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RESEARCH CONTRIBUTION

Newton–Okounkov Polytopes of Flag Varieties for Classical Groups

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Abstract

For classical groups $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})$, we define uniformly geometric valuations on the corresponding complete flag varieties. The valuation in every type comes from a natural coordinate system on the open Schubert cell, and is combinatorially related to the Gelfand–Zetlin pattern in the same type. In types A and C, we identify the corresponding Newton–Okounkov polytopes with the Feigin–Fourier–Littelmann–Vinberg polytopes. In types B and D, we compute low-dimensional examples and formulate open questions.

Keywords Newton–Okounkov convex body · Flag variety · FFLV polytope

1 Introduction

Toric geometry and theory of Newton polytopes exhibited fruitful connections between algebraic geometry and convex geometry. After the Kushnirenko and Bernstein–Khovanskii theorems were proved in the 1970-s (for a reminder see Sect. 1.1), Askold Khovanskii asked how to extend these results to the setting where a complex torus is replaced by an arbitrary connected reductive group. In particular, he advertised widely the problem of finding the right analogs of Newton polytopes for non-toric varieties such as spherical varieties (classical examples of spherical varieties are reviewed in Sect. 1.2). The notion of a Newton polytope was extended to spherical varieties by

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To my teacher R. K. Gordin with gratitude and admiration.

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Andrei Okounkov in the 1990-s Okounkov (1997, 1998). Later, his construction was developed systematically in Kaveh and Khovanskii (2012), Lazarsfeld and Mustata (2009), and the resulting theory of Newton–Okounkov convex bodies is now an active field of algebraic geometry.

While Newton–Okounkov convex bodies can be defined for line bundles on arbitrary varieties (without a group action), they are easier to describe for varieties with a reductive group action. In the latter case, theory of Newton–Okounkov convex bodies is closely related with representation theory. For instance, Gelfand–Zetlin (GZ) polytopes and Feigin–Fourier–Littelmann–Vinberg (FFLV) polytopes (see Sect. 2 for a reminder) arise naturally as Newton–Okounkov polytopes of flag varieties for various geometric valuations Okounkov (1998), Kaveh (2015), Feigin et al. (2017), Fujita and Oya (2017), Kiritchenko (2017). In the present paper, we introduce a new uniform geometric valuation on complete flag varieties for classical groups, and exhibit FFLV polytopes in types A and C as Newton–Okounkov polytopes with respect to this valuation (Theorems 3.2 and 3.3). In type C, the result appears to be new. In type A, it can be deduced from [Kiritchenko (2017), Theorem 2.1]. We provide a proof that works simultaneously in types A and C (see the proof of Theorem 3.3). We hope that in types B and D the valuation and the corresponding Newton–Okounkov polytopes might give a tool for describing FFLV bases (that have been described so far only in types A and C).

1.1 Newton–Okounkov Convex Bodies

In this section, we recall construction of Newton–Okounkov convex bodies for the general mathematical audience. Let us start from the definition of Newton polytopes.

Definition 1 Let $f = \sum_{\alpha\in\mathbb{Z}^n} c_\alpha x^\alpha$ be a Laurent polynomial in n variables (here the multiindex notation x^α for $x = (x_1, \ldots, x_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ stands for $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$). The *Newton polytope* $\Delta_f \subset \mathbb{R}^n$ is the convex hull of all $\alpha \in \mathbb{Z}^n$ such that $c_\alpha \neq 0$.

By definition, Newton polytope is a lattice polytope, that is, its vertices lie in $\mathbb{Z}^n$.

Example 1.1 For $n = 2$ and $f = 1 + 2x_1 + x_2 + 3x_1x_2$, the Newton polytope Δ_f is the square with the vertices (0, 0), (1, 0), (0, 1) and (1, 1).

Note that Laurent polynomials with complex coefficients are well-defined functions at all points $(x_1, \ldots, x_n) \in \mathbb{C}^n$ such that $x_1, \ldots, x_n \neq 0$. They are regular functions on the complex torus $(\mathbb{C}^*)^n := \mathbb{C}^n \setminus \bigcup_{i=1}^n \{x_i = 0\}$.

Theorem 1.2 (Kushnirenko (1976)) *For a given lattice polytope* $\Delta \subset \mathbb{R}^n$*, let* $f_1(x_1, \ldots, x_n)$*, …,* $f_n(x_1, \ldots, x_n)$ *be a generic collection of Laurent polynomials with the common Newton polytope* Δ*. Then the system* $f_1 = \ldots = f_n = 0$ *has* $n!\mathrm{Volume}(\Delta)$ *solutions in the complex torus* $(\mathbb{C}^*)^n$.

The Kushnirenko theorem can be viewed as a generalization of the classical Bezout theorem. The Newton polytope serves as a refinement of the degree of a polynomial.

This makes the Kushnirenko theorem applicable to collections of polynomials which are not generic among all polynomials of given degree but only among polynomials with given Newton polytope. For instance, the Kushnirenko theorem applied to a pair of generic polynomials with Newton polytope as in Example 1.1 yields the correct answer 2 while Bezout theorem yields an incorrect answer 4 (because of two extraneous solutions at infinity). A more geometric viewpoint on the Bezout theorem and its extensions stems from enumerative geometry and will be discussed in Sect. 1.2. The Koushnirenko theorem was extended to the systems of Laurent polynomials with distinct Newton polytopes by David Bernstein and Khovanskii using mixed volumes of polytopes Bernstein (1975). Further generalizations include explicit formulas for the genus and Euler characteristic of complete intersections $\{f_1 = 0\} \cap \ldots \cap \{f_m = 0\}$ in $(\mathbb{C}^*)^n$ for $m < n$ Khovanskii (1978).

The theory of Newton–Okounkov bodies generalizes the Kushnirenko theorem to subspaces of functions not necessarily spanned by monomials. We now consider a much more general situation. Fix a finite-dimensional vector space $V \subset \mathbb{C}(x_1, \ldots, x_n)$ of rational functions on $\mathbb{C}^n$. Let $X_0 \subset \mathbb{C}^n$ be an open dense subset obtained by first removing poles of functions from V and then removing common zeroes of all functions in V. Let $f_1,\ldots, f_n$ be a generic collection of functions from V. How many solutions does a system $f_1 = \ldots = f_n = 0$ have in X_0? For instance, if V is the space spanned by all Laurent polynomials with a given Newton polytope, and $X_0 = (\mathbb{C}^*)^n$, then the answer is given by the Kushnirenko theorem. Here is a simple non-toric example from representation theory.

