal{X} is chosen. Then, for every $a \in \mathcal{B}$ and every $g \in \pi$, there are elements $a^* \in \mathcal{B}$ and $g^* \in \pi$ with $ag = g^*a^*$. Thus, we have a well-defined map $\Sigma_{\mathcal{B}} : \mathcal{B} \times \pi \rightarrow \pi \times \mathcal{B}$ taking (a, g) to (g^*, a^*) . By definition, this is an automaton with π being the set of states. We will refer to this automaton as the *full automaton* of \mathcal{X} in the basis \mathcal{B} . Clearly, the full automaton defines \mathcal{X} up to isomorphism. On the other hand, the full automaton carries excessive information. It is enough to know the values $\Sigma_{\mathcal{B}}(a, g)$ for all g in some generating set of π . If \mathcal{B} is finite and π is generated by a finite set S , then the image of $\mathcal{B} \times S$ under $\Sigma_{\mathcal{B}}$ is finite. In particular, this image lies in $S^* \times \mathcal{B}$, where S^* is also a finite subset of π . Thus, in order to describe the biset, it suffices to indicate the map $\Sigma_{\mathcal{B}} : \mathcal{B} \times S \rightarrow S^* \times \mathcal{B}$ between finite sets. This map is called a (finite) *presentation* of \mathcal{X} . We see that finitely presented bisets can be efficiently described, and computations with them are easy to implement. However, the isomorphism problem for bisets is not easy, cf. Bartholdi and Dudko (2017).

We now go back to the biset $\mathcal{X}_f(y)$ of a quadratic Thurston map f . In a number of important situations, there is a finite generating set $\mathcal{E} \subset \pi_f$ and a basis $[\alpha_0], [\alpha_1]$ with the following property. For $\varepsilon \in \{0, 1\}$ and any element $a \in \mathcal{E}$, we have $[\alpha_\varepsilon].a = a^*[\alpha_{\varepsilon^*}]$ for some $\varepsilon^* \in \{0, 1\}$ and $a^* \in \mathcal{E}$ depending on a and ε . Define an automaton $\Sigma : \{0, 1\} \times \mathcal{E} \rightarrow \mathcal{E} \times \{0, 1\}$ taking (ε, a) to (a^*, ε^*) . This automaton has then a finite set of states. Such automata are practically important and are called *finite state automata*. Observe that Σ defines a finite presentation of $\mathcal{X}_f(y)$. We will see that a simple presentation of $\mathcal{X}_f(y)$ by a finite state automaton can be associated with every invariant spanning tree of f . This observation was also made in Hlushchanka (2017) in a more general context but with a less explicit description of the automaton.

5.2 A Base Edge and Labels

We now assume that (T^*, T) is a dynamical tree pair for f . Let Z be the smallest subarc of T containing both v_1 and v_2 . (In Fig. 1, top left, this is the union of the arcs $\infty 0, 0x_\alpha$, and $x_\alpha v$.) Then $f^{-1}(Z)$ is a Jordan curve containing the critical points c_1 and c_2 . (In Fig. 1, bottom left, this is the only simple cycle in the graph.) We will regard both Z and $f^{-1}(Z)$ as graphs in the sphere whose vertices are the vertices of T and $f^{-1}(T)$, respectively, contained in Z and $f^{-1}(Z)$, respectively. Since the tree T^* cannot contain the Jordan curve $f^{-1}(Z)$, there is at least one edge e'_b of $f^{-1}(Z)$ not contained in T^* . (In Fig. 1, we removed an edge of $f^{-1}(Z)$ when passing from the bottom left to the bottom right picture. We may set e'_b to be this removed edge.) Choose one such edge, and call $e_b = f(e'_b)$ the *base edge* of T . There may be several ways of choosing a base edge.

The two arcs with endpoints c_1, c_2 mapping onto Z will be denoted by Z^0 and Z^1 . Here Z^1 is chosen to include e'_b . Then Z^0 includes the other pullback of e_b .

Set $G = f^{-1}(T)$. Suppose now that some post-critical pseudoaccesses (i.e., pseudoaccesses at critical values) are chosen for T . Intermediate steps in the computation of an automaton for $\mathcal{X}_f(y)$, but not the final result, will depend on this choice. The choice of the post-critical pseudoaccesses gives rise to a representation $G = T^0 \cup T^1$. Here T^0, T^1 are two trees mapping homeomorphically onto T under f . In Sect. 3.2,

we defined T^0 and T^1 using a Jordan arc C connecting v_1 with v_2 outside of T . However, it is easy to see that T^i depend only on the pseudoaccesses defined by C . To fix the labeling, we assume that $Z^i \subset T^i$ for $i = 0, 1$. In fact, Z^i is an “invariant” part of T^i , independent of the choice of the pseudoaccesses.

Modify T^* so that the critical points of f lying in T^* become vertices. To distinguished the new (modified) tree from T^* , we denote it by \overline{T}^* . Let e be an edge of \overline{T}^* . Then e lies in $T^{\ell(e)}$, where $\ell(e) = 0$ or 1 . The number $\ell(e)$ is called the *label* of e , cf. the proof of Lemma 3.6. We now reproduce the combinatorial definition of labels.

Definition 5.4 (*The label of an edge*) Define the *critical pseudoaccesses* of $G = f^{-1}(T)$ as the preimages of the post-critical pseudoaccesses of T . There is a unique function $\ell : E(G) \rightarrow \{0, 1\}$ with the following properties:

1. we have $\ell(e'_b) = 1$;
2. suppose that edges e_1, e_2 share a vertex; then $\ell(e_1) = \ell(e_2)$ if and only if e_1, e_2 are not separated by the critical pseudoaccesses.

The function ℓ with these properties is called the *labeling*. For $e \in E(G)$, the value $\ell(e)$ is called the *label of the edge* e . An edge of \overline{T}^* may consist of several edges of G . These edges have the same label since the critical points of f in T^* are vertices of \overline{T}^* . The label of an edge of \overline{T}^* is defined as the label of any edge of G contained in it. Thus the labeling is also defined on $E(\overline{T}^*)$.

Note that the labeling may not be well defined on $E(T^*)$ if there are edges of T^* subdivided by critical points of f . This was the reason for passing from T^* to \overline{T}^* .

5.3 Signatures

As before, T is a spanning tree for $P(f)$ with specified pseudoaccesses at the critical values. We also need a function on the edges of T .

Definition 5.5 (*Signatures of edges*) Let $C(T)$ be the only boundary circuit of T . Informally: if a particle x loops around T in a small neighbourhood of T so that T

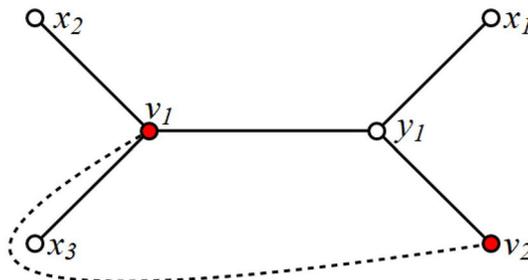


Fig. 2 The sequences $S^0(T)$ and $S^1(T)$. In this example, the sequence $S^0(T)$ consists of the oriented edges $v_1x_2, x_2v_1, v_1y_1, y_1x_1, x_1y_1, y_1v_2$, taken in this order. The sequence $S^1(T)$ consists of the edges $v_2y_1, y_1v_1, v_1x_3, x_3v_1$, taken in this order. The signature of v_1x_2 and y_1x_1 is $(0, 0)$. The signature of v_1y_1 and y_1v_2 is $(0, 1)$. The opposite edges y_1v_1 and v_2y_1 have signature $(1, 0)$. The signature of v_1x_3 is $(1, 1)$. The dashed line is the curve C corresponding to the chosen pseudoaccesses at the critical values

is kept on the *right*, then the cyclically ordered sequence of oriented edges, along which x moves, coincides with $C(T)$. Even more informally: $C(T)$ corresponds to walking around T *clockwise*. The choice of the direction is explained as follows: as we walk around T clockwise, we walk around $\mathbb{S} - T$ counterclockwise. The postcritical pseudoaccesses divide all oriented edges from $C(T)$ into two groups (segments) $S^0(T)$ and $S^1(T)$. The labeling of $S^0(T)$ and $S^1(T)$ is chosen as follows. By definition, $S^0(T)$ originates at v_1 and terminates at v_2 . Then $S^1(T)$ originates at v_2 and terminates at v_1 . See Fig. 2 for an illustration. We can now assign *signatures* to all edges of T . We say that an oriented edge e of T is of signature (i, j) if e appears in $S^i(T)$, and e^{-1} appears in $S^j(T)$. Here, for an oriented edge e , we let e^{-1} denote the same edge with the opposite orientation. Thus there are four possible signatures: $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$.

