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RESEARCH CONTRIBUTION

On a Characterization of Polynomials Among Rational Functions in Non-Archimedean Dynamics

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Abstract
We study a question on characterizing polynomials among rational functions of degree > 1 on the projective line over an algebraically closed field that is complete with respect to a non-trivial and non-archimedean absolute value, from the viewpoint of dynamics and potential theory on the Berkovich projective line.

Keywords Canonical measure · Equilibrium mass distribution · Non-archimedean dynamics · Potential theory

Mathematics Subject Classification Primary 37P50; Secondary 11S82 · 31C15

1 Introduction

Let K be an algebraically closed field that is complete with respect to a non-trivial and non-archimedean absolute value $|\cdot|$. The *Berkovich* projective line $\mathsf{P}^1 = \mathsf{P}^1(K)$ is, as a topological augmentation of the (classical) projective line $\mathbb{P}^1 = \mathbb{P}^1(K) = K \cup \{\infty\}$, a compact, locally compact, uniquely arcwise connected, and Hausdorff topological space. The set $\mathsf{H}^1 := \mathsf{P}^1 \setminus \mathbb{P}^1$ is called the Berkovich upper half space in P^1.

Let $f \in K(z)$ be a rational function of degree $d > 1$. For every $n \in \mathbb{N}$, set $f^n := f \circ f^{n-1}$, where $f^0 := \mathrm{Id}_{\mathbb{P}^1}$. The action of f on $\mathbb{P}^1$ uniquely extends to a continuous endomorphism on P^1, which is still open, surjective, and fiber-discrete, and preserves both $\mathbb{P}^1$ and H^1. Let us define the *Berkovich* Julia set $\mathsf{J}(f)$ of f by the set of all points $\mathcal{S} \in \mathsf{P}^1$ such that for any open neighborhood U of $\mathcal{S}$ in P^1,

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$$\mathsf{P}^1 \setminus E(f) \subset \bigcup_{n \in \mathbb{N}} f^n(U),$$

where the set $E(f) := \{a \in \mathbb{P}^1 : \# \bigcup_{n \in \mathbb{N}} f^{-n}(a) < \infty\}$ is called the (classical) exceptional set of f and is at most countable subset in $\mathbb{P}^1$. The local degree function $\deg_{\cdot} f$ on $\mathbb{P}^1$ also canonically extends to P^1, and this extended local degree function $\deg_{\cdot}(f)$ induces a canonical pullback operator f^* from the space of all Radon measures on P^1 to itself (see Sect. 2.2 below). Corresponding to the construction of the unique maximal entropy measure in complex dynamics (studied since Lyubich [20], Freire–Lopes–Mañé [15], Mañé [23]), the *f-canonical measure* μ_f on P^1 has been constructed as the unique probability Radon measure ν on P^1 such that

$$f^* \nu = d \cdot \nu \text{ on } \mathsf{P}^1 \quad \text{and that} \quad \nu(E(f)) = 0,$$

so in particular μ_f is invariant under f in that $f_* \mu_f = \mu_f$ on P^1. The support of μ_f coincides with $\mathsf{J}(f)$ and is the minimal non-empty and closed subset in P^1 backward invariant under f [14]. The *Berkovich* Fatou set of f is defined by

$$\mathsf{F}(f) := \mathsf{P}^1 \setminus \mathsf{J}(f),$$

and each component of $\mathsf{F}(f)$ is called a *Berkovich Fatou component* of f. We note that $E(f) \subset \mathsf{F}(f)$. A Berkovich Fatou component of f is mapped properly to a Berkovich Fatou component of f under f, and the preimage of a Berkovich Fatou component of f under f is the union of at most d Berkovich Fatou components of f.

Notation 1.1 For every $z \in \mathsf{F}(f) \cap \mathbb{P}^1$, let $D_z = D_z(f)$ be the Berkovich Fatou component of f containing z.

For any $z \in \mathsf{F}(f) \cap \mathbb{P}^1$, the compact subset $\mathsf{P}^1 \setminus D_z$ in P^1 is of logarithmic capacity > 0 with pole z, or equivalently, there is the unique *equilibrium mass distribution* $\nu_{z, \mathsf{P}^1 \setminus D_z}$ on $\mathsf{P}^1 \setminus D_z$ with pole z, which is in fact supported by $\partial D_z \subset \mathsf{J}(f)$ (we will recall some details on the logarithmic potential theory on P^1 in Sect. 2.4 below). If $f(\infty) = \infty \in \mathsf{F}(f)$, then $\nu_{\infty, \mathsf{P}^1 \setminus D_\infty}$ is invariant under f in that

$$f_*(\nu_{\infty, \mathsf{P}^1 \setminus D_\infty}) = \nu_{\infty, \mathsf{P}^1 \setminus D_\infty} \quad \text{on } \mathsf{P}^1$$

(see Lemma 4.7 below). If moreover $f \in K[z]$ or equivalently $f^{-1}(\infty) = \{\infty\}$, then $\infty \in E(f)$, $f^{-1}(D_\infty) = D_\infty$, and we can see

$$\mu_f = \nu_{\infty, \mathsf{P}^1 \setminus D_\infty} \quad \text{on } \mathsf{P}^1$$

(since Brolin [9] in complex dynamics). Let $\delta_{\mathcal{S}}$ be the Dirac measure on P^1 at $\mathcal{S} \in \mathsf{P}^1$.

Our aim is to study whether polynomials can be characterized among rational functions of degree > 1 using potential theory in non-archimedean setting, corresponding to the studies [19,21,22,25,29,30] in complex dynamics. Concretely, we study the following question on a characterization of polynomials among rational functions in non-archimedean dynamics.

Question Let $f \in K(z)$ be a rational function of degree > 1, and suppose that $f(\infty) = \infty \in \mathsf{F}(f)$ (so in particular $f(D_\infty) = D_\infty$) and that $\mathsf{J}(f) \not\subset \mathsf{H}^1$. Then, are the statements

$$\text{(i) } f \in K[z] \quad \text{and} \quad \text{(ii) } \mu_f = \nu_{\infty,\mathsf{P}^1 \setminus D_\infty} \textit{ on } \mathsf{P}^1$$

equivalent?

The corresponding question in complex dynamics has been answered affirmatively (Lopes[21]).

Here are a few comments on this Question. We already mentioned that (i) implies (ii) (without assuming $\mathsf{J}(f) \not\subset \mathsf{H}^1$). It is not difficult to construct such $f \in K(z) \setminus K[z]$ of degree > 1 that $f(D_\infty) = D_\infty$, that $f(\infty) \neq \infty \in \mathsf{F}(f)$, that $\mathsf{J}(f) \not\subset \mathsf{H}^1$, and that $\mu_f = \nu_{\infty,\mathsf{P}^1 \setminus D_\infty}$ on P^1 (e.g., Remark 6.5 below). On the other hand, if $\mathsf{J}(f) \subset \mathsf{H}^1$, then for any $g \in K(z)$ of the same degree as that of f which is close enough to f (in the coefficients topology), both the Berkovich Julia set $\mathsf{J}(g)$ of g and the action of g on $\mathsf{J}(g)$ are *same* as those of f (cf. [14, Sect. 5.3]). Since there is $f \in K[z]$ of degree > 1 satisfying $\mathsf{J}(f) \subset \mathsf{H}^1$ (e.g., such f that has a potentially good reduction, see below a characterization of this condition), for any such f and any $b \in K$, if $0 < |b| \ll 1$, then the small perturbation $f_b(z) := f(z)/(bz+1) \in K(z) \setminus K[z]$ of $f = f/1$ in $K(z)$ is of the same degree as that of f and satisfies that $f_b(\infty) = \infty \in \mathsf{F}(f_b)$, that $\mathsf{J}(f_b) = \mathsf{J}(f) \subset \mathsf{H}^1$, and that $\mu_{f_b} = \nu_{\infty,\mathsf{P}^1 \setminus D_\infty(f_b)}$ on P^1.

Recall that f *has a potentially good reduction* if and only if there exists a point $\mathcal{S} \in \mathsf{H}^1$ such that

$$f^{-1}(\mathcal{S}) = \{\mathcal{S}\};$$

then $\mathsf{J}(f) = \{\mathcal{S}\} (\subset \mathsf{H}^1$ so $\infty \in \mathsf{F}(f))$ and $\mu_f = \nu_{\infty,\mathsf{P}^1 \setminus D_\infty} = \delta_{\mathcal{S}}$ on P^1 (see also Remark 3.2 below). We say f has no potentially good reductions if f does not have a potentially good reduction.

We already mentioned that the total invariance $f^{-1}(D_\infty) = D_\infty$ of D_∞ under f is a necessary condition for $f \in K[z]$. Our first result is the following more general statement, under no potentially good reductions:

Theorem 1 *Let K be an algebraically closed field that is complete with respect to a non-trivial and non-archimedean absolute value. Let $f \in K(z)$ be a rational function of degree > 1. If $\infty \in \mathsf{F}(f)$, $f(D_\infty) = D_\infty$, $\mu_f = \nu_{\infty,\mathsf{P}^1 \setminus D_\infty}$ on P^1, and f has no potentially good reductions, then*

$$f^{-1}(D_\infty) = D_\infty.$$

Our second result is that even if we assume in addition $\mathsf{J}(f) \subset \mathbb{P}^1$, the latter statement (ii) does not necessarily imply the former (i) in Question.

Pick a prime number p. The p-adic norm $|\cdot|_p$ on $\mathbb{Q}$ is normalized so that for any $m, \ell \in \mathbb{Z} \setminus \{0\}$ not divisible by p and any $r \in \mathbb{Z}$, $\left|\frac{m}{\ell} p^r\right|_p = p^{-r}$. The completion $\mathbb{Q}_p$ of $(\mathbb{Q}, |\cdot|_p)$ is still a field, and the extended norm $|\cdot|_p$ on $\mathbb{Q}_p$ extends to an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ as a norm. The completion $\mathbb{C}_p$ of $(\overline{\mathbb{Q}_p}, |\cdot|_p)$ is still an algebraically closed field, and the extended norm $|\cdot|_p$ on $\mathbb{C}_p$ is a non-trivial and non-archimedean absolute value on $\mathbb{C}_p$. The completion $\mathbb{Z}_p$ of $(\mathbb{Z}, |\cdot|_p)$ is a complete discrete valued local ring and has the unique maximal ideal $p\mathbb{Z}_p$, and coincides with the ring of $\mathbb{Q}_p$-integers $\{z \in \mathbb{Q}_p : |z|_p \le 1\}$. In particular, the residual field of $\mathbb{Q}_p$ is $\mathbb{F}_p$.

