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RESEARCH CONTRIBUTION

# On Mutually Semiconjugate Rational Functions

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**Abstract**

We characterize pairs of rational functions $A$, $B$ such that $A$ is semiconjugate to $B$, and $B$ is semiconjugate to $A$.

**Keywords** Semiconjugate rational functions · Commuting rational functions · Lattès maps

## 1 Introduction

Let $A$ and $B$ be rational functions of degree at least two on the Riemann sphere. The function $B$ is said to be *semiconjugate* to the function $A$ if there exists a non-constant rational function $X$ such that the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ \downarrow X & & \downarrow X \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \qquad (1)$$

commutes. If $X$ is invertible, the functions $A$ and $B$ are called *conjugate*. In terms of dynamical systems, the conjugacy condition means that the dynamical systems $A^{\circ k}$, $k \geqslant 1$, and $B^{\circ k}$, $k \geqslant 1$, on $\mathbb{CP}^1$ are equivalent, while the more general condition (1) means that the first of these systems is a factor of the second. In particular, (1) implies that $X$ sends attracting, repelling, and indifferent periodic points of $B$ to periodic points of $A$ of the same character. Note that the semiconjugacy relation is not symmetric. However, it is clear that if $B$ is semiconjugate to $A$, and $C$ is semiconjugate to $B$, then

To Rafail Kalmanovich Gordin, on the occasion of his 70th birthday.

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$C$ is semiconjugate to $A$. Therefore, the semiconjugacy relation is a preorder on the set of rational functions.

Although semiconjugate rational functions appear naturally in complex and arithmetic dynamics (see e.g. the papers Buff and Epstein 2007; Eremenko 2012; Medvedev and Scanlon 2014; Pakovich 2017), the problem of describing such functions started to be systematically studied only recently in the series of papers (Pakovich 2016, 2018, 2019a, b, d). In this paper we address the following related question: under what conditions rational functions $A$ and $B$ are *mutually* semiconjugate, that is $A$ is semiconjugate to $B$, and $B$ is semiconjugate to $A$ ? Such functions are of interest since they exhibit very similar although not identical dynamics. In fact, the mutual semiconjugacy relation can be considered as a weaker form of the classical conjugacy relation.

Examples of mutually semiconjugate rational functions can be obtained by the following construction. Let $A$ be a rational function. For any decomposition $A = U \circ V$ of $A$ into a composition of rational functions, we say that the rational function $\widetilde{A} = V \circ U$ is an *elementary transformation* of $A$. We say that rational functions $A$ and $B$ are *equivalent* and write $A \sim B$ if there exists a chain of elementary transformations between $A$ and $B$. Since obviously

$$\widetilde{A} \circ V = V \circ A, \qquad A \circ U = U \circ \widetilde{A},$$

elementary transformations are mutually semiconjugate, implying inductively that functions $A$ and $B$ are mutually semiconjugate whenever $A \sim B$. Moreover, the corresponding semiconjugacy map $X$ preserves not only the character of periodic points but also their exact periods and multipliers (see Pakovich 2019a).

Roughly speaking, the main result of this paper states that rational functions $A$ and $B$ are mutually semiconjugate *only* if $A \sim B$, unless $A$ and $B$ belong to the class of *Lattès maps*, which is known to be a source of exceptional examples in complex dynamics. A typical example $A_{n,L}$ of such a map is obtained from the "multiplication theorem" for the Weierstrass function:

$$\wp_L(nz) = A_{n,L} \circ \wp_L(z),$$

where $\wp_L$ is the Weierstrass function with period lattice $L$, and $n \geqslant 2$ is an integer. More precisely, we show that if mutually semiconjugate rational functions $A$ and $B$ are not equivalent, then they are Lattès maps with orbifold signature $(2, 2, 2, 2)$.

**Theorem 1.1** *Let $A$ and $B$ be mutually semiconjugate rational functions of degree at least two. Then either $A \sim B$, or there exist orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ with signature $(2, 2, 2, 2)$ on the Riemann sphere such that $A : \mathcal{O}_1 \to \mathcal{O}_1$ and $B : \mathcal{O}_2 \to \mathcal{O}_2$ are covering maps between orbifolds.*

Theorem 1.1 implies that, apart from the very special class of Lattès maps, the equivalence relation induced by the mutual semiconjugacy coincides with the equivalence $\sim$ defined above. In particular, for a rational function $A$ that is not a Lattès map there exist at most finitely many conjugacy classes of rational functions mutually semiconjugate to $A$, since a similar statement is true for equivalence classes of $\sim$ (see Pakovich 2019a).

The paper is organized as follows. In the second section, we recall some definitions and results concerning Riemann surface orbifolds, Lattès maps, and commuting rational functions. We also prove a result concerning mutually semiconjugate Lattès maps with signatures distinct from (2, 2, 2, 2). In the third section, we review results about the equivalence $\sim$ and semiconjugate rational functions, and prove Theorem 1.1. Finally, in the fourth section we consider mutually semiconjugate Lattès maps with orbifold signature (2, 2, 2, 2), and construct examples of such maps that are not equivalent.