Example 1.3 Let $n = 3$. Consider the adjoint representation of $SL_3(\mathbb{C})$ on the space $\mathrm{End}(\mathbb{C}^3)$ of all linear operators on $\mathbb{C}^3$. That is, $g \in SL_3(\mathbb{C})$ acts on an operator $X \in \mathrm{End}(\mathbb{C}^3)$ as follows:

$$\mathrm{Ad}(g) : X \mapsto gXg^{-1}.$$

Let $U^- \subset SL_3(\mathbb{C})$ be the subgroup of lower triangular unipotent matrices:

$$U^- = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & x_3 & 1 \end{pmatrix} \mid (x_1, x_2, x_3) \in \mathbb{C}^3 \right\}. \tag{1}$$

We now define the subspace $V \subset \mathbb{C}(x_1, x_2, x_3)$ to be the restriction of functions from the dual space $\mathrm{End}^*(\mathbb{C}^3)$ to the U^--orbit $\mathrm{Ad}(U^-)E_{13}$ of the operator $E_{13} := e_1 \otimes e_3^* \in \mathrm{End}(\mathbb{C}^3)$ (here (e_1, e_2, e_3) is the standard basis in $\mathbb{C}^3$). More precisely, a linear function $f \in \mathrm{End}^*(\mathbb{C}^3)$ yields the polynomial $\hat{f}(x_1, x_2, x_3)$ as follows:

$$\hat{f}(x_1, x_2, x_3) := f\left(\begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & x_3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & x_3 & 1 \end{pmatrix}^{-1} \right)$$

It is easy to check that the space V is spanned by 8 polynomials: $1, x_1, x_2, x_3, x_1x_2 - x_1^2x_3, x_1x_3, x_2x_3, x_2^2 - x_1x_2x_3$. It will be clear from the next section that the Kushnirenko theorem does not apply to the space V, that is, the normalized volume of the Newton polytope of a generic polynomial from V is bigger than the number of solutions of a generic system $f_1 = f_2 = f_3 = 0$ with $f_i \in V$. In particular, V is not spanned by a collection of monomials in the variables x_1, x_2 and x_3.

To assign the *Newton–Okounkov convex body* to V we need an extra ingredient. Choose a translation-invariant total order on the lattice $\mathbb{Z}^n$ (e.g., we can take the lexicographic order). Consider a map

$$v : \mathbb{C}(x_1, \ldots, x_n)\backslash\{0\} \to \mathbb{Z}^n,$$

that behaves like the lowest order term of a polynomial, namely: $v(f + g) \geq \min\{v(f), v(g)\}$ and $v(fg) = v(f) + v(g)$ for all nonzero f, g. Recall that maps with such properties are called *valuations*. A straightforward construction of valuations is shown in Example 1.5 below.

Definition 2 The *Newton–Okounkov convex body* $\Delta_v(V)$ is the closure of the convex hull of the set

$$\bigcup_{k=1}^{\infty} \left\{ \frac{v(f)}{k} \mid f \in V^k \right\} \subset \mathbb{R}^n.$$

By V^k we denote the subspace spanned by all the k-fold products of functions from V.

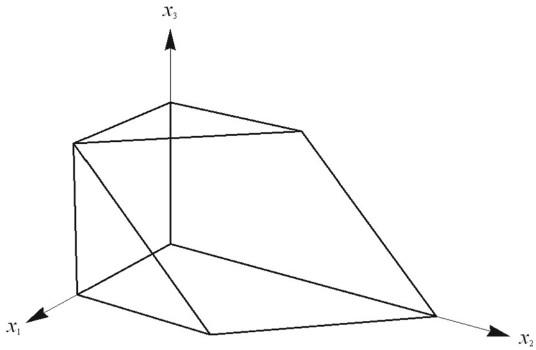
Different valuations might yield different Newton–Okounkov convex bodies. An important application of Newton–Okounkov bodies is the following analog of Kushnirenko theorem. Recall that by $X_0 \subset \mathbb{C}^n$ we denoted an open dense subset where all functions from V are regular (that is, do not have poles).

Theorem 1.4 (Kaveh and Khovanskii (2012), Lazarsfeld and Mustata (2009)) *If V is sufficiently big, then a generic system $f_1 = \ldots = f_n = 0$ with $f_i \in V$ has $n!\mathrm{Volume}(\Delta_v(V))$ solutions in X_0.*

In particular, it follows that all Newton–Okounkov convex bodies for V have the same volume. For more details (in particular, for the precise meaning of "sufficiently big") we refer the reader to (Kaveh and Khovanskii 2012, Theorem 4.9).

Example 1.5 Let V be the space from Example 1.3. Define a valuation v by assigning to a polynomial $f \in \mathbb{C}[x_1, x_2, x_3]$ its lowest order term with respect to the lexicographic ordering of monomials. More precisely, we say that $x_1^{k_1}x_2^{k_2}x_3^{k_3} \succ x_1^{l_1}x_2^{l_2}x_3^{l_3}$ iff there exists $j \leq 3$ such that $k_i = l_i$ for $i < j$ and $k_j > l_j$. It is easy to check that $v(V)$ consists of 8 lattice points $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, $(0, 2, 0)$. Their convex hull is depicted on Fig. 1. This is the FFLV polytope $FFLV(1, 0, -1)$ for the adjoint representation of SL_3 (in this case, it happens to be unimodularly equivalent to a GZ polytope). In particular, $FFLV(1, 0, -1) \subset \Delta_v(V)$. In Example 1.6, we show that $FFLV(1, 0, -1) = \Delta_v(V)$ by comparing the volumes of both convex bodies.

Fig. 1 FFLV polytope for the adjoint representation of SL_3



1.2 Enumerative Geometry

In this section, we give a brief introduction to enumerative geometry for the general mathematical audience. Enumerative geometry motivated the study of Grassmannians, flag varieties and more general spherical varieties. Enumerative geometry has roots in antiquity and asks for number of geometric objects, such as points, lines, planes or circles, that satisfy certain incidence or tangency conditions. For instance, the famous Apollonius problem asks to construct circles that are tangent to three given circles in a plane. Below we recall two classical problems of enumerative geometry from the 19-th century.

Problem 1 (Schubert) *How many lines in a 3-space intersect four given lines in general position?*

We can identify lines in $\mathbb{CP}^3$ with planes through the origin in $\mathbb{C}^4$, that is, a line can be viewed as a point on the Grassmannian $Gr(2, 4)$, parameterizing 2-planes through the origin in $\mathbb{C}^4$. The condition that a line $l \in Gr(2, 4)$ intersects a fixed line l_1 defines a hypersurface $H_1 \subset Gr(2, 4)$. Hence, the problem reduces to computing the number of intersection points of four hypersurfaces in $Gr(2, 4)$. It is not hard to check that the hypersurface H_1 is just a hyperplane section of the Grassmannian under the Plücker embedding $Gr(2, 4) \hookrightarrow \mathbb{P}(\Lambda^2\mathbb{C}^4) \simeq \mathbb{CP}^5$. The image of the Grassmannian is a quadric in $\mathbb{CP}^5$. The number of intersection points of a quadric in $\mathbb{CP}^5$ with four hyperplanes in general position is equal to 2 by the Bezout theorem. Hence, the answer to the Schubert problem is 2.

