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RESEARCH CONTRIBUTION

Algebroidally Integrable Bodies

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Abstract
V. Arnold's problem 1987–14 from his Problems book asks whether there exist bodies with smooth boundaries in $\mathbb{R}^N$ (other than the ellipsoids in odd-dimensional spaces) for which the volume of the segment cut by any hyperplane from the body depends algebraically on the hyperplane. We present a series of very realistic candidates for the role of such bodies, and prove that the corresponding volume functions are at least algebroid, in particular their analytic continuations are finitely valued; to prove their algebraicity it remains to check the condition of finite growth.

Keywords Integral geometry · Monodromy · Analytic continuation · Picard–Lefschetz theory · Integrability · Stratified Morse theory

Mathematics Subject Classification 44A99 · 4403

1 Introduction

Any compact body with regular boundary in $\mathbb{R}^N$ defines a two-valued function on the space of affine hyperplanes: its values are the volumes of two parts into which the hyperplanes cut the body.

A body in $\mathbb{R}^N$ (and also the hypersurface bounding it) is called *algebraically integrable* if this function is algebraic, i.e. there is a non-zero polynomial F in $N+2$ variables such that $F(V, a_1, \ldots, a_N, b) = 0$ any time when V equals the volume cut (to some side) from this body by the hyperplane defined by the equation $a_1x_1 + \cdots + a_Nx_N + b = 0$.

Archimedes [2] has proved that spheres in $\mathbb{R}^3$ are algebraically integrable; today, it is easy to check this property also for all ellipsoids in odd-dimensional spaces.

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Newton ([10], Lemma XXVIII) has proved that bounded convex bodies with smooth boundaries in $\mathbb{R}^2$ never are algebraically integrable. V. Arnold asked (see [6], Problems 1987–14, 1988–13, and 1990–27, and also [5]) whether there exist algebraically integrable bodies with smooth boundaries other than collections of ellipsoids in odd-dimensional spaces. It was proved in [12] that there are no such bodies in even-dimensional spaces. The main obstruction to the integrability is the monodromy action on homology groups related to hyperplane sections of our hypersurfaces: this action controls the ramification of the analytic continuation of volume functions, and usually provides infinitely many values of such a continuation at one and the same hyperplane. This obstruction implies even a more strong assertion: the volume function of a body with smooth boundary in $\mathbb{R}^N$, N even, cannot be algebroid. It provides also many strong restrictions on the local geometry of algebraically integrable bodies for odd N (see [11]), which caused the conjecture that such bodies other than collections of ellipsoids do not exist.

We present below a new series of bodies in all spaces of odd dimensions greater than three, for which this obstruction vanishes: those are tubular neighborhoods of standard even-dimensional spheres (and also any bodies affine equivalent to them). It is extremely plausible that these bodies are algebraically integrable: we prove that the analytic continuations of their volume functions are algebroid (in particular, finitely valued), so it remains only to prove that they have finite order of growth at their singular points.

1.1 Main Results

Let n be odd, m be even, and $\varepsilon \in (0, 1)$. Consider the Euclidean space $\mathbb{R}^{n+m} \equiv \mathbb{R}^n \oplus \mathbb{R}^m$ with orthogonal coordinates $x_1, \dots, x_n$ in $\mathbb{R}^n$ and $y_1, \dots, y_m$ in $\mathbb{R}^m$, and the hypersurface in it defined by equation

$$\left(\sqrt{x_1^2 + \cdots + x_n^2} - 1\right)^2 + y_1^2 + \cdots + y_m^2 = \varepsilon^2. \tag{1}$$

Denote by $\mathcal{P}$ (respectively, by $\mathcal{P}^{\mathbb{C}}$) the space of all affine hyperplanes in $\mathbb{R}^{n+m}$ (respectively, in $\mathbb{C}^{n+m}$), and by $\mathcal{R}eg$ the subset in $\mathcal{P}$ consisting of hyperplanes transversal to the hypersurface (1). Identifying any real hyperplane with its complexification, we consider $\mathcal{P}$ as a subset of $\mathcal{P}^{\mathbb{C}}$.

Theorem 1 *For any $\varepsilon \in (0, 1)$, there is a four-valued analytic function on the space $\mathcal{P}^{\mathbb{C}}$ such that one of volumes cut by hyperplanes from the body bounded by hypersurface (1) in $\mathbb{R}^{n+m}$ coincides in any point $X \in \mathcal{P}$ with a sum of at most two (maybe zero) values of this function. The monodromy group of this function is the Klein four-group generated by permutations $\binom{1234}{2143}$ and $\binom{1234}{4321}$ of values.*

Conjecture 1 This four-valued analytic function is in fact algebraic.

Remark 1 If $n = 1$ or $m = 0$, then (1) defines a reducible hypersurface, and hence does not fit the Arnold's question.

