



0-Cycles on Grassmannians as Representations of Projective Groups

R. Bezrukavnikov^{1,3} · M. Rovinsky^{2,3}

Received: 8 November 2018 / Revised: 26 September 2019 / Accepted: 3 October 2019 /
Published online: 5 November 2019
© Institute for Mathematical Sciences (IMS), Stony Brook University, NY 2019

Abstract

Let F be an infinite division ring, V be a left F -vector space, $r \geq 1$ be an integer. We study the structure of the representation of the linear group $\mathrm{GL}_F(V)$ in the vector space of formal finite linear combinations of r -dimensional vector subspaces of V with coefficients in a field. This gives a series of natural examples of irreducible infinite-dimensional representations of projective groups. These representations are non-smooth if F is locally compact and non-discrete.

Let F be a division ring (a.k.a. a skew field), and V be a left F -vector space. Define multiplication in the associative unital ‘matrix’ ring $\mathrm{End}_F(V)$ so that V becomes a left $\mathrm{End}_F(V)$ -module. In particular, $\mathrm{End}_F(V)$ is opposite to F if $\dim V = 1$.

Assume that $\dim V = r + r' > 1$ for a pair of cardinals $\mathbf{r} = (r, r')$. Denote by $\mathrm{Gr}(\mathbf{r}, V)$ the set of all F -vector subspaces of V of dimension r and of codimension r' (\mathbf{r} -subspaces for brevity). If $r < \dim V + 1$ we set $\mathrm{Gr}(r, V) := \mathrm{Gr}(\mathbf{r}, V)$ for $r' = \dim V - r$. For instance, $\mathrm{Gr}(1, V)$ is the projective space $\mathbb{P}(V) := F^\times \backslash (V \setminus \{0\})$; $\mathrm{Gr}(0, V)$ and $\mathrm{Gr}(\dim V, 0, V)$ are points.

For any associative ring A , denote by $A[\mathrm{Gr}(\mathbf{r}, V)]$ the set of all finite formal linear combinations $\sum_{j=1}^N a_j [L_j]$ with coefficients a_j in A of F -vector subspaces L_j of V in $\mathrm{Gr}(\mathbf{r}, V)$.

The set $A[\mathrm{Gr}(\mathbf{r}, V)]$ carries a natural structure of an A -bimodule: $a \cdot (\sum_i a_i [L_i]) \cdot a' := \sum_i aa_i a' [L_i]$.

Рафилу Калмановичу Гордину.

✉ M. Rovinsky
marat@mccme.ru

R. Bezrukavnikov
bezrukav@math.mit.edu

¹ MIT, Office 2-470, 77 Massachusetts Ave, Cambridge, MA 02139, USA

² AG Laboratory, HSE University, 6 Usacheva Str., Moscow, Russia 119048

³ Institute for Information Transmission Problems of Russian Academy of Sciences, Moscow, Russia

Let $G := GL_F(V) := Aut_F(V)$ be the group of invertible elements of $End_F(V)$. The natural G -action on $Gr(\mathbf{r}, V)$ is transitive and gives rise to an A -linear G -action on $A[Gr(\mathbf{r}, V)]$.

Obviously, the module $A[Gr(\mathbf{r}, V)]$ admits a proper submodule $A[Gr(\mathbf{r}, V)]^\circ$ formed by all finite formal linear combinations $\sum_j a_j [L_j]$ with $\sum_j a_j = 0$, which is nonzero if $r \neq 0$ and $r' \neq 0$.

Our goal is to describe, for any coefficient field K , the structure of the $K[G]$ -module $K[Gr(\mathbf{r}, V)]$. Namely, for any infinite F , we show (in Theorem 4.3) that a canonical nonzero submodule M_1 (constructed in Lemma 1.1) of $\mathbb{Z}[Gr(\mathbf{r}, V)]^\circ$ has the property that $K \otimes M_1$ is the only simple submodule of each nonzero $K[G]$ -submodule of $K[Gr(\mathbf{r}, V)]$.

This irreducibility result is deduced from the case of $\dim V = 2$ (§3). It is also shown in Lemma 3.3 the irreducibility of the representation of $SL_2(\mathbb{Q})$ induced by any non-trivial one-dimensional representation of a proper parabolic subgroup in $SL_2(\mathbb{Q})$.

The module $K[Gr(\mathbf{r}, V)]^\circ$ coincides with $K \otimes M_1$ if and only if either r or r' is finite.

Several remarks on the case of finite F are collected in §5.

When F is a local field, one usually studies either unitary or *smooth* (i.e. with open stabilizers; they are called *algebraic* in Bernstein and Zelevinsky (1976)) representations, while the representations considered here are non-smooth. However, the latter representations are smooth if F is discrete and r is finite; for a field F , they arise as direct summands of “restrictions” of certain geometrically meaningful representations of automorphisms groups of universal domains over F , cf. (Rovinsky 2018, §4).

1 Generators of $A[Gr(r, V)]^\circ$ for an integer r

For any ring A and any set Γ , denote by $A[\Gamma]$ the set of all finite formal linear combinations $\sum_{j=1}^N a_j [g_j]$ with coefficients a_j in A of elements $g_j \in \Gamma$.

If Γ is a group, we consider $A[\Gamma]$ as associative ring with evident relations $[g][g'] = [gg']$, $a[g] = [g]a$ for all $g, g' \in \Gamma$ and $a \in A$. The element $[1]$ is the unit of the ring.

The $A[G]$ -module $A[Gr(\mathbf{r}, V)] = A \otimes \mathbb{Z}[Gr(\mathbf{r}, V)]$ is generated by $[L]$ for any $L \in Gr(\mathbf{r}, V)$.

The following lemma shows that, for an integer r , the $A[GL_F(V)]$ -module $A[Gr(r, V)]^\circ = A \otimes \mathbb{Z}[Gr(r, V)]^\circ$ is generated by $[L] - [L']$ for any $L, L' \in Gr(r, V)$ with $\dim(L \cap L') = r - 1$.

Lemma 1.1 *Let \mathbf{r} be a pair of cardinals. Let L, L' be \mathbf{r} -subspaces in V with $\dim(L/L \cap L') = \dim(L'/L \cap L') = 1$. Then the G -submodule $M_1 = M_1(\mathbf{r}, V)$ of $\mathbb{Z}[Gr(\mathbf{r}, V)]^\circ$ generated by the difference $[L] - [L']$ contains all differences $[L_0] - [L_1]$ of \mathbf{r} -subspaces with $\dim(L_0/L_0 \cap L_1) = \dim(L_1/L_0 \cap L_1) < \infty$, but does not contain differences $[L_0] - [L_1]$ of other pairs of \mathbf{r} -subspaces.*

In particular, M_1 coincides with $\mathbb{Z}[Gr(\mathbf{r}, V)]^\circ$ if and only if at least one of r and r' is finite.