## 2 Orbifolds and Commuting Functions

The problem of describing mutually semiconjugate rational functions is closely related to the problem of describing *commuting* rational functions. Indeed, if $A$ and $B$ are mutually semiconjugate rational functions, then there exist rational functions $X$ and $Y$ such that the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \\ \downarrow Y & & \downarrow Y \\ \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ \downarrow X & & \downarrow X \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \end{array} \tag{2}$$

commutes, implying that the rational function $X \circ Y$ commutes with $A$. Similarly, the rational function $Y \circ X$ commutes with $B$. Commuting rational functions were investigated already by Julia (1922), Fatou (1923), and Ritt (1923). The most complete result, obtained by Ritt, states roughly speaking that commuting rational functions *having no iterate in common* reduce either to powers, or to Chebyshev polynomials, or to Lattès maps. A proof of the Ritt theorem based on modern dynamical methods was given by Eremenko (1989). Commuting rational functions that *do* have a common iterate were studied in Pakovich (2019c).

In this paper, we will use the Ritt theorem in its modern formulation, given in Eremenko (1989). This formulation uses the notion of *orbifold*. Recall that a Riemann surface orbifold is a pair $\mathcal{O} = (R, \nu)$ consisting of a Riemann surface $R$ together with a ramification function $\nu : R \to \mathbb{N} \cup \{\infty\}$ which takes the value $\nu(z) = 1$ except at isolated points. For an orbifold $\mathcal{O} = (R, \nu)$, the *Euler characteristic* of $\mathcal{O}$ is the number

$$\chi(\mathcal{O}) = \chi(R) + \sum_{z \in R} \left( \frac{1}{\nu(z)} - 1 \right), \tag{3}$$

the set of *singular points* of $\mathcal{O}$ is the set

$$c(\mathcal{O}) = \{z_1, z_2, \ldots, z_s, \ldots\} = \{z \in R \mid \nu(z) > 1\},$$

and the *signature* of $\mathcal{O}$ is the set

$$\nu(\mathcal{O}) = \{\nu(z_1), \nu(z_2), \dots, \nu(z_s), \dots\}.$$

This definition of orbifold (see e. g. Douady and Hubbard 1993; Eremenko 1989) differs slightly from the definition given in Milnor (2006b), say, where it is assumed that $\nu$ takes only *finite* values. To pass from the first definition to the second it is necessary to change the surface $R$ in the definition of $\mathcal{O}$ removing all points $z$ where $\nu(z) = \infty$. The same remark concerns other related definitions given below. Note that since removing a point from a surface $R$ reduces the Euler characteristic $\chi(R)$ by one, this passage does not change the Euler characteristic $\chi(\mathcal{O})$ defined by (3).

If $R_1$, $R_2$ are Riemann surfaces provided with ramification functions $\nu_1$, $\nu_2$, then a holomorphic branched covering map

$$f : R_1\backslash\{z : \nu_1(z) = \infty\} \rightarrow R_2\backslash\{z : \nu_1(z) = \infty\}$$

is called *a covering map* $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ between orbifolds $\mathcal{O}_1 = (R_1, \nu_1)$ and $\mathcal{O}_2 = (R_2, \nu_2)$ if for any $z \in R_1$ the equality

$$\nu_2(f(z)) = \nu_1(z)\deg_z f \tag{4}$$

holds. It follows from the chain rule that if $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ and $g : \mathcal{O}_2 \rightarrow \mathcal{O}_3$ are covering maps between orbifolds, then $g \circ f : \mathcal{O}_1 \rightarrow \mathcal{O}_3$ is also a covering map. If $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ is a covering map of finite degree between orbifolds with compact $R_1$ and $R_2$, then the Riemann-Hurwitz formula implies that

$$\chi(\mathcal{O}_1) = \chi(\mathcal{O}_2)\deg f. \tag{5}$$

*A universal covering* of an orbifold $\mathcal{O}$ is a covering map between orbifolds $\theta_{\mathcal{O}} : \widetilde{\mathcal{O}} \rightarrow \mathcal{O}$ such that $\widetilde{R}$ is simply connected and $\widetilde{\nu}(z) \equiv 1$. If $\theta_{\mathcal{O}}$ is such a map, then there exists a group $\Gamma_{\mathcal{O}}$ of conformal automorphisms of $\widetilde{R}$ such that the equality $\theta_{\mathcal{O}}(z_1) = \theta_{\mathcal{O}}(z_2)$ holds for $z_1, z_2 \in \widetilde{R}$ if and only if $z_1 = \sigma(z_2)$ for some $\sigma \in \Gamma_{\mathcal{O}}$. A universal covering exists and is unique up to a conformal isomorphism of $\widetilde{R}$, unless $\mathcal{O}$ is the Riemann sphere with one ramified point or with two ramified points $z_1, z_2$ such that $\nu(z_1) \neq \nu(z_2)$. Furthermore, $\widetilde{R} = \mathbb{D}$ if and only if $\chi(\mathcal{O}) < 0$, $\widetilde{R} = \mathbb{C}$ if and only if $\chi(\mathcal{O}) = 0$, and $\widetilde{R} = \mathbb{CP}^1$ if and only if $\chi(\mathcal{O}) > 0$. Any covering map $f : \mathcal{O}_1 \rightarrow \mathcal{O}_2$ between orbifolds lifts to an isomorphism $\varphi : \widetilde{R}_1 \rightarrow \widetilde{R}_2$ which makes the diagram

$$\begin{array}{ccc} \widetilde{R}_1 & \xrightarrow{\varphi} & \widetilde{R}_2 \\ \downarrow{\theta_{\mathcal{O}_1}} & & \downarrow{\theta_{\mathcal{O}_2}} \\ \mathcal{O}_1 & \xrightarrow{f} & \mathcal{O}_2 \end{array} \tag{6}$$

commutative, and maps points that are in the same orbit of $\Gamma_{\mathcal{O}_1}$ to points that are in the same orbit of $\Gamma_{\mathcal{O}_2}$. The isomorphism $\varphi$ is defined up to a transformation $\varphi \rightarrow g \circ \varphi$, where $g \in \Gamma_{\mathcal{O}_2}$. In the other direction, for any isomorphism $\varphi$ which maps any orbit of $\Gamma_{\mathcal{O}_1}$ to an orbit of $\Gamma_{\mathcal{O}_2}$ there exists a uniquely defined covering map between orbifolds $f : \mathcal{O}_1 \to \mathcal{O}_2$ such that diagram (6) commutes (see Milnor 2006b, Appendix E, and Pakovich 2016, Section 3).

