



Ramanujan's Theorem and Highest Abundant Numbers

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Abstract

In 1915, Ramanujan proved asymptotic inequalities for the sum of divisors function, assuming the Riemann hypothesis (RH). We consider a strong version of Ramanujan's theorem and define highest abundant numbers that are extreme with respect to the Ramanujan and Robin inequalities. Properties of these numbers are very different depending on whether the RH is true or false.

Keywords Ramanujan theorem · Riemann hypothesis · Robin's criterion · Colossally abundant number

1 Introduction

The function $\sigma(n) = \sum_{d|n} d$ is the *sum of divisors* function. In 1913 Grönwall (see Hardy and Wright 1979, Theorem 323) proved that the asymptotic maximal size of $\sigma(n)$ satisfies

$$\limsup_{n \rightarrow \infty} G(n) = e^\gamma, \quad G(n) := \frac{\sigma(n)}{n \log \log n}, \quad n \geq 2,$$

where $\gamma \approx 0.5772$ is the Euler–Mascheroni constant. Robin (1984) showed that the Riemann hypothesis (RH) is true if and only if

$$\sigma(n) < e^\gamma n \log \log n \quad \text{for all } n > 5040. \quad (R)$$

Briggs' computation of the colossally abundant numbers implies (R) for $n < 10^{(10^{10})}$ (Briggs 2006). According to Morrill and Platt (2018), (R) holds for all integers $5040 < n < 10^{(10^{13})}$.

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A positive integer n is called *superabundant* (SA) if

$$\frac{\sigma(k)}{k} < \frac{\sigma(n)}{n} \quad \text{for all integer } k \in [1, n - 1].$$

Colossally abundant numbers (CA) are those numbers n for which there is $\varepsilon > 0$ such that

$$\frac{\sigma(k)}{k^{1+\varepsilon}} \leq \frac{\sigma(n)}{n^{1+\varepsilon}} \quad \text{for all } k \in \mathbb{N}.$$

Bachmann (see Hardy and Wright 1979, Theorem 324) showed that on average, $\sigma(n)$ is around $\pi^2 n/6$. Bachmann and Grönwall’s results ensure that for every $\varepsilon > 0$ the function $\sigma(n)/n^{1+\varepsilon}$ has a maximum and that as ε tends to zero these maxima will increase. Thus there are infinitely many CA numbers.

SA and CA numbers were studied in detail by Alaoglu and Erdős (1944) and Erdős and Nicolas (1975). The study of numbers with $\sigma(n)$ large was initiated by Ramanujan (1915). In fact, SA and CA numbers had been studied by Ramanujan in 1915. Unknown to Alaoglu and Erdős, about 30 pages of Ramanujan’s paper “Highly Composite Numbers” were suppressed. Those pages were finally published in 1997 (Ramanujan 1997).

Let

$$F(x, k) := \frac{\log(1 + 1/(x + \dots + x^k))}{\log x},$$

$$E_p := \{F(p, k) \mid k \geq 1\}, \quad p \text{ is prime,}$$

and

$$E := \bigcup_p E_p = \{\varepsilon_1, \varepsilon_2, \dots\} = \left\{ \log_2 \left(\frac{3}{2} \right), \log_3 \left(\frac{4}{3} \right), \log_2 \left(\frac{7}{6} \right), \dots \right\}.$$

Alaoglu and Erdős (1944, Theorem 10) showed that if ε is not *critical*, i.e. $\varepsilon \notin E$, then $\sigma(k)/k^{1+\varepsilon}$ has a unique maximum attained at the number n_ε . If ε satisfies $\varepsilon_i > \varepsilon > \varepsilon_{i+1}$, $i \in \mathbb{N}$, then n_ε is constant on the interval $(\varepsilon_{i+1}, \varepsilon_i)$ and we call it n_i . Moreover,

$$n_\varepsilon = \prod_{p \in \mathbb{P}} p^{a_\varepsilon(p)}, \quad \text{where } a_\varepsilon(p) = \left\lfloor \frac{\log(p^{1+\varepsilon} - 1) - \log(p^\varepsilon - 1)}{\log p} \right\rfloor - 1.$$

In particular, Alaoglu and Erdős in their 1944 paper found all SA and CA numbers up to 10^{18} . The first 14 CA numbers n_1, n_2, \dots, n_{14} are

2, 6, 12, 60, 120, 360, 2520, 5040, 55440, 720720, 1441440, 4324320, 21621600, 367567200.