Proof Let $c = \dim(L_0/L_0 \cap L_1)$. Fix complete flags $E_0 = 0 \subset E_1 \subset E_2 \subset \dots \subset E_c = L_0/(L_0 \cap L_1)$ and $F_0 = 0 \subset F_1 \subset F_2 \subset \dots \subset F_c = L_1/(L_0 \cap L_1)$ and set

$L'_i = \tilde{E}_{c-i} + \tilde{F}_i$, where \tilde{E} denotes the preimage of a subspace $E \subseteq V/(L_0 \cap L_1)$ under the projection $V \rightarrow V/(L_0 \cap L_1)$. Then L'_0, L'_1, \dots, L'_c are \mathbf{r} -subspaces, while $L'_{i-1} \cap L'_i$ is a hyperplane in both L'_{i-1} and L'_i for each $i, 1 \leq i \leq c$.

As G acts transitively on the set of pairs (S, S') of \mathbf{r} -subspaces of V with $\dim(S/S \cap S') = \dim(S'/S \cap S') = 1$, all $[L'_i] - [L'_{i+1}]$ belong to the G -orbit of $[L] - [L']$. As $[L_0] - [L_1] = ([L'_0] - [L'_1]) + ([L'_1] - [L'_2]) + \dots + ([L'_{c-2}] - [L'_{c-1}]) + ([L'_{c-1}] - [L'_c])$, we see that M_1 contains $[L_0] - [L_1]$.

If either of r and r' is finite then for any pair L_0, L_1 of \mathbf{r} -subspaces of V one has $\dim(L_0/L_0 \cap L_1) = \dim(L_1/L_0 \cap L_1) < \infty$, so it is clear from the above that $[L] - [L']$ generates the G -module $\mathbb{Z}[\text{Gr}(\mathbf{r}, V)]^\circ$.

If $[L_0] - [L_1] \in M_1$, i.e., $[L_0] - [L_1] = \sum_{i=1}^N a_i ([L'_i] - [L''_i])$ with $\dim(L'_i/L'_i \cap L''_i) = \dim(L''_i/L'_i \cap L''_i) = 1$, then rename L'_i and L''_i in a way to get a sequence $L_0 = L'_0, L'_1, \dots, L'_{n-1}, L'_n = L_1$ with $\dim(L'_{i-1}/L'_{i-1} \cap L'_i) = \dim(L'_i/L'_{i-1} \cap L'_i) = 1$. Then $\dim(L'_0/\bigcap_{i=0}^n L'_i) \leq n$, and thus, $\dim(L_0/L_0 \cap L_1)$ is finite.

In particular, if $[L] - [L']$ generates $\mathbb{Z}[\text{Gr}(\mathbf{r}, V)]^\circ$ then $L_0/(L_0 \cap L_1)$ is finite-dimensional for any pair L_0, L_1 of \mathbf{r} -subspaces of V , so at least one of r and r' should be finite. □

2 (Endo)morphisms and Decomposability

Let F be a division ring, V be a left F -vector space, $\mathbf{r}_0 = (r_0, r'_0), \mathbf{r}_1 = (r_1, r'_1)$ be two pairs of cardinals such that $r_0 + r'_0 = r_1 + r'_1 = \dim V$; we may omit r'_i if $r_i < \dim V + 1$. For an \mathbf{r}_0 -subspace L in V , denote by $\text{St}_{[L]}$ the stabilizer of the point $L \in \text{Gr}(\mathbf{r}_0, V)$ in the group $G := \text{GL}_F(V)$.

It is easy to see that the G -orbit of a F -vector subspace L in V is determined by the pair of cardinals $(\dim L, \dim V/L)$; the G -orbit of a pair of F -vector subspaces L, L' in V is determined by the quintuple of cardinals $(\dim(L \cap L'), \dim L/(L \cap L'), \dim L'/(L \cap L'), \dim V/L, \dim V/L')$.

Let A be an associative unital ring. For each triple of cardinals $\mathbf{s} = (s, s', s'')$ with $s + s' = r_0$ and $s + s'' = r_1$ (so s' and s'' may be omitted if $s < \min(r_0, r_1) + 1$), let

$$\eta_{\mathbf{s}}^{\mathbf{r}_0, \mathbf{r}_1} : A[\text{Gr}(\mathbf{r}_0, V)] \rightarrow A[\text{Gr}(\mathbf{r}_1, V)]$$

be the $A[G]$ -morphism given by $[L] \mapsto \sum_{L'} [L']$ if the latter sum is finite and non-empty, where L' runs over all \mathbf{r}_1 -subspaces in V such that $\dim(L \cap L') = s, \dim L/(L \cap L') = s'$ and $\dim L'/(L \cap L') = s''$. Such L' 's form an $\text{St}_{[L]}$ -orbit.

It is clear that $\eta_{(r,0,0)}^{\mathbf{r}, \mathbf{r}}$ is identical on $A[\text{Gr}(\mathbf{r}, V)]$ and $\ker \eta_{\mathbf{0}}^{\mathbf{r}, \mathbf{0}} = \ker \eta_{(r,0,r')}^{\mathbf{r}, (\dim V, 0)} = A[\text{Gr}(\mathbf{r}, V)]^\circ$.

Lemma 2.1 *Set $R := A[G]$. Let $\mathbf{r}_0, \mathbf{r}_1$ be two pairs of cardinals with $r_0 + r'_0 = r_1 + r'_1 = \dim_F V$.*

Then two structures of right A -module on $\text{Hom}_R(A[\text{Gr}(\mathbf{r}_0, V)], A[\text{Gr}(\mathbf{r}_1, V)])$ coincide and it is freely generated by

$\eta_{(0, r_0, 0)}^{r_0, r_1}$	if $\mathbf{r}_1 = (0, \dim V)$;
$\eta_{(r_0, 0, r'_0)}^{r_0, r_1}$	if $\mathbf{r}_1 = (\dim V, 0)$;
$\eta_{(r_0, 0, r'_0 - r'_1)}^{r_0, r_1}$	if V is infinite, while F and $r'_0 \geq r'_1$ are finite;
$\eta_{(r_1, r_0 - r_1, 0)}^{r_0, r_1}$	if V is infinite, while F and $r_0 \geq r_1$ are finite;
the identity $\eta_{(r_0, 0, 0)}^{r_0, r_0} = id_{A[\text{Gr}(\mathbf{r}_0, V)]}$	if V is infinite and $\mathbf{r}_0 = \mathbf{r}_1$,
$\eta_s^{r_0, r_1}$ for all $s, \sigma \leq s \leq \min(r_0, r_1)$,	if V is finite, where $\sigma := \max(0, r_0 - r'_1)$,
0	otherwise.

The ring $\text{End}_{\mathbb{Z}[G]}(\mathbb{Z}[\text{Gr}(\mathbf{r}, V)])$ is commutative. If F is infinite and $\mathbf{r}_0 \neq \mathbf{r}_1$ then, in notation of §1, $\text{Hom}_R(A \otimes M_1(\mathbf{r}_0, V), A \otimes M_1(\mathbf{r}_1, V)) = 0$, while $\text{End}_R(A \otimes M_1(\mathbf{r}_0, V)) = A$ if $r_0 \neq 0$ and $r'_0 \neq 0$, so the R -modules $A[\text{Gr}(\mathbf{r}_0, V)]$ and $A \otimes M_1(\mathbf{r}_0, V)$ are indecomposable if A is a field.