Remark 2 Hypersurfaces defined by similar equations

$$\left(x_1^2 + \cdots + x_n^2 - 1\right)^2 + y_1^2 + \cdots + y_m^2 = \varepsilon^2 \tag{2}$$

with odd n and arbitrary m were considered in [14] as boundaries of *locally* algebraically integrable bodies. The hypersurface (1) has a smaller monodromy group of homology classes of hyperplane sections (which, in fact, controls the ramification of volume function) than (2), because it has singular points in the imaginary domain; these singularities "eat" parabolic points of the hypersurface, which cause a large part of the corresponding monodromy group.

Remark 3 It is important that the second fundamental form of the hypersurface (1) in $\mathbb{R}^{n+m}$ has only even inertia indices at all points where it is non-degenerate, cf. §3 in [13].

Remark 4 Are there any other examples of this sort? Some natural candidates are provided by the iteration of the previous construction: we take a flag of odd-dimensional spaces $\mathbb{R}^k \subset \mathbb{R}^n \subset \mathbb{R}^N$ and a tubular neighbourhood in $\mathbb{R}^N$ of the boundary of a tubular neighbourhood in $\mathbb{R}^n$ of the standard sphere $S^{k-1} \subset \mathbb{R}^k$.

Remark 5 Even if Conjecture 1 will be proved, one of Arnold's problems of this series will remain unsolved: it asks whether there are extra *convex* algebraically integrable bodies in $\mathbb{R}^N$, see [6], problem 1990–27. Also, we do not have good examples for $N = 3$.

Remark 6 See [1,7,9] for some related problems and results.

An obvious problem (whose solution would imply Conjecture 1) is to integrate explicitly in radicals the volume function of the body (1).

1.2 A Detalization of Theorem 1

Denote by W the body in $\mathbb{R}^{n+m}$ bounded by hypersurface (1); denote its volume by C_0, and the corresponding two-valued volume function on $\mathcal{P}$ by V_W. Denote by $\mathcal{P}_2$ (respectively, by $\mathcal{P}_2^{\mathbb{C}}$) the two-dimensional subspace in $\mathcal{P}$ (respectively, in $\mathcal{P}^{\mathbb{C}}$) consisting of all hyperplanes defined by equations of the form

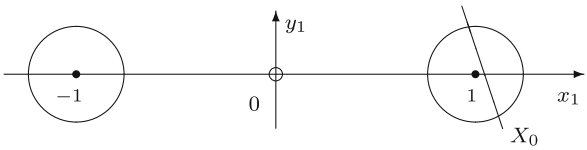
$$px_1 + qy_1 + r = 0 \tag{3}$$

with real (respectively, complex) coefficients p, q, and r; also denote by $\mathcal{R}eg_2$ the space $\mathcal{P}_2 \cap \mathcal{R}eg$.

The group $O(n) \times O(m)$ of independent rotations in $\mathbb{R}^n$ and $\mathbb{R}^m$ acts on the space $\mathcal{P}$. The volume function V_W is constant on the orbits of this action. Any such orbit contains points of the space $\mathcal{P}_2$. In particular, any hyperplane defined by equation

$$\alpha_1 x_1 + \cdots + \alpha_n x_n + \gamma_1 y_1 + \cdots + \gamma_m y_m = \beta, \tag{4}$$

Fig. 1 Planar slice of W



where not all coefficients α_j are equal to 0, can be reduced by this group to a hyperplane with equation

$$x_1 = ay_1 + c\ , \tag{5}$$

where

$$a = \frac{\sqrt{\gamma_1^2 + \cdots + \gamma_m^2}}{\sqrt{\alpha_1^2 + \cdots + \alpha_n^2}}, \quad c = \frac{|\beta|}{\sqrt{\alpha_1^2 + \cdots + \alpha_n^2}}\ , \tag{6}$$

cf. [14].

The space $\mathcal{R}eg_2$ consists of five connected components. Indeed, the coordinate plane $\mathbb{R}^2 \subset \mathbb{R}^{n+m}$ defined by conditions $x_2 = \cdots = x_n = y_2 = \cdots = y_m = 0$ intersects the boundary ∂W of the body W along two circles $(x_1 - 1)^2 + y_1^2 = \varepsilon^2$ and $(x_1 + 1)^2 + y_1^2 = \varepsilon^2$, see Fig. 1. For any hyperplane $X \in \mathcal{R}eg_2$, the line $X \cap \mathbb{R}^2$ can

(1) not intersect these circles leaving them to one side of it,
(2_l) , (2_r) intersect transversally only the left-hand (respectively, right-hand) circle,
(3) not intersect these circles but separate them, or
(4) intersect both of them.

Correspondingly, the space $\mathcal{R}eg$ splits into *four* components filled in by $(O(n) \times O(m))$-orbits of points from these components of $\mathcal{R}eg_2$; the components (2_l) and (2_r) of $\mathcal{R}eg_2$ define one and the same component (2) of $\mathcal{R}eg$. Then Theorem 1 has the following detalization.