For $i, j = 0, 1$, we write $S^i(T^j)$ for the pullback of $S^i(T)$ in T^j . The complement of G in \mathbb{S}^2 consists of two disks Ω^0 and Ω^1 . These disks are bounded by $S^0(T^0) \cup S^1(T^1)$ and $S^0(T^1) \cup S^1(T^0)$. We assume that Ω^0 to be the disk bounded by $S^0(T^0) \cup S^1(T^1)$. See Fig. 3 for an illustration. More precisely, the oriented boundary of Ω^0 , regarded as a chain of oriented edges of G , is the concatenation of $S^0(T^0)$ and $S^1(T^1)$. Then Ω^1 is bounded by $S^0(T^1) \cup S^1(T^0)$ in a similar sense. We will assume that $\Omega^0 \ni y$. The problem, however, is that the two assumptions

1. that $y \in \Omega^0$, and
2. that Ω^0 is bounded by the concatenation of $S^0(T^0)$ and $S^1(T^1)$

may not be compatible. There are two ways of making them both hold. On the one hand, we can choose y differently. Although this is easy in theory, we will not do this

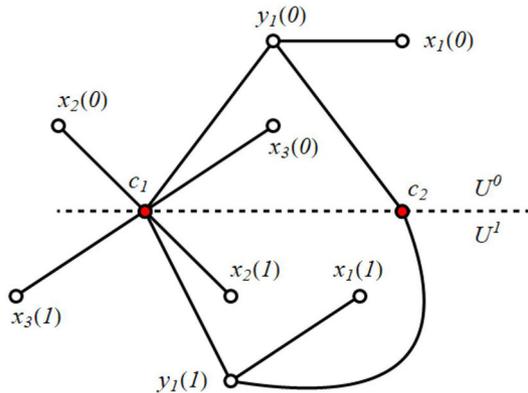


Fig. 3 The graph G and the disks Ω^0 and Ω^1 . Here $G = f^{-1}(T)$, where T is the tree from Fig. 2. The preimages of v_1, v_2 are the critical points c_1, c_2 , respectively. The preimages of other vertices of T are denoted as their images followed by a label 0 or 1 in the parentheses. All preimages in U^0 (the upper half-plane) are labeled 0 and all preimages in U^1 (the lower half-plane) are labeled 1. Then T^0 and T^1 are the copies of T in U^0 and U^1 , respectively. These copies are deformed but topologically the same as T . The sequence $S^0(T^0)$ goes through the vertices $c_1, x_2(0), c_1, y_1(0), x_1(0), y_1(0), c_2$. The sequence $S^1(T^1)$ goes through the vertices $c_2, y_1(1), c_1, x_3(1), c_1$. The disk Ω^0 bounded by $S^0(T^0) \cup S^1(T^1)$ is the exterior of the quadrilateral $c_1 y_1(0) c_2 y_1(1)$ with the arcs $c_1 x_3(1), c_1 x_2(0)$ and $y_1(0) x_1(0)$ removed

in practice. A basepoint will be fixed once and for all (see Assumption 6.3 for the principle of choosing the basepoint). On the other hand, we may relabel T^0 and T^1 by choosing e'_b differently. The edge e'_b of G is one of the two pullbacks of e_b ; the one not in T^* . If we change T^* , then we can also replace e'_b with the other pullback of e_b . In this way, we can satisfy both assumptions. This is how we will act in practice. At each step of our iterative process, we will define T^* (and e'_b) so that both assumptions hold. The exact procedure will be described later. For now, we just assume that both assumptions are satisfied.

5.4 The Choice of Paths α_0, α_1

We assume that (T^*, T) is a dynamical tree pair for f . As before, T comes with a specific choice of pseudoaccesses at the critical values. Recall that $f^{-1}(y) = \{y_0, y_1\}$. We label the preimages y_0, y_1 of y so that $y_i \in \Omega^i$ for $i = 0, 1$. Choose a path α_0 connecting y with y_0 outside of T^* . Similarly, choose a path α_1 connecting y with y_1 outside of T^* . Then $\mathcal{B} = \{[\alpha_0], [\alpha_1]\}$ is a basis of $\mathcal{X}_f(y)$. The basis \mathcal{B} is well defined and depends only on T^* and y . This description of \mathcal{B} is sufficient for now. However, for later use, we will need a more accurate description of α_0 and α_1 . We describe them up to a homotopy rel $f^{-1}(V(T))$ rather than rel $V(T)$. Choose a path α_0 connecting y to y_0 so that it is disjoint from G . This is possible. Indeed, according to assumption (1) made in Sect. 5.3, we have $y \in \Omega_0$. Recall also that Ω_0 is a topological disk, and that $y_0 \in \Omega_0$ by definition of y_0 . Therefore, y can be connected with $y_0 \in \Omega_0$ by a path in Ω_0 . This path is automatically disjoint from G ; and we take this path as α_0 . The path α_1 should be chosen so that it crosses G only once in a point of e'_b . We may arrange that α_1 is smooth and that the intersection is transverse. Since e'_b is not included into T^* , this description of α_1 is consistent with the earlier description.

For example, in Fig. 3, the path α_1 goes from the outside of the quadrilateral $c_1y_1(0)c_2y_1(1)$ to the inside. It may cross either $c_2y_1(1)$ or $y_1(1)c_1$; thus there are two possible choices for e'_b .

5.5 A More Precise Statement of Theorem B

In this section, we restate Theorem B more precisely and in a greater generality. Recall that $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is a Thurston map of degree two. We assume that f has a dynamical tree pair (T^*, T) . Let \mathcal{E}_T and \mathcal{E}_{T^*} be the generating sets of π_f defined as in Sect. 4.1. We will describe a map $\Sigma : \{0, 1\} \times \mathcal{E}_T \rightarrow \mathcal{E}_{T^*} \times \{0, 1\}$. By definition, $\Sigma(\varepsilon, [\gamma])$ is $([\alpha_\varepsilon \gamma^* \alpha_{\varepsilon^*}^{-1}], \varepsilon^*)$, where γ^* is an f -pullback of γ originating at y_ε and terminating at y_{ε^*} . Note that ε^* and γ^* are determined by ε and γ . The map Σ defines a presentation of $\mathcal{X}_f(y)$.

Recall our assumption on the basepoint y : the complementary component Ω^0 of $f^{-1}(T)$ containing y is bounded by $S^0(T^0)$ and $S^1(T^1)$.

Theorem 5.6 *We use the terminology and notation introduced above. Suppose that $\varepsilon \in \{0, 1\}$ and $g \in \mathcal{E}_T$. Then we have*

$$\Sigma(\varepsilon, g) = (g^*, \varepsilon^*),$$

where ε^* and g^* are defined as follows. If $g = id$, then $\varepsilon^* = \varepsilon$ and $g^* = id$. Suppose now that $g = g_e$, where e is an oriented edge of T of signature $(\varepsilon + \delta, \varepsilon^* + \delta)$. Here the addition is mod 2; observe that δ and ε^* are determined by ε and the signature of e . If there is an oriented edge e^* of \overline{T}^* labeled δ that maps over e preserving the orientation, then $g^* = g_{e^*}$. If there is no such edge, then $g^* = id$.

An edge e^* mapping over e preserving the orientation means that $f(e^*)$ is an oriented Jordan arc containing e , and that the orientation of $f(e^*)$ is consistent with that of e . Note that an element g_{e^*} for $e^* \in E(\overline{T}^*)$ is also an element of \mathcal{E}_{T^*} . If, say, a critical point divides an edge of T^* into two edges of \overline{T}^* , then these two edges give rise to the same pair of mutually inverse elements of \mathcal{E}_{T^*} . Suppose that $T^* = T$, then T is an invariant spanning tree for f . In this case, we obtain a finite state automaton $(\{0, 1\}, \mathcal{E}_T, \Sigma)$. Theorem 5.6 gives an explicit description of this automaton. Thus it provides a specification of Theorem B.

Corollary 5.7 *A dynamical tree pair (T^*, T) for f determines the map $\Sigma : \{0, 1\} \times \mathcal{E}_T \rightarrow \mathcal{E}_{T^*} \times \{0, 1\}$ that provides a presentation for the biset of f . In particular, the isomorphism class of the biset $\mathcal{X}_f(y)$ and hence the homotopy class of f are determined by (T^*, T) .*

Note that the description provided in Theorem 5.6 depends on the choice of pseudoaccesses. However, the end result, i.e., the map $\Sigma : \{0, 1\} \times \mathcal{E}_T \rightarrow \mathcal{E}_{T^*} \times \{0, 1\}$, is obviously independent of these choices.

Example 5.8 (An automaton for the basilica polynomial) Recall that the basilica polynomial is $p(z) = z^2 - 1$. It is easy to find a presentation for the biset of p directly (cf. Nekrashevych 2005, Section 5.2.2). However, we will use Theorem 5.6 in order to illustrate its statement. Let T be the invariant spanning tree for p defined in Example 2.1:



Since T is invariant, we may take $T^* = T$. The tree T has three vertices $-1, 0, \infty$ and two edges: $A = [-1, 0]$ and $B = [0, \infty]$. Note that A is the Hubbard tree for p . Orient these two edges from left to right (in the picture, the orientations are represented by arrows). Observe that A maps onto A reversing the orientation, and B maps onto $A \cup B$ preserving the orientation. We may represent this symbolically as

$$A \rightarrow A^{-1}, \quad B \rightarrow A, B.$$

Observe also that B contains the β -fixed point x_β of p , i.e., the landing point of the invariant external ray (the latter ray is also a part of B). Set $a = g_A$ and $b = g_B$. Thus we have $\mathcal{E} = \mathcal{E}_T = \{id, a, b, a^{-1}, b^{-1}\}$.

Theoretically, we have to make some choices. Observe that the location of the basepoint is irrelevant since the complement of T in the sphere is simply connected. The only possible choice for a base edge is B since $f^{-1}(A) \subset T$. Also, we need to choose two pseudoaccesses at the critical values $v_1 = -1$ and $v_2 = \infty$. However,

since both critical values are endpoints of T , these pseudoaccesses are unique. In order to implement the algorithm described in Theorem 5.6, we need to find labels and signatures. Since B is the base edge, and B maps over B , we have $\ell(B) = 0$. Indeed, the edge e'_b of $f^{-1}(G)$ not in T but mapping also over B has label 1 by definition. The two critical pseudoaccesses at 0 separate A from B , hence we have $\ell(A) = 1$. The boundary circuit $C(T)$ is $[A, B, B^{-1}, A^{-1}]$ (square brackets denote a cyclically ordered set). Here A^{-1}, B^{-1} stand for the edges A, B equipped with the opposite orientation. The two post-critical pseudoaccesses divide $C(T)$ into $S^0(T) = (A, B)$ and $S^1(T) = (B^{-1}, A^{-1})$. By Definition 5.5, both A and B have signature $(0, 1)$. The oriented edges A^{-1}, B^{-1} have signature $(1, 0)$. We can now compute (g^*, ε^*) for each pair $(\varepsilon, g) \in \{0, 1\} \times \mathcal{E}$ according to Theorem 5.6.