Proof The cases $\mathbf{r}_1 \in \{(0, \dim V), (\dim V, 0)\}$ are trivial, since then $\text{Gr}(\mathbf{r}_1, V)$ reduces to a single point, so we may further assume $\mathbf{r}_1 \notin \{(0, \dim V), (\dim V, 0)\}$.

Fix some $L \in \text{Gr}(\mathbf{r}_0, V)$, and suppose that the $\text{St}_{[L]}$ -orbit of a point $L' \in \text{Gr}(\mathbf{r}_1, V)$ is finite.

We, thus, assume that $L' \neq 0$ and $L' \neq V$.

- If $L \cong F^{\oplus r_0}$ is infinite then either (i) $L \subseteq L'$ or (ii) $L \cap L' = 0$. In the case (ii), $L' \subseteq L$, since adding different elements of L to a basis element of L' one gets different L' 's, and therefore, $L' = 0$. If, in the case (i), $V/L \cong F^{\oplus r'_0}$ is finite then all L' containing L form a single finite orbit; if $V/L \cong F^{\oplus r'_0}$ is infinite and $L' \neq V$ then $L' = L$.
- If $V/L \cong F^{\oplus r'_0}$ is infinite then the image of L' in V/L should be either 0 or V/L , i.e., either (i) $L' \subseteq L$ or (ii) $V = L + L'$. In the case (i), either (a) $L' = L$, or (b) $L \cong F^{\oplus r_0}$ is finite and then all L' contained in L form a single finite orbit. In the case (ii), either (a) $L' \supseteq L$, or (b) $L \cap L' \neq L$. (iia): $L' = V$, which is excluded. (iib): any orbit is infinite. Namely, choose a vector $v \in L \setminus L'$ and a collection $\{e_i\}_{i \in I}$ presenting a basis of V/L ; then the subspaces $L_j := L \cap L' + \langle e_i^j \mid i \in I \rangle_F$, where $e_i^j = e_i$ if $i \neq j$, while $e_j^j = e_j + v$, are pairwise distinct.
- If V is finite then all orbits are finite and they are parametrized by $s = \dim(L \cap L')$, where $\sigma := \max(0, r_0 - r'_1) \leq s \leq \min(r_0, r_1)$.

As the R -module $A[\text{Gr}(\mathbf{r}_0, V)]$ is generated by $[L]$, any R -module morphism $A[\text{Gr}(\mathbf{r}_0, V)] \rightarrow A[\text{Gr}(\mathbf{r}_1, V)]$ is determined by the image of $[L]$, which in turn is an element of $A[\text{Gr}(\mathbf{r}_1, V)]^{\text{St}_{[L]}}$, i.e., a linear combination of sums of the elements of several finite $\text{St}_{[L]}$ -orbits in $\text{Gr}(\mathbf{r}_1, V)$.

One has $\eta_{s'}^{r'_1, r''} \eta_s^{r, r'} [L] = \sum_{L'} \eta_{s'}^{r'_1, r''} [L'] = \sum_{L'} \sum_{L''} [L''] = \sum_{L'' \in \text{Gr}(r'', V)} N_{L, L''} [L'']$, where

$$N_{L, L''} = |\{L' \in \text{Gr}(r', V) \mid \dim(L \cap L') = s, \dim(L' \cap L'') = s'\}|.$$

It follows that $\eta_{s'}^{r'_1, r''} \eta_s^{r, r'} = \eta_s^{r, r'} \eta_{s'}^{r'_1, r''}$. In other words, the algebra $\text{End}_R(A[\text{Gr}(r, V)])$ is commutative if V is finite, as soon as so is A . If V is infinite then $\text{End}_R(A[\text{Gr}(\mathbf{r}, V)]) = A$.

The R -module $A \otimes M_1(\mathfrak{r}_0, V)$ is generated by $[L] - [L']$ for any $L, L' \in \text{Gr}(\mathfrak{r}_0, V)$ with $\dim L/(L \cap L') = \dim L'/(L \cap L') = 1$, so any morphism φ from the R -module $A \otimes M_1(\mathfrak{r}_0, V)$ is determined by the image of $[L] - [L']$, which in turn is an element of $(A[\text{Gr}(\mathfrak{r}_1, V)]^\circ)^{\text{St}_{[L]} \cap \text{St}_{[L'']}}$.

If F is infinite then the only proper subspaces in V fixed by $\text{St}_{[L]} \cap \text{St}_{[L']}$ are $L, L', L \cap L'$, while the $\text{St}_{[L]} \cap \text{St}_{[L']}$ -orbits of other proper subspaces are infinite. This means that $\varphi([L] - [L']) = a[L] + b[L'] + c[L \cap L']$ for some $a, b, c \in A$. Consider $g \in G$ such that $g(L) \subset L + L', g(L) \notin \{L, L'\}, g(L \cap L') = L \cap L'$ and $g(L') = L'$. Then $\varphi([L] - [g(L)]) = \varphi([L] - [L']) - g\varphi([L] - [L']) = a([L] - [g(L)])$. As $\dim L/(L \cap g(L)) = \dim g(L)/(L \cap g(L)) = 1$, the element $[L] - [g(L)]$ is another generator of $A \otimes M_1$, so we get $\text{End}_R(A \otimes M_1(\mathfrak{r}_0, V)) = A$ and the required vanishing for $\mathfrak{r}_0 \neq \mathfrak{r}_1$. □

Remark 2.2 If V is finite then the morphism $\eta_s^{r, r'}$ of §2 is dual to the morphism $\eta_s^{r', r}$ under the non-degenerate symmetric bilinear pairing on $K[\text{Gr}(\bullet, V)]$, given by $([L], [L]) = 1$ and $([L], [L']) = 0$ if $L \neq L'$: $((\eta_s^{r, r'})^*[L'], [L]) = ([L'], \eta_s^{r', r}[L])$ is 1 if $\dim(L \cap L') = s$, and is 0 otherwise.

3 The One-Dimensional Case

Lemma 3.1 *Let F and K be fields, and V be a two-dimensional F -vector space. If $|F| \leq 3$ assume in addition that F and K are of the same characteristic. Let P be a subgroup in $\text{SL}(V)$ and $\rho : P \rightarrow K^\times$ be a character. Suppose that the induced $K[\text{SL}(V)]$ -module $W_\rho := K[\text{SL}(V)] \otimes_{K[P]} \rho$ admits a submodule W such that $\dim_K(W_\rho/W) = 1$. Then $\rho = 1$ and $W = K[\text{SL}(V)/P]^\circ$.*

Proof As index of commutator subgroup of $\text{SL}(V)$ is 1 if $|F| > 3$, and it is $|F|$ if $|F| \leq 3$, any one-dimensional K -representation of $\text{SL}(V)$ is trivial. On the other hand, $\text{Hom}_{K[\text{SL}(V)]}(W_\rho, K) = \text{Hom}_{K[P]}(\rho, K)$, so the $K[\text{SL}(V)]$ -module W_ρ admits a one-dimensional quotient if and only if $\rho = 1$, while W is the kernel of the degree morphism. □

Remark 3.2 We are particularly interested in the case of a proper parabolic subgroup P .