The following counterexample of the implication (ii)⇒(i) in Question is suggested to the authors by Juan Rivera-Letelier:

Theorem 2 *Pick a prime number p, and set*

$$f(z) := \frac{z^p - 1}{p} \in \mathbb{Q}[z] \quad \textit{and} \quad A(z) := \frac{az+b}{cz+d} \in \mathrm{PGL}(2, \mathbb{Z}_p).$$

If $c \neq 0$ and (a, b, c, d) is close enough to $(1, 0, 0, 1)$ in $(\mathbb{Z}_p)^4$, then there is an attracting fixed point z_A of $f \circ A$ in $\mathbb{C}_p \setminus \mathbb{Z}_p$ (so $z_A \in \mathsf{F}(f \circ A)$) such that

$$\mathsf{J}(f \circ A) = \mathbb{Z}_p = \mathsf{P}^1(\mathbb{C}_p) \setminus D_{z_A}(f \circ A) \quad \textit{and}$$
$$\nu_{z_A, \mathbb{Z}_p} = \nu_{\infty, \mathbb{Z}_p} \quad \textit{on } \mathsf{P}^1(\mathbb{C}_p).$$

Then setting $m_A(z) := 1/(z - z_A) \in \mathrm{PGL}(2, \mathbb{C}_p)$, the rational function $g_A(z) := m_A \circ (f \circ A) \circ m_A^{-1} \in \mathbb{C}_p(z)$ is of degree p and satisfies $g_A \notin \mathbb{C}_p[z]$, $g_A(\infty) = \infty \in \mathsf{F}(g_A)$, $\mathsf{J}(g_A) \subset \mathbb{P}^1(\mathbb{C}_p)$, and

$$\mu_{g_A} = \nu_{\infty, \mathsf{P}^1(\mathbb{C}_p) \setminus D_\infty(g_A)} \quad \textit{on } \mathsf{P}^1(\mathbb{C}_p).$$

1.1 Organization of this Article

In Sects. 2 and 3, we prepare background material from potential theory and dynamics, respectively. In Sect. 4, we make preparatory computations from potential theory and give a proof of the invariance of $\nu_{\infty, \mathsf{P}^1 \setminus D_\infty}$ under f when $f(\infty) = \infty \in \mathsf{F}(f)$. In Sects. 5 and 6, we show Theorems 1 and 2, respectively.

2 Background from Potential Theory on P^1

Let K be an algebraically closed field that is complete with respect to a non-trivial and non-archimedean absolute value $|\cdot|$; in general, a norm $|\cdot|$ on a field k is non-trivial if $|k| \not\subset \{0, 1\}$, and is non-archimedean if $|\cdot|$ satisfies the strong triangle inequality

$$|x+y| \le \max\{|x|, |y|\} \quad \text{for any } x, y \in k.$$

For the foundation of potential theory on $\mathsf{P}^1 = \mathsf{P}^1(K)$, see [5, Sects. 5, 8] , [12, Sect. 7], [13, Sect. 3], [33], and the survey [18, Sects. 1–4], and the book [6, Sect. 13]. In what follows, we adopt a presentation from [28, Sects. 2, 3].

Notation 2.1 Let

$$\pi : K^2 \setminus \{(0,0)\} \to \mathbb{P}^1 = \mathbb{P}^1(K) = K \cup \{\infty\}$$

be the canonical projection such that

$$\pi(p_0, p_1) = \begin{cases} p_1/p_0 & \text{if } p_0 \neq 0, \\ \infty & \text{if } p_0 = 0, \end{cases}$$

following the convention on coordinate of $\mathbb{P}^1$ from the book [16].

On K^2, let $\|(p_0, p_1)\|$ be the maximum norm $\max\{|p_0|, |p_1|\}$. With the wedge product $(p_0, p_1) \wedge (q_0, q_1) := p_0 q_1 - p_1 q_0$ on K^2, the normalized chordal metric $[z, w]$ on $\mathbb{P}^1$ is the function

$$[z, w] := \frac{|p \wedge q|}{\|p\| \cdot \|q\|} (\le 1)$$

on $\mathbb{P}^1 \times \mathbb{P}^1$, where $p \in \pi^{-1}(z), q \in \pi^{-1}(w)$.

2.1 Berkovich Projective Line P^1

A (K-closed) *disk in* K is a subset in K written as $\{z \in K : |z-a| \le r\}$ for some $a \in K$ and some $r \ge 0$. By the strong triangle inequality, two decreasing infinite sequences of disks in K either *infinitely nest* or *are eventually disjoint*. This alternative induces the *cofinal* equivalence relation among decreasing (or more precisely, nesting and non-increasing) infinite sequences of disks in K, and the set of all cofinal equivalence classes $\mathcal{S}$ of decreasing infinite sequences (B_n) of disks in K together with $\infty \in \mathbb{P}^1$ is, as a set, nothing but P^1 ([7, p. 17]); if $B_{\mathcal{S}} := \bigcap_n B_n \neq \emptyset$, then $B_{\mathcal{S}}$ is itself a disk in K, and we also say $\mathcal{S}$ is represented by $B_{\mathcal{S}}$. For example, the *canonical (or Gauss) point* $\mathcal{S}_{\text{can}}$ in P^1 is represented by the the ring of K-integers

$$\mathcal{O}_K := \{z \in K : |z| \le 1\},$$

and each $z \in K$ is represented by the disk $\{z\}$ in K. The above alternative between two (decreasing infinite sequences of) disks in K also induces a canonical ordering $\succeq$ on P^1 so that ∞ is the unique maximal element in $(\mathsf{P}^1, \succeq)$ and that for every $\mathcal{S}, \mathcal{S}' \in \mathsf{P}^1 \setminus \{\infty\}$ satisfying $B_{\mathcal{S}}, B_{\mathcal{S}'} \neq \emptyset$, $\mathcal{S} \succeq \mathcal{S}'$ iff $B_{\mathcal{S}} \supset B_{\mathcal{S}'}$ (the description of $\succeq$ is a little complicated unless $B_{\mathcal{S}}, B_{\mathcal{S}'} \neq \emptyset$), and equips P^1 with a (profinite) tree structure. The topology of P^1 coincides with the weak (or observer) topology on P^1 as a (profinite) tree, so that P^1 is compact and uniquely arcwise-connected, and contains both $\mathbb{P}^1$ and H^1 as dense subsets. For the details on the tree structure on P^1, see e.g. [18, Sect. 2].

2.2 Action of Rational Functions on P^1

Let $h \in K(z)$ be a rational function. The action of h on $\mathbb{P}^1$ uniquely extends to a continuous endomorphism on P^1. Suppose in addition that $\deg h > 0$. Then the extended action of h on P^1 is surjective and open, has discrete (so finite) fibers, and preserves both $\mathbb{P}^1$ and H^1, and the *local degree* function $z \mapsto \deg_z h$ on $\mathbb{P}^1$ also canonically extends to P^1 so that for every $\mathcal{S} \in \mathsf{P}^1$,

$$\sum_{\mathcal{S}' \in h^{-1}(\mathcal{S})} \deg_{\mathcal{S}'} h = \deg h.$$

The action of h on P^1 induces the push-forward operator h_* on the space of all continuous functions on P^1 to itself and, by duality, also the pullback operator h^* on the space of all Radon measures on P^1 to itself; for every continuous test function ϕ on P^1, $(h_*\phi)(\cdot) = \sum_{\mathcal{S}' \in h^{-1}(\cdot)} (\deg_{\mathcal{S}'} h) \cdot \phi(\mathcal{S}')$ on P^1, and for every $\mathcal{S} \in \mathsf{P}^1$, $h^*\delta_{\mathcal{S}} = \sum_{\mathcal{S}' \in h^{-1}(\mathcal{S})} (\deg_{\mathcal{S}'} h) \cdot \delta_{\mathcal{S}'}$ on P^1. For more details, see [5, Sect. 9], [14, Sect. 2.2].

2.3 Kernel Functions and the Laplacian on P^1

The *generalized Hsia kernel* $[\mathcal{S}, \mathcal{S}']_{\mathrm{can}}$ on P^1 *with respect to* $\mathcal{S}_{\mathrm{can}}$ is a unique upper semicontinuous and separately continuous extension of the chordal distance function $\mathbb{P}^1 \times \mathbb{P}^1 \ni (z, z') \mapsto [z, z']$ to $\mathsf{P}^1 \times \mathsf{P}^1$.

More generally, for every $z_0 \in \mathbb{P}^1$, the *generalized Hsia kernel*

$$[\mathcal{S}, \mathcal{S}']_{z_0} := \begin{cases} \dfrac{[\mathcal{S}, \mathcal{S}']_{\mathrm{can}}}{[\mathcal{S}, z_0]_{\mathrm{can}} \cdot [\mathcal{S}', z_0]_{\mathrm{can}}} & \text{on } (\mathsf{P}^1 \setminus \{z_0\}) \times (\mathsf{P}^1 \setminus \{z_0\}) \\ +\infty & \text{on } (\{z_0\} \times \mathsf{P}^1) \cup (\mathsf{P}^1 \times \{z_0\}) \end{cases}$$

on P^1 *with respect to* z_0 is a unique upper semicontinuous and separately continuous extension of the function $(\mathbb{P}^1 \setminus \{z_0\}) \times (\mathbb{P}^1 \setminus \{z_0\}) \ni (z, z') \mapsto [z, z']/([z, z_0] \cdot [z', z_0])$ as a function $\mathsf{P}^1 \times \mathsf{P}^1 \to [0, +\infty]$. In particular, the function

$$|\mathcal{S} - \mathcal{S}'|_\infty := [\mathcal{S}, \mathcal{S}']_\infty$$

on $\mathsf{P}^1 \times \mathsf{P}^1$ extends the distance function $K \times K \ni (z, z') \mapsto |z - z'|$ to $(\mathsf{P}^1 \setminus \{\infty\}) \times (\mathsf{P}^1 \setminus \{\infty\})$, jointly upper semicontinuously and separately continuously, and the function

$$|\mathcal{S}|_\infty := |\mathcal{S} - 0|_\infty (= [\mathcal{S}, 0]_\infty) \quad \text{on } \mathsf{P}^1$$

extends the norm function $K \ni z \mapsto |z|$ to $\mathsf{P}^1 \setminus \{\infty\}$ continuously (see [13, Sect. 3.4], [5, Sect. 4.4]).

Let Ω_{can} be the Dirac measure $\delta_{\mathcal{S}_{\text{can}}}$ on P^1 at $\mathcal{S}_{\text{can}}$. The Laplacian Δ on P^1 is normalized so that for each $\mathcal{S}' \in \mathsf{P}^1$,

$$\Delta \log[\cdot, \mathcal{S}']_{\text{can}} = \delta_{\mathcal{S}'} - \Omega_{\text{can}}$$

on P^1, and then, for every $z_0 \in \mathbb{P}^1$ and every $\mathcal{S}' \in \mathsf{P}^1 \setminus \{z_0\}$, $\Delta \log[\cdot, \mathcal{S}']_{z_0} = \delta_{\mathcal{S}'} - \delta_{z_0}$ on P^1. For the details on the construction and properties of Δ, see [5, Sect. 5], [12, Sect. 7.7], [14, Sect. 2.4], [33, Sect. 3]; in [5,33], the opposite sign convention for Δ is adopted.