Schubert's problem can also be solved for real lines in $\mathbb{R}^3$ by elementary metods (for instance, by using two families of lines on a hyperboloid of one sheet). In this context, Schubert's problem was recently applied to experimental physics (Belyaev et al. 2019).

Problem 2 (Steiner) *How many smooth conics are tangent to five given conics?*

Similarly to the Schubert problem, we can identify conics with points in $\mathbb{CP}^5$, namely, the conic given by an equation $ax^2+bxy+cy^2+dxz+eyz+fz^2 = 0$ corresponds to the point $(a : b : c : d : e : f) \in \mathbb{CP}^5$. Smooth conics form an open subset $C \subset \mathbb{CP}^5$ (the complement $\mathbb{CP}^5 \backslash C$ is the zero set of the discriminant). The condition that a conic is tangent to a given conic defines a hypersurface in $\mathbb{CP}^5$ of degree 6. Using Bezout theorem in $\mathbb{CP}^5$ one might guess (as Jacob Steiner himself did) that the answer to the Steiner problem is 6^5. However, the correct answer is much smaller. This is similar to the difference between the Bezout and Kushnirenko theorems: the former yields extraneous solutions that have no enumerative meaning. The correct answer was found by Michel Chasles who used (in modern terms) a *wonderful compactification* of C, namely, the space of complete conics.

Hermann Schubert developed a powerful general method (calculus of conditions) for solving problems of enumerative geometry such as Problems 1, 2. In a sense, his method was based on an informal version of intersection theory. The 15-th Hilbert problem asked for a rigorous foundation of Schubert calculus.[1] In the first half of the 20-th century, these foundations were developed both in the topological (cohomology rings) and algebraic (Chow rings) settings. However, Schubert's version of intersection theory was formalized only in the 1980-s by De Concini and Procesi (1985).

In particular, many problems of enumerative geometry (including Problems 1 and 2) reduce to computation of the self-intersection index of a hypersurface in a homogeneous space G/H where G is a reductive group such as $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$ or $Sp_{2n}(\mathbb{C})$. In the toric case ($G = (\mathbb{C}^*)^n$), the Kushnirenko theorem yields an explicit formula for the self-intersection index of a hypersurface $\{f = 0\}$ in a homogeneous space where f is a generic polynomial with a given Newton polytope. In the reductive case, explicit formulas were obtained by Kazarnovskii (1987) (case of $(G \times G)/G^{\text{diag}}$) and Michel Brion (general case) (Brion 1989, Theorem 4.1). Though the Brion–Kazarnovskii formula was originally stated in different terms, it can be reformulated using Newton–Okounkov polytopes (Kaveh and Khovanskii 2012, Corollaries 6.9, 6.10).

Example 1.6 We now place Example 1.3 into the context of enumerative geometry. Let $X = \{(V^1 \subset V^2 \subset \mathbb{C}^3) \mid \dim V^i = i\}$ be the variety of complete flags in $\mathbb{C}^3$. This is a homogeneous space under the action of $SL_3(\mathbb{C})$, namely, $X = SL_3(\mathbb{C})/B$ where B is the subgroup of upper-triangular matrices. It is easy to check that B acts on X with an open dense orbit $U^-B/B \simeq U^-$ where U^- is the subgroup in (1) of Example 1.3.

We say that two flags $V^1 \subset V^2$ and $W^1 \subset W^2$ in $\mathbb{C}^3$ are in general position if $V^1 \not\subset W^2$ and $W^1 \not\subset V^2$. How many flags in $\mathbb{C}^3$ are not in general position with three given flags? By taking projectivizations of subspaces $V^1 \subset V^2 \subset \mathbb{C}^3$ we can regard a flag as $a \in l \subset \mathbb{CP}^2$, where $a = \mathbb{P}(V^1)$ is a point and $l = \mathbb{P}(V^2)$ is a line on the projective plane. Hence, we can reduce the question to the following elementary problem.

Problem 3 (High school geometry) *Consider a triangle ABC on the plane. Take points A', B', C' that lie on the lines BC, AC and AB, respectively. Find all configurations (X, YZ) (where X lies on a line YZ) such that (X, YZ) is not in general position with the configurations (A', BC), (B', AC) and (C', AB).*

[1] Das Problem besteht darin, *diejenigen geometrischen Anzahlen strenge und unter genauer Feststellung der Grenzen ihrer Gültigkeit zu beweisen, die insbesondere Schubert auf Grund des sogenannten Princips der speciellen Lage mittelst des von ihm ausgebildeten Abzählungskalküls bestimmt hat* (Hilbert).

It is easy to show that there are 6 such configurations.

On the other hand, the same answer can be found using the simplest projective embedding of X:

$$p : X \hookrightarrow \mathbb{P}(\mathbb{C}^3) \times \mathbb{P}(\Lambda^2\mathbb{C}^3) \overset{\text{Segre}}{\hookrightarrow} \mathbb{P}(\mathrm{End}(\mathbb{C}^3)); \quad p : (V^1, V^2) \mapsto V^1 \times V^2 \mapsto V^1 \otimes \Lambda^2 V^2,$$

and counting the number of intersection points of $p(X)$ with 3 generic hyperplanes in $\mathbb{CP}^8$ (that is, the *degree* of $p(X)$). Restricting the map p to the open dense B-orbit $U^- \subset X$ we get that the latter problem reduces to the problem from Example 1.3. In particular, we can show that the inclusion $FFLV(1, 0, -1) \subset \Delta_v(V)$ is an equality. Indeed, by Theorem 1.4 the volume of $\Delta_v(V)$ times 3! is equal to the degree of $p(X)$, that is, 6. Hence, the volume of $\Delta_v(V)$ is equal to 1. Since the volume of $FFLV(1, 0, -1)$ is also equal to 1, the inclusion $FFLV(1, 0, -1) \subset \Delta_v(V)$ of convex bodies implies the exact equality.