1. Denote by ∞ the point of $\mathbb{P}(V)$ fixed by P . Then $\text{SL}(V)/P \rightarrow \mathbb{P}(V), [g] \mapsto g\infty$ is an isomorphism of $\text{SL}(V)$ -sets (so that $[P/P] \mapsto \infty$) inducing an isomorphism of $K[\text{SL}(V)]$ -modules $W_\rho \xrightarrow{\sim} K[\mathbb{P}(V)]$ if $\rho = 1$.
2. Note that (i) $P \cong F^\times \ltimes F$ where F^\times acts on F by squares: $[a : b \mapsto a^2b]$; (ii) $[P, P] = P^u$ if $|F| > 3$ and $[P, P] = 1$ if $|F| \leq 3$, where $P^u \cong F$ is the unipotent radical of P . This shows that ρ factors through P/P^u .

If considered as $K[P]$ -module, W_ρ splits as $K[\text{SL}(V)\backslash P] \otimes_{K[P]} \rho \oplus \rho$, where $K[\text{SL}(V)\backslash P] \otimes_{K[P]} \rho$ is a free module over the group algebra $K[P^u]$ of P^u . Namely, as the affine line $\mathbb{P}(V)\backslash\{\infty\}$ is a principal homogeneous space over P^u , a choice of an element $O \in \text{SL}(V)\backslash P$ gives rise to an isomorphisms of left $K[P^u]$ -modules $\iota_P : K[P^u] \otimes_K \rho \xrightarrow{\sim} K[\text{SL}(V)\backslash P] \otimes_{K[P]} \rho, [u] \otimes e \mapsto [uO] \otimes e$.

Let $T \cong P/P^u$ be the torus in P fixing the class o of O on the affine line $\mathbb{P}(V) \setminus \{\infty\}$. The set $\{g \in OP : g^2 = -1\}$ consists of all elements of $SL(V)$ interchanging the points o and ∞ of the projective line $SL(V)/P$. This is a principal homogeneous space over T . As it is non-empty, we may further assume that $O^2 = -1$. As $(Ot)^2 = -1$ for any $t \in T$, we get $tO = Ot^{-1}$, so the T -action on the target of ι_P corresponds to the T -action on $K[P^u] \otimes_K \rho$ given by $t : [u] \otimes e \mapsto [tut^{-1}] \otimes t^{-1}e$, since $t \circ \iota_P : [u] \otimes e \mapsto [tuO] \otimes e = [(tut^{-1})tO] \otimes e = [(tut^{-1})O] \otimes t^{-1}e$.

This means that the $K[P]$ -submodules of $K[SL(V) \setminus P] \otimes_{K[P]} \rho$ correspond to the T -invariant ideals in $K[P^u]$.

3.1 The Case of $SL_2(\mathbb{Q})$

Lemma 3.3 *Let V be a two-dimensional \mathbb{Q} -vector space, K be a field. Let P be a proper parabolic subgroup in $SL(V)$ and $\rho : P \rightarrow K^\times$ be a character.*

Then (i) the $K[SL(V)]$ -module $W_\rho := K[SL(V)] \otimes_{K[P]} \rho$ is simple if $\rho \neq 1$, (ii) $K[\mathbb{P}(V)]^\circ$ is a unique simple submodule of $K[\mathbb{P}(V)] = W_1$.

Proof By Lemma 3.1, it suffices to show that any nonzero $K[SL(V)]$ -submodule W in W_ρ has K -codimension ≤ 1 . We keep the setting of Remark 3.2 with $F = \mathbb{Q}$. We identify $P^u \cong \mathbb{Q}$ with the rational powers $X^\mathbb{Q}$ of an indeterminate X , so that $K[P^u] \cong \bigcup_{N \geq 1} K[X^{1/N}, X^{-1}]$.

Any ideal in $K[P^u]$ is determined by its intersections with each subalgebra $K[X^{1/N}, X^{-1}]$, and thus, is generated by a collection of polynomials of minimal degree $P_N(X^{1/N})$ for all $N \geq 1$ such that $P_N(0) = 1$ and $P_N(X) | P_M(X^{N/M})$ in $K[X]$ if $M | N$.

As $P/P^u \cong \mathbb{Q}^\times$ acts evidently on $P^u \cong X^\mathbb{Q}$, any P/P^u -invariant proper ideal $I = (P_N(X^{1/N}))_{N \geq 1}$ in $\bigcup_{N \geq 1} K[X^{1/N}, X^{-1}]$ contains $P_N(X^{M_1^2/(M_2^2 N)})$ for all $M_1, M_2, N \geq 1$. In particular, $P_N(X)$ divides $P_N(X^{M^2})$ for all $M, N \geq 1$, which implies that if $P_N(\lambda) = 0$ then $P_N(\lambda^{M^2}) = 0$ for all $M \geq 1$, so the set $\{\lambda^{M^2}\}_{M \geq 1}$ is finite, i.e., λ is a root of unity, say $\lambda^N = 1$. Then $P_N(X) | (X^{M_N} - 1)^{m_N}$ for some $M_N, m_N \geq 1$, and thus, I contains $(X^{M_N/N} - 1)^{m_N}$, and therefore, I contains $(X^{M_N^2/N} - 1)^{m_N} \in ((X^{M_N/N} - 1)^{m_N})$, and consequently, $(X^{1/N} - 1)^{m_N} \in I$ for all $N \geq 1$. Let $s \geq 1$ be such that $m_s \leq m_N$ for all $N \geq 1$. Then $(X^{1/(N^{2s})} - 1)^{m_s} \in I$ for all $N \geq 1$, and therefore, $(X^{1/N} - 1)^{m_s} \in ((X^{1/(N^{2s})} - 1)^{m_s}) \subset I$ for all $N \geq 1$.

Let $J \subset \bigcup_{N \geq 1} K[X^{1/N}, X^{-1}]$ be the ideal generated by $X^{1/N} - 1$ for all $N \geq 1$. As J^n/J^{n+1} is a K -vector space of dimension $\max(\delta_{n,0}, \delta_{\text{char}(K),0}) \leq 1$ with the group P/P^u acting by n -th powers, we conclude that I coincides with J^n for some $n \geq 0$.

The splitting $W_\rho = \rho \dot{+} K[SL(V) \setminus P] \otimes_{K[P]} \rho$ induces two projections $\pi_\infty : W \rightarrow \rho$ and $\pi_{\text{aff}} : W \rightarrow K[SL(V) \setminus P] \otimes_{K[P]} \rho$. As $SL(V)$ is transitive on $\mathbb{P}(V)$, the projection π_∞ is surjective. As $\text{Hom}_{K[P]}(W_\rho, \rho) = K$, if π_∞ splits as a morphism of $K[P]$ -modules then W contains $[1] \otimes \rho$, and therefore, $W = W_\rho$.