Robin (1984, Sect. 3: Prop. 1) showed that if the Riemann hypothesis is false, then there exists a counterexample to the Robin criterion (R) which is a colossally abundant number. Thus, it suffices to check (R) only for CA numbers.

Ramanujan (1997, p. 143), proved that if n is a CA number (he called CA numbers as *generalized superior highly composite*) then under the RH the following inequalities hold

$$\limsup_{n \rightarrow \infty} \left(\frac{\sigma(n)}{n} - e^\gamma \log \log n \right) \sqrt{\log n} \leq -c_1, \quad c_1 := e^\gamma (2\sqrt{2} - 4 - \gamma + \log 4\pi) \approx 1.3932, \quad (1)$$

$$\liminf_{n \rightarrow \infty} \left(\frac{\sigma(n)}{n} - e^\gamma \log \log n \right) \sqrt{\log n} \geq -c_2, \quad c_2 := e^\gamma (2\sqrt{2} + \gamma - \log 4\pi) \approx 1.5578. \quad (2)$$

Denote

$$T(n) := \left(e^\gamma \log \log n - \frac{\sigma(n)}{n} \right) \sqrt{\log n}.$$

It is easy to see that Ramanujan’s inequalities (1) and (2) yield the following fact: *If the RH is true, then there is i_0 such that for all CA numbers n_i , $i \geq i_0$, we have*

$$1.393 < T(n_i) < 1.558. \quad (3)$$

Note that (2) does not hold for all integers. Indeed, if p_i is prime, then $\sigma(p_i) = p_i + 1$. Therefore,

$$\limsup_{i \rightarrow \infty} T(p_i) = \infty.$$

However, (1) holds for all numbers. In Sect. 2 we prove the following theorem.

Theorem 1 (The strong Ramanujan theorem) *If the RH is true, then*

$$\liminf_{n \rightarrow \infty} T(n) \geq c_1 > 1.393.$$

It is an interesting open problem: *Can Ramanujan’s constant c_1 be improved?*

Theorem 1 implies the following inequality (see Corollary 1 in Sect. 2):

If the RH is true, then there is m_0 such that for all $n > m_0$ we have

$$\sigma(n) + \frac{1.393 n}{\sqrt{\log n}} < e^\gamma n \log \log n \quad (4)$$

which is stronger than *Ramanujan’s theorem* (Broughan 2017, Theorem 7.2):

If the RH is true, then there is m_0 such that for all $n > m_0$ we have

$$\sigma(n) < e^\gamma n \log \log n. \quad (5)$$

Note that, for fixed $\varepsilon > 0$, CA numbers n may be viewed as maximizers of

$$Q(k) - \varepsilon \log k = \log(\sigma(k)/k^{1+\varepsilon}), \quad Q(k) := \log \sigma(k) - \log k.$$

Equivalently, n is CA if $(x_n, A(x_n))$ is a vertex of the convex envelope of A on D , where

$$x_k := \log k, \quad A(x_k) := x_k - \log \sigma(k) = -Q(k), \quad D := \{x_k\}, \quad k \geq 2,$$

see details in Sect. 3, Example 1.

Let $n \geq 2$ and s be a real number. Denote

$$R_s(n) := (e^\gamma n \log \log n - \sigma(n)) (\log n)^s.$$

Now we define *Highest Abundant* (HA) numbers. We say that $n \in D \subset \mathbb{N}$ is HA with respect to R_s and write $n \in HA_s(D)$ if for some real a

$$R_s(k) - ak$$

attains its minimum on D at n . For $D = \{n \in \mathbb{N} \mid n \geq 5040\}$ we denote $HA_s(D)$ by HA_s .

Actually, if D is infinite, then $HA_s(D)$ can be empty or contain only one number m_0 . It is clear that m_0 is the minimum number in $D = \{m_0 = x_0, x_1, \dots\}$. Then there is a_0 such that m_0 is defined by any $a \leq a_0$.