The computations can be organized as follows. Draw the following table:

	$a(0, 1)$	$b(0, 1)$	$a^{-1}(1, 0)$	$b^{-1}(1, 0)$
0				
1	$a^{-1}(1)$ $b(0)$	$b(0)$	$a(1)$ $b^{-1}(0)$	$b^{-1}(0)$

In the top row, we list all elements of $\mathcal{E} - \{id\}$. After each element, we indicate its signature. Thus, columns of the table (except for the leftmost one) are marked by oriented edges of T . These are the A -column, then the B -column, etc. The last row is temporarily filled as follows. In the A -column, we write all elements of the form g_e , where e is mapped over A preserving the orientation. In our case, these elements are $a^{-1} = g_{A^{-1}}$ and $b = g_B$. We proceed similarly with other columns. After each element of $\mathcal{E} - \{id\}$ in the last row, we indicate in the parentheses the label of the corresponding edge.

Now we can fill the second and the third rows of the table. For example, look at the A -column. Take one of the entries in the last row, say, $a^{-1}(1)$. Here a^{-1} is an element of \mathcal{E} and 1 is the label. Add the label to both components of the signature written in the same column. In our case, we obtain $(0 + 1, 1 + 1) = (1, 0)$. This means that $1.a = a^{-1}0$ by Theorem 5.6. We write $a^{-1}0$ at the intersection of the A -column with the row marked 1. Now take the remaining entry in the last row, $b(0)$. Adding the label to the signature, we obtain $(0 + 0, 1 + 0) = (0, 1)$. This means that $0.a = b1$. We write $b1$ at the intersection of the A -column with the row marked 0.

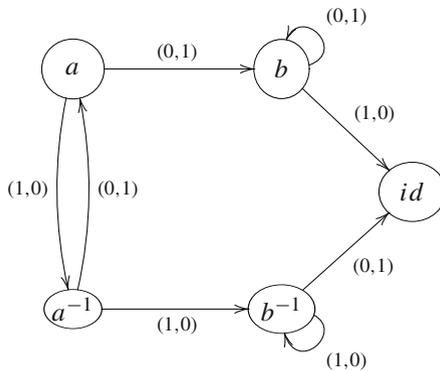
More generally, consider the column marked by an oriented edge e of T . In the first row, we indicated the signature $(\varepsilon + \delta, \varepsilon^* + \delta)$ of this edge, right after the corresponding element g_e . In the last row, we indicated an edge e^* mapping over e in an orientation preserving fashion, and the label δ of e^* . We add δ to both components of $(\varepsilon + \delta, \varepsilon^* + \delta)$ to obtain $(\varepsilon, \varepsilon^*)$. Then we have $\varepsilon.g_e = g_{e^*}\varepsilon^*$ by Theorem 5.6. We write $g_{e^*}\varepsilon^*$ at the intersection of the e -column with the row marked ε . If there is no edge e^* of label δ mapping over e preserving orientation, then we write ε^* .

Acting in this way, we obtain the following table (from which we removed the last row as it was not needed anymore).

	a	b	a^{-1}	b^{-1}
0	$b1$	$b1$	$a1$	1
1	$a^{-1}0$	0	$b^{-1}0$	$b^{-1}0$

Here, in order to evaluate $\Sigma(\varepsilon, g_e)$, one has to look at the intersection of the row marked $\varepsilon \in \{0, 1\}$ with the column marked $g_e \in \mathcal{E}$. The cell of the table at the given position contains $g_{e'}\varepsilon'$ or ε' . In the former case, we have $\Sigma(\varepsilon, g_e) = (g_{e'}, \varepsilon')$. In the latter case, we have $\Sigma(\varepsilon, g_e) = (id, \varepsilon')$.

The following is the Moore diagram for the obtained automaton.



5.6 Proof of Theorem 5.6

Set $G = f^{-1}(T)$. Since $f(T^*) \subset T$, we have $T^* \subset G$ set-theoretically. Moreover, all vertices of T^* are also vertices of G but, in general, not the other way around. Recall that complementary components of G correspond to boundary circuits of G . We will write $C^i(G)$ for the boundary circuit corresponding to Ω^i . Recall that the labeling of Ω^i was defined so that $C^0(G)$ is the concatenation of $S^0(T^0)$ and $S^1(T^1)$. The boundary circuit $C^1(G)$ is then the concatenation of $S^0(T^1)$ and $S^1(T^0)$.

Let e be an edge of T . Then $f^{-1}(e)$ can be represented as a union $e^0 \cup e^1$, where $e^0 \in E(T^0)$ and $e^1 \in E(T^1)$. The following proposition is an alternative description of the boundary circuits $C^i(G)$, where $i = 0, 1$.

Proposition 5.9 *Let e be an oriented edge of T of signature (i, j) . Then e^0 belongs to the boundary circuit $C^i(G)$ and e^1 belongs to the boundary circuit $C^{1-i}(G)$. In other words, e^δ belongs to $C^{i+\delta}(G)$, where $\delta = 0, 1$ and the addition is mod 2.*

Proof Suppose that the signature of e is (i, j) . It follows by definition of a signature that $e \in S^i(T)$. The edge e^0 of G is a part of T^0 hence also of $S^i(T^0)$. By definition, this means that $e^0 \in C^i(G)$. The proof of the claim that $e^1 \in C^{1-i}(G)$ is similar. \square

We are now ready to prove Theorem 5.6.

Proof of Theorem 5.6 Suppose that we are given $\varepsilon \in \{0, 1\}$ and $g \in \mathcal{E}_T$. If $g = id$, then the conclusion is obvious. Thus we may assume that $g = g_e$ for some $e \in E(T)$. Let g^* and ε^* be as in the statement of Theorem 5.6. Namely, let (i, j) be the signature of e . Set $\delta = \varepsilon + i \pmod 2$. Then we also have $i = \varepsilon + \delta \pmod 2$. Define ε^* as $j + \delta \pmod 2$. Then we also have $j = \varepsilon^* + \delta$. Thus the signature of e can be written as $(\varepsilon + \delta, \varepsilon^* + \delta)$. If there is an edge e^* of \overline{T}^* labeled δ that maps over e preserving the orientation, then we set $g^* = g_{e^*}$. Note that, if an edge e^* exists, then it is unique. Indeed, there is only one edge e^δ of T^δ mapping to e . This edge e^δ may or may not be a subset of T^* . If it is, then it is contained in a unique edge e^* of \overline{T}^* . We equip e^* with the orientation induced from the orientation of e by the map $f : e^\delta \rightarrow e$. Thus e^* is uniquely determined as an oriented edge of \overline{T}^* . If e^δ is not a subset of T^* , then we set $g^* = id$.

We now need to prove that $\Sigma(\varepsilon, g) = (g^*, \varepsilon^*)$, i.e., that $[\alpha_\varepsilon].g = g^*[\alpha_{\varepsilon^*}]$. Let γ_e be a smooth loop based at y , crossing T just once transversely and approaching it from the left. Thus $[\gamma_e] = g$. By definition $[\alpha_\varepsilon].g$ is (the homotopy class of) the concatenation of α_ε and a pullback γ_e^* of γ_e . The pullback γ_e^* should start at y_ε , where α_ε ends. The path γ_e^* approaches some boundary edge e' of Ω^ε . Thus e' is an edge of G . Equip e' with an orientation such that γ_e^* approaches e' from the left. Then e' is an element of the boundary circuit $C^\varepsilon(G)$ corresponding to the boundary of Ω^ε . Observe that e' must be a pullback of e , hence it must coincide with e^0 or with e^1 . We need to find which one. By Proposition 5.9, the edge e^δ belongs to $C^{i+\delta}(G) = C^\varepsilon(G)$. Therefore, we have $e' = e^\delta$. Since e is of signature $(\varepsilon + \delta, \varepsilon^* + \delta)$, the two sides of the arc e' belong to Ω^ε and Ω^{ε^*} . Indeed, the left side of e' is Ω^ε , as we already know. On the other hand, by Proposition 5.9, the opposite edge $(e')^{-1}$ belongs to the boundary circuit $C^{j+\delta}(G) = C^{\varepsilon^*}(G)$. It follows that the right side of e' is Ω^{ε^*} . When crossing e' , the path γ_e^* leaves Ω^ε and enters Ω^{ε^*} (it may be that $\varepsilon = \varepsilon^*$).

It follows that the path $\alpha_\varepsilon.\gamma_e$ terminates in Ω_{ε^*} . We must have then

$$[\alpha_\varepsilon].[\gamma] = [\alpha_\varepsilon\gamma_e^*\alpha_{\varepsilon^*}^{-1}][\alpha_{\varepsilon^*}],$$

and it remains to show that $[\alpha_\varepsilon\gamma_e^*\alpha_{\varepsilon^*}^{-1}] = g^*$.

Suppose first that e' is not a subset of T^* (then $g^* = id$). Then γ_e^* is disjoint from T^* . Since, by our assumption, α_0, α_1 are also disjoint from T^* , the loop $\alpha_\varepsilon\gamma_e^*\alpha_{\varepsilon^*}^{-1}$ lies entirely in $\mathbb{S}^2 - T^*$. The set $\mathbb{S}^2 - T^*$ is simply connected, therefore, this loop is contractible in $\mathbb{S}^2 - T^*$ and in $\mathbb{S}^2 - P(f) \supset \mathbb{S}^2 - T^*$. Thus both sides of the equality $[\alpha_\varepsilon\gamma_e^*\alpha_{\varepsilon^*}^{-1}] = g^*$ equal id , and the equality holds.

Finally, suppose that e' is a subset of T^* . Then, since the edge e^* of \overline{T}^* has label δ , we have $e^* \supset e'$. The path γ_e^* intersects G once, and approaches e^* from the left. Therefore, $[\alpha_\varepsilon\gamma_e^*\alpha_{\varepsilon^*}^{-1}] = g_{e^*}$, which proves the desired. □

6 The Ivy Iteration

We start with a geometric explanation of the process, after which we provide a formal combinatorial implementation.