2 GZ Patterns and FFLV Polytopes

In this section, we recall the definitions of GZ patterns in types A, B, C, D and FFLV polytopes in types A and C. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ denote a non-increasing sequence of integers. In what follows, we regard λ as a dominant weight of a classical group. GZ polytopes for classical groups G were constructed using representation theory, namely, lattice points in the polytope $GZ(\lambda)$ parameterize elements of the GZ basis in the irreducible representation V_λ of G with the highest weight λ (see Molev (2006) for a survey on GZ bases). Lattice points in FFLV polytopes $FFLV(\lambda)$ parameterize elements of a different basis in the same representation [see Feigin et al. (2011a, b)]. In particular, $GZ(\lambda)$ and $FFLV(\lambda)$ have the same Ehrhart and volume polynomials.

2.1 GZ Patterns

2.1.1 Type A

We now regard λ as a dominant weight of SL_n. In convex geometric terms, the GZ polytope $GZ(\lambda) \subset \mathbb{R}^d$, where $d := \frac{n(n-1)}{2}$, is defined as the set of all points $(u_1^1, u_2^1, \ldots, u_{n-1}^1; u_1^2, \ldots, u_{n-2}^2; \ldots; u_1^{n-1}) \in \mathbb{R}^d$ that satisfy the following interlacing inequalities:

$$\begin{array}{ccccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & & \ldots & & & \lambda_n \\
& \boxed{u_1^1} & & \boxed{u_2^1} & & & \ldots & & & \boxed{u_{n-1}^1} & \\
& & \boxed{u_1^2} & & \ldots & & & & \boxed{u_{n-2}^2} & & \\
& & & \ddots & & & \ddots & & & & \\
& & & & \boxed{u_1^{n-2}} & & & \boxed{u_2^{n-2}} & & & \\
& & & & & & \boxed{u_1^{n-1}} & & & &
\end{array} \qquad (GZ_A)$$

where the notation

$$\begin{array}{ccc} a & & b \\ & c & \end{array}$$

means $a \geq c \geq b$ (the table encodes $2d$ inequalities). Lattice points in $GZ(\lambda)$ parameterize elements of the Gelfand–Zetlin basis in the irreducible representation V_λ of SL_n with the highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ Molev (2006). In particular, the number of lattice points in $GZ(\lambda)$ is equal to dim V_λ and can be easily computed using the Weyl dimension formula.

2.1.2 Types B and C

Let λ be a dominant weight of $Sp_{2n}(\mathbb{C})$, that is, all the λ_i are non-negative. Put $d = n^2$ and denote coordinates in $\mathbb{R}^d$ by $(x_1^1, \ldots, x_n^1; y_1^1, \ldots, y_{n-1}^1; \ldots; x_1^{n-1}, x_2^{n-1}, y_1^{n-1}; x_1^n)$. For every λ, define the *symplectic GZ polytope* $SGZ(\lambda) \subset \mathbb{R}^d$ for $Sp_{2n}(\mathbb{C})$ by the following interlacing inequalities:

$$\begin{array}{ccccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & \ldots & & \lambda_n & & 0 \\
& \boxed{x_1^1} & & \boxed{x_2^1} & & \ldots & & & & \boxed{x_n^1} & \\
& & \boxed{y_1^1} & & \boxed{y_2^1} & & \ldots & & \boxed{y_{n-1}^1} & & 0 \\
& & & \boxed{x_1^2} & & \ldots & & & & \boxed{x_{n-1}^2} & \\
& & & & \boxed{y_1^2} & & \ldots & & \boxed{y_{n-2}^2} & & 0 \\
& & & & & \ddots & & & \vdots & & \vdots \\
& & & & & & \boxed{x_1^{n-1}} & & & \boxed{x_2^{n-1}} & \\
& & & & & & & & \boxed{y_1^{n-1}} & & 0 \\
& & & & & & & & & \boxed{x_1^n} &
\end{array} \qquad (GZ_C)$$

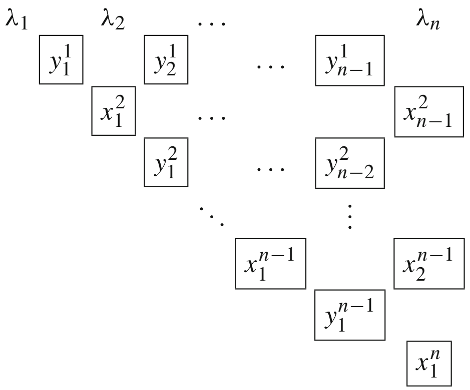
Again, every coordinate in this table is bounded from above by its upper left neighbor and bounded from below by its upper right neighbor (the table encodes $2d$ inequalities). Roughly speaking, $SGZ(\lambda)$ is the polytope defined using half of the GZ pattern (GZ_A) for $SL_{2n}(\mathbb{C})$.

To define the GZ polytope in type B (that is, for $G = SO_{2n+1}(\mathbb{C})$) we use the same pattern and inequalities but choose a bigger lattice $L \subset \mathbb{R}^d$ so that the standard lattice $\mathbb{Z}^d \subset L$ has index 2 in L [see (Berenstein and Zelevinsky 1989, Sect. 4) for more details]. Similarly to type A case, lattice points in this polytope parameterize the Gelfand–Zetlin basis in the corresponding irreducible representations of $SO_{2n+1}(\mathbb{C})$ Molev (2006).

Note that the construction of Gelfand–Zetlin bases in type C becomes considerably more complicated in type C than in types A, B and D Zhelobenko (1987). This motivated Okounkov to find another geometric connection between GZ polytopes in type C and irreducible representaions of Sp_{2n}. As a byproduct, he introduced the construction of what is now called a Newton–Okounkov body. In particular, he performed the first explicit computation of a Newton–Okounkov polytope of a flag variety Okounkov (1998).

2.1.3 Type D

Let λ be a dominant weight of $SO_{2n}(\mathbb{C})$. Put $d = n(n-1)$. Denote coordinates in $\mathbb{R}^d$ by $(y_1^1, \ldots, y_{n-1}^1; \ldots; x_1^{n-1}, x_2^{n-1}, y_1^{n-1}; x_1^n)$. For every λ, define the *even orthogonal GZ polytope* $OGZ(\lambda) \subset \mathbb{R}^d$ for $SO_{2n}(\mathbb{C})$ using the following table:



(GZ_D)

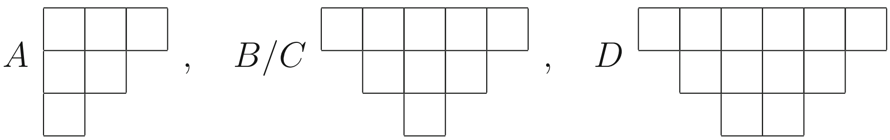
Again, every coordinate in this table is bounded from above by its upper left neighbor and bounded from below by its upper right neighbor. There are also extra inequalities for every $i = 1, \ldots, n-2$:

$$x_{n-i}^i + x_{n+1-i}^i + x_{n-i}^{i+1} \geq y_{n-i}^i; \quad x_{n-i-1}^{i+1} + x_{n+1-i}^i + x_{n-i}^{i+1} \geq y_{n-i}^i,$$

and inequality $x_1^{n-1} + x_2^{n-1} + x_1^n \geq y_1^{n-1}$ [see (Berenstein and Zelevinsky 1989, Sect. 4) for more details].