It can be shown, see Proposition 1 in Sect. 3, if $HA_s(D) = \{m_0, m_1, \dots\}$ contains at least two numbers, then there is a set of *critical* values a , $\mathcal{A}_s(D) = \{a_1, a_2, \dots\}$, $a_1 < a_2 < \dots$, such that if a is not critical, then $R_s(n) - na$ has a unique minimum on D attained at the number m_a . If $a \in (a_i, a_{i+1})$, $i = 1, 2, \dots$, then m_a is constant on the interval (a_i, a_{i+1}) and $m_a = m_i$. In fact, a_i is the slope of R_s on $[m_{i-1}, m_i]$, i.e.

$$a_i = \frac{R_s(m_i) - R_s(m_{i-1})}{m_i - m_{i-1}}.$$

We see that definitions of CA and HA numbers are similar, in both cases numbers can be determined through the vertices of the convex envelopes of certain functions. In Example 3 (Sect. 3) is considered HA numbers with respect to R_s , $s = 1$, on $D = [2, n_{13} = 21621600]$. There are 13 HA numbers in this interval, 12 of them are CA numbers (except $n_6 = 360$) and one more $m = 2162160$ is SA but m is not CA. However, properties of HA and CA numbers are different. The property that HA_s is infinite depending on whether the RH is true or false.

Theorem 2 (i) Let $s > 1/2$. If the RH is true, then HA_s is infinite and $\lim_{n \rightarrow \infty} a_n = \infty$.

If the RH is false, then HA_s is empty.

(ii) Let $s \leq 0$. If the RH is false, then HA_s is infinite, all $a_i < 0$ and $\lim_{n \rightarrow \infty} a_n = 0$.

If the RH is true, then $HA_s = \{5040\}$ and $\mathcal{A}_s = \{0\}$.

In Sect. 4 (Theorems 3 and 4) we consider extensions of Theorem 2. Proofs of these theorems rely on Robin’s inequalities (7) and (8) (Sect. 4), the strong Ramanujan theorem and his inequality (2), namely on Corollary 2 in Sect. 2.

Let $h_n := \sum_{i=1}^n 1/i$ denote the harmonic sum. Using (R) Lagarias (2002) showed that the Riemann hypothesis is equivalent to the following inequality

$$L_0(n) := h_n + \exp(h_n) \log(h_n) - \sigma(n) > 0 \quad \text{for all } n > 1. \tag{L}$$

In Sect. 4 we consider an analog of Theorem 2 for (L) on $D = \mathbb{N}$.

2 The Strong Ramanujan Theorem

Ramanujan’s theorem in the form of (5) is present in Broughan (2017, Theorem 7.2), Nicolas and Sondow (2014), Ramanujan (1997, p. 152) and other places. This theorem can be easily derived from (1) for the CA numbers. Theorem 1 extends (1) for all $n \in \mathbb{N}$ and is a strong version of Ramanujan’s theorem, see (4). However, we could not find a proof of Theorem 1 for arbitrary integers. In this section we fill this gap.

Proof of Theorem 1 Let

$$f(n) := \sqrt{\log n} \log \log n, \quad g(n) := e^\gamma - G(n).$$

Then $T(n) = f(n) g(n)$.

Let S be the set of all non-CA integers $n > 2$. Then for every $n \in S$ there is $i = i(n) > 1$ such that $n_{i-1} < n < n_i$, where n_{i-1} and n_i are two consecutive CA numbers. Robin (1984, Proposition 1) showed that

$$G(n) \leq \max(G(n_{i-1}), G(n_i)).$$

We divide S into two disjoint subsets S_1 and S_2 :

$$S_1 := \{n \in S \mid G(n) \leq G(n_{i-1})\}, \quad S_2 := \{n \in S \mid G(n_{i-1}) < G(n) \leq G(n_i)\}.$$

(1) Suppose $n \in S_1$. Then $g(n) \geq g(n_{i-1})$, where $i = i(n)$. Since f is a monotonically increasing function, we have $f(n) > f(n_{i-1})$ and $T(n) > T(n_{i-1})$. Thus,

$$\liminf_{n \in S_1, n \rightarrow \infty} T(n) \geq \liminf_{i \rightarrow \infty} T(n_{i-1}) = \liminf_{i \rightarrow \infty} T(n_i) \geq c_1.$$

(2) Suppose $n \in S_2$. Then $g(n) \geq g(n_i)$ and $f(n) > f(n_{i-1})$. That yields

$$T(n) > f(n_{i-1})g(n_i) = T(n_i)F(i), \quad F(i) := \frac{f(n_{i-1})}{f(n_i)}.$$