6.1 A Geometric Description of the Iterative Process

Consider a marked Thurston map $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ of degree 2 with critical values v_1 and v_2 .

We now describe a procedure that, given a spanning tree T for f , allows to recover a dynamical tree pair (T^*, T) . Take the full preimage $G = f^{-1}(T)$. The basic idea is to select a spanning tree T^* in G . More precisely, we select some subtree of G containing $P(f)$ and then erase some of its vertices. Thus the choice of T^* is in general not unique. Below, we will give more precise comments on what it involves to make this choice, in terms of combinatorics.

Recall that the basepoint y is assumed to be outside of $T \cup G$. We also assume that y and y_0 are in the same component of $\mathbb{S}^2 - G$. Choose a base edge e_b of T . As above, we assume that e_b separates v_1 from v_2 . Having chosen a base edge e_b , we can recover T^* . There are two pullbacks of e_b in $G = f^{-1}(T)$. We choose one of the two pullbacks e'_b so that the following properties hold:

- The edge e'_b of G is oriented so that $f : e'_b \rightarrow e_b$ preserves the orientation.
- Consider a path in $\mathbb{S}^2 - G$ originating at y and approaching e'_b . This path approaches e'_b from the left.

Recall that Z is the smallest arc in T connecting v_1 with v_2 . Then the Jordan curve $f^{-1}(Z)$ consists of two pullbacks of Z . Both pullbacks of e_b are in $f^{-1}(Z)$. They are oriented both from c_1 to c_2 or both from c_2 to c_1 . Thus, one of them, e'_b , is oriented as the boundary of the component of $\mathbb{S}^2 - f^{-1}(Z)$ containing y . This shows that e'_b is well defined.

We can now define a spanning tree T^* . Clearly, $f^{-1}(Z)$ is the only simple loop in G . Thus, removing e'_b from G leads to a tree. We set \widehat{T} to be the smallest subtree of this tree containing $P(f)$. (In particular, all endpoints of \widehat{T} must be in $P(f)$). Finally, define T^* as the tree obtained from \widehat{T} by erasing all vertices of \widehat{T} that are not in $P(f)$ and are not branch points of \widehat{T} . The erased vertices become points in the edges of T^* . Then (T^*, T) is a dynamical tree pair for f . By definition of labels given in Sect. 5.2, we have $\ell(e'_b) = 1$, equivalently, $e'_b \in T^1$. By the properties of e'_b listed above, the orientation of e'_b corresponds to the orientation of $S^1(T^1)$. Hence the boundary circuit $C^0(G)$ corresponding to Ω^0 is the concatenation of $S^0(T^0)$ and $S^1(T^1)$. Thus our assumption made in Sect. 5.3 is fulfilled, and Theorem 5.6 applies to (T^*, T) .

The *topological ivy iteration* is aimed at finding an invariant spanning tree for f , up to homotopy, or, more generally, at finding periodic (also up to homotopy, to be made precise later) spanning trees. Note that an invariant (or periodic), up to homotopy, spanning tree for f yields a genuine invariant (or periodic) spanning tree for some map homotopic to f . Consider a dynamical tree pair (T^*, T) as above. Since there are finitely many choices for e_b , there are also finitely many choices for T^* .

Definition 6.1 (*Topological ivy object*) A (topological) *ivy object* is defined as a homotopy class of spanning trees for $P(f)$. We will write $\text{Ivy}(f)$ for the set of all ivy objects for f . For a fixed f with $|P(f)| \geq 4$, there are countably many ivy objects. There is a free action of the pure mapping class group of $(\mathbb{S}^2, P(f))$ on the set of ivy objects

of f . In general, this action is not transitive as, for example, spanning trees may have different combinatorics.

Note that, as a set, $\text{Ivy}(f)$ depends only on $P(f)$, not on f . However, we will introduce a relation on $\text{Ivy}(f)$ that will depend on the dynamics of f . If T is a spanning tree for f , then $[T]$ will denote the corresponding ivy object. Suppose that (T^*, T) is a dynamical tree pair as above. Then we will write $[T] \dashrightarrow [T^*]$, and call the thus defined relation on $\text{Ivy}(f)$ the *pullback relation*. The pullback relation on $\text{Ivy}(f)$ can be represented by a structure of an abstract directed graph. There are finitely many arrows originating at each element of $\text{Ivy}(f)$. The set $\text{Ivy}(f)$ equipped with the graph structure just described is called the *ivy graph* of f .

Recall that, a subset $C \subset \text{Ivy}(f)$ is *pullback invariant* if the following property holds: whenever $[T] \in C$ and $[T] \dashrightarrow [T^*]$, we also have $[T^*] \in C$. In other words, in the associated directed graph, there are no edges originating in C and terminating outside of C . Finding pullback invariant subsets of $\text{Ivy}(f)$ is obviously related to finding periodic spanning trees. By the way, we can now rigorously define a *periodic object* $\tau \in \text{Ivy}(f)$ as an element of some directed cycle in $\text{Ivy}(f)$. The length of any simple cycle containing τ is called a *period* of τ . (Note that τ may have several periods according to this definition. Moreover, as a rule, periodic ivy objects do have several different periods.) If T is a spanning tree such that $[T]$ is periodic of period p , then we say that the spanning tree T is periodic of period p , up to homotopy.

There is no way of defining the forward image of an ivy object $[T]$ under f simply because homotopies $\text{rel. } P(f)$ are not preserved by f . This is also confirmed by the existence of a periodic ivy object with several different periods.

6.2 A Formal Description of the Ivy Iteration

The purpose of this section is to give a compact and precise description of the computational scheme. The scheme has been implemented (as a Wolfram Mathematica code) according to this description. Motivations and geometric explanations are given above. However, we will still need some work to relate the geometric story to the combinatorial story.

Push Forward of a Generating Set

Let \mathcal{X} be an abstract left free biset over a group π . Consider a basis \mathcal{B} of \mathcal{X} and a symmetric generating set \mathcal{E} of π . (Recall that *symmetric* means that $g^{-1} \in \mathcal{E}$ for every $g \in \mathcal{E}$). We will assume that $1 \in \mathcal{E}$. Consider the full automaton $\Sigma : \mathcal{B} \times \pi \rightarrow \pi \times \mathcal{B}$ of \mathcal{X} . Set $\Sigma = (\sigma, \iota)$, so that $\Sigma(a, g) = (\sigma(a, g), \iota(a, g))$ for all $a \in \mathcal{B}$, $g \in \pi$. Define the *push forward* $P\mathcal{E}$ of \mathcal{E} as the set $\sigma(\mathcal{B}, \mathcal{E})$. In other words, $P\mathcal{E}$ consists of all elements of the form $\sigma(a, g)$, where $a \in \mathcal{B}$ and $g \in \mathcal{E}$. Note also that the set $P\mathcal{E}$ coincides with the set of all restrictions (sections) of generators from \mathcal{E} under the associated wreath recursion, in Nekrashevych's (2005) terminology. It is easy to see that $P\mathcal{E}$ is also a symmetric set containing 1. Note that a push forward of \mathcal{E} will correspond to a pullback of a tree and, in our case, it will also be a generating set.

For our purposes, we need to combine the push forward operation with a change of a basis.

Push Forwards with Simultaneous Basis Changes

As before, \mathcal{X} is an abstract left free biset over a group π . Let \mathcal{B} be a basis of \mathcal{X} . Any function $\lambda : \mathcal{B} \rightarrow \pi$ defines a basis change: the new basis consists of $\lambda(a)a$, where $a \in \mathcal{B}$. Denote this basis change by C_λ . We will also write a_λ for $\lambda(a)a$. The new basis obtained from \mathcal{B} through C_λ will be denoted by $C_\lambda\mathcal{B}$. Let $C_\lambda\Sigma$ be the full automaton of \mathcal{X} in the new basis $C_\lambda\mathcal{B}$. We will now see how the maps Σ and $C_\lambda\Sigma$ are related to each other. The following simple computation solves the problem. Take any $a \in \mathcal{B}$ and any $g \in \pi$, and set $(g^*, a^*) = \Sigma(a, g)$. Then the following equalities hold in \mathcal{X} :

$$a_\lambda \cdot g = \lambda(a)a \cdot g = \lambda(a)g^*a^* = \lambda(a)g^*\lambda(a^*)^{-1}a_\lambda^*.$$

This computation shows that $C_\lambda\Sigma = (C_\lambda\sigma, C_\lambda\iota)$ is given by

$$C_\lambda\sigma(a_\lambda, g) = \lambda(a)g^*\lambda(a^*)^{-1}, \quad C_\lambda\iota(a_\lambda, g) = a_\lambda^*.$$

Given a symmetric generating set \mathcal{E} of π containing 1, set $P_\lambda\mathcal{E}$ to be the generating set obtained as the push forward of \mathcal{E} under $C_\lambda\Sigma$.