Commuting rational functions having no iterate in common can be described in terms of orbifolds $\mathcal{O} = (\mathbb{CP}^1, \nu)$ with $\chi(\mathcal{O}) = 0$. The signature of such an orbifold has one of the following forms

$$(\infty, \infty), \quad (2, 2, \infty), \quad (2, 2, 2, 2), \quad (3, 3, 3), \quad (2, 4, 4), \quad (2, 3, 6). \tag{7}$$

Correspondingly, the group $\Gamma_{\mathcal{O}}$ is conjugate in $Aut(\mathbb{C})$ to

$$\begin{aligned}
&z \to z + im, \ m \in \mathbb{Z};\\
&z \to \pm z + m, \ m \in \mathbb{Z};\\
&z \to \pm z + m + n\tau, \ m, n \in \mathbb{Z};\\
&z \to \omega^{2k} z + m + n\omega, \quad m, n \in \mathbb{Z}, \quad 0 \leqslant k \leqslant 2;\\
&z \to i^k z + m + ni, \quad m, n \in \mathbb{Z}, \quad 0 \leqslant k \leqslant 3;\\
&z \to \omega^k z + m + n\omega, \quad m, n \in \mathbb{Z}, \quad 0 \leqslant k \leqslant 5,
\end{aligned} \tag{8}$$

where $\tau$ is a complex number with $\Im(\tau) > 0$, and $\omega = e^{\pi i/3}$. Finally, the universal covering of $\mathcal{O}$ with $\Gamma_{\mathcal{O}}$ from the list (8), up to the transformation $\theta_{\mathcal{O}} \to \mu \circ \theta_{\mathcal{O}}$, where $\mu \in Aut(\mathbb{CP}^1)$, is

$$exp(2\pi z), \quad cos(2\pi z), \quad \wp(z, 1, \tau), \quad \wp'(z, 1, \omega), \quad \wp^2(z, 1, i), \quad \wp'^2(z, 1, \omega),$$

where $\wp = \wp(z, \omega_1, \omega_2)$ denotes the Weierstrass functions with periods $\omega_1, \omega_2$ (see Douady and Hubbard 1993; Milnor 2006a).

In terms of orbifolds, the Ritt theorem can be formulated as follows (Eremenko 1989).

**Theorem 2.1** *Let $A$ and $C$ be commuting rational functions of degree at least two having no iterate in common. Then there exists an orbifold $\mathcal{O} = (\mathbb{CP}^1, \nu)$ with $\chi(\mathcal{O}) = 0$ such that $A : \mathcal{O} \to \mathcal{O}$ and $C : \mathcal{O} \to \mathcal{O}$ are covering maps between orbifolds.* □

If $\mathcal{O} = (\mathbb{CP}^1, \nu)$ is an orbifold with $\chi(\mathcal{O}) = 0$, and $f$ is a rational function such that $f : \mathcal{O} \to \mathcal{O}$ is a covering map between orbifolds, then $\widetilde{R} = \mathbb{C}$, and $f$ lifts to an affine map $\varphi = az + b$, $a, b \in \mathbb{C}$, which makes the diagram

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\varphi = az+b} & \mathbb{C}\\
\Big\downarrow \theta_{\mathcal{O}} & & \Big\downarrow \theta_{\mathcal{O}}\\
\mathcal{O} & \xrightarrow{f} & \mathcal{O},
\end{array} \tag{9}$$

commutative. Thus, on one hand, the Ritt theorem reduces describing pairs of commuting rational functions $A$ and $C$ having no iterate in common to describing pairs of affine maps $\varphi$ and $\psi$ that map any orbit of some group $\Gamma$ from list (8) to another orbit and satisfy the equality

$$\varphi \circ \psi = g \circ \psi \circ \varphi$$

for some $g \in \Gamma$. On the other hand, the Ritt theorem imposes restrictions on possible ramifications of $A$ and $C$ resulting from the definition of covering map (4) and list (7). Note that if $\mathcal{O} = (\mathbb{CP}^1, \nu)$ is an orbifold and $f : \mathcal{O} \to \mathcal{O}$ is a covering map of degree at least two, then (5) implies that $\chi(\mathcal{O}) = 0$. In particular, the condition $\chi(\mathcal{O}) = 0$ in the formulation of the Ritt theorem is actually redundant.

If $\nu(\mathcal{O}) = (\infty, \infty)$, then any rational function $f$ of degree at least two such that $f : \mathcal{O} \to \mathcal{O}$ is a covering map between orbifolds is conjugate to $z^{\pm n}$, $n \geqslant 2$, while if $\nu(\mathcal{O}) = (2, 2, \infty)$, then any such function is conjugate to $\pm T_n$, $n \geqslant 2$. Rational functions $f$ of degree at least two such that $f : \mathcal{O} \to \mathcal{O}$ is a covering map for an orbifold $\mathcal{O}$ whose signature is $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$, or $(2, 2, 2, 2)$ are called *Lattès maps*. Such rational functions possess a number of remarkable features (see Milnor 2006a; Pakovich 2019b).