We have

$$\lim_{i \rightarrow \infty} \frac{\log(n_{i-1})}{\log(n_i)} = 1. \tag{6}$$

Indeed, let $P(n)$ denote the largest prime factor of n . Alaoglu & Erdős (1944, Theorem 7) proved that $P(n) \sim \log n$ for all SA numbers. Then, in particular, it holds for CA numbers. The quotient of two consecutive CA numbers is either a prime or the product of two distinct primes (Alaoglu and Erdős 1944, page 455; Broughan 2017, Lemma 6.15), i.e. $n_i \leq n_{i-1}P^2(n_i) \sim n_{i-1} \log^2(n_i)$. Then we have

$$1 > \frac{\log(n_{i-1})}{\log(n_i)} > \frac{\log(n_i) - 2 \log(P(n_i))}{\log(n_i)} \sim 1 - \frac{2 \log \log n_i}{\log n_i} \sim 1.$$

It is not hard to see that (6) implies $\lim_{i \rightarrow \infty} F(i) = 1$. That yields

$$\liminf_{n \in S_2, n \rightarrow \infty} T(n) \geq \liminf_{i \rightarrow \infty} T(n_i)F(i) = \liminf_{i \rightarrow \infty} T(n_i) \geq c_1.$$

Thus, we have (1) for CA, S_1 and S_2 , i.e. for all numbers. □

Remark In the first version of this paper our proof of Case (2) relies on Wu (2019, Theorem 1). I am very grateful to Xiaolong Wu for the idea of proving this case using (6). Note that (6) is easily derived from the results of the classical paper of Alaoglu and Erdős (1944).

Corollary 1 *If the RH is true, then for every $\varepsilon > 0$ there is m_0 such that for all $n > m_0$ we have*

$$\sigma(n) + (c_1 - \varepsilon) \frac{n}{\sqrt{\log n}} < e^\gamma n \log \log n.$$

In particular, if $\varepsilon \leq 1.393$, then $\sigma(n) < e^\gamma n \log \log n$ for all $n > m_0$.

From (2) for CA numbers n_i we have

$$\limsup_{i \rightarrow \infty} T(n_i) \leq c_2 < 1.558.$$

This fact and Corollary 1 yield the following corollary:

Corollary 2 *If the RH is true, then for every $\varepsilon > 0$ there is m_0 such that a set*

$$M(\varepsilon) := \{n > m_0 \mid T(n) < c_2 + \varepsilon\}$$

is infinite and for all $n \in M(\varepsilon)$ we have $T(n) > c_1 - \varepsilon$.

3 Convex Envelope of Functions

Let $D = \{x_n\}$ be an increasing sequence. Let $h : D \rightarrow \mathbb{R}$ be a function on D . We say that h is *convex* (or *concave upward*) on D if for all $a, x, b \in D$ such that $a < x < b$ we have

$$h(x) \leq \frac{(b - x)h(a) + (x - a)h(b)}{b - a}.$$

Denote by $\Omega(f)$ the set of all convex functions $h : D \rightarrow \mathbb{R}$ such that $h(x) \leq f(x)$ for all $x \in D$. Suppose $\Omega(f) \neq \emptyset$. The *lower convex envelope* \check{f} of a function f on D is defined at each point of D as the supremum of all convex functions that lie under that function, i.e.

$$\check{f}(x) := \sup\{h(x) \mid h \in \Omega(f)\}.$$

Alternatively, \check{f} can be defined as follows. Let

$$\Gamma_f := \{(x, f(x)) \in D \times \mathbb{R} \subset \mathbb{R}^2\}$$

be the graph of f . The convex hull of Γ_f in \mathbb{R}^2 is the set of all convex combinations of points in Γ_f :

$$\text{conv}(\Gamma_f) := \{c_1 p_1 + \dots + c_k p_k \mid p_i \in \Gamma_f, c_i \geq 0, i = 1, \dots, k, c_1 + \dots + c_k = 1\}.$$

Then the graph $\{(x, \check{f}(x)) \in D \times \mathbb{R}\}$ is the *lower convex hull* of $\text{conv}(\Gamma_f)$.

It is clear, if D is finite, then $\Omega(f)$ is not empty. However, if D is infinite, then $\Omega(f)$ can be empty, for instance if $f(n) = -n^2$ and $D = \mathbb{N}$.