In what follows, we will use not GZ polytopes themselves but the GZ tables.

Remark 2.1 If we rotate GZ tables in types A, B/C and D by $\frac{3\pi}{4}$ clockwise we will get the following tables:



We will use this presentation of GZ tables in the proof of Theorem 3.3.

2.2 FFLV Polytopes

2.2.1 Type A

For every dominant weight λ of $SL_n(\mathbb{C})$, we now define the FFLV polytope $FFLV(\lambda)$. Put $d := \frac{n(n-1)}{2}$. Label coordinates in $\mathbb{R}^d$ by $(u^1_{n-1}; u^2_{n-2}, u^1_{n-2}; \ldots; u^{n-1}_1, u^{n-2}_1, \ldots, u^1_1)$. and organize them using the GZ table (GZ_A). The polytope $FFLV(\lambda)$ in type A is defined by inequalities $u^l_m \geq 0$ and

$$\sum_{(l,m)\in D} u^l_m \leq \lambda_i - \lambda_j$$

for all Dyck paths D going from λ_i to λ_j in table (GZ_A) where $1 \leq i < j \leq n$. A *Dyck path* is a broken line whose segments either connect u^i_j with u^{i+1}_j or connect u^i_j with u^i_{j+1} (the notion of Dyck paths originated in enumerative combinatorics in relation with Catalan numbers). Note that $FFLV(\lambda)$ only depends on the differences $(\lambda_1 - \lambda_2)$,..., $(\lambda_{n-1} - \lambda_n)$. An example of FFLV polytope for $n = 3$ and $\lambda = (1, 0, -1)$ is depicted on Figure 1. It is given by six inequalities

$$0 \leq u^1_1 \leq \lambda_1 - \lambda_2; \quad 0 \leq u^1_2 \leq \lambda_2 - \lambda_3; \quad 0 \leq u^2_1; \quad u^1_1 + u^2_1 + u^1_2 \leq \lambda_1 - \lambda_3.$$

The lattice points in $FFLV(\lambda)$ parameterize the FFLV basis in the irreducible representation V_λ of SL_n. This basis was constructed in Feigin et al. (2011a) in type A and in Feigin et al. (2011b) in type C following a conjecture by Ernest Vinberg. While $GZ(\lambda)$ and $FFLV(\lambda)$ have the same number of lattice points for all λ (in particular, the same Ehrhart polynomial) they are not combinatorially equivalent. An interesting combinatorial proof of the equality of Ehrhart polynomials of $GZ(\lambda)$ and $FFLV(\lambda)$ (in types A and C) is given in Ardila et al. (2011).

2.2.2 Type C

Similarly to type A case, we define FFLV polytopes in type C using the corresponding GZ table (GZ_C). The only difference with type A case is that we allow Dyck paths to end at one of the 0 entry in the rightmost column of the table (see Feigin et al. (2011b), Ardila et al. (2011) for more details). It is interesting that $FFLV(\lambda)$ in type C was first constructed from a combinatorial viewpoint in Ardila et al. (2011) and then interpreted from a representation-theoretic viewpoint in Feigin et al. (2011b).

3 Valuations on Flag Varieties

We now construct uniformly a valuation v on flag varieties in types A, B, C and D. In types A and C, we identify the corresponding Newton–Okounkov polytopes with FFLV polytopes. In type B_2, we get a *symplectic DDO polytope* (Kiritchenko 2016, Sect. 4), which is not combinatorially equivalent to either the FFLV or the GZ polytope in type C_2. In type D_3, we get a polytope that is different from both GZ and FFLV polytopes in type A_3, however, the question of combinatorial equivalence is open.

Fix a complete flag of subspaces $F^\bullet := (F^1 \subset F^2 \subset \ldots \subset F^{n-1} \subset \mathbb{C}^n)$, and a basis $e_1,\ldots, e_n$ in $\mathbb{C}^n$ compatible with $F^\bullet$, that is, $F^i = \langle e_1, \ldots, e_i \rangle$. Define a non-degenerate symmetric bilinear $(\cdot, \cdot)$ form on $\mathbb{C}^n$ as follows:

$$(e_i, e_j) = \begin{cases} 1 & \text{if } i + j = n + 1; \\ 0 & \text{otherwise .} \end{cases}$$

Similarly, we define a non-degenerate skew symmetric form $\omega(\cdot, \cdot)$ for even n. For $i < j$, put

$$\omega(e_i, e_j) = -\omega(e_j, e_i) = \begin{cases} 1 & \text{if } i + j = n + 1; \\ 0 & \text{if } i + j \neq n + 1. \end{cases}$$

Let $B \subset SL_n(\mathbb{C})$ be a subgroup of upper triangular matrices with respect to the basis $e_1,\ldots, e_n$. Recall that the complete flag variety $SL_n(\mathbb{C})/B$ can be defined as the variety of complete flags of subspaces $M^\bullet = (\{0\} \subset V^1 \subset V^2 \subset \ldots \subset V^{n-1} \subset \mathbb{C}^n)$. Similarly, we regard $SO_n(\mathbb{C})/B$ for any n and Sp_n/B for even n as subvarieties of *orthogonal* and *isotropic* flags in $SL_n(\mathbb{C})/B$ and SL_n/B, respectively. A complete flag $M^\bullet$ in $\mathbb{C}^n$ is *orthogonal* if V^i is orthogonal to to V^{n-i} with respect to $(\cdot, \cdot)$. A complete flag $M^\bullet$ in $\mathbb{C}^n$ is called *isotropic* if the restriction of ω to $V^{\frac{n}{2}}$ is zero, and $V^{n-i} = \{v \in \mathbb{C}^n \mid \omega(v, u) = 0 \text{ for all } u \in V^i\}$. In particular, the flag $F^\bullet$ is orthogonal and isotropic by our choice of the forms $(\cdot, \cdot)$ and ω.