Assume now that the projection π_∞ does not split, so then (i) $0 \rightarrow J^m \otimes_{K[P]} \rho \rightarrow W \rightarrow \rho \rightarrow 0$, (ii) the projection π_{aff} is injective, i.e., the $K[P]$ -module W is isomorphic to a power of J . Since the quotient of the $K[P]$ -module W by J^m is

ρ , we get $m = 1$, so $W \cong K[\mathrm{SL}(V)\backslash P] \otimes_{K[P]} \rho$ as $K[P]$ -module, and thus, the $K[\mathrm{SL}(V)]$ -module W contains the linear combination $([1] + a[O]) \otimes e$ for some $a \in K$ and $O \in \mathrm{SL}(V)\backslash P$. Then W contains $([1] + a[uO]) \otimes e$ for all $u \in P^u$, and therefore, $\dim_K(W_\rho/W) \leq 1$. \square

3.2 The Case of $\mathrm{SL}_2(F)$ in Equal Positive Characteristics

Lemma 3.4 *Let F and K be fields of characteristic $p > 0$, and V be a two-dimensional F -vector space. Then the $K[\mathrm{PSL}(V)]$ -module $K[\mathbb{P}(V)]^\circ$ is simple.*

Proof Let $M \neq 0$ be a $K[\mathrm{PSL}(V)]$ -submodule in $K[\mathbb{P}(V)]^\circ$, and $\alpha = \sum_{i=1}^n a_i[x_i] \in M$ for some $n \geq 1$, pairwise distinct $x_i \in \mathbb{P}(V)$ and some $a_i \in K^\times$ with $\sum_{i=1}^n a_i = 0$. Let $P \cong F^{\times 2} \ltimes F$ be the stabilizer of x_1 in $\mathrm{PSL}(V)$.

For a choice of a point $O \in \mathbb{P}(V)\backslash\{x_1\}$, denote by $\beta = \sum_{i=2}^n a_i[x_i - O]$ the element of $K[P^u]$ such that $\beta O = \alpha - a_1[x_1]$. Then $\beta^{p-1}\alpha = a_1^p \cdot ([x_1] - [O])$. As O can be chosen to be an arbitrary point of $\mathbb{P}(V)\backslash\{x_1\}$, α generates $K[\mathbb{P}(V)]^\circ$ as $K[\mathrm{PSL}(V)]$ -module. \square

4 The Case of an Infinite Base Skew Field

Lemma 4.1 *Let V be a left vector space over a division ring F , and λ be an F -linear functional on V . Then, for any $w \in V$, the endomorphism $1 - \lambda tw \in \mathrm{End}_F(V)$, $x \mapsto x - \lambda(x)tw$, is not invertible for at most one value of $t \in F$.*

Proof Let $\mu := \lambda(w)$. Then, for any $t \in F$ such that $\mu t \neq 1$, one has $(1 - \lambda tw)(1 + \lambda t(1 - \mu t)^{-1}w) = 1 - \lambda tw + \lambda t(1 - \mu t)^{-1}(1 - \mu t)w = 1$. \square

Lemma 4.2 *For each integer $s, N \geq 1$, let $D_s(N) := \sum_{m \equiv s \pmod p} (-1)^m \binom{N}{m} \in \mathbb{Z}$. Then the matrix*

$$\Delta = \begin{bmatrix} D_1(N) & \cdots & D_{p-1}(N) \\ \cdots & \cdots & \cdots \\ D_1(N + p - 2) & \cdots & D_{p-1}(N + p - 2) \end{bmatrix}$$

is invertible over $\mathbb{Z}[1/p]$.

Proof Denote by μ_p the set of complex p -th roots of unity. Then

$$D_s(N) = \frac{1}{p} \sum_{\zeta \in \mu_p} \zeta^{-s} (1 - \zeta)^N.$$

Fix a primitive $\zeta \in \mu_p$. Then the matrix $p\Delta$ coincides with the product

$$\begin{bmatrix} (1-\zeta)^N & (1-\zeta^2)^N & \dots & (1-\zeta^{p-1})^N \\ (1-\zeta)^{N+1} & (1-\zeta^2)^{N+1} & \dots & (1-\zeta^{p-1})^{N+1} \\ \dots & \dots & \dots & \dots \\ (1-\zeta)^{N+p-2} & (1-\zeta^2)^{N+p-2} & \dots & (1-\zeta^{p-1})^{N+p-2} \end{bmatrix} \begin{bmatrix} \zeta^{-1} & \zeta^{-2} & \dots & \zeta^{-(p-1)} \\ \zeta^{-2} & \zeta^{-4} & \dots & \zeta^{-2(p-1)} \\ \dots & \dots & \dots & \dots \\ \zeta^{-(p-1)} & \zeta^{-2(p-1)} & \dots & \zeta^{-(p-1)(p-1)} \end{bmatrix},$$

so $\det(p\Delta)$ is product of Vandermonde determinants:

$$\det(p\Delta) = \prod_{j=1}^{p-1} (1-\zeta^j)^N \prod_{1 \leq j < s \leq p-1} (\zeta^j - \zeta^s) \prod_{j=1}^{p-1} \zeta^{-j} \prod_{1 \leq j < s \leq p-1} (\zeta^s - \zeta^j).$$

As the norm in the extension $\mathbb{Q}(\zeta)/\mathbb{Q}$ of the element $1-\zeta^j$ is p , we see that $\det \Delta = \pm p^m$ for an integer $m \geq 0$, so $\det \Delta$ is invertible in $\mathbb{Z}[1/p]$. □

4.1 A Filtration

For a left vector space V over a division ring F , let $G := GL_F(V) := \text{Aut}_F(V)$.

Theorem 4.3 *Let $\mathbf{r} = (r, r')$ be a pair of cardinals ≥ 1 , F be an infinite division ring of characteristic $p \geq 0$, V be a left F -vector space of dimension $r + r'$. Set $M_0 := \mathbb{Z}[\text{Gr}(\mathbf{r}, V)]^\circ$ and $G := GL_F(V)$. Then there is a sequence of nonzero G -submodules $M_0 \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \dots$ such that*

- (1) M_0/M_1 is a free abelian group, vanishing if and only if at least one of r and r' is finite;
- (2) if $p > 0$ then the natural map $\mathbb{Z}[1/p] \otimes M_n \rightarrow \mathbb{Z}[1/p] \otimes M_1$ is surjective for any $n \geq 1$;
- (3) for any $n \geq 1$ and any field K , the natural map $K \otimes M_n \rightarrow K \otimes M_1$ is surjective;
- (4) if, for an associative unital ring A , a $A[G]$ -submodule M of $A \otimes M_0$ contains an element $\alpha = \sum_{i=0}^N a_i [L_i]$, where $L_0 \not\subseteq \bigcup_{i=1}^N L_i$, then M contains $Aa_0 \otimes M_n$ for some n depending on N .

Proof Fix an arbitrary subspace $U \subset V$ of dimension $r - 1$ and of codimension $r' + 1$. Fix some $e_0, e_1 \in V$ that are F -linearly independent in V/U . For each sequence $(t_i)_{i \geq 1}$ in F^\times and all $n \geq 1$, define $\gamma_n((t_i)_{i \geq 1}) := \sum_{I \subseteq \{1, \dots, n\}} (-1)^{|I|} [U + F(e_0 + \sum_{i \in I} t_i e_1)] \in M_0$.

Note that (i) the G -orbit of $\gamma_n((t_i)_{i \geq 1})$ is independent of a particular choice of U, e_0, e_1 , (ii) $\gamma_{n+1}((t_i)_{i \geq 1}) = (1-\xi)\gamma_n((t_i)_{i \geq 1})$ for any $\xi \in G$ identical on $U + F \cdot e_1$ and such that $\xi e_0 = e_0 + t_{n+1} e_1$, (iii) all $\gamma_n((t_i)_{i \geq 1})$ are nonzero if, e.g., t_1, t_2, \dots are either linearly independent over the prime subfield, or all equal to 1 if $p = 0$.