Let f be a function on $D = \{x_0, x_1, \dots\}$ with $\Omega(f) \neq \emptyset$. Then \check{f} is a piecewise linear convex function on D . Hence, there is a subset

$$H_f := \{m_0 = x_0, m_1, \dots\} \subset D$$

such that \check{f} is a linear function on $[m_{i-1}, m_i]$, $\check{f}(m_i) = f(m_i)$ for all i , and the sequence of slopes $A_f := \{a_1, a_2, \dots\}$ is strictly monotonic increasing, i.e. $a_1 < a_2 < \dots$, where

$$a_i := \frac{f(m_i) - f(m_{i-1})}{m_i - m_{i-1}}.$$

Let \tilde{H}_f be a subset in D such that $m \in \tilde{H}_f$ if for some $a \in \mathbb{R}$ the function $f(x) - ax$ attains its minimum on D at m , i.e.

$$\tilde{H}_f := \{m \in D \mid \exists a \in \mathbb{R}, \forall x \in D, f(m) - ma \leq f(x) - ax\}.$$

The next proposition can be easily derived from the above definitions.

Proposition 1 *Let f be a function on $D = \{x_n\}$ with $\Omega(f) \neq \emptyset$. Then \tilde{H}_f coincides with H_f and every $m_i \in H_f, i \geq 1$, is uniquely determined by any $a \in (a_{i-1}, a_i)$.*

Example 1 Let $D := \{x_n\}$, where $x_n := \log n, n \in \mathbb{N}$. Let $f(x_n) := x_n - \log \sigma(n)$. Then $f(x_n) = -Q(n)$, where Q is defined in Sect. 1. It is easy to see that in this case Proposition 1 yields that H_f is the set of CA numbers and $A_f = \{-\varepsilon_i\}$.

If $D = \{x_0, x_1, \dots, x_l\}$ is finite, then $H_f := \{m_0 = x_0, m_1, \dots, m_k \leq x_l\}$ and the cardinality $|A_f| = k$. If D is infinite, then A_f can be (i) infinite or (ii) finite. It is not

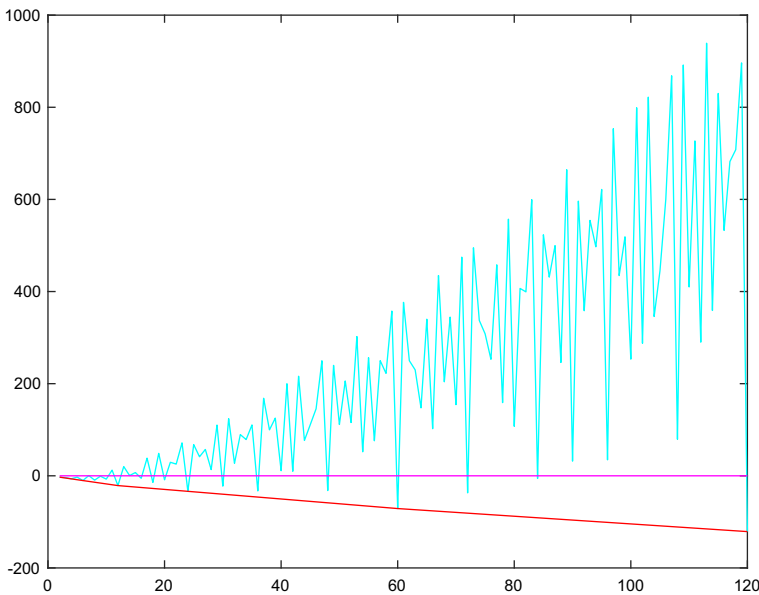


Fig. 1 Graphs of R_1 and \check{R}_1 on $D = \{2, \dots, 120\}$

hard to see that in case (ii) $H_f := \{m_0, m_1, \dots, m_k\}$ and $A_f = \{a_1, \dots, a_k, a_{k+1}\}$, where

$$\check{f}(n) = \check{f}(m_k) + (n - m_k) a_{k+1} \quad \text{for all } n \in D, \quad n \geq m_k.$$

Let $m_{k+1} := \infty$. Then for both cases we have that a_i is the slope of \check{f} on $[m_{i-1}, m_i]$.

Example 2 Let $f(n) = R_1(n) = (e^\gamma n \log \log n - \sigma(n)) \log n$ on $D = \{2, \dots, 120\}$. Then $H_f = \{2, 6, 12, 60, 120\}$. (Note that H_f consists of the first five CA numbers.) In this case \check{R}_1 is a convex monotonically decreasing function on D , see Fig. 1.