In this paper all considered orbifolds (except for universal coverings) will be defined on $\mathbb{CP}^1$, and we simply will write $\mathcal{O}$ instead of $\mathcal{O} = (\mathbb{CP}^1, \nu)$. The following statement describes compositional properties of rational functions $C$ that are self-covering maps $C : \mathcal{O} \to \mathcal{O}$ (cf. Pakovich 2019b, Theorem 4.1).

**Lemma 2.1** *Let $\mathcal{O}$ be an orbifold and $C$ a rational function such that $C : \mathcal{O} \to \mathcal{O}$ is a covering map between orbifolds. Assume that $C = X \circ Y$ is a decomposition of $C$ into a composition of rational functions. Then there exists an orbifold $\mathcal{O}^*$ with $\nu(\mathcal{O}^*) = \nu(\mathcal{O})$ such that $Y : \mathcal{O} \to \mathcal{O}^*$ and $X : \mathcal{O}^* \to \mathcal{O}$ are covering maps between orbifolds.*

***Proof*** Since

$$\nu((X \circ Y)(z)) = \nu(z)\deg_z(X \circ Y) = \nu(z)\deg_z Y \deg_{Y(z)} X \tag{10}$$

and the value $(X \circ Y)(z)$ depends only on the value $Y(z)$, defining for $z \in \mathbb{CP}^1$ the value $\nu^*(z)$ by the formula

$$\nu^*(z) = \nu(z')\deg_{z'} Y,$$

where $z'$ is any point such that $Y(z') = z$, we obtain a well-defined orbifold $\mathcal{O}^*$ such that $Y : \mathcal{O} \to \mathcal{O}^*$ and $X : \mathcal{O}^* \to \mathcal{O}$ are covering maps. Moreover, applying formula (5) to any of these maps we see that $\chi(\mathcal{O}^*) = 0$. Finally, it is not hard to prove that $\nu(\mathcal{O}^*) = \nu(\mathcal{O})$. Indeed, if $\nu(\mathcal{O}) = (\infty, \infty)$, then (10) implies easily that $\mathcal{O}^*$ has exactly two points with ramification $\infty$. Therefore, since $\mathcal{O}^*$ belongs to list (7), the equality $\nu(\mathcal{O}^*) = (\infty, \infty)$ holds. Similarly, we obtain that if $\nu(\mathcal{O}) = (2, 2, \infty)$, then $\nu(\mathcal{O}^*) = (2, 2, \infty)$. Assume now that $\nu(\mathcal{O}) = (2, 3, 6)$. Since $X : \mathcal{O}^* \to \mathcal{O}$ is a covering map, it follows from (4) that

$$\nu^*(z) \mid \nu(X(z)), \quad z \in \mathbb{CP}^1,$$

implying that either $\nu(\mathcal{O}^*) = (2, 3, 6)$, or $\nu(\mathcal{O}^*) = (3, 3, 3)$, or $\nu(\mathcal{O}^*) = (2, 2, 2, 2)$. However, in the last two cases $Y : \mathcal{O} \to \mathcal{O}^*$ cannot be a covering map, since

$$\nu(z) \mid \nu^*(Y(z)), \quad z \in \mathbb{CP}^1.$$

The rest of the cases are considered similarly. □

Let us list several properties of Lattès maps used in the following. First, if $f$ is a Lattès map, then an orbifold $\mathcal{O}$ such that $f : \mathcal{O} \to \mathcal{O}$ is a covering map, is defined in a unique way by dynamical properties of $f$ (see Milnor 2006a and also Pakovich 2019b, Theorem 6.1). We will use the notation $\mathcal{O}_f$ for this orbifold and the notation $l = l(f)$ for the least common multiple of numbers in the signature of $\mathcal{O}_f$. Secondly, although the functions $\theta_{\mathcal{O}}$ and $\varphi$ in diagram (9) are not defined in a unique way by $f$, the number $a^l$ depends on $f$ only, and the numbers $a$ and $\deg f$ are related by the equality

$$\deg f = |a|^2 \tag{11}$$

(see Milnor 2006a, Lemma 5.1). Thirdly, if $f$ satisfies (9) and $z \in \mathbb{CP}^1$ is a fixed point of $f$, then the multiplier of $f$ at $z$ is given by the formula

$$\mu = (\omega a)^{\nu(z)}, \tag{12}$$

where $\omega$ is some $l$th root of unity, and $\nu$ is the ramification function for $\mathcal{O}_f$ (see Milnor 2006a, Corollary 3.9).

Finally, we need the following rigidity property of Lattès maps which states, roughly speaking, that if $l \geqslant 3$, then for fixed $a^l$ there exist at most two conjugacy classes of rational functions $f$ which make diagram (9) commutative, and these classes can be distinguished by their dynamical properties (see Milnor 2006a, Theorem 5.2).

**Theorem 2.2** *Let $f$ be a Lattès map with $l = l(f) \geqslant 3$. Then the conjugacy class of $f$ is completely determined by the numbers $l$ and $a^l$ together with the information as to whether $f$ does or does not have a fixed point of multiplier $\mu = a^l$.* □

Note that in view of formula (12) the property of $f$ to have a fixed point of multiplier $\mu = a^l$ is equivalent to the following property:

($\star$) *there exists a fixed point $z$ of $f$ with $\nu(z) = l$.*

Theorem 2.2 results in the following statement.