Push Forwards of Tree Structures

In contrast to the above, we now explicitly assume that \mathcal{B} consists of two elements. As always, we assume that \mathcal{E} is symmetric and contains 1. We say that \mathcal{E} is *tree-like* if there exists a tree structure \mathcal{V} on \mathcal{E} , cf. Sect. 4.2. Fix a tree-like generating set \mathcal{E} . Below, we will describe a certain set of basis changes λ for which $P_\lambda\mathcal{E}$ are also tree-like. For each such λ , the corresponding tree structure $P_\lambda\mathcal{V}$ on $P_\lambda\mathcal{E}$ is defined below. The map $\Sigma : \mathcal{B} \times \mathcal{E} \rightarrow P\mathcal{E} \times \mathcal{B}$ can be extended to a map

$$\Sigma^* = (\sigma^*, \iota^*) : \mathcal{B} \times \mathcal{E}^* \rightarrow (P\mathcal{E})^* \times \mathcal{B}.$$

The map Σ^* is defined inductively as follows. If \emptyset denotes the empty word in \mathcal{E}^* , then we set $\Sigma^*(a, \emptyset) = (\emptyset, a)$. Suppose now that an element of \mathcal{E}^* has the form $g \cdot w$, where $g \in \mathcal{E}$ and $w \in \mathcal{E}^*$. Set $(g^*, a^*) = \Sigma(a, g)$. Then we set

$$\sigma^*(a, g \cdot w) = g^*\sigma^*(a^*, w), \quad \iota^*(a, g \cdot w) = \iota^*(a^*, w).$$

Replacing \mathcal{B} with $C_\lambda\mathcal{B}$, we may assume that $\lambda \equiv 1$. Suppose that $\mathcal{B} = \{a, b\}$. In order to define the new vertex set $P\mathcal{V} \subset (P\mathcal{E})^*$, we first consider the following three sets:

$$\begin{aligned} \mathcal{V}(a) &= \{\sigma^*(a, v) \mid v \in \mathcal{V}, \iota^*(a, v) = a\}, \\ \mathcal{V}(b) &= \{\sigma^*(b, v) \mid v \in \mathcal{V}, \iota^*(b, v) = b\}, \\ \mathcal{V}(a, b) &= \{\sigma^*(a, v) \cdot \sigma^*(b, v) \mid v \in \mathcal{V}, \iota^*(a, v) = b\}. \end{aligned}$$

Now take the union of these three sets, remove the trivial element $1 \in \mathcal{E}^*$ from it as well as all elements of the form $g \cdot g^{-1}$, where $g \in \mathcal{E}$. The remaining set is $P\mathcal{V}$.

The Combinatorial Ivy Iteration

Thus, for every tree-like generating set \mathcal{E} of π , there are several tree-like generating sets of the form $P_\lambda \mathcal{E}$. Set $\mathcal{B} = \{a, b\}$. We will only consider basis changes associated with functions $\lambda : \mathcal{B} \rightarrow \pi$ such that $\lambda(a) = 1$. This is equivalent to saying that the first basis element a will be fixed once and for all. In our implementation, this will be the class of a constant loop.

Suppose that $g \in \mathcal{E}$ is an element with the property that $\iota(0, g) = 1$. Any such element g is called a *base element*. With any base element g , we associate the function $\lambda_g : \mathcal{B} \rightarrow \pi$ such that $\lambda_g(b) = \sigma(0, g)$. (Recall that $\lambda(a) = 1$.) The *combinatorial ivy iteration* is the process of passing from \mathcal{E} to $P_{\lambda_g} \mathcal{E}$.

Define a *combinatorial ivy object* as a tree-like generating set, up to conjugacy. More precisely, a combinatorial ivy object is a conjugacy class of tree-like generating sets. If \mathcal{E} is such a generating set, then its conjugacy class will be denoted by $[\mathcal{E}]$. Let $\text{Ivy}_c(f)$ be the set of all combinatorial ivy objects with $\pi = \pi_f$. For a pair of generating sets \mathcal{E} and $\mathcal{E}' = P_{\lambda_g} \mathcal{E}$ as above, connect $[\mathcal{E}]$ with $[\mathcal{E}']$ by a directed edge. Each pair $([\mathcal{E}], [\mathcal{E}'])$ yields only one directed edge, no matter in how many ways $[\mathcal{E}']$ can be represented in the form $[P_{\lambda_g} \mathcal{E}]$. In this way, $\text{Ivy}_c(f)$ becomes a directed graph. We will show that the combinatorial ivy iteration represents the topological ivy iteration. In particular, the graph $\text{Ivy}(f)$ is isomorphic to a subgraph of $\text{Ivy}_c(f)$. A more precise statement is given in the following theorem:

Theorem 6.2 *There is an isomorphic embedding of $\text{Ivy}(f)$ into $\text{Ivy}_c(f)$. This embedding takes a class $[T]$ of a spanning tree T to the conjugacy class of the corresponding generating set \mathcal{E}_T . Let T and a base edge e_b of T define a dynamical tree pair (T^*, T) as in Sect. 6.1. If g is the element of \mathcal{E}_T corresponding to e_b , then T^* corresponds to $P_{\lambda_g} \mathcal{E}_T$. There is a canonical tree structure \mathcal{V} on \mathcal{E}_T such that $G(\mathcal{V})$ is isomorphic to T . The tree structure \mathcal{V}^* on \mathcal{E}_{T^*} corresponding to T^* is obtained as $P_{\lambda_g} \mathcal{V}$.*

6.3 Translation from Geometry to Combinatorics

Proposition 4.3 implies that there is at least a set-theoretic embedding of $\text{Ivy}(f)$ into $\text{Ivy}_c(f)$. Our symbolic implementation of the ivy iteration will rely on the following assumption.

Assumption 6.3 Whenever we consider a spanning tree T for $P(f)$, we assume that there is a fixed point y_0 of f outside of T . Moreover, as long as we deal with spanning trees that eventually map to T , the point y_0 is kept the same. The point y_0 will be used as the basepoint for π_f .

Assumption 6.3 can always be fulfilled if we replace f with a homotopic Thurston map. Indeed, choose a point $y_0 \notin T$ with $f(y_0) \notin T$ and a path β connecting $f(y_0)$ to y_0 outside of T . Then the assumption is satisfied if we replace f with $\sigma_\beta \circ f$. Here σ_β is a path homeomorphism introduced in Sect. 2.3. Now, if y_0 is fixed under f , then

y_0 is disjoint from $f^{-n}(T)$ for all $n \geq 0$. In particular, if (T^*, T) is a dynamical tree pair for f , then $y_0 \in \mathbb{S}^2 - T^*$. Thus we can keep the same f -fixed base point during the ivy iteration.

The Basis Change Associated with a Dynamical Tree Pair

Consider a spanning tree T for f , and choose a base edge e_b of T . The choice of e_b defines a dynamical tree pair (T^*, T) as in Sect. 6.1. Consider the basis $\mathcal{B} = \{[\alpha_0], [\alpha_1]\}$ of $\mathcal{X}_f(y_0)$ associated with T . According to the convention introduced in Sect. 5.4, the element $[\alpha_0]$ is the class of the constant path. The path α_1 connects y_0 with another preimage y_1 of y_0 outside of T . This property defines \mathcal{B} uniquely. Let now $\mathcal{B}^* = \{[\alpha_0^*], [\alpha_1^*]\}$ be the basis associated with T^* . We need to show that $[\alpha_1^*] = \sigma(0, g)[\alpha_1]$, where g is the element of \mathcal{E}_T corresponding to e_b . Then we will have $\mathcal{B}^* = C_{\lambda_g} \mathcal{B}$. This is a part of the correspondence between the geometric and the combinatorial ivy iterations.

Set $h = \sigma(0, g)$. Let e'_b be the edge of $f^{-1}(T)$ of label 1 that is not in T^* . Recall that α_1^* is defined as a path from y_0 to y_1 crossing e'_b and disjoint from $f^{-1}(T)$ otherwise. In Sect. 6.1, we have chosen e'_b so that α_1^* approaches it from the left. It follows that $[\alpha_0].g = [\alpha_1^*]$ in $\mathcal{X}_f(y_0)$. Thus we can define α_1^* as $\alpha_0.\gamma_{e_b}$. On the other hand, we have $\Sigma(0, g) = (h, 1)$, therefore, $[\alpha_1^*] = [\alpha_0].g = h[\alpha_1]$, as desired.

6.4 Pullback and Vertex Words

In this section, we study the effect of the pullback relation on vertex words.

Proposition 6.4 *Let x be a vertex of T , and $v \in \mathcal{E}^*$ be the corresponding vertex word. The vertex x is a critical value of f if and only if $\iota^*(0, v) = 1$. In this case, we also have $\iota^*(1, v) = 0$.*

Proof Let $\Pi : \mathcal{E}^* \rightarrow \pi_f$ be the evaluation map. Note that $\iota^*(\varepsilon, v) = \iota(\varepsilon, \Pi(v))$ for each $\varepsilon \in \{0, 1\}$. The element $\Pi(v) \in \pi_f$ is represented by a loop around x that crosses T only in a small neighborhood of x . We may assume that this loop γ is smooth and simple. Then it bounds a disk D such that $D \cap V(T) = \{x\}$. The two f -pullbacks of γ are loops or not loops depending on whether x is a critical value or not. On the other hand, these pullbacks are loops if and only if $\iota(\varepsilon, \Pi(v)) = \varepsilon$ for all $\varepsilon = 0, 1$. □

Suppose now that $v = a_0 \cdots a_{k-1}$. Consider elements $\sigma^*(0, v) = b_0 \cdots b_{k-1}$ and $\sigma^*(1, v) = c_0 \cdots c_{k-1}$ of $(P\mathcal{E})^*$. Here b_i and c_i are elements of π_f , for $i = 0, \dots, k-1$. Set $\varepsilon_0 = 0$ and $\varepsilon_i = \iota^*(0, a_0 \cdots a_{i-1})$ for $i = 1, \dots, k$. Similarly, we set $\delta_0 = 0$ and $\delta_i = \iota^*(1, a_0 \cdots a_{i-1})$ for $i = 1, \dots, k$. Suppose first that $\varepsilon_k = 1$ (then also $\delta_k = 0$). Then we define the word $w(0, v) = w(1, v) \in (P\mathcal{E})^*$ as

$$\sigma^*(0, v)\sigma^*(1, v) = b_0 \cdots b_{k-1} \cdot c_0 \cdots c_{k-1}.$$

Suppose now that $\varepsilon_k = 0$ (then also $\delta_k = 1$). Then we define

$$w(0, v) = \sigma^*(0, v) = b_0 \cdots b_{k-1}, \quad w(1, v) = \sigma^*(1, v) = c_0 \cdots c_{k-1}.$$

Proposition 6.5 *Suppose that the basis \mathcal{B} of $\mathcal{X}_f(y_0)$ corresponds to T^* . Then the tree structure \mathcal{V}^* on \mathcal{PE} corresponding to T^* coincides with the set of $w(\varepsilon, v) \in (\mathcal{PE})^*$, where ε runs through $\{0, 1\}$, and v runs through \mathcal{V} , except that we omit $w(\varepsilon, v)$ if it is empty or it has the form $a \cdot a^{-1}$ for some $a \in \mathcal{E}$. In other words, we have $\mathcal{V}^* = P\mathcal{V}$.*

Proof Let x be a vertex of T , and v be the corresponding vertex word. Let A_0, \dots, A_{k-1} be the oriented edges of T such that $a_i = g_{A_i}$ for $i = 0, \dots, k - 1$. Define β_i as a smooth path disjoint from T , starting at y , and ending in a small neighborhood of x . Choose β_i so that it approaches x between A_{i-1} and A_i . Here $i - 1$ is understood modulo k so that for $i = 0$ we have $A_{i-1} = A_{k-1}$. Choose the paths γ_{A_i} with $a_i = [\gamma_{A_i}]$ as $\beta_i \gamma_i \beta_{i+1}^{-1}$, where γ_i is a short path in a small neighborhood of x crossing A_i just once and transversely. We may assume that γ_i is disjoint from all other edges of T . For $\varepsilon \in \{0, 1\}$, consider the pullback β_i^ε of β_i originating at y_ε .