Recall that if G is a connected complex semisimple group (e.g., a classical group), then the Picard group of the complete flag variety G/B can be identified with the weight lattice of G (Brion 2005, 1.4.2). In particular, there is a bijection between dominant weights λ and globally generated line bundles L_λ. Recall also that the space of global sections $H^0(G/B, L_\lambda)$ is isomorphic to V_λ^* where V_λ is the irreducible representation of G with the highest weight λ. Let $v_\lambda \in V_\lambda$ be a highest weight vector, i.e., the line $\langle v_\lambda \rangle \subset V_\lambda$ is B-invariant. There is a well-defined map

$$p_\lambda : G/B \to \mathbb{P}(V_\lambda), \quad gB \mapsto \langle gv_\lambda \rangle \subset \mathbb{P}(V_\lambda).$$

For instance, if $G = SL_3$ and $\lambda = (1, 0, -1)$, then p_λ coincides with the map p of Example 1.6. Similarly to Example 1.3 we may identify V_λ^* with a subspace of $\mathbb{C}(G/B)$. This amounts to fixing a global section $s_0 \in H^0(G/B, L_\lambda)$ and identifying $s \in H^0(G/B, L_\lambda)$ with $\frac{s}{s_0} \in \mathbb{C}(G/B)$. Denote by $\Delta_v(G/B, L_\lambda) \subset \mathbb{R}^d$ the Newton–Okounkov convex body corresponding to G/B, L_λ and v (we denote by d the dimension of G/B). In what follows, we use that the normalized volume of $\Delta_v(G/B, L_\lambda)$ is equal by Theorem 1.4 to the degree of $p_\lambda(G/B) \subset \mathbb{P}(V_\lambda)$. The latter is equal to the volume of $GZ(\lambda)$ and $FFLV(\lambda)$ by the Hilbert's theorem. Indeed, the number of lattice points in both polytopes (regarded as a polynomial in λ) coincides with $\dim V_\lambda = \dim H^0(G/B, L_\lambda)$. Hence, the normalized volume on the one hand and the degree on the other hand are leading coefficients in the same polynomial.

3.1 Type A

Let $G = SL_n(\mathbb{C})$. Put $d = \frac{n(n-1)}{2}$. Recall that the open Schubert cell X° with respect to $F^\bullet$ is defined as the set of all flags $M^\bullet$ that are in general position with the standard flag $F^\bullet$, i.e., all intersections $M^i \cap F^j$ are transverse. We can identify the open Schubert cell $X^\circ \subset G/B$ with an affine space $\mathbb{C}^d$ by choosing for every flag $M^\bullet$ a basis $v_1,\ldots,$ v_n in $\mathbb{C}^n$ of the form:

$$v_1 = e_n + x_1^{n-1}e_{n-1} + \ldots + x_1^1 e_1,$$
$$v_2 = e_{n-1} + x_2^{n-2}e_{n-2} + \ldots + x_2^1 e_1, \quad \ldots \quad , v_{n-1} = e_2 + x_{n-1}^1 e_1, \quad v_n = e_n,$$

so that $V^i = \langle v_1, \ldots, v_i \rangle$. Such a basis is unique, hence, the coefficients $(x_j^i)_{i+j<n}$ are coordinates on the open cell. In other words, every flag $M^\bullet \in X^\circ$ gets identified with a triangular matrix:

$$\begin{pmatrix} x_1^1 & x_2^1 & \ldots & x_{n-1}^1 & 1 \\ x_1^2 & x_2^2 & \ldots & 1 & 0 \\ \vdots & \vdots & & & \vdots \\ x_1^{n-1} & 1 & \ldots & 0 & 0 \\ 1 & 0 & & 0 & 0 \end{pmatrix}. \qquad (*)$$

We order the coefficients $(x_j^i)_{i+j<n}$ of this matrix by starting from column $(n-1)$ and going from top to bottom in every column and from right to left along columns. More precisely, put $(y_1, \ldots, y_d) := (x_{n-1}^1; x_{n-2}^1, x_{n-2}^2; \ldots; x_1^1, x_1^2, \ldots, x_1^{n-1})$. For instance, if $n = 4$ we get the ordering:

$$\begin{pmatrix} y_4 & y_2 & y_1 & 1 \\ y_5 & y_3 & 1 & 0 \\ y_6 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We fix the lexicographic ordering on monomials in coordinates $y_1, \ldots, y_d$ so that $y_1 \succ y_2 \succ \ldots \succ y_d$. By the lexicographic ordering we mean that $y_1^{k_1} \cdots y^{k_d} \succ y^{l_1} \cdots y^{l_d}$ iff there exists $j \le d$ such that $k_i = l_i$ for $i < j$ and $k_j > l_j$.

Remark 3.1 Put $\overline{w_0} = (s_1)(s_2s_1)(s_3s_2s_1)\ldots(s_{n-1}\ldots s_1)$, and denote by w_k the k-th terminal subword of $\overline{w_0}$, that is, $w_{d-1} = s_1$, $w_{d-2} = s_2s_1$ and so on. It is not hard to check that coordinates $(y_1, \ldots, y_d)$ are compatible with the flag of translated Schubert subvarieties:

$$w_0 X_{\mathrm{id}} \subset w_0 w_{d-1}^{-1} X_{w_{d-1}} \subset w_0 w_{d-2}^{-1} X_{w_{d-2}} \subset \ldots \subset w_0 w_1^{-1} X_{w_1} \subset SL_n/B,$$

i.e., $w_0 w_k^{-1} X_{w_k} \cap X^\circ = \{y_1 = \ldots = y_k = 0\}$. In (Kiritchenko 2017, Sect. 2.2), there is a geometric construction of different coordinates that are compatible with the same flag.

Let v denote the lowest order term valuation on $\mathbb{C}(G/B)$, that is, if $y_1^{k_1} \cdots y_d^{k_d}$ is the lowest order term of a polynomial $f \in \mathbb{C}(G/B)$ then $v(f) := (k_1, \ldots, k_d) \in \mathbb{Z}^d$. For the ratio $\frac{f}{g}$ of two polynomials we put $v(\frac{f}{g}) := v(f) - v(g)$. Let L_λ be the line bundle on G/B corresponding to a dominant weight $\lambda := (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ of G.

Theorem 3.2 (Kiritchenko 2017, Theorem 2.1) *In type A, the Newton–Okounkov convex body $\Delta_v(G/B, L_\lambda)$ coincides with the FFLV polytope $FFLV(\lambda)$.*

For instance, the computation of the polytope $\Delta_v(SL_n/B, L_\lambda)$ for $n = 3$ and $\lambda = (1, 0, -1)$ is illustrated in Examples 1.3, 1.5, 1.6. Using Remark 3.1 we could deduce Theorem 3.2 directly from (Kiritchenko 2017, Theorem 2.1). Below we give another proof that works simultaneously for types A and C.