**Theorem 2.3** *Let $A$ and $B$ be mutually semiconjugate rational functions of degree at least two, and $X$, $Y$ rational functions such that diagram (2) commutes. Assume that there exists an orbifold $\mathcal{O}$ with signature distinct from $(2, 2, 2, 2)$ such that $A : \mathcal{O} \to \mathcal{O}$ and $X \circ Y : \mathcal{O} \to \mathcal{O}$ are covering maps between orbifolds. Then $B$ is conjugate to $A$.*

***Proof*** By Lemma 2.1, there exists an orbifold $\mathcal{O}^*$ with $\nu(\mathcal{O}^*) = \nu(\mathcal{O})$ such that $Y : \mathcal{O} \to \mathcal{O}^*$ and $X : \mathcal{O}^* \to \mathcal{O}$ are covering maps between orbifolds. Furthermore, since

$$\nu(\mathcal{O}^*) = \nu(\mathcal{O}) \neq (2, 2, 2, 2),$$

changing in diagram (2) the function $Y$ to the function $\mu \circ Y$, the function $X$ to the function $X \circ \mu^{-1}$, and the function $B$ to the function $\mu^{-1} \circ B \circ \mu$ for convenient $\mu \in Aut(\mathbb{CP}^1)$, we may assume that $\mathcal{O}^* = \mathcal{O}$.

If $\nu(\mathcal{O}) = (\infty, \infty)$, then without loss of generality we may assume that

$$\nu(0) = \infty, \quad \nu(\infty) = \infty,$$

implying that

$$A = az^n, \quad Y = bz^m,$$

where $a, b \in \mathbb{C}$ and $n, m \in \mathbb{Z}$. It follows now from the equality $B \circ Y = Y \circ A$ that $B = a^m b^{1-n} z^n$. Thus, in this case $A$ and $B$ are conjugate.

Similarly, if $\nu(\mathcal{O}) = (2, 2, \infty)$ and

$$\nu(1) = 2, \quad \nu(-1) = 2, \quad \nu(\infty) = \infty,$$

then

$$A = \pm T_n, \quad Y = \pm T_{m_1},$$

implying that $B = \pm T_n$. However, in this case a further investigation is needed, since the functions $T_n$ and $-T_n$ are conjugate for even $n$, but not conjugate for odd. To finish the proof, we observe that the equality

$$-T_n \circ \pm T_m = \pm T_m \circ T_n \tag{13}$$

for odd $n$ is impossible. Thus, if $A = T_n$, then $B = T_n$. In turn, this implies that if $A = -T_n$, then $B = -T_n$, for otherwise the lower square in (2) would provide a solution of (13).

Finally, assume that $\nu(\mathcal{O})$ is $(2, 4, 4)$, $(3, 3, 3)$, or $(2, 3, 6)$. Let us complete diagram (2) to the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\varphi = az+b} & \mathbb{C} \\ \downarrow \theta_{\mathcal{O}} & & \downarrow \theta_{\mathcal{O}} \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 \\ \downarrow Y & & \downarrow Y \\ \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ \downarrow X & & \downarrow X \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1 . \end{array} \tag{14}$$

Since $\theta_{\mathcal{O}} : \mathbb{C} \to \mathcal{O}$ and $Y : \mathcal{O} \to \mathcal{O}$ are covering maps, their composition $Y \circ \theta_{\mathcal{O}} : \mathbb{C} \to \mathcal{O}$ is also a covering map (here $\mathbb{C}$ stands for the orbifold $(\mathbb{C}, \nu)$ with $\nu \equiv 1$). Thus, $Y \circ \theta_{\mathcal{O}}$ along with $\theta_{\mathcal{O}}$ is a universal covering of $\mathcal{O}$, implying by the chain rule that $B : \mathcal{O} \to \mathcal{O}$ is a covering map. Moreover, (14) implies that $A$ and $B$ have the same invariant $a^l$. Therefore, by Theorem 2.2, it is enough to show that property ($\star$) holds for $A$ if and only if it holds for $B$.

Consider the semiconjugacy in the upper square in (2). Clearly, $Y$ maps fixed points of $A$ to fixed points of $B$. Therefore, since the equality

$$\nu(Y(z)) = \nu(z)\deg_z Y$$

implies that $\nu(Y(z)) = l$ whenever $\nu(z) = l$, if property ($\star$) holds for $A$, then it holds for $B$. Moreover, arguing as in the case $\nu(\mathcal{O}) = (2, 2, \infty)$, we conclude that if the property ($\star$) does not hold for $A$, then it does not hold for $B$. □

## 3 Equivalence and Semiconjugacy

Let $A$ be a rational function. We recall that for any decomposition $A = U \circ V$ of $A$ into a composition of rational functions the rational function $\widetilde{A} = V \circ U$ is called an *elementary transformation* of $A$, and rational functions $A$ and $B$ are called *equivalent* if there exists a chain of elementary transformations between $A$ and $B$. Since for any Möbius transformation $\mu$ the equality

$$A = (A \circ \mu) \circ \mu^{-1}$$

holds, each equivalence class $[A]$ is a union of conjugacy classes. Thus, like the mutual semiconjugacy relation, the relation $\sim$ is a weaker form of the classical conjugacy relation. Moreover, equivalent rational functions have similar dynamic characteristics. To make the last statement precise, we recall that the *multiplier spectrum* of a rational function $A$ of degree $d$ is a function which assigns to each $s \geqslant 1$ the unordered list of multipliers at all $d^s + 1$ fixed points of $A^{\circ s}$ taken with appropriate multiplicity. Two rational functions are called *isospectral* if they have the same multiplier spectrum. In this notation, the following statement is true (see Pakovich 2019a, Corollary 2.1).