For each $n \geq 1$, let M_n be the G -submodule in M_0 generated by the elements $\gamma_n((t_i)_{i \geq 1})$ for all sequences $(t_i)_{i \geq 1}$ in F^\times . In particular, we have inclusions $M_n \supseteq M_{n+1}$ for all n .

Then (1) follows from Lemma 1.1: M_0/M_1 is the group of formal finite linear combinations $\sum_i a_i [L_i]'$, where $a_i \in \mathbb{Z}$ and $[L_i]'$ are classes of ‘commensurable’ \mathbf{r} -subspaces, i.e. $L_0 \sim L_1$ if $\dim(L_0/L_0 \cap L_1) = \dim(L_1/L_0 \cap L_1) < \infty$.

Set $\gamma_n := \gamma_n(1, 1, 1, \dots) = \sum_{s=0}^n (-1)^s \binom{n}{s} [U + F(e_0 + se_1)]$. As any nonzero $\gamma_1((t_i)_{i \geq 1})$ belongs to the G -orbit of γ_1 , the G -module M_1 is generated by γ_1 .

If $p > 0$, it follows from Lemma 4.2 that the $\mathbb{Z}[1/p]$ -submodule in $\mathbb{Z}[1/p] \otimes M_0$ generated by $\gamma_n = \sum_{s=0}^{p-1} D_s(n)[U + F(e_0 + se_1)]$, $\gamma_{n+1}, \dots, \gamma_{n+p-2}$ contains γ_1 , which implies (2).

If $\ell := \text{char}(K) \neq p$ is a prime then $\gamma_{\ell N} \equiv [U + F \cdot e_0] - [U + F(e_0 + \ell^N e_1)] = g_N \gamma_1 \pmod{\ell M_1}$ for any integer $N \geq 1$ and some $g_N \in G$, so $M_n + \ell M_1$ contains γ_1 for any $n \geq 1$, which proves (3) for $K = \mathbb{Z}/\ell$, and thus, for all fields K of characteristic ℓ .

By Lemmas 3.3 and 3.4, if characteristic of F is 0, or if characteristics of F and K coincide, then the $K[G]$ -submodule in $K \otimes M_0$ generated by γ_n coincides with $K \otimes M_1$ for any $n \geq 1$.

As (2) implies the case of $\text{char}(K) \neq p > 0$, this completes the proof of (3).

For (4), we are going to show that, together with $\alpha = \sum_{i=0}^N a_i [L_i]$, any $A[G]$ -submodule M of $A \otimes M_0$ contains $a_0 \otimes \gamma_n$ for some n if L_0 is contained in neither of L_1, \dots, L_N .

It is a folklore result that a vector space over an infinite division ring cannot be a finite union of proper linear subspaces, see e.g. (Roman 2008, Theorem 1.2). Fix some $v \in L_0 \setminus \bigcup_{i=1}^N L_i$, some $w \in V \setminus L_0$. For each $1 \leq i \leq N$, fix some F -linear morphism $\xi_i : V \rightarrow F \cdot w \subset V$ vanishing on L_i but not on v .

For each subset $I \subseteq \{1, \dots, N\}$, the image of the endomorphism $\xi_I := \sum_{i \in I} \xi_i$ of V is contained in $F \cdot w$, i.e., $\xi_I = \lambda_I \cdot w$ for a linear functional $\lambda_I := \sum_{i \in I} \lambda_i$ on V , so by Lemma 4.1 there is at most one value of t such that $1 + \lambda_I t w$ is not invertible. As F is infinite, we may therefore replace w with a nonzero multiple, so that $1 + \xi_I = 1 + \lambda_I \cdot w$ become invertible for all subsets $I \subseteq \{1, \dots, N\}$.

Then the element $\Xi := \sum_{I \subseteq \{1, \dots, N\}} (-1)^{|I|} [1 + \xi_I] \in \mathbb{Z}[G]$ annihilates all $[L_i]$ for $1 \leq i \leq N$, and

$$\Xi[L_0] = \sum_{I \subseteq \{1, \dots, N\}} (-1)^{|I|} \left[L_0 \cap \ker \xi_I + F \cdot \left(v + \left(\sum_{i \in I} \lambda_i(v) \right) w \right) \right].$$

In particular, (i) if $r = 1$ and $\lambda_i = t_i$ for all $1 \leq i \leq N$ then $g \Xi \alpha = a_0 \otimes \gamma_N((t_i)_{i \geq 1})$ for any $g \in G$ such that $gv = e_0$ and $gw = e_1$; (ii) if $\sum_{i \in I} \lambda_i \neq 0$ for any non-empty I then $\Xi[L_0] = [L_0] + \sum_{n=1}^M b_n [L'_n]$ for some hyperplanes $L'_n \neq L_0$ in $V' := L_0 + F \cdot w$ and some $b_n \in \mathbb{Z}$.

Set $V'' := V' / \bigcap_{n=1}^M \ker \xi_n$, which is of dimension $\leq N$. Denote by V''^\vee the dual space of V'' . Consider the canonical identification $\psi : \text{Gr}(\dim V'' - 1, V'') \xrightarrow{\sim} \mathbb{P}(V''^\vee)$, sending each hyperplane L in V'' to the line of all linear functionals vanishing on L . Then $\psi(g^{-1}[L]) = g^* \psi([L])$ for all $g \in \text{GL}_F(V'')$ (or their lifts in $\text{GL}_F(V'')$), where $(g^* \lambda)(x) := \lambda(gx)$ for all $\lambda \in V''^\vee$ and $g \in \text{GL}_F(V'')$.

Then ψ identifies the image of $\Xi[L_0] = [L_0] + \sum_{n=1}^M b_n [L'_n]$ in $\mathbb{Z}[\text{Gr}(\dim V'' - 1, V'')]^\circ$ with $[q_0] + \sum_{n=1}^M b_n [q_n] \in \mathbb{Z}[\mathbb{P}(V''^\vee)]^\circ$. As we have just seen in the case $r = 1$ (with V''^\vee instead of V), for any sequence $(t_i)_{i \geq 1}$ in F^\times , there is an element $\beta \in \mathbb{Z}[\text{GL}_F(V'')]$ (similar to the element Ξ) such that $\beta([q_0] + \sum_{n=1}^M b_n [q_n]) = \sum_{I \subseteq \{1, \dots, M\}} (-1)^{|I|} [F \cdot (e''_0 + (\sum_{i \in I} t_i) e''_i)]$ for some F -linearly

independent $e''_0, e''_1 \in V''$. Denote by β^* the image of β under the anti-involution $\mathbb{Z}[\mathrm{GL}_F(V'')] \rightarrow \mathbb{Z}[\mathrm{GL}_F(V'')]$, $[g] \mapsto [g^{-1}]$. Then, for any linear combination β' of elements of G identical on $\mathbb{P}(\bigcap_{n=1}^N \ker \xi_n)$ and extending β^* , one has $\beta'(\Xi[L_0]) = \gamma_M((t_i)_{i \geq 1})$, and thus, $\beta'(\Xi\alpha) = a_0 \otimes \gamma_M((t_i)_{i \geq 1})$. \square

Corollary 4.4 *Let $\mathbf{r} = (r, r')$ be a pair of cardinals ≥ 1 , F be an infinite division ring, K be a field, V be a left F -vector space of dimension $r + r'$.*

Then any nonzero K -subrepresentation of $\mathrm{GL}_F(V)$ in $K[\mathrm{Gr}(\mathbf{r}, V)]$ contains $K \otimes M_1$.