Theorem 2 *There is a four-valued analytic function Ψ on the space $\mathcal{P}^{\mathbb{C}}$ such that*

1) *its restriction to $\mathcal{P}$ is $(O(n) \times O(m))$-invariant,*
2) *the sum of its four values is everywhere equal to C_0;*
3) *any value of the volume function V_W on any hyperplane $X \in \mathcal{R}eg$ coincides with*
   - *a value of this function Ψ, or C_0 less such a value, if X belongs to the domain* (2) *of $\mathcal{R}eg$,*
   - *the sum of some two values of Ψ if X belongs to one of domains* (3) *or* (4)*;*
4) *the monodromy group of this analytic function is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and is generated by permutations $\binom{1234}{2143}$ and $\binom{1234}{4321}$ of values;*
5) *the restriction of this function Ψ to the subspace $\mathcal{P}_2^{\mathbb{C}}$ splits into the product of two two-valued algebroid functions, one of which has the form*

$$\Phi(V, a, c) \equiv V^2 - S(a, c)V + P(a, c) \tag{7}$$

in coordinates a, c (see (5)), where S and P are single-valued analytic functions invariant under reflection $a \mapsto -a$, and the other one is equal to $\Phi(V, -a, -c)$.

The restrictions of the volume function V_W to domains (3) *and* (4) *coincide there with some two-valued analytic functions, whose two values can be obtained one from the other by analytic continuation along paths lying inside these domains. On the contrary, two leaves of the function V_W in the domain* (2) *of $\mathcal{R}eg$ are not analytic continuations of one another even through the complex space $\mathcal{P}^{\mathbb{C}}$.*

1.3 Scheme of the Proof of Theorem 2 (cf. [12])

Denote by A the hypersurface in $\mathbb{C}^{n+m}$ defined by Eq. (1) or by (obviously equivalent to it) equation

$$\left(x_1^2 + \cdots + x_n^2 + y_1^2 + \cdots + y_m^2 + (1 - \varepsilon^2)\right)^2 - 4(x_1^2 + \cdots + x_n^2) = 0\,. \tag{8}$$

Denote by $\mathcal{R}eg^{\mathbb{C}}$ the subset in $\mathcal{P}^{\mathbb{C}}$ consisting of all complex hyperplanes in $\mathbb{C}^{n+m}$ generic with respect to A (that is, of hyperplanes whose closures in the compactification $\mathbb{CP}^{n+m}$ of $\mathbb{C}^{n+m}$ are transversal to the stratified variety consisting of the hypersurface A and the "infinitely distant" plane $\mathbb{C}P^{n+m} \setminus \mathbb{C}^{n+m}$). Denote by $\mathcal{R}eg_2^{\mathbb{C}}$ the intersection $\mathcal{R}eg^{\mathbb{C}} \cap \mathcal{P}_2^{\mathbb{C}}$.

By Thom isotopy lemma (see, e.g. [8]), groups $H_{n+m}(\mathbb{C}^{n+m}, A \cup X)$ are isomorphic to one another for all $X \in \mathcal{R}eg^{\mathbb{C}}$; moreover, they form a covering over $\mathcal{R}eg^{\mathbb{C}}$ with canonical flat connection (Gauss–Manin connection, see, e.g. [3,11]). This connection naturally defines the monodromy action $\pi_1(\mathcal{R}eg^{\mathbb{C}}, X) \to \mathrm{Aut}(H_*(\mathbb{C}^{n+m}, A \cup X))$. Integrals of the volume form

$$\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge dy_1 \wedge \cdots \wedge \mathrm{d}y_m \tag{9}$$

along the elements of all these homology groups are well defined and provide linear functions $H_{n+m}(\mathbb{C}^{n+m}, A \cup X; \mathbb{C}) \to \mathbb{C}$.

Let X_0 be a distinguished point in $\mathcal{R}eg^{\mathbb{C}}$, and Ξ an element of the group

$$H_{n+m}(\mathbb{C}^{n+m}, A \cup X_0)\,. \tag{10}$$

The pair (X_0, Ξ) defines an analytic function on $\mathcal{R}eg^{\mathbb{C}}$: the value of its continuation along a path l in $\mathcal{R}eg^{\mathbb{C}}$ connecting X_0 with some point X' is equal to the integral of the form (9) along the cycle in $H_{n+m}(\mathbb{C}^{n+m}, A \cup X')$ obtained from Ξ by the Gauss–Manin connection over our path l.

If $X_0 \in \mathcal{R}eg$ and Ξ is the homology class of a part of the body W cut from it by hyperplane X_0, then in a neighbourhood of the point X_0 in $\mathcal{P}$, this function coincides with the volume function; hence, the analytic continuations of these two functions into the complex domain also coincide. The ramification of the analytic continuation of the volume function is thus controlled by the monodromy action of the group $\pi_1(\mathcal{R}eg^{\mathbb{C}}, X_0)$ on the group (10).