2.4 Logarithmic Potential Theory on P^1

For every $z \in \mathbb{P}^1$ and every positive Radon measure ν on P^1 supported by $\mathsf{P}^1 \setminus \{z\}$, the *logarithmic potential* of ν on P^1 with pole z is the function

$$p_{z,\nu}(\cdot) := \int_{\mathsf{P}^1} \log[\cdot, \mathcal{S}']_z \, \nu(\mathcal{S}') \quad \text{on } \mathsf{P}^1,$$

and the *logarithmic energy* of ν with pole z is defined by

$$I_{z,\nu} := \int_{\mathsf{P}^1} p_{z,\nu} \nu \in [-\infty, +\infty).$$

Then $p_{z,\nu} : \mathsf{P}^1 \to [-\infty, +\infty]$ is upper semicontinuous, and in fact is *strongly* upper semicontinuous in that for every $\mathcal{S} \in \mathsf{P}^1$,

$$\limsup_{\mathcal{S}' \to \mathcal{S}} p_{z,\nu}(\mathcal{S}') = p_{z,\nu}(\mathcal{S}) \tag{2.1}$$

([5, Proposition 6.12]).

For every non-empty subset C in P^1 and every $z \in \mathbb{P}^1 \setminus C$, we say C is *of logarithmic capacity* > 0 *with pole* z if

$$V_z(C) := \sup_{\nu} I_{z,\nu} > -\infty,$$

where ν ranges over all probability Radon measures on P^1 supported by C; otherwise, we say C is *of logarithmic capacity* 0 *with pole* z. For every non-empty compact subset C in P^1 of logarithmic capacity > 0 with pole $z \in \mathbb{P}^1 \setminus C$, there is a *unique* probability Radon measure ν on P^1, which is called the *equilibrium mass distribution on* C *with pole* z and is denoted by $\nu_{z,C}$, such that $\operatorname{supp} \nu \subset C$ and that $I_{z,\nu} = V_z(C)$, and then (i) $\nu_{z,C}(E) = 0$ for any subset E in C of logarithmic capacity 0 with pole z, (ii) letting D_z be the component of $\mathsf{P}^1 \setminus C$ containing z, we have

$$\operatorname{supp} \nu_{z,C} \subset \partial D_z, \quad p_{z,\nu_{z,C}} \geq I_{z,\nu_{z,C}} \text{ on } \mathsf{P}^1, \quad p_{z,\nu_{z,C}} > I_{z,\nu_{z,C}} \text{ on } D_z, \quad \text{and}$$

$$p_{z,\nu_{z,C}} \equiv I_{z,\nu_{z,C}} \text{ on } \mathsf{P}^1 \setminus (D_z \cup E),$$

where E is a possibly empty F_σ-subset in ∂D_z of logarithmic capacity 0 with pole z, (iii) if in addition $p_{z,\nu_{z,C}}$ is continuous on $\mathsf{P}^1 \setminus \{z\}$, then

$$\operatorname{supp} \nu_{z,C} = \partial D_z \quad \text{and} \quad p_{z,\nu_{z,C}} \equiv I_{z,\nu_{z,C}} \text{ on } \mathsf{P}^1 \setminus D_z,$$

and (iv) for any probability Radon measure ν' supported by C, we have

$$\inf_{\mathcal{S}\in C} p_{z,\nu'} \leq I_{z,\nu_{z,C}} \leq \sup_{\mathcal{S}\in C} p_{z,\nu'} \tag{2.2}$$

(see [5, Sects. 6.2, 6.3]).

We list a few observations:

Observation 2.2 For every $a \in K \setminus \{0\}$ and every $b \in K$, setting $\ell(z) := az + b \in \mathrm{PGL}(2, K)$, we have $\log|\ell(\mathcal{S}) - \ell(\mathcal{S}')|_\infty = \log|\mathcal{S} - \mathcal{S}'|_\infty + \log|a|$ on $K \times K$, and in turn on $\mathsf{P}^1 \times \mathsf{P}^1$. In particular, for every non-empty compact subset C in $\mathsf{P}^1 \setminus \{\infty\}$ of logarithmic capacity > 0 with pole ∞, we have $I_{\infty,\nu_{\infty,\ell(C)}} = I_{\infty,\nu_{\infty,C}} + \log|a|$ and $\ell_*(\nu_{\infty,C}) = \nu_{\infty,\ell(C)}$ on P^1.

Observation 2.3 Since the involution $\iota(z) = 1/z \in \mathrm{PGL}(2, \mathcal{O}_K)$ acts on $(\mathbb{P}^1, [z, w])$ isometrically, for any $z_0 \in \mathbb{P}^1$, we have $[\iota(\mathcal{S}), \iota(\mathcal{S}')]_{\iota(z_0)} = [\mathcal{S}, \mathcal{S}']_{z_0}$ on $\mathbb{P}^1 \times \mathbb{P}^1$, and in turn on $\mathsf{P}^1 \times \mathsf{P}^1$. Hence for any non-empty compact subset C in P^1 and any $z \in \mathsf{P}^1 \setminus C$, if C is of logarithmic capacity > 0 with pole z, then $V_z(C) = V_{\iota(z)}(\iota(C))$ and $\iota_*(\nu_{z,C}) = \nu_{\iota(z),\iota(C)}$ on P^1.

Observation 2.4 For every $z \in \mathbb{P}^1$, the strong triangle inequality $[\mathcal{S}, \mathcal{S}'']_z \leq \max\{[\mathcal{S}, \mathcal{S}']_z, [\mathcal{S}', \mathcal{S}'']_z\}$ for $\mathcal{S}, \mathcal{S}', \mathcal{S}'' \in \mathsf{P}^1$ still holds (see [5, Proposition 4.10]). Hence for every non-empty compact subset C in $\mathsf{P}^1 \setminus \{\infty\}$ and every $z \in \mathbb{P}^1 \setminus C$ so close to ∞ that $[z, \infty] < \inf_{\mathcal{S}\in C}[\mathcal{S}, z]_{\mathrm{can}}$, we have $[\cdot, \infty]_{\mathrm{can}} = [\cdot, z]_{\mathrm{can}}$ on C, which yields $[\mathcal{S}, \mathcal{S}']_\infty = [\mathcal{S}, \mathcal{S}']_z$ on $C \times C$, so if in addition C is of logarithmic capacity > 0 with pole ∞, then $V_\infty(C) = V_z(C)$ and $\nu_{\infty,C} = \nu_{z,C}$ on P^1.

2.5 Potential Theory with a Continuous Weight on P^1

A *continuous weight* g *on* P^1 is a continuous function on P^1 such that

$$\mu^g := \Delta g + \Omega_{\mathrm{can}}$$

is a probability Radon measure on P^1. Then μ^g has no atoms on $\mathbb{P}^1$, or more strongly, $\mu^g(E) = 0$ for any subset E in P^1 of logarithmic capacity 0 with some (indeed any) point in $\mathsf{P}^1 \setminus E$.

For a continuous weight g on P^1, the *g-potential kernel* on P^1 (the negative of an Arakelov Green kernel function on P^1 relative to μ^g [5, Sect. 8.10]) is an upper semicontinuous function

$$\Phi_g(\mathcal{S}, \mathcal{S}') := \log[\mathcal{S}, \mathcal{S}']_{\mathrm{can}} - g(\mathcal{S}) - g(\mathcal{S}') \quad \text{on } \mathsf{P}^1 \times \mathsf{P}^1. \tag{2.3}$$

For every Radon measure ν on P^1, the *g-potential* of ν on P^1 is the function

$$U_{g,\nu}(\cdot) := \int_{\mathsf{P}^1} \Phi_g(\cdot, \mathcal{S}')\nu(\mathcal{S}') \quad \text{on } \mathsf{P}^1,$$

and the *g-energy* of ν is defined by

$$I_{g,\nu} := \int_{\mathsf{P}^1} U_{g,\nu}\nu \in [-\infty, +\infty).$$

The *g-equilibrium energy V_g of (the whole)* P^1 is the supremum of the *g-energy* functional $\nu \mapsto I_{g,\nu}$, where ν ranges over all probability Radon measures on P^1. Then $V_g \in \mathbb{R}$ since $I_{g,\Omega_{\mathrm{can}}} > -\infty$. As in the logarithmic potential theory presented in the previous subsection, there is a unique probability Radon measure ν^g on P^1, which is called the *g-equilibrium mass distribution on* P^1, such that $I_{g,\nu^g} = V_g$. In fact

$$U_{g,\nu^g} \equiv V_g \quad \text{on } \mathsf{P}^1 \quad \text{and} \quad \nu^g = \mu^g \quad \text{on } \mathsf{P}^1$$

(see [5, Theorem 8.67, Proposition 8.70]).

A continuous weight g on P^1 is a *normalized weight on* P^1 if $V_g = 0$. For a continuous weight g on P^1, $\overline{g} := g + V_g/2$ is the unique normalized weight on P^1 satisfying $\mu^{\overline{g}} = \mu^g$.

3 Background from Dynamics on P^1

For a potential-theoretic study of dynamics of a rational function of degree > 1 on $\mathsf{P}^1 = \mathsf{P}^1(K)$, see [5, Sect. 10], [14, Sect. 3], [18, Sect. 5], and [6, Sect. 13]. In the following, we adopt a presentation from [28, Sect. 8.1].

3.1 Canonical Measure and the Dynamical Green Function of *f* on P^1

Let $f \in K(z)$ be a rational function of degree $d > 1$. We call $F \in (K[p_0, p_1]_d)^2$ a *lift* of f if

$$\pi \circ F = f \circ \pi$$

on $K^2 \setminus \{(0, 0)\}$, where for each $j \in \mathbb{N} \cup \{0\}$, $K[p_0, p_1]_j$ is the set of all homogeneous polynomials in $K[p_0, p_1]$ of degree j, as usual. A lift $F = (F_0, F_1)$ of f is unique up to multiplication in $K \setminus \{0\}$. Setting $d_0 := \deg F_0(1, z)$ and $d_1 := \deg F_1(1, z)$ and letting $c_0^F, c_1^F \in K \setminus \{0\}$ be the coefficients of the maximal degree terms of $F_0(1, z), F_1(1, z) \in K[z]$, respectively, the *homogeneous* resultant

$$\operatorname{Res} F = (c_0^F)^{d-d_1} \cdot (c_1^F)^{d-d_0} \cdot R\big(F_0(1, \cdot), F_1(1, \cdot)\big) \in K$$

of F does not vanish, where $R(P, Q) \in K$ is the usual resultant of $(P, Q) \in (K[z])^2$ (for the details on Res F, see e.g. [32, Sect. 2.4]).

Let F be a lift of f, and for every $n \in \mathbb{N} \cup \{0\}$, set $F^n = F \circ F^{n-1}$ where $F^0 := \mathrm{Id}_{K^2}$. Then for every $n \in \mathbb{N}$, F^n is a lift of f^n, and the function

$$T_{F^n} := \log \|F^n\| - d^n \cdot \log \| \cdot \|$$

on $K^2 \setminus \{(0,0)\}$ descends to $\mathbb{P}^1$ and in turn extends continuously to P^1, satisfying the equality $\Delta T_{F^n} = (f^n)^* \Omega_{\mathrm{can}} - d^n \cdot \Omega_{\mathrm{can}}$ on P^1 (see, e.g., [26, Definition 2.8]). The dynamical Green function of F on P^1 is the uniform limit $g_F := \lim_{n\to\infty} T_{F^n}/d^n$ on P^1, which is a continuous weight on P^1. The *energy* formula

$$V_{g_F} = -\frac{\log |\mathrm{Res}\, F|}{d(d-1)}$$

is due to DeMarco [11] for archimedean K by a dynamical argument, and due to Baker–Rumely [4] when f is defined over a number field; see Baker [2, Appendix A] or the present authors [29, Appendix] for a simple and potential-theoretic proof of this remarkable formula, for general K. The *f-canonical measure* is the probability Radon measure

$$\mu_f := \Delta g_F + \Omega_{\mathrm{can}} \quad \text{on } \mathsf{P}^1.$$

The measure μ_f is independent of the choice of the lift F of f, has no atoms in $\mathbb{P}^1$, and satisfies the f-balanced property $f^*\mu_f = d \cdot \mu_f$ (so in particular $f_*\mu_f = \mu_f$) on P^1. For more details, see [5, Sect. 10], [10, Sect. 2], [14, Sect. 3.1].