In particular, $K \otimes M_1$ is the only irreducible K -subrepresentation of $\mathrm{GL}_F(V)$ in $K[\mathrm{Gr}(\mathbf{r}, V)]$.

The following conditions are equivalent: (i) at least one of r and r' is finite, (ii) $M_1 = \mathbb{Z}[\mathrm{Gr}(\mathbf{r}, V)]^\circ$, (iii) $K \otimes M_1 = K[\mathrm{Gr}(r, V)]^\circ$, (iv) $K[\mathrm{Gr}(\mathbf{r}, V)]^\circ$ is irreducible, (v) $K[\mathrm{Gr}(\mathbf{r}, V)]/K \otimes M_1$ is irreducible, (vi) $K[\mathrm{Gr}(\mathbf{r}, V)]/K \otimes M_1 \cong K$.

5 Some Remarks on the Case of a Finite Base Field

There is an extensive literature on representations of finite Chevalley groups, see e.g. Alperin (1986), Benson (1984). For this reason we do not treat in detail the case where F is a finite field.

5.1 The Case of Characteristic 0 Coefficient Field

Proposition 5.1 *Let $F = \mathbb{F}_q$ be a finite field, V be a finite F -vector space, K be a field where $q^n \neq q$ for $2 \leq n \leq \dim V + 1$ (e.g., $\mathrm{char}(K) = 0$), $r \geq 1$ be an integer. Then the $K[\mathrm{PGL}(V)]$ -module $K[\mathrm{Gr}(r, V)]$ is a sum of $\min(r, \dim V - r) + 1$ pairwise distinct simple submodules.*

Proof As the $K[\mathrm{PGL}(V)]$ -modules are semisimple, it suffices to apply Lemma 2.1 asserting that the algebra $\mathrm{End}_{K[\mathrm{PGL}(V)]}(K[\mathrm{Gr}(r, V)])$ is commutative and of dimension $\min(r, \dim V - r) + 1$ as K -vector space. \square

Proposition 5.2 *Let F be a finite field, V be an infinite F -vector space, K be a field of characteristic 0, $r \geq 1$ be an integer. Then both $K[\mathrm{PGL}(V)]$ -modules, $K[\mathrm{Gr}(r, V)]$ and $K[\mathrm{Gr}((\dim V, r), V)]$, admit unique composition series, both of length $r + 1$.*

This follows from a description of the smooth K -representations of $\mathrm{GL}(V)$ for a countable F -vector space V given in Rovinsky (2018, Theorem A.17). The corresponding composition series are

$$0 \subset K \otimes \ker \eta_{r-1}^{r,r-1} \subset K \otimes \ker \eta_{r-2}^{r,r-2} \subset \dots \subset K \otimes \ker \eta_1^{r,1} \\ \subset K \otimes \ker \eta_0^{r,0} \subset K[\mathrm{Gr}(r, V)] \quad \text{and}$$

$$0 \subset \Phi_1 \subset \Phi_2 \subset \dots \subset \Phi_{r-1} \subset \Phi_r \subset K[\mathrm{Gr}((\dim V, r), V)], \text{ where } \Phi_n = \\ K \otimes \ker \eta_{(\dim V, 0, n)}^{(\dim V, r), (\dim V, r-n)}.$$

5.2 The Case of Positive Characteristic Coefficient Field

Proposition 5.3 *Let K be a field of characteristic ℓ . Let F be a union of finite fields.*

- (1) *Suppose that V is finite and either $\dim \mathbb{P}(V) = 1$ or ℓ is not characteristic of F .*
 - *If ℓ does not divide $|\mathbb{P}(V)|$ then $K[\mathbb{P}(V)] = K[\mathbb{P}(V)]^\circ \oplus K \cdot \sum_{x \in \mathbb{P}(V)} [x]$ is the sum of two simple $K[\text{PGL}(V)]$ -submodules.*
 - *If ℓ divides $|\mathbb{P}(V)|$ then $K \cdot \sum_{x \in \mathbb{P}(V)} [x]$ is the only simple submodule of $K[\mathbb{P}(V)]$, while $K[\mathbb{P}(V)]^\circ / K \cdot \sum_{x \in \mathbb{P}(V)} [x]$ is the only simple submodule of $K[\mathbb{P}(V)] / K \cdot \sum_{x \in \mathbb{P}(V)} [x]$.*
- (2) *If $\mathbb{P}(V)$ is infinite and either $\dim V = 2$ or characteristic of F is not ℓ then $K[\mathbb{P}(V)]^\circ$ is the only simple $K[\text{PGL}(V)]$ -submodule of $K[\mathbb{P}(V)]$.*

Proof We have to show that any $\alpha = \sum_{x \in \mathbb{P}(V)} a_x [x] \in K[\mathbb{P}(V)]$ generates a $K[\text{PGL}(V)]$ -submodule containing $K[\mathbb{P}(V)]^\circ$, whenever not all a_x are equal. As $\text{PGL}(V)$ is 2-transitive on $\mathbb{P}(V)$, it suffices to show that the $K[\text{PGL}(V)]$ -submodule generated by α contains a difference of two distinct points.

For each x with $a_x \neq 0$ fix its lift $\tilde{x} \in V$. Choose a maximal subset B consisting of F -linearly independent elements among \tilde{x} 's. We replace F with the subfield of F generated by the coefficients of the elements \tilde{x} in the base B , and replace $\mathbb{P}(V)$ with the projectivization of the space spanned by the \tilde{x} 's over the new F . We thus assume that $\mathbb{P}(V)$ is finite. Then we proceed by induction of the dimension n of $\mathbb{P}(V)$.

For each hyperplane $H \subset \mathbb{P}(V)$, let $U_H \subset \text{PGL}(V)$ be the translation group of the affine space $\mathbb{P}(V) \setminus H$. Then $(\sum_{h \in U_H} h)\alpha = (\sum_{x \notin H} a_x) \sum_{x \notin H} [x] + q^{\dim \mathbb{P}(V)} \sum_{x \in H} a_x [x]$, where q is order of F .

For the induction step in the case $\ell \nmid q$ (and $\dim \mathbb{P}(V) > 1$), fix some hyperplane H containing points y, z with $a_y \neq a_z$ and fix some $\eta \in \text{PGL}(V)$ such that $\eta(H) = H, \eta(y) = z$ and $\eta(u) = u$ for some $u \in H$. Then $(\eta - 1)(\sum_{h \in U_H} h)\alpha = q^{\dim \mathbb{P}(V)} \sum_{x \in H} (a_{\eta^{-1}x} - a_x)[x] = q^{\dim \mathbb{P}(V)} (\dots + (a_y - a_z)[z] + 0[u])$, so we are reduced to the case of dimension $n - 1$.