Suppose first that x is not a critical value of f . Then there are two preimages x_0 and x_1 of x . We will prove that the vertex words of x_0, x_1 are $w(0, v), w(1, v)$ (not necessarily in this order), provided that x_0 and x_1 are vertices of T^* . Without loss of generality, we may assume that β_0^0 ends near x_0 . (Otherwise, simply swap x_0 and x_1 .) Set γ_i^ε to be the pullback of γ_i near x_ε . An important observation is that a pullback of a short path is short. Therefore, γ_i^ε must indeed stay near one preimage of x rather than wander between the two preimages. Then the induction on i shows that b_i is represented by $\alpha_{\varepsilon_i} \beta_i^{\varepsilon_i} \gamma_i^0 (\beta_{i+1}^{\varepsilon_{i+1}})^{-1} \alpha_{\varepsilon_{i+1}}^{-1}$ and that all $\beta_i^{\varepsilon_i}$ end near x_0 . Note also that $b_i = id$ if and only if γ_i^0 does not cross any edge of T^* . Otherwise it crosses exactly one edge. The composition of all γ_i^0 is a small loop around x_0 . Moreover, this loop is a pullback of the small loop γ around x , which is the composition of all γ_i . It follows that the oriented edges of T^* coming out of x_0 correspond precisely to non-identity elements b_i . This means that $w(0, v)$ is the vertex word for x_0 ; the proof of $w(1, v)$ being the vertex word for x_1 is similar.

Suppose now that x is a critical value of f . Then there is just one preimage $x_0 = x_1$ of x . We will prove that $w(0, v) = w(1, v)$ is the vertex word for $x_0 = x_1$ provided that x_0 is a vertex of T^* . Set γ_i^0 be the pullback of γ_i originating where $\beta_i^{\varepsilon_i}$ ends, and γ_i^1 be the pullback of γ_i originating where $\beta_i^{\delta_i}$ ends. Since $\delta_i \neq \varepsilon_i$, the paths γ_i^0 and γ_i^1 are always different pullbacks of γ_i . Similarly to the above, b_i is represented by $\alpha_{\varepsilon_i} \beta_i^{\varepsilon_i} \gamma_i^0 (\beta_{i+1}^{\varepsilon_{i+1}})^{-1} \alpha_{\varepsilon_{i+1}}^{-1}$ and c_j is represented by $\alpha_{\delta_j} \beta_j^{\delta_j} \gamma_j^1 (\beta_{j+1}^{\delta_{j+1}})^{-1} \alpha_{\delta_{j+1}}^{-1}$. The composition of all γ_i^0 is a pullback of γ but it is not a loop; it is only a ‘‘half’’ of a loop. The other half is the composition of all γ_i^1 , which is also the other pullback of γ . It follows that oriented edges of T^* coming out of x_0 correspond precisely to non-identity elements b_i or non-identity elements c_j . This means that $w(0, v) = w(1, v)$ is the vertex word for x_0 .

To conclude the proof, we observe that any vertex of T^* is mapped to a vertex of T . Thus any vertex of T^* can be obtained as described above. If $w(\varepsilon, v)$ is empty, then obviously, the corresponding point x_ε of $f^{-1}(T)$ does not belong to T^* . If $w(\varepsilon, v)$ has the form $a \cdot a^{-1}$, then x_ε belongs to an edge of T^* corresponding to a (thus, in particular, x_ε is not a vertex of T^*). Conversely, if x_ε is not a vertex of T^* , then this may be due to one of the following reasons. Firstly, we may have $x_\varepsilon \notin T^*$, then

$w(\varepsilon, v)$ is empty. Secondly, x_ε may belong to some edge of T^* . In this case, $w(\varepsilon, v)$ must have the form $a \cdot a^{-1}$, where $a \in P\mathcal{E}$ corresponds to this edge. \square

Proposition 6.5 concludes the proof of Theorem 6.2.

7 Examples of the Ivy Iteration

In this section, we describe some particular computations of the ivy graphs made according to the ivy iteration. We consider only the simplest examples, for which other, sometimes more efficient, computational approaches to distinguishing Thurston equivalence classes are available. In particular, in most examples, particular invariant spanning trees are known. We find (conjecturally) all periodic spanning trees in these examples. Also, we find some pullback invariant sets of ivy objects and show their combinatorial structure.

In Kelsey and Lodge (2018), all non-Euclidean Thurston maps with at most 4 post-critical points are classified, and an algorithm is suggested for solving the twisting problem for such maps. However, invariant spanning trees for rational maps from Kelsey and Lodge (2018) are not immediate from the provided description.

More complicated examples will be worked out in a separate publication.

7.1 The Basilica Polynomial

Let us go back to Example 5.8. This example deals with the basilica polynomial $f(z) = z^2 - 1$. We started with an invariant spanning tree

$$\bullet \xrightarrow{A} \circ \xrightarrow{B} \bullet$$

and deduced the corresponding presentation of the biset $\mathcal{X}_f(y)$, see Fig. 4. This is enough to start the combinatorial ivy iteration. This process leads to a pullback invari-

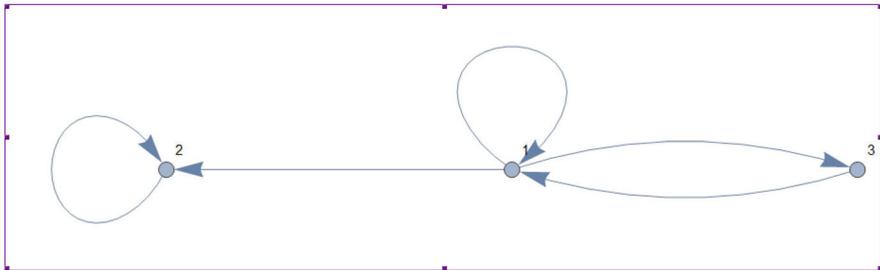


Fig. 4 A pullback invariant subset of 3 elements in $Ivy(f)$, where $f(z) = z^2 - 1$ is the basilica polynomial. Arrows represent the pullback relation. Vertex 1 corresponds to the tree T . Vertex 2 corresponds to an invariant spider for f . Vertex 3 corresponds to a spanning tree that is not invariant up to homotopy; it is periodic of period 2. Note that vertex 1 has periods 1 and 2

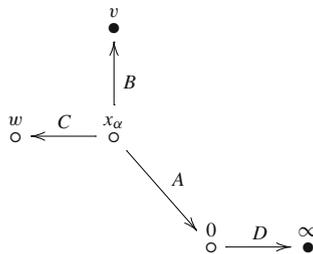
ant subset of $\text{Ivy}_c(f)$ consisting of 3 combinatorial ivy objects. Two of these objects correspond to invariant spanning trees.

The two (up to homotopy) invariant spanning trees of f are easy to describe. The first one is the Hubbard tree connected to infinity as described in Sect. 2.1. The second one is a spider in the sense of Hubbard and Schleicher (1994). There is only one remaining ivy object in the given pullback invariant subset of $\text{Ivy}(f)$. It does not correspond to an invariant spanning tree for f .

Note that $\text{Ivy}(f)$ consists of 5 objects. These objects are the same for all f with a given post-critical set $P(f)$ of 3 elements. However, the pullback relations (in particular, pullback invariant subsets) defined by f are different. Three of the elements of $\text{Ivy}(f)$ correspond to the unions of two sides of the triangle with vertices in $P(f)$. The remaining two elements are stars with endpoints in $P(f)$. The two stars differ by the cyclic order of edges at the only branch point. Recall that Teichmüller theory provides powerful invariants of Thurston equivalence classes in form of certain spaces, groups, correspondences between spaces, and virtual homomorphisms between groups. However, all these invariants are trivial in the case $|P(f)| = 3$.

7.2 The Rabbit Polynomial

The rabbit polynomial $p(z) = z^2 + c$ is such that 0 is periodic of period 3, and $\text{Im}(c) > 0$. These conditions determine c uniquely. Indeed, the period 3 assumption leads to a cubic equation on c , which has one real and two complex conjugate roots. An invariant spanning tree T for f constructed as in Sect. 2.1 looks as follows.



The choice of the basepoint y for the fundamental group $\pi_f = \pi_1(\mathbb{S}^2 - P(f), y)$ is irrelevant. Indeed, the complement of T is simply connected. The elements of π_f associated with the edges of T will be denoted by a, b, c, d , so that a small letter denotes g_e , where e is the edge denoted by the corresponding capital letter. Thus \mathcal{E}_T consists of id, a, b, c , and their inverses. The edges of T map forward as follows:

$$A \rightarrow B, \quad B \rightarrow C, \quad C \rightarrow A, \quad D \rightarrow B^{-1}AD.$$

Let us compute the map $\Sigma : \{0, 1\} \times \mathcal{E}_T \rightarrow \mathcal{E}_T \times \{0, 1\}$ from Theorem 5.6. We choose D as the base edge. Then we have the following labels:

$$\ell(A) = \ell(B) = \ell(C) = 1, \quad \ell(D) = 0.$$

The two pseudoaccesses of T at the critical values $v_1 = v$ and $v_2 = \infty$ are unique. We have

$$S^0(T) = (B^{-1}, A, D), \quad S^1(T) = (D^{-1}, A^{-1}, C, C^{-1}, B).$$

By Definition 5.5, the edges A and D have signature $(0, 1)$, the edge B has signature $(1, 0)$, the edge C has signature $(1, 1)$.