The *dynamical Green function g_f of f on* P^1 is the unique normalized weight on P^1 such that $\mu^{g_f} = \mu_f$. By the above energy formula on V_{g_F} and

$$\mathrm{Res}(cF) = c^{2d} \cdot \mathrm{Res}\, F \quad \text{for every } c \in K \setminus \{0\},$$

there is a lift F of f normalized so that $V_{g_F} = 0$ or equivalently that $g_F = g_f$ on P^1, and such a *normalized lift* F of f is unique up to multiplication in $\{z \in K : |z| = 1\}$. By $g_f = g_F = \lim_{n\to\infty} T_{F^n}/d^n$ on P^1 for a normalized lift F of f, for every $n \in \mathbb{N}$, we have $g_{F^n} = g_{f^n} = g_f$ on P^1 and $\mu_{f^n} = \mu_f$ on P^1. We note that $g_f \circ f = d \cdot \lim_{n\to\infty} T_{F^{n+1}}/d^{n+1} - T_F = d \cdot g_f - T_F$ on $\mathbb{P}^1$, that is,

$$d \cdot g_f - g_f \circ f = T_F \tag{3.1}$$

on $\mathbb{P}^1$, and in turn on P^1 by the density of $\mathbb{P}^1$ in P^1 and the continuity of both sides on P^1 (cf. [27, Proof of Lemma 2.4]).

3.2 Fundamental Properties of μ_f

Recall the definition of $\mathsf{J}(f)$ in Sect. 1. The characterization of μ_f as the unique probability Radon measure ν on P^1 such that $\nu(E(f)) = 0$ and that $f^*\nu = d \cdot \nu$ on

P^1 is a consequence of the following equidistribution theorem: *for every probability Radon measure* μ *on* P^1, *if* $\mu(E(f)) = 0$, *then*

$$\lim_{n\to\infty} \frac{(f^n)^*\mu}{d^n} = \mu_f \quad \textit{weakly on } \mathsf{P}^1. \tag{3.2}$$

This foundational result is due to Favre and Rivera-Letelier [14] (for a purely potential-theoretic proof, see also Jonsson [18]) and is a non-archimedean counterpart to Brolin [9], Lyubich [20], Freire et al. [15].

Remark 3.1 The *classical Julia set* $\mathsf{J}(f) \cap \mathbb{P}^1$ of f coincides with the set of all points in $\mathbb{P}^1$ at each of which the family $\left(f^n : (\mathbb{P}^1, [z, w]) \to (\mathbb{P}^1, [z, w])\right)_{n\in\mathbb{N}}$ is not locally equicontinuous (see, e.g., [5, Theorem 10.67]).

The equality $\operatorname{supp} \mu_f = \mathsf{J}(f)$ holds; the inclusion $\mathsf{J}(f) \subset \operatorname{supp} \mu_f$ follows from the definition of $\mathsf{J}(f)$, the balanced property $f^*\mu_f = d \cdot \mu_f$ on P^1, and $\operatorname{supp} \mu_f \not\subset E(f)$ (or more precisely, recalling that $E(f)$ is an at most countable subset in $\mathbb{P}^1$ and that μ_f has no atoms in $\mathbb{P}^1$). The opposite inclusion $\operatorname{supp} \mu_f \subset \mathsf{J}(f)$ follows from the definition of $\mathsf{J}(f)$ and the above equidistribution theorem.

Remark 3.2 (see, e.g., [5, Corollary 10.33]) If μ_f has an atom in P^1, then f has a potentially good reduction, so in particular $\mathsf{J}(f)$ is a singleton in H^1.

For every $n \in \mathbb{N}$, by $\operatorname{supp} \mu_f = \mathsf{J}(f)$ and $\mu_{f^n} = \mu_f$ on P^1, we also have $\mathsf{J}(f^n) = \mathsf{J}(f)$. For every $m \in \mathrm{PGL}(2, K)$, we have $m_*\mu_f = \mu_{m\circ f\circ m^{-1}}$ on P^1, $m(\mathsf{J}(f)) = \mathsf{J}(m \circ f \circ m^{-1})$, and $m(\mathsf{F}(f)) = \mathsf{F}(m \circ f \circ m^{-1})$.

3.3 Root Divisors on $\mathbb{P}^1$ and the Proximity Functions on P^1

For any distinct $h_1, h_2 \in K(z)$, let $[h_1 = h_2]$ be the effective (K-)divisor on $\mathbb{P}^1$ defined by all solutions to the equation $h_1 = h_2$ in $\mathbb{P}^1$ taking into account their multiplicities, which is also regarded as the Radon measure

$$\sum_{w\in\mathbb{P}^1} (\operatorname{ord}_w[h_1 = h_2]) \cdot \delta_w$$

on P^1. The function $\mathbb{P}^1 \ni z \mapsto [h_1(z), h_2(z)]$ between h_1 and h_2 uniquely extends to a continuous function $\mathcal{S} \mapsto [h_1, h_2]_{\mathrm{can}}(\mathcal{S})$ on P^1 (see, e.g., [26, Proposition 2.9]), so that for every continuous weight g on P^1, (the exp of) the function

$$\Phi(h_1, h_2)_g(\mathcal{S}) := \log[h_1, h_2]_{\mathrm{can}}(\mathcal{S}) - g(h_1(\mathcal{S})) - g(h_2(\mathcal{S})) \quad \text{on } \mathsf{P}^1 \tag{3.3}$$

is a unique continuous extension of (the exp of) the function $\mathbb{P}^1 \ni z \mapsto \Phi_g(h_1(z), h_2(z))$.

4 Potential-Theoretic Computations

Let $f \in K(z)$ be a rational function of degree $d > 1$.

Lemma 4.1 (Riesz's decomposition for the pullback of an atom) *For every* $\mathcal{S} \in \mathsf{P}^1$,

$$\Phi_{g_f}(f(\cdot), \mathcal{S}) = U_{g_f, f^*\delta_{\mathcal{S}}}(\cdot) \quad on\ \mathsf{P}^1. \tag{4.1}$$

Proof Fix a lift F of f normalized so that $g_F = g_f$ on P^1. Fix $w \in \mathbb{P}^1$ and $W \in \pi^{-1}(w)$. Choose a sequence $(q_j)_{j=1}^d$ in $K^2 \setminus \{(0,0)\}$ such that $F(p_0, p_1) \wedge W \in K[p_0, p_1]_d$ factors as $F(p_0, p_1) \wedge W = \prod_{j=1}^d ((p_0, p_1) \wedge q_j)$ in $K[p_0, p_1]$. This together with (3.1) and the definition of T_F implies

$$\begin{aligned}
&\Phi_{g_f}(f \circ \pi, w) - U_{g_f, f^*\delta_w} \circ \pi \\
&\quad = \big(\log|F(\cdot) \wedge W| - \log\|F\| - \log\|W\| - (g_f \circ f)(\pi(\cdot)) - g_f(w)\big) \\
&\qquad - \sum_{j=1}^d \big(\log|\cdot \wedge q_j| - \log\|\cdot\| - \log\|q_j\| - g_f \circ \pi - g_f(\pi(q_j))\big) \\
&\quad = \Big(\log|F(\cdot) \wedge W| - \sum_{j=1}^d \log|\cdot \wedge q_j|\Big) - \big((g_f \circ f)(\pi(\cdot)) + d \cdot g_f \circ \pi\big) \\
&\qquad - (\log\|F\| - d \cdot \log\|\cdot\|) \\
&\qquad - (g_f(w) + \log\|W\|) + \sum_{j=1}^d (g_f(\pi(q_j)) + \log\|q_j\|) \\
&\quad \equiv -(g_f(w) + \log\|W\|) + \sum_{j=1}^d (g_f(\pi(q_j)) + \log\|q_j\|) =: C \quad \text{on } K^2 \setminus \{0\},
\end{aligned}$$

so $\Phi_{g_f}(f(\cdot), w) - U_{g_f, f^*\delta_w}(\cdot) \equiv C$ on $\mathbb{P}^1$, and in turn on P^1 by the density of $\mathbb{P}^1$ in P^1 and the continuity of (the exp of) both sides on P^1. Integrating both sides against μ_f over P^1, since $\int_{\mathsf{P}^1} U_{g_f, f^*\delta_w} \mu_f = \int_{\mathsf{P}^1} U_{g_f, \mu_f}(f^*\delta_w) = 0$ (by $U_{g_f, \mu_f} \equiv 0$) and $f_*\mu_f = \mu_f$, we have

$$C = \int_{\mathsf{P}^1} \Phi_{g_f}(f(\cdot), w)\mu_f = U_{g_f, f_*\mu_f}(w) = U_{g_f, \mu_f}(w) = 0.$$

This completes the proof of (4.1) in the case $\mathcal{S} = w \in \mathbb{P}^1$.

Fix $\mathcal{S}_0 \in \mathsf{H}^1$. By the density of $\mathbb{P}^1$ in P^1, we can choose a sequence (w_n) in $\mathbb{P}^1$ tending to $\mathcal{S}_0$ as $n \to \infty$. Then $\lim_{n\to\infty} f^*\delta_{w_n} = f^*\delta_{\mathcal{S}_0}$ weakly on P^1 and, for every $n \in \mathbb{N}$, applying (4.1) to $\mathcal{S} = w_n \in \mathbb{P}^1$, we have $\Phi_{g_f}(f(\cdot), w_n) = U_{g_f, f^*\delta_{w_n}}(\cdot)$ on P^1. Hence, for each $\mathcal{S}' \in \mathsf{H}^1$, by the continuity of both $\Phi_{g_f}(f(\mathcal{S}'), \cdot)$ and $\Phi_{g_f}(\mathcal{S}', \cdot)$ on P^1, we have

$$\Phi_{g_f}(f(\mathcal{S}'),\mathcal{S}_0)=\lim_{n\to\infty}\Phi_{g_f}(f(\mathcal{S}'),w_n)=\lim_{n\to\infty}U_{g_f,f^*\delta_{w_n}}(\mathcal{S}')=U_{g_f,f^*\delta_{\mathcal{S}_0}}(\mathcal{S}').$$

This completes the proof of (4.1) by the density of H^1 in P^1 and the continuity of (the exp of) both $\Phi_{g_f}(f(\cdot),\mathcal{S}_0)$ and $U_{g_f,f^*\delta_{\mathcal{S}_0}}(\cdot)$ on P^1. □

The following computation is an application of Lemma 4.1. We include a proof of it although it will not be used in this article.