In particular, suppose that this hyperplane X_0 belongs to $\mathcal{R}eg_2$ and is sufficiently close to the hyperplane given by equation $x_1 = 1$ (but is not equal to it, because the latter hyperplane is not generic "at infinity"), so that the line $X_0 \cap \mathbb{R}^2$ intersects the right-hand circle of ∂W only, see Fig. 1. It is easy to see that the component of $W \backslash X_0$ containing the piece of $\mathbb{R}^2 \cap W$ placed to the right from X_0 in Fig. 1 is a *vanishing cycle* in the group (10); it contracts to a point when we move the plane X_0 to the right parallel to itself until the tangency with ∂W. We prove below that the orbit of this element of (10) under the action of the group $\pi_1(\mathcal{R}eg^{\mathbb{C}}, X_0)$ consists of four elements, and this orbit splits into two two-element orbits of the action of the group $\pi_1(\mathcal{R}eg_2^{\mathbb{C}}, X_0)$. Therefore, the analytic continuation of the corresponding volume function from a neighbourhood of X_0 to entire $\mathcal{R}eg^{\mathbb{C}}$ (respectively, to $\mathcal{R}eg_2^{\mathbb{C}}$) is four-valued (respectively, two-valued).

These two analytic continuations are exactly the functions Ψ and Φ promised in Theorem 2. The coefficients $S(a, c)$ and $P(a, c)$ in (7) are, respectively, the sum and the product of both values of the second analytic continuation at the point $X(a, c) \in \mathcal{R}eg_2^{\mathbb{C}}$ defined by Eq. (5). By definition, these functions S and P are single-valued.

If a hyperplane X with Eq. (5) intersects only the right-hand (respectively, left-hand) circle of $\partial W \cap \mathbb{R}^2$ then the volume of one of parts cut by it from the body W is equal to one of roots of the polynomial $\Phi(\cdot, a, c)$ (respectively, $\Phi(\cdot, -a, -c)$). If X separates two circles of $\partial W \cap \mathbb{R}^2$ then any part of W cut by X is equal to the sum of both elements of the same orbit of the action of $\pi_1(\mathcal{R}eg_2^{\mathbb{C}}, X_0)$ on the $\pi_1(\mathcal{R}eg^{\mathbb{C}}, X_0)$-orbit of the vanishing cycle, and hence its volume is equal to $S(a, c)$ or to $C_0 - S(a, c) \equiv S(-a, -c)$. A rotation of X inside this component of $\mathcal{R}eg$ permutes the half-spaces of $\mathbb{R}^{n+m}$ separated by X, and hence permutes also these two leaves of the function V_W. If X intersects both these circles, then any of these two parts is equal to the sum of two elements of the $\pi_1(\mathcal{R}eg^{\mathbb{C}}, X_0)$-orbit of our vanishing cycle which belong to different $\pi_1(\mathcal{R}eg_2^{\mathbb{C}}, X_0)$-orbits, so that the volume function at X is equal to the sum of one root of $\Phi(\cdot, a, c)$ and one root of $\Phi(\cdot, -a, -c)$. The basic element of $\pi_1(\mathcal{R}eg_2^{\mathbb{C}}, X_0)$ permutes these elements within any of two pairs, and hence also moves one of the leaves of V_W in the domain (4) of $\mathcal{R}eg$ to the other.

All these facts will be proved in Sects. 2–4.

1.4 On Functions *S* and *P*

The functions $S(a, c)$ and $P(a, c)$ defined in this way are regular in $\mathbb{C}^2$ outside the divisor given by equation

$$a^2 + 1 = 0\,, \tag{11}$$

as follows from the next proposition.

Proposition 1 *Let D be a compact subset in the space $\mathbb{C}^2$ of hyperplanes (5), and be separated from the divisor (11). Then*

1) there is a constant $C = C(D)$ such that for any hyperplane $X \in D$, the function

$$\|(x, y)\|^2 \equiv |x_1|^2 + \cdots + |x_n|^2 + |y_1|^2 + \cdots + |y_m|^2$$

is regular on the stratified variety $A \cup X$ everywhere outside the ball $B_C \subset \mathbb{C}^{n+m}$ in which this function takes the values $\leq C$;

2) the functions $|S|$ and $|P|$ are bounded on D.

Proof. Statement 1) follows immediately from equation (8).

We can assume that D is convex: otherwise, we cover it by finitely many convex compact sets and prove our estimates for any of these sets separately. By considerations of stratified Morse theory, all homology classes in $H_{n+m}(\mathbb{C}^{n+m}, A \cup X)$, $X \in D$, which can be obtained by Gauss–Manin connection from our vanishing cycle, can be realized by compact cycles contained in B_C and depending continuously on X. Therefore, the integrals of the form (9) along all of them are uniformly bounded. □

Conjecture 2 These functions S and P are rational; they have poles of orders, respectively, $\leq (n+m-1)/2$ and $\leq (n+m)$ on the variety (11), and poles of orders $n+m$ and $2(n+m)$ on the divisor $\{c = \infty\}$, and no other singularities in $\mathcal{P}_2^{\mathbb{C}}$. In particular, single-valued entire analytic functions $S(a,c) \times (a^2+1)^{(n+m-1)/2}$ and $P(a,c) \times (a^2+1)^{n+m}$ grow only polynomially in $\mathbb{C}^2$; hence, they are polynomials, and function (7) is algebraic.