**Lemma 3.1** *Let $A$ and $B$ be rational functions such that $A \sim B$. Then $A$ and $B$ are isospectral.* □

Lemma 3.1 has two implications. On one hand, it permits to conclude that two functions are *not* equivalent if they have different multiplier spectrum. On the other hand, by the fundamental result of McMullen (1987), the conjugacy class of any rational function $A$ that is not *a flexible Lattès map* (see e.g. Milnor 2006a or Silverman 2007 for the definition) is defined up to finitely many choices by its multiplier spectrum. Thus, Lemma 3.1 implies that for any function $A$ that is not a flexible Lattès map the number of conjugacy classes in the equivalence class $[A]$ is finite. More precisely, the following statement holds (see Pakovich 2019a, Theorem 1.1).

**Theorem 3.1** *Let $A$ be a rational function. Then its equivalence class $[A]$ contains infinitely many conjugacy classes if and only if $A$ is a flexible Lattès map.* □

Note that there exists no *absolute* bound for the number of conjugacy classes in $[A]$, and one can construct rational functions $A$ of degree $n$ for which $[A]$ contains $\approx \log_2 n$ conjugacy classes (see Pakovich 2016, p. 1241). On the other hand, although the proof of the McMullen theorem is non-effective, Theorem 3.1 can be deduced from effective results of the paper Pakovich (2019d), implying that the number of conjugacy classes in $[A]$ can be bounded in terms of degree of $A$ only.

Finally, we mention that to our best knowledge only three types of examples of isospectral rational functions are known: flexible or not flexible Lattès maps (see McMullen 1987; Milnor 2006a; Silverman 2007), and equivalent functions. So, the following question is of great interest.

**Problem 3.1** *Do there exist isospectral rational functions that are neither Lattès maps nor equivalent?*

It was already mentioned in the introduction that the equivalence $\sim$ is closely related to the semiconjugacy. Moreover, using elementary transformations one can reduce any solution of (1) to a so-called primitive solution. We say that a solution $A, X, B$ of functional equation (1) is *primitive* if

$$\mathbb{C}(X, B) = \mathbb{C}(z),$$

that is if the functions $X$ and $B$ generate the whole field of rational functions. It was shown in Pakovich (2016) (see also Pakovich 2018), that for any primitive solution of (1) there exist orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $A : \mathcal{O}_1 \to \mathcal{O}_1$, $B : \mathcal{O}_2 \to \mathcal{O}_2$, and $X : \mathcal{O}_1 \to \mathcal{O}_2$ are *minimal holomorphic maps* between orbifolds (see Pakovich 2016 for the definition). This condition generalizes the condition provided by the Ritt theorem, and implies strong restrictions on a possible form of $A$, $B$ and $X$.

In what follows, we will not use the description of primitive solutions given in Pakovich (2016). However, we will need the following reduction: for an arbitrary solution $A, X, B$ of (1) there exists a decomposition $X = X_0 \circ W$ and a rational function $B_0 \sim B$ such that the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{B} & \mathbb{CP}^1 \\ W\downarrow & & \downarrow W \\ \mathbb{CP}^1 & \xrightarrow{B_0} & \mathbb{CP}^1 \\ X_0\downarrow & & \downarrow X_0 \\ \mathbb{CP}^1 & \xrightarrow{A} & \mathbb{CP}^1, \end{array} \tag{15}$$

commutes and $A, X_0, B_0$ is a primitive solution of (1). To see that this is true, we observe that if $A, X, B$ is a primitive solution, then we can set $W = z$, $X_0 = X$, $B_0 = B$. On the other hand, if the solution $A, X, B$ is not primitive, then by the

Lüroth theorem there exists a rational function $W$ of degree greater than one such that $\mathbb{C}(X, B) = \mathbb{C}(W)$ and the equalities

$$X = X' \circ W, \quad B = B' \circ W$$

hold for some rational functions $X'$ and $B'$ with $\mathbb{C}(X', B') = \mathbb{C}(z)$. Clearly, the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{\ B\ } & \mathbb{CP}^1 \\ W\downarrow & & \downarrow W \\ \mathbb{CP}^1 & \xrightarrow{W \circ B'} & \mathbb{CP}^1 \\ X'\downarrow & & \downarrow X' \\ \mathbb{CP}^1 & \xrightarrow{\ A\ } & \mathbb{CP}^1, \end{array}$$

commutes. Thus, if the solution $A, X', W \circ B'$ of (1) is primitive, we are done. Otherwise, we can apply the above transformation to this solution. Since $\deg X' < \deg X$, it is clear that after a finite number of steps we will obtain required functions $X_0$, $B_0$, $W$ (see Pakovich (2019b), Section 3, for more details).

In addition to the above reduction, to prove Theorem 1.1 we need the following two lemmas (see Pakovich (2019c), Lemma 2.4 and Lemma 2.5).