Lemma 4.2 (Riesz's decomposition for the fixed points divisor on $\mathbb{P}^1$)

$$\Phi(f,\mathrm{Id}_{\mathbb{P}^1})_{g_f}=U_{g_f,[f=\mathrm{Id}_{\mathbb{P}^1}]}\quad on\ \mathsf{P}^1. \tag{4.2}$$

Proof Fix a lift F of f normalized so that $g_F=g_f$ on P^1. Choose a sequence $(q_j)_{j=1}^{d+1}$ in $K^2\setminus\{(0,0)\}$ so that $(F\wedge\mathrm{Id}_{\mathbb{P}^1})(p_0,p_1)\in K[p_0,p_1]_{d+1}$ factors as $(F\wedge\mathrm{Id}_{\mathbb{P}^1})(p_0,p_1)=\prod_{j=1}^{d+1}((p_0,p_1)\wedge q_j)$ in $K[p_0,p_1]$, which with (3.1) implies

$$\Phi(f,\mathrm{Id}_{\mathbb{P}^1})_{g_f}-U_{g_f,[f=\mathrm{Id}_{\mathbb{P}^1}]}\equiv\sum_{j=1}^{d+1}(g_f(\pi(q_j))+\log\|q_j\|)=:C$$

on $\mathbb{P}^1$, and in turn on P^1 by the density of $\mathbb{P}^1$ in P^1 and the continuity of (the exp of) both sides on P^1. Integrating both sides against μ_f over P^1, since $\int_{\mathsf{P}^1}U_{g_f,[f=\mathrm{Id}_{\mathbb{P}^1}]}\mu_f=\int_{\mathsf{P}^1}U_{g_f,\mu_f}[f=\mathrm{Id}_{\mathbb{P}^1}]=0$ (by $U_{g_f,\mu_f}\equiv 0$), we have $C=\int_{\mathsf{P}^1}\Phi(f,\mathrm{Id}_{\mathbb{P}^1})_{g_f}\mu_f$, so that we first have

$$\Phi(f,\mathrm{Id}_{\mathbb{P}^1})_{g_f}=U_{g_f,[f=\mathrm{Id}_{\mathbb{P}^1}]}+\int_{\mathsf{P}^1}\Phi(f,\mathrm{Id}_{\mathbb{P}^1})_{g_f}\mu_f\quad\text{on }\mathsf{P}^1.$$

Fix $z_0\in\mathbb{P}^1\setminus(\mathrm{supp}[f=\mathrm{Id}_{\mathbb{P}^1}])$. Using the above equality twice, by $f_*[f=\mathrm{Id}_{\mathbb{P}^1}]=[f=\mathrm{Id}_{\mathbb{P}^1}]$ on P^1 and (4.1), we have

$$\begin{aligned}
&\Phi_{g_f}(f(z_0),z_0)-\int_{\mathsf{P}^1}\Phi(f,\mathrm{Id}_{\mathbb{P}^1})_{g_f}\mu_f\\
=&U_{g_f,[f=\mathrm{Id}_{\mathbb{P}^1}]}(z_0)=U_{g_f,f_*[f=\mathrm{Id}_{\mathbb{P}^1}]}(z_0)=\int_{\mathsf{P}^1}\Phi_{g_f}(z_0,\cdot)(f_*[f=\mathrm{Id}_{\mathbb{P}^1}])(\cdot)\\
=&\int_{\mathsf{P}^1}\Phi_{g_f}(z_0,f(\cdot))[f=\mathrm{Id}_{\mathbb{P}^1}](\cdot)=\int_{\mathsf{P}^1}U_{g_f,f^*\delta_{z_0}}[f=\mathrm{Id}_{\mathbb{P}^1}]\\
=&\int_{\mathsf{P}^1}U_{g_f,[f=\mathrm{Id}_{\mathbb{P}^1}]}(f^*\delta_{z_0})=\int_{\mathsf{P}^1}\Big(\Phi(f,\mathrm{Id}_{\mathbb{P}^1})_{g_f}-\int_{\mathsf{P}^1}\Phi(f,\mathrm{Id}_{\mathbb{P}^1})_{g_f}\mu_f\Big)(f^*\delta_{z_0})\\
=&\int_{\mathsf{P}^1}\Phi(f,\mathrm{Id}_{\mathbb{P}^1})_{g_f}(f^*\delta_{z_0})-d\cdot\int_{\mathsf{P}^1}\Phi(f,\mathrm{Id}_{\mathbb{P}^1})_{g_f}\mu_f,
\end{aligned}$$

and moreover, $\int_{\mathsf{P}^1} \Phi(f, \mathrm{Id}_{\mathbb{P}^1})_{g_f}(f^*\delta_{z_0}) = U_{g_f, f^*\delta_{z_0}}(z_0) = \Phi_{g_f}(f(z_0), z_0)$ by (4.1). Hence $(d-1)\int_{\mathsf{P}^1} \Phi(f, \mathrm{Id}_{\mathbb{P}^1})_{g_f}\mu_f = 0$, and in turn since $d > 1$,

$$\int_{\mathsf{P}^1} \Phi(f, \mathrm{Id}_{\mathbb{P}^1})_{g_f}\mu_f = 0. \tag{4.3}$$

This completes the proof. □

From now on, we focus on the case where $\infty \in \mathsf{F}(f)$. We adopt the following convention when no confusion would be caused:

Convention For every probability Radon measure ν supported by $\mathsf{P}^1 \setminus \{\infty\}$, we denote $p_{\infty,\nu}$ and $I_{\infty,\nu}$ by p_ν and I_ν, respectively, for simplicity.

Since $\operatorname{supp}\mu_f = \mathsf{J}(f) \subset \mathsf{P}^1 \setminus D_\infty$, the equality (4.5) below implies that $\mathsf{P}^1 \setminus D_\infty$ is of logarithmic capacity > 0 with pole ∞.

Lemma 4.3 *Suppose that* $\infty \in \mathsf{F}(f)$. *Then*

$$p_{\mu_f} = g_f - \log[\cdot, \infty]_{\mathrm{can}} + \frac{I_{\mu_f}}{2} \quad \text{on } \mathsf{P}^1, \tag{4.4}$$

$$I_{\mu_f} = -2 \cdot g_f(\infty) > -\infty, \quad \text{and} \tag{4.5}$$

$$\Phi_{g_f}(\cdot, \infty) = -p_{\mu_f} + I_{\mu_f} \quad \text{on } \mathsf{P}^1. \tag{4.6}$$

Proof Suppose $\infty \in \mathsf{F}(f)$. Then we have $\operatorname{supp}\mu_f = \mathsf{J}(f) \subset \mathsf{P}^1 \setminus D_\infty$ and

$$0 = V_{g_f} = \int_{\mathsf{P}^1 \times \mathsf{P}^1} \Phi_{g_f}(\mu_f \times \mu_f) = I_{\mu_f} - 2 \cdot \int_{\mathsf{P}^1} (g_f - \log[\cdot, \infty]_{\mathrm{can}})\mu_f,$$

so that $I_{\mu_f} = 2 \cdot \int_{\mathsf{P}^1} (g_f - \log[\cdot, \infty]_{\mathrm{can}})\mu_f$, which with

$$0 \equiv U_{g_f, \mu_f} = p_{\mu_f} - (g_f - \log[\cdot, \infty]_{\mathrm{can}}) - \int_{\mathsf{P}^1} (g_f - \log[\cdot, \infty]_{\mathrm{can}})\mu_f \quad \text{on } \mathsf{P}^1$$

yields (4.4). By (4.4) and $\log[z, \infty] = \log[z, 0] - \log|z|$ on $\mathbb{P}^1 \setminus \{\infty\}$, we have

$$g_f(\infty) = \lim_{z \to \infty} \big((p_{\mu_f}(z) - \log|z|) + \log[z, 0]\big) - \frac{I_{\mu_f}}{2} = -\frac{I_{\mu_f}}{2},$$

so that (4.5) holds. By (4.4) and (4.5), we have $\Phi_{g_f}(\cdot, \infty) = \log[\cdot, \infty]_{\mathrm{can}} - g_f - g_f(\infty) = (-p_{\mu_f} + I_{\mu_f}/2) + I_{\mu_f}/2 = -p_{\mu_f} + I_{\mu_f}$ on P^1, so (4.6) also holds. □

Let $F = (F_0, F_1) \in (K[p_0, p_1]_d)^2$ be a normalized lift of f, and $c_0^F, c_1^F \in K \setminus \{0\}$ be the coefficients of the maximal degree terms of $F_0(1, z), F_1(1, z) \in K[z]$, respectively. No matter whether $\infty \in \mathsf{F}(f)$, by the equality $[z, \infty] = 1/\|(1, z)\|$ on $\mathbb{P}^1$ and the definition of T_F, we have

$$T_F = -\log[f(\cdot), \infty]_{\mathrm{can}} + \log|F_0(1, \cdot)|_\infty + d \cdot \log[\cdot, \infty]_{\mathrm{can}}$$

on $\mathbb{P}^1 \setminus (\{\infty\} \cup f^{-1}(\infty))$, and in turn on $\mathsf{P}^1 \setminus (\{\infty\} \cup f^{-1}(\infty))$ by the density of $\mathbb{P}^1$ in P^1 and the continuity of both sides on $\mathsf{P}^1 \setminus (\{\infty\} \cup f^{-1}(\infty))$. By (3.1), this equality is rewritten as

$$d \cdot (g_f - \log[\cdot, \infty]_{\mathrm{can}}) - (g_f \circ f - \log[f(\cdot), \infty]_{\mathrm{can}}) = \log |F_0(1, \cdot)|_\infty \tag{4.7}$$

on $\mathsf{P}^1 \setminus (\{\infty\} \cup f^{-1}(\infty))$.