Example 3 Let $f = R_1$ on $D = \{2, 3, \dots, n_{13} = 21621600\}$. Then

$$H_f = \{2, 6, 12, 60, 120, 2520, 5040, 55440, 720720, 1441440, 2162160, 4324320, 21621600\}.$$

In this list of 13 numbers m_0, \dots, m_{12} there are 12 out of the first 13 CA numbers except $n_6 = 360$. However, m_{10} is an SA number $2162160 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ but is not CA. \check{R}_1 on H_f has a minimum at $m_5 = 2520$ and is positive for $m_i > m_6 = 5040$. We have

$$a_1 < \dots < a_5 < 0 < a_6 < \dots < a_{12}.$$

Now we prove the main results of this section.

Lemma 1 *Let $D \subset \mathbb{N}$ be infinite and $n_0 \in D$. Let f and g be functions on D such that*

$$f(n) \geq g(n) \quad \text{for all } n \geq n_0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{g(n)}{n} = \infty.$$

Then A_f is infinite and $\lim_{n \rightarrow \infty} a_n = \infty$.

Proof By assumption, for any real a there is n_a such that $g(n) > an$ for all $n \geq n_a$. This fact yields that for any linear function $l(x) = ax + b$ there is no or there are finitely many $n \in D$ such that $f(n) \leq l(n)$.

Let $D = \{x_0, x_1, \dots\}$, $m_0 := x_0$ and $H_f^{(0)} := \{m_0\}$. Suppose $H_f^{(i)} = \{m_0, \dots, m_i\}$ and $\mathcal{A}_f^{(i)} = \{a_1, \dots, a_i\}$. Let $l(x)$ be a linear function given by two points $(m_i, f(m_i))$ and $(x_{k+1}, f(x_{k+1}))$, where $m_i = x_k$. Denote

$$D_l = \{n \in D \mid n > m_i, f(n) \leq l(n)\}.$$

We have $1 \leq |D_l| < \infty$. Let x_j be a number in D_l such that the slope of a linear function given by two points $(m_i, f(m_i))$ and $(n, f(n))$, $n \in D_l$, attains its minimum at x_j . We denote the correspondent linear function by l_{i+1} . It is clear that $f(n) \geq l_{i+1}(n)$ for all $n \in D_l$. Hence, $m_{i+1} = x_j$ and a_{i+1} is the slope of l_{i+1} . We can continue this process. Since $f(n)/n \rightarrow \infty$ as $n \rightarrow \infty$, we have $a_i \rightarrow \infty$ as $i \rightarrow \infty$. □

Lemma 2 *Let g_1 and g_2 be functions on $D \subset \mathbb{N}$ such that for all $n \in D$ we have*

$$g_2(n) \geq g_1(n) \quad \text{and} \quad \lim_{n \rightarrow \infty} g_1(n) = \infty, \quad \lim_{n \rightarrow \infty} \frac{g_2(n)}{n} = 0.$$

Suppose for a function f on D there is $n_0 \in D$ such that $f(n) \geq g_1(n)$ for all $n \geq n_0$. If there are infinitely many $n \in D$ such that $f(n) \leq g_2(n)$, then $A_f = \{a_1, \dots, a_k\}$ is finite and

$$a_1 < \dots < a_k = 0.$$

Proof Denote $D_0 := \{n \in D \mid n < n_0\}$ and $D_1 := \{n \in D \mid n \geq n_0, f(n) \leq g_2(n)\}$. By assumption, D_1 is infinite and for any linear function $l(x) = ax + b$ with $a > 0$ there is no or there are finitely many $n \in D_1$ such that $f(n) \geq l(n)$. Hence, all $a_i \leq 0$. Since $f(n) \rightarrow \infty$ as $n \rightarrow \infty$, we have that A_f is finite and the largest $a_k = 0$. □

Lemma 3 *Let g_1 and g_2 be functions on $D \subset \mathbb{N}$ such that for all $n \in D$ we have*

$$g_2(n) \geq g_1(n) \quad \text{and} \quad \lim_{n \rightarrow \infty} g_2(n) = -\infty, \quad \lim_{n \rightarrow \infty} \frac{g_1(n)}{n} = 0.$$

Suppose for a function f on D there is $n_0 \in D$ such that $f(n) \geq g_1(n)$ for all $n \geq n_0$. If there are infinitely many $n \in D$ such that $f(n) \leq g_2(n)$, then A_f is infinite and $\lim_{n \rightarrow \infty} a_n = 0$.