This conjecture obviously implies Conjecture 1.

2 Geometry and Topology of Variety (1)

Lemma 1 *1. The singular locus* $\operatorname{sing} A$ *of variety A is distinguished by the system of equations*

$$x_1 = \cdots = x_n = 0, \;\; y_1^2 + \cdots + y_m^2 + 1 - \varepsilon^2 = 0\,, \tag{12}$$

in particular it is a smooth $(m-1)$-dimensional complex manifold.

2. Let D be a small $2(n+1)$-dimensional open disc in $\mathbb{C}^{n+m}$ transversal to the manifold $\operatorname{sing} A$ *(e.g. a fiber of its tubular neighbourhood). Then the variety $A \cap D$ is diffeomorphic to the zero set of a non-degenerate quadratic form in $\mathbb{C}^{n+1}$, i.e. to the variety defined by equation*

$$z_1^2 + \cdots + z_{n+1}^2 = 0\,. \tag{13}$$

Proof is immediate. □

Corollary 1 *In conditions of Lemma 1, the group $H_i(A \cap D, A \cap \partial D)$ is equal to $\mathbb{Z}$ if i is equal to $n+1$ or n, and is trivial for all other i.* □

Corollary 2 *The* complex link *(see [8]) of* $\operatorname{sing} A$ *is homotopically equivalent to S^n.* □

Lemma 2 *$H_{n+m-1}(A) \simeq \mathbb{Z}^2$. This group is generated by*

1) the fundamental cycle of the manifold $\partial W \equiv A \cap \mathbb{R}^{n+m}$, and

2) the set of points $(x_1, \ldots, x_n, y_1, \ldots, y_m) \in A$ such that all x_j are real, $x_1^2 + \cdots + x_n^2 \leq (1-\varepsilon)^2$, the real parts of all y_j are equal to 0, and $y_1^2 + \cdots + y_m^2 \geq \varepsilon^2 - 1$.

The group $H_{n+m-2}(A)$ is trivial.

Proof. Two described cycles are independent in $H_{n+m-1}(A)$ since they bound relative cycles in $H_{n+m}(\mathbb{C}^{n+m}, A)$ whose volumes (i.e. the integrals of the form (9) along them) are not equal to zero and are incomparable for generic ε. Indeed, the volume of the tube W tends to zero when ε tends to 0, and the volume of the second cycle tends to a real non-zero constant (whose sign is equal to $(-1)^{m/2}$); therefore, the ratios of these two numbers run a continuum of values.

Denote by X_1 the (non-generic) hyperplane $\{x_1 = 0\}$ in $\mathbb{C}^{n+m}$. Consider the exact sequence of the pair $(A, A \cap X_1)$:

$$H_{n+m-1}(A \cap X_1) \to H_{n+m-1}(A) \to H_{n+m-1}(A, A \cap X_1) \to \tag{14}$$

$$H_{n+m-2}(A \cap X_1) \to H_{n+m-2}(A) \to H_{n+m-2}(A, A \cap X_1)\ . \tag{15}$$

The left-hand group in (14) is trivial as $A \cap X_1$ is a $(n + m - 2)$-dimensional Stein space. The restriction of the real function $|x_1|$ to the non-singular variety $A \backslash X_1$ has exactly four Morse critical points with coordinates $x_1 = \pm 1 \pm \varepsilon$, $x_2 = \cdots = x_n = y_1 = \cdots = y_m = 0$; their Morse indices are equal to $n + m - 1$. Therefore, the concluding group $H_{n+m-2}(A, A \cap X_1)$ in (15) is trivial, and the concluding group $H_{n+m-1}(A, A \cap X_1)$ in (14) is isomorphic to $\mathbb{Z}^4$ and is generated by intersections of the above-described two cycles with half-spaces where $x_1 \leq 0$ or $x_1 \geq 0$. The boundaries of two generators lying in the half-space $\{x_1 \geq 0\}$ are the cycles in $A \cap X_1$ which are independent in $H_{n+m-2}(A \cap X_1)$ by exactly the same reasons by which two cycles discussed in the present lemma are independent in $H_{n+m-1}(A)$. So the rank of the arrow connecting (14) and (15) is equal to 2, and its kernel (isomorphic to $H_{n+m-1}(A)$) is two dimensional, i.e. $H_{n+m-1}(A) \simeq \mathbb{Z}^2$. The space $A \cap X_1$ is analogous to A in the space $X_1 \equiv \mathbb{C}^{n+m-1}$ instead of $\mathbb{C}^{n+m}$, which implies the equality $H_{n+m-2}(A \cap X_1) \simeq \mathbb{Z}^2$. Two generators of this group belong to the image of the boundary map of our exact sequence; hence, both arrows in (15) are trivial, and $H_{n+m-2}(A) \simeq 0$. □

Denote by $\mathring{A}$ the regular part $A \backslash \mathrm{sing} A$ of variety A.