3.2 Type *C*

Let $n = 2r$ be even, and $G = Sp_n(\mathbb{C})$. Put $d = r^2$. We define the open Schubert cell X° with respect to $F^\bullet$ as the set of all isotropic flags $M^\bullet$ that are in general position with the standard flag $F^\bullet$. Again, we can identify the open Schubert cell $X^\circ \subset G/B$ with an affine space $\mathbb{C}^d$ using matrix $(*)$. Since $M^\bullet$ is isotropic, the coefficients $(x_j^i)_{i+j<n}$ are no longer independent variables. It is not hard to check that exactly d coefficients, namely, $(x_j^i)_{i+j<n, i\le j}$ are independent. Again, we order the coordinates by starting from column $(n-1)$ and going from top to bottom in every column and from right to left along columns. That is, put $(y_1, \ldots, y_d) := (x_{n-1}^1; x_{n-2}^1, x_{n-2}^2; \ldots; x_r^1, x_r^2, \ldots, x_r^r; \ldots; x_2^1, x_2^2; x_1^1)$.

It is easy to check that every x_j^i for $i > j$ can be expressed as a polynomial in coordinates $y_1, \ldots, y_d$ with the lowest order term x_i^j. In particular, there is a table inside the matrix $(*)$ whose coefficients are coordinates on the Schubert cell X°. Here is an example for $n = 6$ (coefficients inside the table are boxed):

$$\begin{pmatrix} \boxed{x_1^1} & \boxed{x_2^1} & \boxed{x_3^1} & \boxed{x_4^1} & \boxed{x_5^1} & 1 \\ x_1^2 & \boxed{x_2^2} & \boxed{x_3^2} & \boxed{x_4^2} & 1 & 0 \\ x_1^3 & x_2^3 & \boxed{x_3^3} & 1 & 0 & 0 \\ x_1^4 & x_2^4 & 1 & 0 & 0 & 0 \\ x_1^5 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that the table is shaped exactly as the GZ pattern in type C rotated by $\frac{3\pi}{4}$ clockwise (see Remark 2.1).

Similarly to the type A case, let v be the lowest term valuation on $\mathbb{C}(G/B)$ associated with this ordering. Let L_λ be the line bundle on G/B corresponding to a dominant weight $\lambda := (\lambda_1, \dots, \lambda_r) \in \mathbb{Z}^r$ of G. As before, denote by $\Delta_v(G, L_\lambda) \subset \mathbb{R}^d$ the Newton–Okounkov convex body corresponding to G/B, L_λ and v.

Theorem 3.3 *In type C, the Newton–Okounkov convex body $\Delta_v(G/B, L_\lambda)$ coincides with the FFLV polytope $FFLV(\lambda)$ in type C.*

Proof We will provide a uniform proof for types A and C. Note that in both types the irreducible representation corresponding to the fundamental weight ω_k is contained in the k-th exterior power of the tautological representation $G \subset GL_n(\mathbb{C})$ (Fulton and Harris 2004, Exercise 15.14, Theorem 17.5). Hence, the space $H^0(G/B, L_{\omega_k}) = V^*_{\omega_k}$ is spanned by the restrictions of Plücker coordinates of the Grassmannian $Gr(k, n) \subset \mathbb{P}(\Lambda^k\mathbb{C}^n)$ to X°. More precisely, there is a map $X^\circ \subset G/B \to Gr(k, n) \to \mathbb{P}(\Lambda^k\mathbb{C}^n) \supset \mathbb{P}(V_{\omega_k})$, which allows us to identify $V^*_{\omega_k}$ with a subspace of $\mathbb{C}(X^\circ) = \mathbb{C}(G/B)$ spanned by certain $k \times k$ minors of matrix $(*)$. Namely, we take all $k \times k$ minors of the $k \times n$ submatrix of $(*)$ formed by the first k rows. This is equivalent to taking all minors of $k \times (n-k)$ submatrix $A_{k,n-k}$ of $(*)$ with coefficients x^i_j where $i \le k$ and $j \le (n-k)$.

It follows easily from the definition of the valuation v that the lowest order term in any minor of matrix $A_{k,n-k}$ is the diagonal term. Hence, $v(V^*_{\omega_k})$ consists precisely of those points with coordinates (u^i_j) in $\mathbb{R}^d$ such that $u^i_j = 0, 1$ and two nonzero u^i_j never lie on the same Dyck path. Hence, the convex hull of $v(V^*_{\omega_k})$ coincides with the FFLV polytope $FFLV(\omega_i)$ and we get the inclusion $FFLV(\omega_i) \subset \Delta_v(G/B, L_{\omega_i})$. By the superadditivity of Newton–Okounkov convex bodies (Kaveh and Khovanskii 2012, Proposition 2.32) we also have that if $\lambda = \sum m_i\omega_i$ then

$$\sum m_i \Delta_v(G/B, L_{\omega_i}) \subset \Delta_v(G/B, L_\lambda),$$

where the addition in the left hand side is Minkowski sum. From definition one can show $FFLV(\lambda) = \sum m_i FFLV(\omega_i)$. Hence we get the inclusion

$$FFLV(\lambda) \subset \Delta_v(G/B, L_\lambda).$$

This inclusion is equality because both convex bodies have the same volume. □

Remark 3.4 The proof relies on the fact that the volume of $FFLV(\lambda)$ is equal to the degree of $p_\lambda(G/B) \subset \mathbb{P}(V_\lambda)$. In types A and C, this fact has both representation theoretic Feigin et al. (2011a, b) and combinatorial proofs Ardila et al. (2011). In type A, there is also a convex geometric proof (Kiritchenko 2017, Sect. 4). It would be interesting to check whether this proof extends to type C.

Similarly to the type A case, the valuation v in type C can be defined using a flag of translated Schubert subvarieties, however, they no longer correspond to terminal subwords of any decomposition of the longest element in the Weyl group of G. For instance, if $n = 4$ we get subvarieties corresponding to elements $s_2s_1s_2$, s_1s_2 and s_1 of the Weyl group.