Lemma 4.4 (Pullback formula for p_{μ_f} under f) *If* $\infty \in \mathsf{F}(f)$*, then*

$$\log |F_0(1, \cdot)|_\infty = d \cdot p_{\mu_f} - p_{\mu_f} \circ f - (d-1)\frac{I_{\mu_f}}{2} \tag{4.8}$$

on $\mathsf{P}^1 \setminus (\{\infty\} \cup f^{-1}(\infty))$*; moreover, for every* $\mathcal{S}' \in \mathsf{P}^1 \setminus \{\infty, f(\infty)\}$*,*

$$\begin{aligned} &p_{\mu_f}(\mathcal{S}') - \int_{\mathsf{P}^1 \setminus \{\infty\}} p_{\mu_f}(f^*\delta_{\mathcal{S}'}) + (d-1) I_{\mu_f} \\ &= - \int_{\mathsf{P}^1} \log |F_0(1, \cdot)|_\infty \frac{f^*\delta_{\mathcal{S}'}}{d} + (d-1)\frac{I_{\mu_f}}{2}, \end{aligned} \tag{4.9}$$

and similarly

$$\int_{\mathsf{P}^1 \setminus \{\infty\}} p_{\mu_f}(f^*\delta_\infty) - (d-1) I_{\mu_f} = -\log |c_0^F| - (d-1)\frac{I_{\mu_f}}{2}. \tag{4.10}$$

Proof Suppose $\infty \in \mathsf{F}(f)$. Then for every $\mathcal{S}' \in \mathsf{P}^1 \setminus \{\infty, f(\infty)\}$, by (4.7) and (4.4), we have (4.8). Integrating both sides in (4.8) against $f^*\delta_{\mathcal{S}'}/d$ over P^1, we have (4.9). Similarly, integrating both sides in (4.8) against μ_f over P^1, also by $f_*\mu_f = \mu_f$ and $I_{\mu_f} := \int_{\mathsf{P}^1} p_{\mu_f} \mu_f$, we have

$$\begin{aligned} \log |c_0^F| + \int_{\mathsf{P}^1 \setminus \{\infty\}} p_{\mu_f}(f^*\delta_\infty) &= \int_{\mathsf{P}^1} \log |F_0(1, \cdot)|_\infty \mu_f \\ &= d \cdot I_{\mu_f} - \int_{\mathsf{P}^1} (p_{\mu_f} \circ f)\mu_f - (d-1)\frac{I_{\mu_f}}{2} = (d-1)\frac{I_{\mu_f}}{2}, \end{aligned}$$

so (4.10) also holds. □

If $f(\infty) = \infty$, then $F(0,1) = (0, c_1^F)$, so that by the homogeneity of F, for every $n \in \mathbb{N}$, $F^n(0,1) = (0, (c_1^F)^{(d^n-1)/(d-1)})$ and that

$$g_f(\infty) = \lim_{n\to\infty} \frac{T_{F^n}(\infty)}{d^n} = \lim_{n\to\infty} \frac{\log \|F^n(0,1)\|}{d^n} - \log \|(0,1)\| = \frac{\log |c_1^F|}{d-1}.$$

Lemma 4.5 *If $f(\infty)=\infty\in\mathsf{F}(f)$, then*

$$I_{\mu_f}=-\frac{2}{d-1}\log|c_1^F| \tag{4.11}$$

and, for every $\mathcal{S}'\in\mathsf{P}^1$,

$$\int_{\mathsf{P}^1\setminus\{\infty\}} p_{\mu_f}(f^*\delta_{\mathcal{S}'})-(d-1)I_{\mu_f}=\begin{cases} p_{\mu_f}(\mathcal{S}') & \text{if } \mathcal{S}'\neq\infty,\\ \log\left|\dfrac{c_1^F}{c_0^F}\right| & \text{if } \mathcal{S}'=\infty.\end{cases} \tag{4.12}$$

Proof Suppose that $f(\infty)=\infty\in\mathsf{F}(f)$. Then by the above computation of $g_f(\infty)$ and (4.5), we have (4.11). Moreover, for every $\mathcal{S}'\in\mathsf{P}^1\setminus\{\infty\}$, using (4.6) twice and (4.1) (and the assumption $f(\infty)=\infty$), we compute

$$\begin{aligned}-p_{\mu_f}(\mathcal{S}')+I_{\mu_f}&=\Phi_{g_f}(\infty,\mathcal{S}')=\Phi_{g_f}(f(\infty),\mathcal{S}')\\ &=\int_{\mathsf{P}^1}\Phi_{g_f}(\infty,\cdot)(f^*\delta_{\mathcal{S}'})=-\int_{\mathsf{P}^1}p_{\mu_f}(f^*\delta_{\mathcal{S}'})+d\cdot I_{\mu_f},\end{aligned}$$

so (4.12) holds for $\mathcal{S}'\in\mathsf{P}^1\setminus\{\infty\}$. Finally, (4.12) for $\mathcal{S}'=\infty$ holds by (4.10) and (4.11). □

Let us now focus on $\nu_\infty=\nu_{\infty,\mathsf{P}^1\setminus D_\infty}$ when $\infty\in\mathsf{F}(f)$. Then $f(\infty)\in\mathsf{F}(f)$ and, since $\operatorname{supp}\nu_\infty\subset\partial D_\infty\subset\mathsf{J}(f)=\operatorname{supp}\mu_f$, we have

$$\operatorname{supp}(f_*\nu_\infty)\subset f(\mathsf{J}(f))=\mathsf{J}(f)=\operatorname{supp}\mu_f\subset\mathsf{P}^1\setminus D_\infty.$$

Lemma 4.6 *Suppose that $\infty\in\mathsf{F}(f)$. Then for every $\mathcal{S}'\in\mathsf{P}^1\setminus\{\infty,f(\infty)\}$,*

$$\begin{aligned}&p_{f_*\nu_\infty}(\mathcal{S}')-\int_{\mathsf{P}^1}p_{\nu_\infty}(f^*\delta_{\mathcal{S}'})+d\cdot I_{\nu_\infty}-\int_{\mathsf{P}^1}(p_{f_*\nu_\infty})\mu_f\\ &\quad=p_{\mu_f}(\mathcal{S}')-\int_{\mathsf{P}^1}p_{\mu_f}(f^*\delta_{\mathcal{S}'})+(d-1)I_{\mu_f}\end{aligned} \tag{4.13}$$

and, if in addition ν_∞ is invariant under f in that $f_*\nu_\infty=\nu_\infty$ on P^1, then

$$\begin{aligned}&p_{\nu_\infty}(\mathcal{S}')-\int_{\mathsf{P}^1}p_{\nu_\infty}(f^*\delta_{\mathcal{S}'})+(d-1)\cdot I_{\nu_\infty}\\ &\quad=p_{\mu_f}(\mathcal{S}')-\int_{\mathsf{P}^1}p_{\mu_f}(f^*\delta_{\mathcal{S}'})+(d-1)I_{\mu_f}.\end{aligned} \tag{4.14}$$

Proof Suppose that $\infty\in\mathsf{F}(f)$. Then for every $\mathcal{S}'\in\mathsf{P}^1\setminus\{\infty,f(\infty)\}$, using (4.4) repeatedly and (4.1), we have

$$p_{f_*\nu_\infty}(\mathcal{S}')=\int_{\mathsf{P}^1}\log|\mathcal{S}'-\cdot|_\infty(f_*\nu_\infty)=\int_{\mathsf{P}^1}\log|\mathcal{S}'-f(\cdot)|_\infty\nu_\infty$$

$$
\begin{aligned}
&= \int_{\mathsf{P}^1} \Big(\Phi_{g_f}(f(\cdot), \mathcal{S}') + \big(p_{\mu_f}(f(\cdot)) - \frac{I_{\mu_f}}{2}\big) + \big(p_{\mu_f}(\mathcal{S}') - \frac{I_{\mu_f}}{2}\big)\Big)\nu_\infty \\
&= \int_{\mathsf{P}^1} \Big(\int_{\mathsf{P}^1} \Phi_{g_f}(\cdot, \mathcal{S})(f^*\delta_{\mathcal{S}'})(\mathcal{S})\Big)\nu_\infty + \int_{\mathsf{P}^1}(p_{\mu_f} \circ f)\nu_\infty + p_{\mu_f}(\mathcal{S}') - I_{\mu_f} \\
&= \int_{\mathsf{P}^1} \Big(\int_{\mathsf{P}^1} \Big(\log|\mathcal{S} - \cdot|_\infty - \big(p_{\mu_f}(\mathcal{S}) - \frac{I_{\mu_f}}{2}\big) - \big(p_{\mu_f}(\cdot) - \frac{I_{\mu_f}}{2}\big)\Big)(f^*\delta_{\mathcal{S}'})(\mathcal{S})\Big)\nu_\infty \\
&\quad + \int_{\mathsf{P}^1}(p_{\mu_f} \circ f)\nu_\infty + p_{\mu_f}(\mathcal{S}') - I_{\mu_f} \\
&= \int_{\mathsf{P}^1} p_{\nu_\infty}(f^*\delta_{\mathcal{S}'}) + \int_{\mathsf{P}^1}(p_{\mu_f} \circ f - d \cdot p_{\mu_f})\nu_\infty \\
&\quad + p_{\mu_f}(\mathcal{S}') - \int_{\mathsf{P}^1} p_{\mu_f}(f^*\delta_{\mathcal{S}'}) + (d-1)I_{\mu_f}.
\end{aligned}
$$

Moreover, by Fubini's theorem and $p_{\nu_\infty} \equiv I_{\nu_\infty}$ on $\mathsf{P}^1 \setminus D_\infty$, we also have

$$
\begin{aligned}
&\int_{\mathsf{P}^1}(p_{\mu_f} \circ f - d \cdot p_{\mu_f})\nu_\infty \\
&\quad = \int_{\mathsf{P}^1} p_{\mu_f}(f_*\nu_\infty) - d \cdot \int_{\mathsf{P}^1} p_{\mu_f}\nu_\infty = \int_{\mathsf{P}^1}(p_{f_*\nu_\infty})\mu_f - d \cdot I_{\nu_\infty},
\end{aligned}
$$

which completes the proof of (4.13).

If in addition $f_*\nu_\infty = \nu_\infty$ on P^1, then by the identity $p_{\nu_\infty} \equiv I_{\nu_\infty}$ on $\mathsf{P}^1 \setminus (D_\infty \cup E)$, where E is an F_σ-subset in ∂D_∞ of logarithmic capacity 0 with pole ∞, and by the vanishing $\mu_f(E) = 0$ (from (4.5)), we also have

$$
\int_{\mathsf{P}^1}(p_{f_*\nu_\infty})\mu_f = \int_{\mathsf{P}^1}(p_{\nu_\infty})\mu_f = I_{\nu_\infty}, \tag{4.15}
$$

which completes the proof of (4.14). □

Lemma 4.7 (Invariance of ν_∞ under f) *If $f(\infty) = \infty \in \mathsf{F}(f)$, then $f_*\nu_\infty = \nu_\infty$ on P^1 and, for every $\mathcal{S}' \in \mathsf{P}^1$,*

$$
\int_{\mathsf{P}^1 \setminus \{\infty\}} p_{\nu_\infty}(f^*\delta_{\mathcal{S}'}) - (d-1)I_{\nu_\infty} = \begin{cases} p_{\nu_\infty}(\mathcal{S}') & \text{if } \mathcal{S}' \neq \infty, \\ \log\left|\dfrac{c_1^F}{c_0^F}\right| & \text{if } \mathcal{S}' = \infty. \end{cases} \tag{4.16}
$$

Proof Suppose that $f(\infty) = \infty \in \mathsf{F}(f)$. Then for every $\mathcal{S}' \in \mathsf{P}^1 \setminus \{\infty\}$, by (4.13) and (4.12), we have

$$
p_{f_*\nu_\infty}(\mathcal{S}') = \int_{\mathsf{P}^1} p_{\nu_\infty}(f^*\delta_{\mathcal{S}'}) - d \cdot I_{\nu_\infty} + \int_{\mathsf{P}^1}(p_{f_*\nu_\infty})\mu_f. \tag{4.13′}
$$

We claim that

$$p_{f_*\nu_\infty} \equiv \int_{\mathsf{P}^1} (p_{f_*\nu_\infty})\mu_f \quad \text{on } \mathsf{J}(f); \tag{4.17}$$

for, by the equality (4.13') and $p_{\nu_\infty} \geq I_{\nu_\infty}$ on P^1 (and Fubini's theorem and (4.4)), we have

$$p_{f_*\nu_\infty} \geq \int_{\mathsf{P}^1} (p_{f_*\nu_\infty})\mu_f > -\infty \quad \text{on } \mathsf{P}^1 \setminus \{\infty\},$$

so that $p_{f_*\nu_\infty} \equiv \int_{\mathsf{P}^1} p_{\mu_f}(f_*\nu_\infty)\, \mu_f$-a.e. on P^1. Hence the claim follows by the strong upper semicontinuity (2.1) of $p_{f_*\nu_\infty}$ on P^1 and $\mathsf{J}(f) = \operatorname{supp} \mu_f$, also recalling Remark 3.2.