**Lemma 3.2** *A solution $A, X, B$ of (1) is primitive if and only if the algebraic curve*

$$A(x) - X(y) = 0$$

*is irreducible.* □

**Lemma 3.3** *Let $A, X, B$ be a primitive solution of (1). Then for any $s \geqslant 1$ the triple $A^{\circ s}, X, B^{\circ s}$ is also a primitive solution of (1).* □

***Proof of Theorem 1.1*** Let $A, B$ be mutually semiconjugate rational functions, and $X$, $Y$ corresponding rational functions which make diagram (2) commutative. Then by the Ritt theorem, either there exists an orbifold $\mathcal{O}$ with $\chi(\mathcal{O}) = 0$ such that $A : \mathcal{O} \to \mathcal{O}$ and $X \circ Y : \mathcal{O} \to \mathcal{O}$ are covering maps between orbifolds, or there exist $s, k \geqslant 1$ such that

$$A^{\circ s} = (X \circ Y)^{\circ k}. \tag{16}$$

In the first case, the statement of the theorem follows from Theorem 2.3. On the other hand, to prove the theorem in the second case it is enough to show that in diagram (15), constructed for $A, X, B$ from the lower square in (2), the equality $\deg X_0 = 1$ holds (cf. Pakovich (2019c), Theorem 2.5). Indeed, in this case $B_0$ is conjugate to $A$, and hence

$$B \sim B_0 \sim A.$$

Assume to the contrary that $\deg X_0 \geqslant 2$. Set

$$F = W \circ Y \circ (X \circ Y)^{\circ k-1},$$

where $k$ is defined by (16). Then

$$A^{\circ s} = X_0 \circ F$$

by (16), implying that the curve

$$F(x) - y = 0$$

is a component of the curve

$$A^{\circ s}(x) - X_0(y) = 0.$$

Moreover, since $\deg X_0 > 1$, this component is proper. Therefore, the triple $A^{\circ s}, X_0, B_0^{\circ s}$ is not a primitive solution of (1) by Lemma 3.2. On the other hand, this triple must be a primitive solution by Lemma 3.3. The contradiction obtained shows that $\deg X_0 = 1$. □

## 4 Case of Signature (2, 2, 2, 2)

We recall that any Lattès map with the invariant $l$ equal to 2 is conjugate to a rational function $f$ such that the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\varphi=az+b} & \mathbb{C} \\ \downarrow{\scriptstyle \wp_L} & & \downarrow{\scriptstyle \wp_L} \\ \mathcal{O} & \xrightarrow{f} & \mathcal{O} \end{array} \tag{17}$$

commutes for some lattice $L$ of rank two in $\mathbb{C}$ and affine map $\varphi$. The Weierstrass function $\wp_L$ is the universal covering of $\mathcal{O}_f$, and the corresponding group $\Gamma_{\mathcal{O}}$ is generated by translations by elements of $L$ and the transformation $z \to -z$. Furthermore, the function $\varphi = az + b$ in (17) maps any orbit of $\Gamma_{\mathcal{O}}$ to another orbit, implying that $aL \subset L$ (see e.g. Milnor 2006a, Lemma 5.1). For most lattices $L$ the condition $aL \subset L$ implies that $a \in \mathbb{Z}$. In particular, for such $L$ the degree of $f$ in (17) is a perfect square by formula (11). Lattices for which there exists a non-integer $a$ satisfying $aL \subset L$ are called lattices with *complex multiplication*.

For an integer $n \geqslant 2$ and a lattice $L$ we define a Lattès map $A_{n,L}$ by the commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\varphi = nz} & \mathbb{C} \\ \downarrow \wp_L & & \downarrow \wp_L \\ \mathbb{O} & \xrightarrow{A_{n,L}} & \mathbb{O}. \end{array}$$

Clearly, the rational functions $A_{n,L}$ and $A_{m,L}$ commute for any $n, m \geqslant 2$. Let $L'$ be a lattice satisfying

$$L \subset L' \subset L/n. \tag{18}$$

For example, if $L = \langle \omega_1, \omega_2 \rangle$, we can set $L' = \langle \omega_1, \frac{\omega_2}{n} \rangle$. With such $L'$ we can associate a functional decomposition

$$A_{n,L} = X_{L'} \circ Y_{L'} \tag{19}$$

as follows. Since any even doubly periodic meromorphic function with period lattice $L$ is a rational function in $\wp_L$, it follows from (18) that there exist rational functions $X, Y, F$ such that

$$\wp_{L/n} = X \circ \wp_{L'}, \quad \wp_{L'} = Y \circ \wp_L, \quad \wp_{L/n} = F \circ \wp_L,$$

and it is clear that $F = X \circ Y$. On the other hand, since

$$\wp_{L/n}(z/n) = n^2 \wp_L(z),$$

we have:

$$\wp_{L/n} = n^2 \wp_L(nz) = n^2 A_{n,L} \circ \wp_L.$$

Therefore, $F = n^2 A_{n,L}$, and hence (19) holds for $X_{L'} = X/n^2$ and $Y_{L'} = Y$. Another way to obtain decomposition (19) is to consider the projections of the isogeny $\mathbb{C}/L \to \mathbb{C}/L'$ and its dual (see Pakovich 2019a, Section 3). Note that

$$\deg X_{L'} = [L/n : L'], \quad \deg Y_{L'} = [L' : L],$$

and both these numbers are greater than one since $L'$ is distinct from $L$ and $L/n$. Finally, it is clear that