By Theorem 5.6, a presentation for $\mathcal{X}_f(y_0)$ looks as follows

	a	b	c	d	a^{-1}	b^{-1}	c^{-1}	d^{-1}
0	$d1$	$a1$	$b0$	$d1$	$c^{-1}1$	$d1$	$b^{-1}0$	1
1	$c0$	$d^{-1}0$	1	0	$d^{-1}0$	$a^{-1}0$	1	$d^{-1}0$

Note that $a = b^{-1}c^{-1}$, so that it is enough to use only b, c and d as generators of π_f . We will identify π_f with the free group generated by b, c , and d . Then the tree structure on \mathcal{E}_T consists of $b^{-1}c^{-1} \cdot c \cdot b, cb \cdot d, b^{-1}, c^{-1}$, and d^{-1} . Using the combinatorial ivy iteration, we found a pullback invariant subset of $\text{Ivy}(p)$ consisting of 10 ivy objects, see Fig. 5. This is the pullback invariant subset containing the class of the tree T .

We see that, similarly to the basilica, there are two invariant spanning trees for f , up to homotopy, among the trees representing objects in the found pullback invariant subset. One tree corresponds to the Hubbard tree connected to ∞ . The other tree is an invariant spider.

7.3 Simple Capture of the Basilica at $\sqrt{2}$

Let $p(z) = z^2 - 1$ be the basilica polynomial. The point $\sqrt{2}$ is preperiodic under p of preperiod 2: it maps to 1, and 1 maps to -1 . It follows that the simple capture f of p at $\sqrt{2}$ has the following invariant spanning tree T :

$$\bullet \xrightarrow{A} \circ \xrightarrow{B} \circ \xrightarrow{C} \bullet$$

We oriented the edges of T from left to right, and labeled them A, B, C . The corresponding symmetric generating set of π_f is $\mathcal{E}_T = \{1, a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\}$, where $a = g_A, b = g_B, c = g_C$.

Since -1 and $\sqrt{2}$ are incident each to a unique edge of T , there are unique pseudoaccesses at -1 and $\sqrt{2}$. As the base edge of T , we take C . Note that 0 is the only critical point in T , thus it separates edges of different labels. We may assume that

$$\ell(A) = 0, \quad \ell(C) = \ell(B) = 1$$

although the opposite assignment of labels is also possible. In fact, there are two edges of $G = f^{-1}(T)$ mapping onto C . They are separated by a critical point mapping to

$\sqrt{2}$. One or the other assignment of labels depends on which of the two edges is chosen as e'_b . The latter, in turn, depends on which complementary component of G contains the point y . The oriented edges A, B, C have signature $(0, 1)$. The opposite oriented edges A^{-1}, B^{-1}, C^{-1} have signature $(1, 0)$.

By Theorem 5.6, the biset of f is represented as follows:

	a	b	c	a^{-1}	b^{-1}	c^{-1}
0	$a^{-1}1$	1	1	$b^{-1}1$	$c^{-1}1$	1
1	$b0$	$c0$	0	$a0$	0	0

With the help of the ivy iteration, we found the pullback invariant subset of $\text{Ivy}(f)$ of order 40 containing $[T]$. Within this subset, there are three invariant spanning trees for f , up to homotopy, see Fig. 6. Vertex 3 corresponds to the invariant spanning tree

$$\bullet \text{---} \overset{-1}{\circ} \text{---} \overset{0}{\circ} \text{---} \overset{1}{\circ} \text{---} \overset{\sqrt{2}}{\bullet}$$

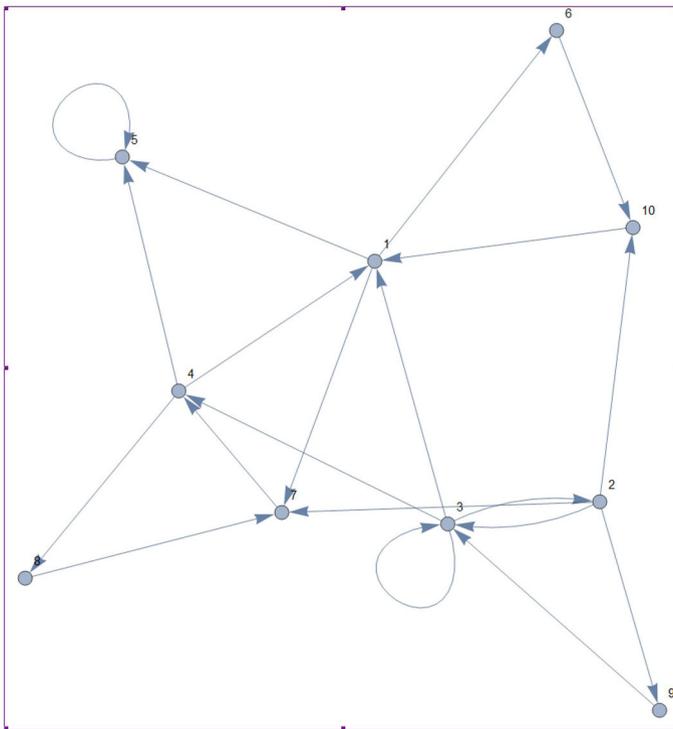


Fig. 5 The pullback invariant subset of $\text{Ivy}(f)$ containing $[T]$, where f is the rabbit polynomial, and T is the invariant spanning tree for f obtained by connecting the Hubbard tree to ∞ . This subset consists of 10 elements. Vertex 5 represents an invariant spider, and vertex 3 represents T

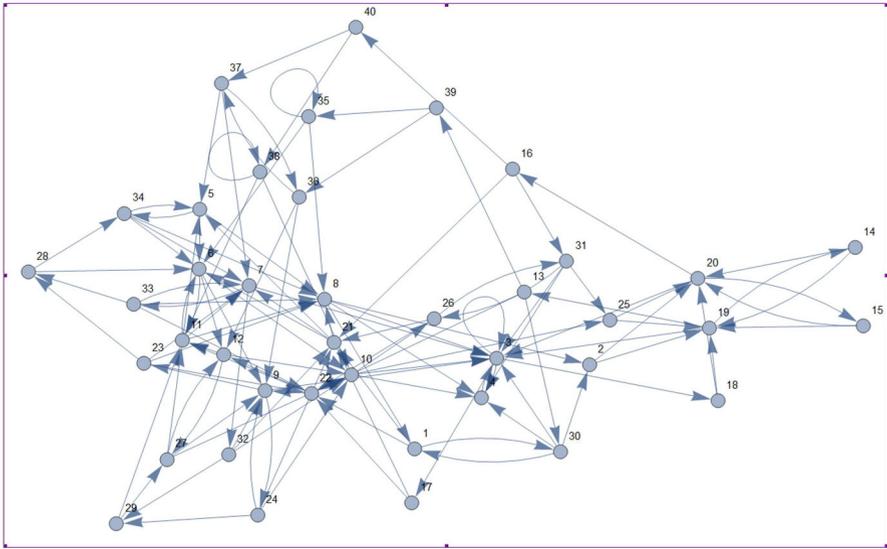
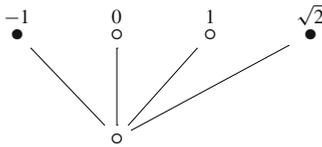
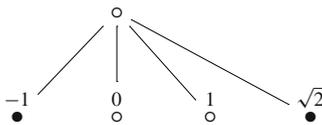


Fig. 6 The pullback invariant subset of 40 elements in $Ivy(f)$ containing $[T]$. Here f is a simple capture of the basilica at $\sqrt{2}$, and T is the invariant spanning tree for f introduced in Sect. 7.3. Vertices 3, 35 and 38 represent invariant spanning trees for f

Vertex 35 corresponds to the invariant spanning tree



Finally, vertex 38 corresponds to the invariant spanning tree



7.4 A Capture of the Chebyshev Polynomial

Finally, we consider an example, where an invariant spanning tree is not known a priori. Namely, we take a simple capture of the Chebyshev polynomial $p(z) = z^2 - 2$ whose post-critical set has cardinality 4. There are two preimages of 0 under p , namely, $\pm\sqrt{2}$. We restrict our attention to a simple capture of p at $\sqrt{2}$. There are two simple captures of p at $\sqrt{2}$ corresponding to capture paths β_u and β_d (“u” and “d” are from “up” and “down”). We may define β_u as a path along the external ray of argument $\frac{1}{8}$, and β_d as a path along the external ray of argument $\frac{7}{8}$. Clearly, any other simple capture path

for p ending at $\sqrt{2}$ is homotopic to β_u or β_d relative to the set $\{\sqrt{2}, 0, -2, 2\}$ (which is the post-critical set of the captures). The extended Hubbard tree T with vertices in $\{\sqrt{2}, 0, -2, 2\}$ is the following:



(We have marked post-critical points of the capture rather than of p .)

Consider $f_u = \sigma_{\beta_u} \circ p$. To lighten the notation, we will write σ_u instead of σ_{β_u} . The full preimage $G_u = f_u^{-1}(T)$ can be obtained as the full preimage under p of $\sigma_u^{-1}(T)$. The tree $\sigma_u^{-1}(T)$ can be represented (up to homotopy) as follows. Comparing $\sigma_u^{-1}(T)$ to T : the edge $[-2, 0]$ is preserved in $\sigma_u^{-1}(T)$. The edges $[0, \sqrt{2}]$ and $[\sqrt{2}, 2]$ are replaced with the external rays of arguments $\frac{1}{4}$ and 0 , respectively. See Fig. 7, top right, for an illustration of $\sigma_u^{-1}(T)$.