Proof It is not hard to see that the assumptions yield that for any $l(x) = ax + b$ with $a < 0$ there is no or there are finitely many $n \in D$ such that $f(n) \leq l(n)$. Let l_i be the same as in Lemma 1. In this case for $n \in D_1$, that defined in Lemma 2, we have $f(n) \rightarrow -\infty$ and $f(n)/n \rightarrow 0$ as $n \rightarrow \infty$. Thus, $a_i \rightarrow 0$ as $i \rightarrow \infty$. \square

Lemma 4 *Let $D \subset \mathbb{N}$ be infinite. Let g be a function on D such that*

$$\lim_{n \rightarrow \infty} \frac{g(n)}{n} = -\infty.$$

Suppose for a function f on D there are infinitely many $n \in D$ such that $f(n) \leq g(n)$. Then $\Omega(f)$ is empty.

Proof Let $D_g := \{n \in D \mid f(n) \leq g(n)\}$. Let l_n be a linear function given by two points $(x_0, f(x_0))$ and $(n, f(n))$. By assumption for any a there is $n \in D_g$ such that the slope of l_n is less than a . Moreover, there are infinitely many m in D_g with $f(m) < l_n(m)$. This completes the proof. \square

4 Proof of Theorem 2 and Its Extensions

Robin (1984, Theorem 2) showed that for all $n \geq 3$

$$R_0(n) = e^\gamma n \log \log n - \sigma(n) > -0.6482 \frac{n}{\log \log n}. \tag{7}$$

If the RH is false Robin (1984, Theorem 1) proved that there exist constants $b \in (0, 1/2)$ and $c > 0$ such that

$$R_0(n) < -\frac{cn \log \log n}{(\log n)^b} \tag{8}$$

holds for infinitely many n . Thus, if the RH is false there are infinitely many $n \in \mathbb{N}$ such that

$$C_1(n) := -\frac{0.6482n}{\log \log n} < R_0(n) < C_2(n) := -\frac{cn \log \log n}{(\log n)^b}.$$

Let $\tau(n)$ be any positive function on $D \subset \mathbb{N}$. Denote

$$R_\tau(n) := (e^\gamma n \log \log n - \sigma(n)) \tau(n), \quad n \in D.$$

We defined HA numbers with respect to R_τ as follows:

$$HA_\tau(D) := H_{R_\tau}(D) = \{m \in D \mid \exists a \in \mathbb{R}, \quad \forall x \in D, R_\tau(m) - ma \leq R_\tau(x) - ax\}.$$

As above, $\mathcal{A}_\tau(D) = \{a_1, a_2, \dots\}$ are slopes of R_τ on $HA_\tau(D)$ and we denote $HA_\tau(D)$ by HA_τ for $D = \{n \in \mathbb{N} \mid n \geq 5040\}$.

The following theorem extends Theorem 2(i).

Theorem 3 Let $\tau(n) > 0$ for all $n \geq 5040$. Denote

$$\Phi_\tau := \lim_{n \rightarrow \infty} \frac{\tau(n)}{\sqrt{\log n}}.$$

- (a) Assume the RH is true. If $\Phi_\tau = \infty$, then HA_τ is infinite and $\lim_{n \rightarrow \infty} a_n = \infty$.
- (b) If the RH is false and $\Phi_\tau > 0$, then HA_τ is empty.

Proof (a) Suppose the RH is true. Let

$$g(n) := \frac{1.393 n \tau(n)}{\sqrt{\log n}}.$$

By Corollary 1 there is n_0 such that for all $n \geq n_0$ we have

$$R_\tau(n) = \frac{nT(n)\tau(n)}{\sqrt{\log n}} \geq g(n) \text{ and by assumption } \lim_{n \rightarrow \infty} \frac{g(n)}{n} = 1.393\Phi_\tau = \infty.$$

Then Lemma 1 with $f = R_\tau$ yields that $\lim_{n \rightarrow \infty} a_n = \infty$.

- (b) Suppose the RH is false. Since $b < 1/2$ by (8) there are infinitely many $n \in \mathbb{N}$ such that

$$R_\tau(n) = R_0(n)\tau(n) \leq C_2(n)\tau(n) < g(n) := -\frac{cn \tau(n) \log \log n}{\sqrt{\log n}}.$$

Then Lemma 4 with $f = R_\tau$ completes the proof. □

Now we consider a generalization of Theorem 2(ii).