Let Y be a generic hyperplane in $\mathbb{C}^{n+m}$ tangent to the manifold $\mathrm{sing} A$ at some its point s. Let B be a small ball centered at s, and $\tilde{Y} \in \mathcal{R}eg^{\mathbb{C}}$ be a hyperplane parallel to Y and very close to it.

Lemma 3 *Both groups participating in the map*

$$H_{n+m-1}(A \cap B, A \cap B \cap \tilde{Y}) \to H_{n+m-1}(A \cap B, (\mathring{A} \cup \tilde{Y}) \cap B) \tag{16}$$

from the exact sequence of the triple $(A \cap B, (\mathring{A} \cup \tilde{Y}) \cap B, A \cap B \cap \tilde{Y})$ *are isomorphic to* $\mathbb{Z}$*, and this map is an isomorphism.*

Proof. The space of choices of initial data participating in our lemma (consisting of the point $s \in \mathrm{sing} A$, generic tangent plane Y, ball B, and neighbouring plane $\tilde{Y} \in \mathcal{R}eg^{\mathbb{C}}$) is irreducible; therefore, it is enough to consider an arbitrary such collection of them. So we take for s the point with coordinates $y_1 = i\sqrt{1 - \varepsilon^2}$, $y_2 = \cdots = y_m = 0$, choose the radius ρ of the ball B much smaller than ε, and define $\tilde{Y}$ by the equation $y_1 = i(\sqrt{1 - \varepsilon^2} - \zeta)$, where ζ is a positive constant much smaller than ρ. We will

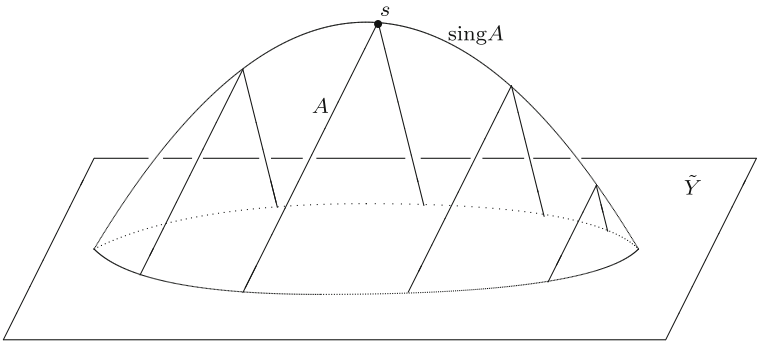


Fig. 2 Fibration of A close to the singular stratum

show that both groups (16) are then generated by one and the same relative cycle, namely by the part of cycle 2) from Lemma 2 placed in the domain, where $\operatorname{Im} y_1 \in \left[\sqrt{1-\varepsilon^2}-\zeta, \sqrt{1-\varepsilon^2}\right]$.

The left-hand group in (16) appears as follows. The group $\tilde{H}_*(A \cap B)$ is trivial, as $A \cap B$ is homeomorphic to the cone over the point s. The group $H_*(A \cap B \cap \tilde{Y})$ can be reduced by a kind of $(m-1)$-fold suspension to the similar homology group in the transversal slice of sing$A$, see [8]. Let us remind a realization of this reduction following [11], §II.4 (see Fig. 2). Fiber the ball B into its sections by a family of parallel $(n+1)$-dimensional complex planes transversal to sing$A$, along any of which the coordinates $y_2, \ldots, y_m$ take some fixed values. Some of these planes are non-generic with respect to the hypersurface $A \cap \tilde{Y}$: this happens when the intersection point of such a plane and the variety sing$A$ belongs to $\tilde{Y}$. By the usual complex Morse lemma (applied to the restriction of the function y_1 to sing$A$), the variety sing$A \cap \tilde{Y} \cap B$ parameterizing the non-generic planes contains a sphere S^{m-2} as a deformation retract. This sphere can be spanned by a $(m-1)$-dimensional disc Ψ^{m-1}, whose interior points lie in sing$A \cap B \backslash \tilde{Y}$; the class of this disc in $H_{m-1}(\text{sing}A \cap B, \text{sing}A \cap B \cap \tilde{Y})$ is not equal to zero.