3.3 Type *B*

Let $n = 2r + 1$ be odd, and $G = SO_n(\mathbb{C})$. Put $d = r^2$. We define the open Schubert cell X° with respect to $F^\bullet$ as the set of all orthogonal flags $M^\bullet$ that are in general position with the standard flag $F^\bullet$. Again, there is a table (shaped as the GZ pattern in type C) inside the matrix $(*)$ whose coefficients are coordinates on the Schubert cell X°. Here is an example for $n = 5$ (coefficients inside the table are boxed):

$$\begin{pmatrix} x_1^1 & \boxed{x_2^1} & \boxed{x_3^1} & \boxed{x_4^1} & 1 \\ x_1^2 & x_2^2 & \boxed{x_3^2} & 1 & 0 \\ x_1^3 & x_2^3 & 1 & 0 & 0 \\ x_1^4 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Put $(y_1, \ldots, y_d) := (x_{n-1}^1; x_{n-2}^1, x_{n-2}^2; \ldots; x_{r+1}^1, x_{r+1}^2, \ldots, x_{r+1}^r; \ldots; x_3^1, x_3^2; x_2^1)$. As before, let v be the lowest term valuation on $\mathbb{C}(G/B)$ associated with the ordering $y_1 \succ \ldots \succ y_d$. It is easy to check that every x_j^i for $i > j$ can be expressed as a polynomial in coordinates y_1,..., y_d with the lowest order term x_i^j, while x_i^i is a polynomial with the lowest order term $(x_{r+1}^i)^2$.

While we may still use Plücker coordinates to compute $\Delta_v(G/B, L_{\omega_k})$ it is no longer true that the lowest order term in any minor of matrix $A_{k,n-k}$ is the diagonal term (because the diagonal coefficients x_i^i might contribute higher order terms). In particular, the defining inequalities for the convex hull P_k of $v(H^0(G/B, L_{\omega_k}))$ will be more intricate. Still, they can be described by generalizing the notion of Dyck paths. It would be interesting to compare these inequalities with those of Backhaus and Kus (2019) (type B_3), see also [Ma]. To check whether the polytope $P_\lambda := \sum m_i P_i$ coincides with the convex body $\Delta_v(G/B, L_\lambda)$ for $\lambda = \sum m_i\omega_i$ we have to compare their volumes. For instance, one could try to construct a volume preserving piecewise linear map between P_λ and the corresponding GZ-polytope in type B extending the construction of (Kiritchenko 2017, Sect. 4.2).

For B_2, it is easy to check using Plücker coordinates that the convex hull P_λ of $v(H^0(G/B, L_\lambda)) \subset \mathbb{R}^4$ for $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$ contains the Minkowski sum $\lambda_1 P_1 + \lambda_2 P_2$, where P_1 is the 3-dimensional simplex with the vertices $(0, 0, 0, 0)$, $(1, 0, 0, 0)$, $(0, 2, 0, 0)$, $(0, 0, 0, 1)$, and P_2 is the 3-dimensional simplex with the vertices $(0, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$. Hence, P_λ is identical to the Newton–Okounkov polytope computed in (Kiritchenko 2016, Proposition 4.1) (up to relabeling of coordinates). Denote coordinates in $\mathbb{R}^4$ by (u_1, u_2, u_3, u_4). Then P_λ is given by inequalities:

$$0 \le u_1, u_2, u_3, u_4; \quad u_1 \le \lambda_1; \quad u_3 \le \lambda_2; \quad 2u_1 + u_2 + 2(u_3 + u_4) \le 2(\lambda_1 + \lambda_2);$$
$$2u_1 + u_2 + u_3 + u_4 \le 2\lambda_1 + \lambda_2.$$

In particular, its volume coincides with the degree of $p_\lambda(G/B) \subset \mathbb{P}(V_\lambda)$. Hence, $P_\lambda = \Delta_v(G/B, L_\lambda)$. Note that P_λ is not combinatorially equivalent to the FFLV polytope in type C_2 [see (Kiritchenko 2017, Sect. 2.4)].

3.4 Type *D*

Let $n = 2r$ be even, and $G = SO_n(\mathbb{C})$. Put $d = r(r-1)$. There is a table (shaped as the GZ pattern in type D) inside the matrix $(*)$ whose coefficients are coordinates on the Schubert cell X°. Here is an example for $n = 6$ (coefficients inside the table are boxed):

$$\begin{pmatrix} x_1^1 & \boxed{x_2^1} & \boxed{x_3^1} & \boxed{x_4^1} & \boxed{x_5^1} & 1 \\ x_1^2 & x_2^2 & \boxed{x_3^2} & \boxed{x_4^2} & 1 & 0 \\ x_1^3 & x_2^3 & 0 & 1 & 0 & 0 \\ x_1^4 & x_2^4 & 1 & 0 & 0 & 0 \\ x_1^5 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Put $(y_1, \ldots, y_d) := (x^1_{n-1}; x^1_{n-2}, x^2_{n-2}; \ldots; x^1_{r+1}, x^2_{r+1}, \ldots, x^{r-1}_{r+1}; x^1_r, x^2_r, \ldots, x^{r-1}_r; \ldots; x^1_3, x^2_3; x^1_2)$, and define v as before. It is easy to check that every x^i_j for $i > j$ can be expressed as a polynomial in coordinates $y_1, \ldots, y_d$ with the lowest order term x^j_i, while x^i_i is a polynomial with the lowest order term $x^i_r x^i_{r+1}$.

For D_3, it is not hard to check using Plücker coordinates that the convex hull P_λ of $v(H^0(G/B, L_\lambda)) \subset \mathbb{R}^6$ for $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3$ contains the Minkowski sum $\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$, where P_1 is the 4-dimensional polytope with the vertices $(0, 0, 0, 0, 0, 0)$, $(1, 0, 0, 0, 0, 0)$, $(0, 1, 0, 0, 0, 0)$, $(0, 0, 0, 1, 0, 0)$, $(0, 0, 0, 0, 0, 1)$, $(0, 1, 0, 1, 0, 0)$, P_2 is the 3-dimensional simplex with the vertices $(0, 0, 0, 0, 0, 0)$, $(0, 1, 0, 0, 0, 0)$, $(0, 0, 1, 0, 0, 0)$, $(0, 0, 0, 0, 0, 1)$, and P_3 is the 3-dimensional simplex with the vertices $(0, 0, 0, 0, 0, 0)$, $(0, 0, 0, 1, 0, 0)$, $(0, 0, 0, 0, 1, 0)$, $(0, 0, 0, 0, 0, 1)$.

By reordering coordinates, we can get that $P_1 = FFLV(\widetilde{\omega}_2)$, $P_2 = FFLV(\widetilde{\omega}_1)$, $P_3 = FFLV(\widetilde{\omega}_3)$ for fundamental weights $\widetilde{\omega}_1$, $\widetilde{\omega}_2$, $\widetilde{\omega}_3$ of $SL_4(\mathbb{C})$. However, these reorderings do not agree for different fundamental weights so it is not clear whether P_λ is unimodularly equivalent to $FFLV(\lambda)$ in type A_3 (or to other known polytopes). To compare P_λ with FFLV and GZ polytopes one might write down the inequalities that define P_λ and use them to count the number of facets of P_λ. It would also be interesting to compute the inequalities for P_λ in the case of D_4 and compare them with those of [G].

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