Next, G_u is obtained as the full p -preimage of $\sigma_u^{-1}(T)$, see Fig. 7, bottom left. The graph G_u consists of the line segments $[-\sqrt{2}, 0]$, $[0, \sqrt{2}]$, and the external rays $R_p(0), R_p(\frac{1}{2}), R_p(\frac{1}{8}), R_p(\frac{5}{8})$. Here $R_p(\theta)$ stands for the external ray of argument θ in the dynamical plane of p . We want to find a spanning tree $T^* \subset G_u$ so that to make (T^*, T) into a dynamical tree pair.

Denote the edges of T as A, B, C and orient them as in Fig. 7, top left. Then $Z = A \cup B$ is a simple arc in T connecting the two critical values (this is consistent with the meaning of the symbol Z in Sects. 5.2 and 6.1). The full preimage $f_u^{-1}(Z)$ is the simple closed curve consisting of the segments $[-\sqrt{2}, 0]$, $[0, \sqrt{2}]$, the rays $R_p(\frac{1}{8}), R_p(\frac{5}{8})$, and the point ∞ . We need to choose an edge e'_b in $f_u^{-1}(Z)$ that will not be included into T^* . Take e'_b to be the edge that goes along the ray $R_p(\frac{5}{8})$. Then the segment $[-\sqrt{2}, 0]$ should also be removed from T^* since $-\sqrt{2}$ is not in $P(f_u)$. Thus T^* is as shown in Fig. 7, bottom right. Denote the edges of T^* by A^*, B^*, C^*, D^* and orient them as in Fig. 7, bottom right. The edges of T^* map over the edges of T as follows:

$$A^* \rightarrow A, \quad B^* \rightarrow B, \quad C^* \rightarrow C^{-1}, \quad D^* \rightarrow C.$$

Thus, in this example, every edge of T^* maps over just one edge of T , which is not the case in general.

We will write a, b, c for the elements of π_{f_u} corresponding to the edges A, B, C , respectively. Thus the generating set $\mathcal{E} = \mathcal{E}_T$ consists of $id, a^{\pm 1}, b^{\pm 1}, c^{\pm 1}$. Similar convention will apply to T^* , so that $\mathcal{E}^* = \mathcal{E}_{T^*}$ consists of id, a^*, b^*, c^*, d^* , and their inverses. Elements of \mathcal{E}_{T^*} are shown through their representatives in Fig. 7, bottom right (dashed loops). Inspecting how the dashed loops cross the edges of T , we can express elements of \mathcal{E}^* through those of \mathcal{E} :

$$a^* = ba^{-1}, \quad b^* = ca^{-1}, \quad c^* = ac^{-1}a^{-1}, \quad d^* = a^{-1}.$$

We now compute the presentation of the biset of f_u associated with (T^*, T) as in Theorem 5.6. To this end, we first need to choose post-critical pseudoaccesses for

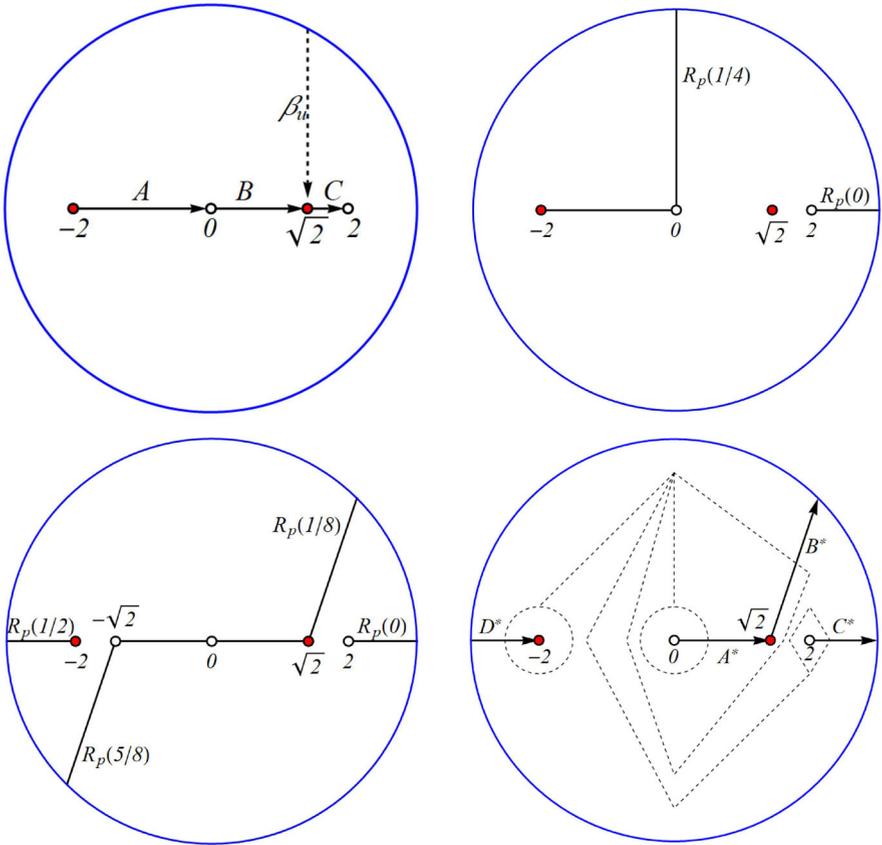


Fig. 7 The simple capture $f_u = \sigma_u \circ p$ of $p(z) = z^2 - 2$. Copies of the sphere are represented as disks, in which the boundary circles are assumed to be collapsed. Thus, in each of the four pictures, one should think of the entire boundary circle as one point

T . The critical values of f_u are $v_1 = -2$ and $v_2 = \sqrt{2}$. Since v_1 is an endpoint of T , there is only one pseudoaccess at v_1 . However, v_2 admits to pseudoaccesses: one is from above, and the other is from below. We choose the one from above. According to this choice of pseudoaccesses, we have $S^0(T) = (A, B)$ and $S^1(T) = (C, C^{-1}, B^{-1}, A^{-1})$. Therefore, the edges A and B have signature $(0, 1)$, and C has signature $(1, 1)$. Next, we need to compute the labels for all edges of T^* . Since e'_b by definition has label 1, we have $\ell(A^*) = \ell(B^*) = 0$ (indeed, we change the label as we pass through the critical point 0). It is slightly harder to figure out the labels of C^* and D^* since we need to look at the critical pseudoaccesses at ∞ . The pseudoaccesses at ∞ separate B^* and C^* from D^* . Therefore, we have $\ell(C^*) = 0$ and $\ell(D^*) = 1$.

We can now write down the presentation of the biset $\mathcal{X}_{f_u}(y)$ associated with (T^*, T) :

With the help of a computer, we found a pullback invariant subset of size 81 in $\text{Ivy}(f)$ containing $[T]$, see Fig. 8. This subset contains an invariant ivy object. Thus, we found an invariant (up to homotopy) spanning tree for f . This invariant tree is a star.

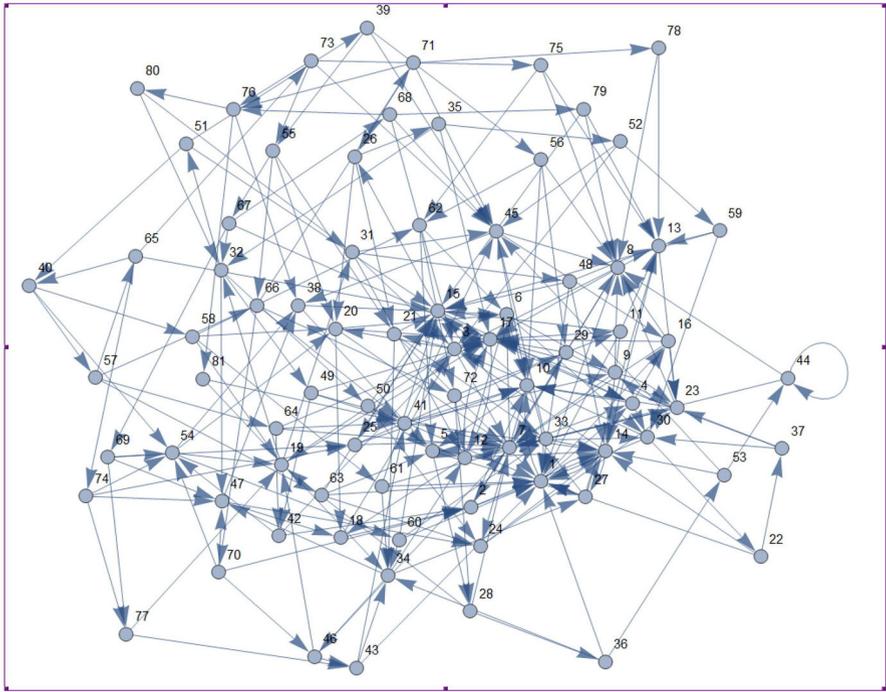


Fig. 8 The pullback invariant subset of $\text{Ivy}(f_u)$ containing $[T]$. There are 81 objects in this subset. Vertex 44 represents an invariant ivy object

	a	b	c	a^{-1}	b^{-1}	c^{-1}
0	a^*1	b^*1	d^*0	1	1	$d^{*-1}0$
1	0	0	$c^{*-1}1$	$a^{*-1}0$	$b^{*-1}0$	c^*1

7.5 Some Open Questions

The following are open questions about the pullback relation on spanning trees that seem important:

- For a quadratic rational Thurston map f , can there be an infinite sequence of pairwise different ivy objects $[T_n]$ such that $[T_n] \dashrightarrow [T_{n+1}]$?
- Is there a uniform upper bound on the number of invariant spanning trees, up to isotopy, for a quadratic rational Thurston map?
- Can there be two disjoint pullback invariant subsets of $\text{Ivy}(f)$, for a quadratic rational Thurston map f ?

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