$$\deg X_{L'} \cdot \deg Y_{L'} = n^2. \tag{20}$$

We now observe that since the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{mz} & \mathbb{CP}^1 \\ \downarrow{\scriptstyle \wp_L} & & \downarrow{\scriptstyle \wp_L} \\ \mathbb{CP}^1 & \xrightarrow{A_{m,L}} & \mathbb{CP}^1 \\ \downarrow{\scriptstyle A_{n,L}} & & \downarrow{\scriptstyle A_{n,L}} \\ \mathbb{CP}^1 & \xrightarrow{A_{m,L}} & \mathbb{CP}^1 \end{array}$$

commutes, it follows from the equalities (19) and $\wp_{L'} = Y_{L'} \circ \wp_L$ that the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{A_{m,L}} & \mathbb{CP}^1 \\ \downarrow{\scriptstyle Y_{L'}} & & \downarrow{\scriptstyle Y_{L'}} \\ \mathbb{CP}^1 & \xrightarrow{A_{m,L'}} & \mathbb{CP}^1 \\ \downarrow{\scriptstyle X_{L'}} & & \downarrow{\scriptstyle X_{L'}} \\ \mathbb{CP}^1 & \xrightarrow{A_{m,L}} & \mathbb{CP}^1 \end{array}$$

also commutes. Thus, $A_{m,L'}$ and $A_{m,L}$ are mutually semiconjugate. Since for any lattice $L$ there exist lattices $L'$ satisfying (18), we obtain in this way a large class of examples of mutually semiconjugate rational functions, and we will show below that at least some of these functions are not equivalent.

Note that the functions $A_{m,L'}$ and $A_{m,L}$ are isospectral for any lattice $L$, since the multiplier spectrum of $A_{m,L}$ depends only on $m$ (see e.g. Silverman 2007, Proposition 6.52(b)). Thus, we cannot use Lemma 3.1 to prove that $A_{m,L'} \not\sim A_{m,L}$. Instead, we use the following observation.

**Lemma 4.1** *Let $A$ be a Lattès map of degree $d$ with $l(A) = 2$, and $B$ a rational function such that $B \sim A$. Then $B$ is a Lattès map with $l(B) = 2$, and there exists a rational function $T$ whose degree divides $d^k$, $k \geqslant 0$, such that $T : \mathcal{O}_A \to \mathcal{O}_B$ is a covering map.*

***Proof*** If $B$ is an elementary transformation of $A$, that is $A = U \circ V$ and $B = V \circ U$ for some rational functions $U$ and $V$, then by Lemma 2.1 there exists an orbifold $\mathcal{O}'$ with $\nu(\mathcal{O}') = (2, 2, 2, 2)$ such that

$$V : \mathcal{O} \to \mathcal{O}', \quad U : \mathcal{O}' \to \mathcal{O}$$

are covering maps. Therefore, $B : \mathcal{O}' \to \mathcal{O}'$ is a covering map, and hence $B$ is a Lattès map with $\mathcal{O}_B = \mathcal{O}'$ and $l(B) = 2$. Moreover, the map $T = V$ satisfies the requirements of the lemma since deg $V$ is a divisor of $d$. Since any $B \sim A$ is obtained from $A$ by a chain of elementary transformations, and elementary transformations do not change the degree, using the above reasoning recursively and composing corresponding functions $V$, we obtain a rational function $T$ with the required properties. □

**Theorem 4.1** *Let $L$ be a lattice without complex multiplication, and $n$, $m$ distinct primes. Then for any lattice $L'$ satisfying $L \subset L' \subset L/n$ the functions $A_{m,L'}$ and $A_{m,L}$ are mutually semiconjugate but non-equivalent.*

***Proof*** Assume that $A_{m,L} \sim A_{m,L'}$, and let $T : \mathcal{O}_{A_{m,L}} \to \mathcal{O}_{A_{m,L'}}$ be a covering map between orbifolds provided by Lemma 4.1. Then $\deg T = m^k$, $k \geqslant 0$, since $\deg A_{m,L} = m^2$ and $m$ is a prime. Applying Lemma 2.1 to decomposition (19), we conclude that there exists an orbifold $\mathcal{O}^*$ with $\nu(\mathcal{O}^*) = (2, 2, 2, 2)$ such that

$$Y_{L'} : \mathcal{O} \to \mathcal{O}^*, \quad X_{L'} : \mathcal{O}^* \to \mathcal{O}$$

are covering maps, and as in the proof of Theorem 2.3 we see that the map $\wp_{L'} = Y_{L'} \circ \wp_L$ is the universal covering of $\mathcal{O}^*$. Since $\wp_{L'}$ is also the universal covering of $\mathcal{O}_{A_{m,L'}}$, this implies that

$$\mathcal{O}^* = \mathcal{O}_{A_{m,L'}}.$$

Thus, $X'_L : \mathcal{O}_{A_{m,L'}} \to \mathcal{O}_{A_{m,L}}$ is a covering map, and hence the composition

$$X'_L \circ T : \mathcal{O}_{A_{m,L}} \to \mathcal{O}_{A_{m,L}}$$

is also a covering map. Since by assumption $L$ is a lattice without complex multiplication, the number $\deg (X'_L \circ T)$ must be a perfect square. On the other hand, since $n$ is a prime, it follows from (18) and (20) that

$$\deg X'_L = [L/n : L'] = n,$$

implying that

$$\deg (X'_L \circ T) = nm^k, \quad k \geqslant 0.$$

Thus, since $n > 1$ and $\gcd(n, m) = 1$, the number $nm^k$ cannot be a perfect square. The contradiction obtained finishes the proof. □

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