Once the identity (4.17) is at our disposal, using also the maximum principle for the subharmonic function $p_{f_*\nu_\infty}$ and the latter inequality in (2.2), we have

$$p_{f_*\nu_\infty} \equiv \int_{\mathsf{P}^1} (p_{f_*\nu_\infty})\mu_f = \sup_{\mathsf{J}(f)} p_{f_*\nu_\infty} \geq \sup_{\mathsf{P}^1 \setminus D_\infty} p_{f_*\nu_\infty} \geq I_{\nu_\infty} \quad \text{on } \mathsf{J}(f),$$

and integrating both sides of this inequality against $f_*\nu_\infty$, we have $I_{f_*\nu_\infty} \geq I_{\nu_\infty}$ or equivalently

$$f_*\nu_\infty = \nu_\infty \quad \text{on } \mathsf{P}^1.$$

Then (4.16) holds for every $\mathcal{S}' \in \mathsf{P}^1 \setminus \{\infty\}$ by (4.14) and (4.12). Finally, integrating both sides in (4.8) against ν_∞ over P^1, by (4.15) and Fubini's theorem, we compute

$$\begin{aligned}
\log|c_0^F| + \int_{\mathsf{P}^1 \setminus \{\infty\}} p_{\nu_\infty}(f^*\delta_\infty) &= \int_{\mathsf{P}^1} \log|F_0(1, \cdot)|_\infty \nu_\infty \\
&= d \cdot I_{\nu_\infty} - \int_{\mathsf{P}^1} (p_{\mu_f} \circ f)\nu_\infty - (d-1)\frac{I_{\mu_f}}{2} \\
&= d \cdot I_{\nu_\infty} - \int_{\mathsf{P}^1} (p_{f_*\nu_\infty})\mu_f - (d-1)\frac{I_{\mu_f}}{2} = (d-1)I_{\nu_\infty} - (d-1)\frac{I_{\mu_f}}{2},
\end{aligned}$$

which with (4.11) yields (4.16) for $\mathcal{S}' = \infty$. □

Remark 4.8 All the computations in this Section are also valid for $K = \mathbb{C}$.

Remark 4.9 The f-invariance of ν_∞ in Lemma 4.7 is a non-archimedean counterpart to Mañé and da Rocha [22, p. 253, before Corollary 1]. Their argument was based on solving Dirichlet problem using the Poisson kernel on $D_\infty \cup \partial D_\infty$. A similar machinery has been only partly developed in the potential theory on P^1 (see [5, Sects. 7.3, 7.6]).

5 Proof of Theorem 1

Let $f \in K(z)$ be a rational function of degree $d > 1$, and $F = (F_0, F_1) \in (K[p_0, p_1]_d)^2$ be a normalized lift of f. When $\infty \in \mathsf{F}(f)$, let us still denote $\nu_{\mathsf{P}^1\setminus D_\infty} = \nu_{\infty,\mathsf{P}^1\setminus D_\infty}$ by ν_∞ for simplicity. If $\mu_f = \nu_\infty$ on P^1, then not only $p_{\mu_f} = p_{\nu_\infty} > I_{\nu_\infty} = I_{\mu_f}$ on D_∞ but, by the continuity of p_{μ_f} on $\mathsf{P}^1 \setminus \{\infty\}$ (by (4.4)), also $p_{\mu_f} = p_{\nu_\infty} \equiv I_{\nu_\infty} = I_{\mu_f}$ on $\mathsf{P}^1 \setminus D_\infty$.

Suppose that $\infty \in \mathsf{F}(f)$, $f(D_\infty) = D_\infty$ (so $D_\infty \subset f^{-1}(D_\infty)$), and $\mu_f = \nu_\infty$ on P^1. Then by (4.8) and $p_{\mu_f} \equiv I_{\mu_f}$ on $\mathsf{P}^1 \setminus D_\infty$, we have

$$\log |F_0(1, \cdot)|_\infty \equiv (d-1)\frac{I_{\mu_f}}{2} =: I_0 \quad \text{on } \mathsf{P}^1 \setminus f^{-1}(D_\infty). \tag{5.1}$$

Let $\mathcal{S}_0$ be the point in H^1 represented by the disk $\{z \in K : |z| \le e^{I_0}\}$ in K.

Suppose also that $f^{-1}(D_\infty) \setminus D_\infty \neq \emptyset$. Then $\deg F_0(1, z) > 0$. The subset

$$U_\infty := \{\mathcal{S} \in \mathsf{P}^1 : |F_0(1, \mathcal{S})|_\infty > e^{I_0}\}$$

in P^1 is the component of $\mathsf{P}^1 \setminus (F_0(1, \cdot))^{-1}(\mathcal{S}_0)$ containing ∞, and $\partial U_\infty = (F_0(1, \cdot))^{-1}(\mathcal{S}_0)$. By (5.1), we have $U_\infty \subset f^{-1}(D_\infty)$, and in turn

$$U_\infty \subset D_\infty.$$

For every $w \in f^{-1}(\infty) \setminus \{\infty\} = (F_0(1, \cdot))^{-1}(0) \subset \{\mathcal{S} \in \mathsf{P}^1 : |F_0(1, \mathcal{S})|_\infty < e^{I_0}\}$, let D_w (resp. U_w) be the component of $f^{-1}(D_\infty)$ (resp. the component of $\{\mathcal{S} \in \mathsf{P}^1 : |F_0(1, \mathcal{S})|_\infty < e^{I_0}\}$) containing w. Then U_w is the component of $\mathsf{P}^1 \setminus (F_0(1, \cdot))^{-1}(\mathcal{S}_0)$ containing w, and ∂U_w is a singleton in $(F_0(1, \cdot))^{-1}(\mathcal{S}_0) = \partial U_\infty$. For every $w \in f^{-1}(\infty) \cap D_\infty$, $D_w = D_\infty$.

We claim that ∂D_∞ is a singleton say $\{\mathcal{S}_\infty\}$ in H^1 and, moreover, that for every $w \in f^{-1}(\infty) \setminus D_\infty (\neq \emptyset$ under the assumption that $f^{-1}(D_\infty) \setminus D_\infty \neq \emptyset)$,

$$\partial D_w = \partial D_\infty (= \{\mathcal{S}_\infty\});$$

indeed, for every $w \in f^{-1}(\infty) \setminus D_\infty$, we not only have $D_w \subset U_w$ (since otherwise, we must have $\emptyset \neq D_w \cap U_\infty \subset D_w \cap D_\infty$ so $D_w = D_\infty$, which contradicts $w \notin D_\infty$) but also $U_w \subset D_w$ (by (5.1)), so that $U_w = D_w$. This together with $\partial U_w \subset \partial U_\infty$ and $U_\infty \subset D_\infty$ yields

$$\partial D_w = \partial U_w \subset \partial D_\infty$$

(since otherwise, we must have $\emptyset \neq U_w \cap D_\infty = D_w \cap D_\infty$ so $D_w = D_\infty$, which contradicts $w \notin D_\infty$). Hence the claim holds since $f(\partial U_w) = f(\partial D_w) = \partial D_\infty$ is a singleton in H^1.

Once the claim is at our disposal, we compute

$$f^{-1}(\{\mathcal{S}_\infty\}) = f^{-1}(\partial D_\infty) \subset \bigcup_{w \in f^{-1}(\infty)} \partial D_w$$
$$= \Big(\bigcup_{w \in f^{-1}(\infty) \cap D_\infty} \partial D_w\Big) \cup \Big(\bigcup_{w \in f^{-1}(\infty) \setminus D_\infty} \partial D_w\Big) = \{\mathcal{S}_\infty\} \cup \{\mathcal{S}_\infty\} = \{\mathcal{S}_\infty\},$$

so f has a potential good reduction. □

6 Proof of Theorem 2

Pick a prime number p, and let us denote $|\cdot|_p$ by $|\cdot|$ for simplicity. Set

$$f(z) := \frac{z^p - z}{p} \in \mathbb{Q}[z] \quad \text{and} \quad A(z) := \frac{az+b}{cz+d} \in \mathrm{PGL}(2, \mathbb{Z}_p).$$

If $|c| < 1$, then $|ad - bc| = |ad| = 1$, so that $|a| = |d| = 1$.

Let $\mathsf{J}(f \circ A)$ and $\mathsf{F}(f \circ A)$ denote the Berkovich Julia and Fatou sets in $\mathsf{P}^1(\mathbb{C}_p)$ of $f \circ A$ as an element of $\mathbb{C}_p(z)$ of degree p, respectively.

6.1 Computing $\mathsf{J}(f \circ A)$

The fact that $\mathsf{J}(f)$ coincides with the classical Julia set of f (see Remark 3.1), which is $\mathbb{Z}_p$, is well known (see e.g., [17, Example 4.11], [6, Example 5.30]). In this subsection, more general facts will be established.

Lemma 6.1 *If $|c| < 1$, then $(f \circ A)^{-1}(\mathbb{Z}_p) = \mathbb{Z}_p$.*

Proof We first claim that for every $z \in \mathbb{Z}$, $p \cdot f(z) = z^p - z \equiv 0$ modulo $p\mathbb{Z}$; indeed, when is obvious if $z = 0$ modulo $p\mathbb{Z}$, and is the case by Fermat's Little Theorem when $z \neq 0$ modulo $p\mathbb{Z}$. By this claim, we have $f(\mathbb{Z}) \subset \mathbb{Z}$ (cf. [34]), and in turn $f(\mathbb{Z}_p) \subset \mathbb{Z}_p$ by the continuity of the action of f on $\mathbb{Q}_p$ and the density of $\mathbb{Z}$ in $\mathbb{Z}_p$. Next, we claim that $f^{-1}(\mathbb{Z}_p) \subset \mathbb{Z}_p$ or equivalently that for every $w \in \mathbb{Z}_p$, $f^{-1}(w) \subset \mathbb{Z}_p$; indeed, setting $W(X) := X^p - X - pw \in \mathbb{Z}_p[X]$ of degree p, we have already seen that the reduction $\overline{W}(X) = X^p - X \in \mathbb{F}_p[X]$ of W modulo $p\mathbb{Z}_p$ has p distinct roots $\overline{0}, \ldots, \overline{p-1}$ in $\mathbb{F}_p$. Hence by Hensel's lemma (see, e.g., [24, Corollary 1 in Sect. 5.1], [8, Sect. 3.3.4, Proposition 3]), $W(X)$ also has p distinct roots in $\mathbb{Z}_p$, and has no other roots in $\overline{\mathbb{Q}_p}$, so the claim holds. We have seen that $f^{-1}(\mathbb{Z}_p) = \mathbb{Z}_p$.