Assume now that $\dim \mathbb{P}(V) = 1$.

- (1) *If ℓ does not divide $q+1$ then there is $y \in \mathbb{P}(V)$ such that $(q+1)a_y \neq \sum_{x \in \mathbb{P}(V)} a_x$. If ℓ divides $q + 1$ then fix an arbitrary $y \in \mathbb{P}(V)$ with $a_y \neq 0$ (so that $(q + 1)a_y = 0 \neq \sum_{x \in \mathbb{P}(V)} a_x$). Fix some involution $\xi \in \text{PGL}(V)$ such that $\xi y \neq y$. Then $(\xi - 1)(\sum_{h \in U_{\{y\}}} h)\alpha = (qa_y - \sum_{x \neq y} a_x)([\xi y] - [y]) = ((q + 1)a_y - \sum_{x \in \mathbb{P}(V)} a_x)([\xi y] - [y])$. Thus, assuming that either ℓ does not divide $q + 1$ or $\sum_{x \in \mathbb{P}(V)} a_x \neq 0$, the $K[\text{PGL}(V)]$ -submodule generated by α contains $[\xi y] - [y]$, and therefore, it contains $K[\mathbb{P}(V)]^\circ$.*

Assume now that ℓ divides $q+1$ and $\sum_{x \in \mathbb{P}(V)} a_x = 0$. Then $-a_y^{-1}(\sum_{h \in U_{\{y\}}} h)\alpha = \sum_{x \in \mathbb{P}(V)} [x]$.

Assuming in addition that $\alpha \notin K \cdot \sum_{x \in \mathbb{P}(V)} [x]$, fix some $z \in \mathbb{P}(V)$ with $a_z \neq a_y$. Let $T \subset \text{PGL}(V)$ be the torus fixing y and z . Then $(\sum_{h \in T} h)\alpha =$

$$(\sum_{x \neq y, z} a_x) \sum_{x \neq y, z} [x] + (q-1)a_y[y] + (q-1)a_z[z] = -(a_y + a_z) \sum_{x \neq y, z} [x] - 2a_y[y] - 2a_z[z].$$

If $\ell = 2$, fix some $\xi \in \text{PGL}(V)$ such that $\xi y = z$ and $\xi z \neq y$. Then $(\sum_{h \in T} h)\alpha = -(a_y + a_z) \sum_{x \in \mathbb{P}(V)} [x] + (a_y + a_z)[y] + (a_y + a_z)[z]$, so $(\xi - 1)(\sum_{h \in T} h)\alpha = (a_y + a_z)([\xi z] - [y])$, and thus, the $K[\text{PGL}(V)]$ -submodule generated by α contains $[\xi z] - [y]$, and therefore, it contains $K[\mathbb{P}(V)]^\circ$.

If $\ell \neq 2$, fix some involution $\xi \in \text{PGL}(V)$ such that $\xi y = z$. Then $(\xi - 1)(\sum_{h \in T} h)\alpha = 2(a_y - a_z)([y] - [z])$, and thus, the $K[\text{PGL}(V)]$ -submodule generated by α contains $[y] - [z]$, and therefore, it contains $K[\mathbb{P}(V)]^\circ$.

- (2) Any nonzero element of $K[\mathbb{P}(V)]$ can be considered as an element $\alpha = \sum_{x \in \mathbb{P}^1(\mathbb{F}_q)} a_x[x]$ for a finite subfield $\mathbb{F}_q \subset F$. Extending \mathbb{F}_q in F if necessary, we may assume that $a_x = 0$ for at least one x . Then it follows from (1) that any $K[\text{PGL}_2(\mathbb{F}_q)]$ -submodule containing α contains $K[\mathbb{P}^1(\mathbb{F}_q)]^\circ$, and thus, any $K[\text{PGL}(V)]$ -submodule containing α contains $K[\mathbb{P}(V)]^\circ$.

□

Proposition 5.4 *Let $F = \mathbb{F}_q$ be a finite field, V be an F -vector space and K be a field extension of F . Then the $K[\text{PGL}(V)]$ -module $K[\mathbb{P}(V)]^\circ$ is simple if and only if $\dim V = 2$.*

Proof Let $\text{Sym}_F^n V := (V^{\otimes n})_{\mathfrak{S}_n}$ be the n -th symmetric power of V , so $\dim_F \text{Sym}_F^n V = \binom{\dim V + n - 1}{\dim V - 1}$. The natural morphism of $K[\text{PGL}(V)]$ -modules $K[\mathbb{P}(V)]^\circ = K[(V \setminus \{0\})/F^\times]^\circ \rightarrow K \otimes_F \text{Sym}_F^{q-1} V, \sum_x a_x[x] \mapsto \sum_x a_x \tilde{x}^{q-1}$, is nonzero. One has $\dim_K K[\mathbb{P}(V)]^\circ = |\mathbb{P}(V)| - 1 = (q^{\dim V} - q)/(q - 1)$.

If $\dim V = 2$ then $\dim_K K[\mathbb{P}(V)]^\circ = \dim_K K \otimes_F \text{Sym}_F^{q-1} V = q$.

Assuming that $(q^n - q)/(q - 1) \geq \binom{n+q-2}{n-1}$ for some $n \geq 2$, let us show that $(q^{n+1} - q)/(q - 1) > \binom{n+q-1}{n}$. Indeed, $\binom{n+q-1}{n} = \binom{n+q-2}{n-1}(n + q - 1)/n < \binom{n+q-2}{n-1}q \leq q(q^n - q)/(q - 1) < (q^{n+1} - q)/(q - 1)$.

This implies that $\dim K[\mathbb{P}(V)]^\circ > \dim K \otimes_F \text{Sym}_F^{q-1} V$ if $\dim V > 2$, and thus, the above morphism is not injective, so $K[\mathbb{P}(V)]^\circ$ is not simple.

The simplicity in the case $\dim V = 2$ is shown in Lemma 3.4.

□

Acknowledgements We are grateful to Leonid Rybnikov and Sasha Kazilo for bringing us together and providing an exceptional environment that made our work possible. The study has been funded within the framework of the HSE University Basic Research Program and the Russian Academic Excellence Project ‘5-100’. R.B. is partially supported by an NSF grant.

References

Alperin, J.L.: Local Representation Theory. Modular Representations as an Introduction to the Local Representation Theory of Finite Groups. Cambridge University Press, Cambridge (1986)

Benson, D.: Modular Representation Theory. New Trends and Methods. Lecture Notes in Mathematics 1081 (2nd print 2006) (1984)

Bernstein, I.N., Zelevinsky, A.V.: Representations of the group $GL(n, F)$, where F is a local non-archimedean field. *Uspehi Matem. Nauk* **31**(3) (189), 5–70 (1976)

Roman, S.: Advanced Linear Algebra, Graduate Texts in Mathematics 135, 3rd edn. Springer, Berlin (2008)

Rovinsky, M.: Semilinear representations of symmetric groups and of automorphism groups of universal domains. *Selecta Math.* **24**(3), 2319–2349 (2018)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.