Let Υ be a generic complex $(n+1)$-dimensional plane from our family, which is transversal to sing$A$ at a point of Ψ^{m-1}, and Ξ be an i-dimensional cycle in $A \cap B \cap \Upsilon \cap \tilde{Y}$. Then we can span an $(i+m-1)$-dimensional cycle in the variety $A \cap B \cap \tilde{Y}$ extending this cycle by the local triviality into all similar slices of this variety by the planes of this family intersecting sing$A$ in interior points of the disc Ψ^{m-1} and contracting them over the boundary points of this disc. This suspension operation defines an isomorphism $H_i(A \cap B \cap \Upsilon \cap \tilde{Y}) \to H_{i+m-1}(A \cap B \cap \tilde{Y})$.

In our case, by Lemma 1 the variety $A \cap B \cap \Upsilon \cap \tilde{Y}$ is homeomorphic to a generic hyperplane section of the cone (13), hence is homotopy equivalent to S^{n-1}. This sphere S^{n-1} can be realized by the intersection of $A \cap \Upsilon \cap \tilde{Y}$ with cycle 2 from Lemma 2. Therefore, $H_{n+m-2}(A \cap B \cap \tilde{Y}) \simeq \mathbb{Z}$, and the equality of the first group in (16) to $\mathbb{Z}$ follows by the exact sequence of the pair $(A \cap B, A \cap B \cap \tilde{Y})$. By the construction, a generator of this group is realized by the corresponding part of cycle 2 of Lemma 2.

Considering the second group (16), we can replace the set $\mathring{A}$ by the complement in A of a small closed tubular neighbourhood T of the singular locus $\operatorname{sing} A$; so we study the relative homology group

$$H_*(A \cap B, ((A \backslash T) \cup (A \cap \tilde{Y})) \cap B) \equiv H_*(A \cap T \cap B, A \cap (\partial T \cup \tilde{Y}) \cap B). \tag{17}$$

By Lemma 1, the variety $A \cap T \cap B$ is homeomorphic to the direct product of the $(2m-2)$-dimensional disc $\operatorname{sing} A \cap B$ and the variety $A \cap D$ considered in Corollary 1. We can arrange the fibration of the tubular neighbourhood of $\operatorname{sing} A$ in B in such a way that each fiber either completely belongs to $\tilde{Y}$ or does not intersect it. Then our homeomorphism turns the pair $(A \cap T \cap B, A \cap (\partial T \cup \tilde{Y}) \cap B)$ from (17) to the pair

$$((\operatorname{sing} A \cap B) \times (A \cap D), ((\operatorname{sing} A \cap \tilde{Y} \cap B) \times (A \cap D)) \cup ((\operatorname{sing} A \cap B) \times (A \cap \partial D))).$$

The corresponding relative homology group (i.e. the right-hand group in (17)) is, therefore, isomorphic to $\mathbb{Z}$ by Corollary 1 and Künneth formula for the direct product of pairs $(\operatorname{sing} A \cap B, \operatorname{sing} A \cap \tilde{Y} \cap B)$ and $(A \cap D, A \cap \partial D)$. A generator of this group is again realized by cycle 2 from Lemma 2 (intersected with the neighbourhood T). □

Lemma 4 *For any affine complex hyperplane $X \in \mathcal{R}eg^{\mathbb{C}}$ in $\mathbb{C}^{n+m}$, we have isomorphisms*

$$H_{n+m}(\mathbb{C}^{n+m}, A \cup X) \simeq H_{n+m-1}(A \cup X) \simeq H_{n+m-1}(A, A \cap X)\,. \tag{18}$$

All these groups are isomorphic to $\mathbb{Z}^6$.

Proof. The isomorphisms (18) follow from exact sequences of pairs.

By Thom isotopy theorem, all homology groups in (18) form coverings over the path-connected space $\mathcal{R}eg^{\mathbb{C}}$, so it is sufficient to calculate these groups for single hyperplane X given by equation $x_1 + y_1 = 0$. We will assume that ε in (1) is small enough, in particular $\varepsilon < 1/\sqrt{2}$.

The restriction of the function $x_1 + y_1$ to the stratified variety A has six critical points. Four of them are Morse critical points placed in $\mathring{A}$, namely those are real points with $(x_1, y_1) = \pm(1, 0) \pm (\varepsilon/\sqrt{2}, \varepsilon/\sqrt{2})$, $x_2 = \cdots = x_n = y_2 = \cdots = y_n = 0$ (the signs $\pm$ are independent). In addition, we have two critical points $\{x_1 = \cdots = x_n = y_2 = \cdots = y_m = 0,\ y_1 = \pm i\sqrt{1-\varepsilon^2}\}$ of the restriction of function $x_1 + y_1$ to the singular locus (12); they are Morse critical points of this function $x_1 + y_1$ on the stratified variety A in the sense of [8], §1.4.