Suppose now that $|c| < 1$. Then for every $z \in \mathbb{Z}_p$, we have $|cz| < 1 = |d|$, so that $|A(z)| = |az+b|/|cz+d| = |az+b| \leq 1$. Hence $A(\mathbb{Z}_p) \subset \mathbb{Z}_p$, and similarly $A^{-1}(\mathbb{Z}_p) \subset \mathbb{Z}_p$ since $A^{-1}(z) = (dz-b)/(-cz+a) \in \mathrm{PGL}(2, \mathbb{Z}_p)$ and $|-c| = |c| < 1$. Now we conclude that $(f \circ A)^{-1}(\mathbb{Z}_p) = A^{-1}(\mathbb{Z}_p) = \mathbb{Z}_p$. □

Lemma 6.2 *If $|b| \ll 1$ and $|c| \ll 1$, then $f \circ A$ has an attracting fixed point z_A in $\mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{Z}_p$, which tends to ∞ as $(a, b, c, d) \to (1, 0, 0, 1)$ in $(\mathbb{Z}_p)^4$. Moreover, if in addition $c \neq 0$, then $z_A \in \mathbb{C}_p \setminus \mathbb{Z}_p$ and $(f \circ A)^{-1}(z_A) \neq \{z_A\}$.*

Proof Since $f^{-1}(\infty) = \{\infty\}$ and $\deg f = p > 1$, the former assertion holds also noting that $(\mathrm{Id}_{\mathbb{P}^1(\mathbb{C}_p)})' \equiv 1 \neq 0$ and applying an implicit function theorem to the equation $(f \circ A)(z) = z$ near $(z, a, b, c, d) = (\infty, 1, 0, 0, 1)$ in $\mathbb{P}^1(\mathbb{C}_p) \times (\mathbb{Z}_p)^4$ (see, e.g., [1, (10.8)]). Moreover, since $f'(z) = z^{p-1} - p^{-1}$ and $f''(z) = (p-1)z^{p-2}$, the point $A^{-1}(\infty) = -d/c$ is the unique point $z \in \mathbb{P}^1(\mathbb{C}_p)$ such that $\deg_z(f \circ A) = p(= \deg(f \circ A))$, and on the other hand, if in addition $c \neq 0$, then the point $A^{-1}(\infty)$ is $\neq \infty$ and is not fixed by $f \circ A$. Hence the latter assertion holds also noting that $(f \circ A)(\infty) \neq \infty$ if in addition $c \neq 0$. □

Consequently, if $|b| \ll 1$ and $|c| \ll 1$, then

$$\mathsf{J}(f \circ A) = \mathbb{Z}_p = \mathsf{P}^1(\mathbb{C}_p) \setminus D_{z_A}(f \circ A); \tag{6.1}$$

indeed, by Lemma 6.1 (and (3.2)), if $|c| < 1$, then $\mathsf{J}(f \circ A) \subset \mathbb{Z}_p$. If in addition $|b| \ll 1$ and $|c| \ll 1$, then by Lemma 6.2 (and $\mathbb{Z}_p \subset \mathbb{C}_p$), we have $\mathsf{F}(f \circ A) = D_{z_A}(f \circ A)$, which is an (immediate) attractive basin of f (see [31, Théorème de Classification]) associated with $z_A \in \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{Z}_p$, and in turn have $\mathsf{J}(f \circ A) = \mathbb{Z}_p$ since $(f \circ A)(\mathbb{Z}_p) \subset \mathbb{Z}_p$ by Lemma 6.1.

6.2 Computing Energies and Measures

Since

$$\mathrm{Res}\big(p^{1/2} \cdot \big(z_0^p, z_0^p f(z_1/z_0)\big)\big) = (p^{1/2})^{2p} \cdot (1^{p-p} \cdot (p^{-1})^{p-0} \cdot 1) = 1,$$

the pair

$$F(z_0, z_1) := p^{1/2} \cdot \big(z_0^p, z_0^p f(z_1/z_0)\big) \in (\mathbb{Q}[z_0, z_1]_p)^2$$

is a normalized lift of f. Noting that $|\mathrm{Res}(az_0+bz_1, cz_0+dz_1)| = |ad-bc| = 1$ and using a formula for the homogeneous resultant of the composition of homogeneous polynomial maps (see, e.g., [32, Exercise 2.12]), we also have $\big|\mathrm{Res}\big(F(az_0+bz_1, cz_0+dz_1)\big)\big| = \big|(\mathrm{Res}\, F)^1 \cdot (\mathrm{Res}(az_0+bz_1, cz_0+dz_1))^{p^2}\big| = 1$, so that

$$\begin{aligned} F_A(z_0, z_1) &:= F(az_0 + bz_1, cz_0 + dz_1) \\ &= p^{1/2} \cdot \left((az_0 + bz_1)^p, \frac{(cz_0 + dz_1)^p - (az_0 + bz_1)^{p-1}(cz_0 + dz_1)}{p}\right) \\ &\in (\mathbb{Q}_p[z_0, z_1]_p)^2 \end{aligned}$$

is a normalized lift of $f \circ A$. For every $n \in \mathbb{N}$, write

$$(F_A)^n = \big(F_{A,0}^{(n)}, F_{A,1}^{(n)}\big) \in (\mathbb{Q}_p[z_0, z_1]_{p^n})^2.$$

Lemma 6.3 *If* $|b| < 1$ *and* $|c| < 1$, *then*

$$g_{f\circ A}(\infty)\Big(=\sum_{j=1}^{\infty}\Big(\frac{\log\|(F_A)^j(0,1)\|}{p^j}-\frac{\log\|(F_A)^{j-1}(0,1)\|}{p^{j-1}}\Big)\Big)=\frac{\log p}{2(p-1)}.$$

Proof Suppose that $|b| < 1$ and $|c| < 1$(, and recall $|p| = p^{-1} < 1$). Then for every $(z_0, z_1) \in \mathbb{C}_p^2$, if $|z_0| < |z_1|$, then

$$|cz_0 + dz_1| = |dz_1| = |z_1| > \max\{|az_0|, |bz_1|\} \geq |az_0 + bz_1|$$

so

$$\begin{aligned}&|F^{(1)}_{A,0}(z_0,z_1)| < |F^{(1)}_{A,1}(z_0,z_1)| \quad \text{and}\\ &\|F_A(z_0,z_1)\| = |F^{(1)}_{A,1}(z_0,z_1)| = p^{1/2}|cz_0+dz_1|^p\\ &\qquad = p^{1/2}|dz_1|^p = p^{1/2}|z_1|^p = p^{1/2}\|(z_0,z_1)\|^p.\end{aligned}$$

Hence inductively, for every $n \in \mathbb{N}$, we have $|F^{(n)}_{A,0}(0,1)| < |F^{(n)}_{A,1}(0,1)|$, and moreover

$$\begin{aligned}&\sum_{j=1}^{n}\Big(\frac{\log\|(F_A)^j(0,1)\|}{p^j}-\frac{\log\|(F_A)^{j-1}(0,1)\|}{p^{j-1}}\Big)=\sum_{j=1}^{n}\frac{\frac{1}{2}\log p}{p^j}\\ &=\Big(\frac{1}{2}\log p\Big)\frac{(1/p)(1-1/p^n)}{1-1/p}\to\Big(\frac{1}{2}\log p\Big)\frac{1}{p-1}\end{aligned}$$

as $n \to \infty$. □

Lemma 6.4 *If* (a, b, c, d) *is close enough to* $(1, 0, 0, 1)$ *in* $(\mathbb{Z}_p)^4$, *then*

$$\mu_{f\circ A} = \nu_{\infty,\mathbb{Z}_p} = \nu_{z_A,\mathbb{Z}_p} \quad \text{on } \mathsf{P}^1(\mathbb{C}_p).$$

Proof If $|b| \ll 1$ and $|c| \ll 1$, then by (6.1) and $\mathbb{Z}_p \subset \mathbb{C}_p$, we have

$$\infty \in \mathsf{F}(f\circ A) = D_{z_A}(f\circ A) = \mathsf{P}^1(\mathbb{C}_p)\setminus\mathbb{Z}_p.$$

Then by (4.5) and Lemma 6.3, we have

$$I_{\infty,\mu_{f\circ A}} = -2\cdot\left(\frac{\log p}{2(p-1)}\right) = \log p^{\frac{-1}{p-1}},$$

and in particular, recalling $\nu_{\infty,\mathbb{Z}_p} = \mu_f$ on $\mathsf{P}^1(\mathbb{C}_p)$, also $I_{\infty,\nu_{\infty,\mathbb{Z}_p}} = I_{\infty,\mu_f} = \log p^{\frac{-1}{p-1}}$ (for a non-dynamical and more direct computation of $I_{\infty,\nu_{\infty,\mathbb{Z}_p}}$, see [3]). Now the first equality holds by the uniqueness of the equilibrium mass distribution on the non-polar compact subset $\mathbb{Z}_p$ in $\mathsf{P}^1(\mathbb{C}_p)$. The second equality holds since z_A

tends to ∞ as $(a, b, c, d) \to (1, 0, 0, 1)$ in $(\mathbb{Z}_p)^4$ (by Lemma 6.2), also recalling Observation 2.4. □

Remark 6.5 If $0 < |c| \ll 1$ and $|b| \ll 1$, then $(f \circ A)(\infty) \neq \infty \in \mathsf{F}(f \circ A)$, $(f \circ A)(D_\infty(f \circ A)) = D_\infty(f \circ A)$, $\mathsf{J}(f \circ A) \not\subset \mathsf{H}^1$ (indeed $\mathsf{J}(f \circ A) \subset \mathbb{C}_p$), and $\mu_{f \circ A} = \nu_{\infty, \mathsf{P}^1 \setminus D_\infty}$ on P^1.

6.3 Conclusion

If $|b| \ll 1$ and $0 < |c| \ll 1$, then setting $m_A(z) := \frac{1}{z - z_A} \in \mathrm{PGL}(2, \mathbb{C}_p)$, the rational function

$$g_A := m_A \circ (f \circ A) \circ m_A^{-1} \in \mathbb{C}_p(z)$$

is of degree p and satisfies $g_A(\infty) = \infty$, $|g_A'(\infty)| < 1$, $g_A^{-1}(\infty) \neq \{\infty\}$, and $\infty \in m_A(D_{z_A}(f \circ A)) = D_\infty(g_A)$. If moreover (a, b, c, d) is close enough to $(1, 0, 0, 1)$ in $(\mathbb{Z}_p)^4$, then also recalling Observations 2.2 and 2.3, we have

$$\begin{aligned} \mu_{g_A} &= (m_A)_* \mu_{f \circ A} = (m_A)_* \nu_{\infty, \mathbb{Z}_p} = (m_A)_* \nu_{z_A, \mathbb{Z}_p} \\ &= (m_A)_* \nu_{z_A, \mathsf{P}^1 \setminus D_{z_A}(f \circ A)} = \nu_{\infty, \mathsf{P}^1 \setminus D_\infty(g_A)} \quad \text{on } \mathsf{P}^1(\mathbb{C}_p). \end{aligned}$$

Now the proof of Theorem 2 is complete. □

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