Theorem 4 Let

$$\tau(n) > 0, n \geq 5040, \lim_{n \rightarrow \infty} \frac{\tau(n)}{\log \log n} = 0 \text{ and } \lim_{n \rightarrow \infty} \frac{\tau(n) n \log \log n}{\sqrt{\log n}} = \infty.$$

- (a) If the RH is false, then HA_τ is infinite, all $a_i < 0$ and $\lim_{n \rightarrow \infty} a_n = 0$.
- (b) If the RH is true, then $HA_\tau = \{5040\}$ and $\mathcal{A}_\tau = \{0\}$.

Proof (a) Suppose the RH is false. Let

$$g_1(n) := C_1(n)\tau(n), \quad g_2(n) := C_2(n)\tau(n).$$

Then by (7) we have that $g_1(n) < R_\tau(n)$ for all $n \in D$ and by (8) the inequality $R_\tau(n) < g_2(n)$ holds for infinitely many n . Since $f = R_\tau$, g_1 and g_2 satisfy the assumption of Lemma 3 we have (a).

(b) Suppose the RH is true. Let

$$g_1(n) := \frac{1.393 n \tau(n)}{\sqrt{\log n}}, \quad g_2(n) := \frac{1.558 n \tau(n)}{\sqrt{\log n}}.$$

Then Corollary 2 yields that $f = R_\tau$, g_1 and g_2 satisfy the assumption of Lemma 2. Since for all $n > 5040$ we have $R_\tau(n) > 0 > R_\tau(5040)$, there are not $a_i \leq 0$. Thus, $HA_\tau = \{5040\}$. \square

Proof of Theorem 2 This theorem immediately follows from Theorems 3 and 4. Indeed, if $\tau(n) = (\log n)^s$, then $R_\tau(n) = R_s(n)$. It clear that $\Phi_\tau = \infty$ in Theorem 3 only if $s > 1/2$ and the assumptions in Theorem 4 hold if $s \leq 0$. \square

From the Lagarias inequalities (Lagarias 2002, Lemmas 3.1, 3.2) for $n > 20$ we have

$$R_0(n) + h_n \leq L_0(n) \leq R_0(n) + \frac{7n}{\log n}. \quad (9)$$

Let $L_\tau(n) := L_0(n)\tau(n)$. Then (9) yields analogs of Theorems 3 and 4 for L_τ . We can just substitute R_τ by L_τ .

Theorem 5 (i) *If the RH is true, $\tau(n) > 0$ and $\Phi_\tau = \infty$, then there are infinitely many HA numbers with respect to L_τ and $\lim_{n \rightarrow \infty} a_n = \infty$.*

(ii) *Let $\tau(n)$ satisfy the assumptions of Theorem 4. If the RH is false, then there are infinitely many HA numbers with respect to L_τ , all $a_i < 0$ and $\lim_{n \rightarrow \infty} a_n = 0$.*

References

- Alaoglu, L., Erdős, P.: On highly composite and similar numbers. *Trans. Am. Math. Soc.* **56**, 448–469 (1944)
- Briggs, K.: Abundant numbers and the Riemann hypothesis. *Exp. Math.* **15**, 251–256 (2006)
- Broughan, K.: *Equivalents of the Riemann Hypothesis*. Cambridge University Press, Cambridge (2017)
- Erdős, P., Nicolas, J.-L.: Répartition des nombres superabondants. *Bull. Math. Soc. France* **103**, 65–90 (1975)
- Hardy, G.H., Wright, E.M.: *An Introduction to the Theory of Numbers*, 5th edn. Oxford University Press, Oxford (1979)
- Lagarias, J.C.: An elementary problem equivalent to the Riemann hypothesis. *Am. Math. Mon.* **109**, 534–543 (2002)
- Morrill, T., Platt, D.: Robin's inequality for 25-free integers and obstacles to analytic improvement (2018). Preprint. [arXiv:1809.10813](https://arxiv.org/abs/1809.10813)
- Nicolas, J.-L., Sondow, J.: Ramanujan, Robin, highly composite numbers, and the Riemann hypothesis. *Contemp. Math.* **627**, 145–156 (2014)
- Ramanujan, S.: Highly composite numbers. *Proc. Lond. Math. Soc.* **14**, 347–407 (1915)
- Ramanujan, S.: Highly composite numbers (annotated and with a foreword by J.-L. Nicolas and G. Robin). *Ramanujan J.* **1**, 119–153 (1997)
- Robin, G.: Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. *J. Math. Pures Appl.* **63**, 187–213 (1984)
- Wu, X.: Subsets of colossally abundant numbers (2019). Preprint [arXiv:1903.03490](https://arxiv.org/abs/1903.03490)

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