Let B_j, $j = 1, \ldots, 6$, be small balls in $\mathbb{C}^{n+m}$ around these six critical points. Consider a graph in $\mathbb{C}^1$ consisting of closed paths connecting the non-critical value 0 with all critical values of the map $x_1 + y_1 : A \to \mathbb{C}^1$ as in Fig. 3. Denote these critical values by z_j, and denote by $\tilde{z}_j$ some points of this graph which are very close to its endpoints z_j; let $\tilde{Y}_j \in \mathcal{R}eg^{\mathbb{C}}$ be six hyperplanes given by equations $x_1 + y_1 = \tilde{z}_j$. Denote by $\ddagger$ the pre-image in A of this graph under the map $x_1 + y_1$, and by $\breve{\ddagger}$ the

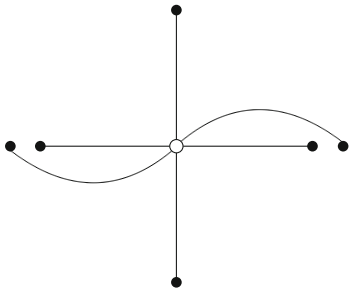


Fig. 3 Vanishing cycles for the model pair $(A, A \cap X)$

pre-image of the same graph less the union of tiny segments $(\tilde{z}_j, z_j]$ at the ends of its legs.

The space $\ddagger$ is a deformation retract of A, and $A \cap X$ is a deformation retract of $\check{\ddagger}$. This implies the first isomorphism of the sequence

$$H_*(A, A \cap X) \simeq H_*(\ddagger, \check{\ddagger}) \simeq \bigoplus_{j=1}^{6} H_*(A \cap (x_1 + y_1)^{-1}([\tilde{z}_j, z_j]), A \cap \tilde{Y}_j) \simeq$$

$$\simeq \bigoplus_{j=1}^{6} H_*(A \cap B_j, A \cap \tilde{Y}_j \cap B_j); \tag{19}$$

here the second equality is excision, and the third one follows from the local triviality of the restrictions of the map $x_1 + y_1$ to the sets $(A \backslash B_j) \cap (x_1 + y_1)^{-1}([\tilde{z}_j, z_j])$.

All six summands in (19) are isomorphic to $\mathbb{Z}$ in dimension $n+m-1$ and are trivial in all other dimensions. For four summands related to non-vertical legs of the graph of Fig. 3, this follows from the usual Morse lemma (and the corresponding generators of the group $H_{n+m-1}(A, A \cap X)$ are standard Lefschetz thimbles, see [11]); for two vertical legs, the same follows from Lemma 3. □

Corollary 3 *For any $X \in \mathcal{R}eg^{\mathbb{C}}$, the group $H_{n+m-2}(A \cap X, \mathbb{Q})$ is isomorphic to $\mathbb{Q}^4$.*

Proof. This follows immediately from Lemmas 2 and 4 and exact sequence of the pair (A, X). □

Denote by $rH_*(A, A \cap X)$ the subgroup in $H_*(A, A \cap X)$ represented by relative cycles avoiding the singular locus of A, i.e. the image of the obvious map $H_*(\mathring{A}, \mathring{A} \cap X) \to H_*(A, A \cap X)$.

Lemma 5 *For any $X \in \mathcal{R}eg^{\mathbb{C}}$, the group $rH_{n+m-1}(A, A \cap X)$ is isomorphic to $\mathbb{Z}^4$. For X considered in the proof of Lemma 4, this group is generated by four Lefschetz thimbles defined by non-vertical paths in Fig. 3.*

Proof. The groups $rH_*(A, A \cap X)$ for all generic X are isomorphic to one another, so let us take X from the proof of Lemma 4. By Lemma 3 (and the exact sequence of triple mentioned in its statement), the image of the group $H_{n+m-1}(\mathring{A}, \mathring{A} \cap X)$ in the group (19) can be at most four-dimensional since it does not contain non-zero linear combinations of the summands corresponding to two critical points from $\operatorname{sing} A$. Four

other generators of this sum can be realized by standard Lefschetz thimbles and hence belong to the image of this group. □

3 Monodromy Action

Fundamental groups $\pi_1(\mathcal{R}eg^{\mathbb{C}}, X)$ and $\pi_1(\mathcal{R}eg_2^{\mathbb{C}}, X)$ act by monodromy operators on all groups (18) and on the subgroup $rH_*(A, A \cap X)$ of the last of them; this action commutes with isomorphisms (18), cf. [12].

Theorem 3 *The image of the group $\pi_1(\mathcal{R}eg^{\mathbb{C}}, X)$ in the group of automorphisms of the lattice $rH_*(A, A \cap X) \sim \mathbb{Z}^4$ under the monodromy representation is the Klein four-group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. For some concordant choice of orientations of basic Lefschetz thimbles generating this lattice, the automorphisms defined by this monodromy action preserve the set of these basic elements and act on it by permutations $\binom{1234}{2143}$ and $\binom{1234}{4321}$ and their compositions. The similar image of $\pi_1(\mathcal{R}eg_2^{\mathbb{C}}, X)$ is isomorphic to $\mathbb{Z}_2$ and is generated by only the first of these permutations.*

A proof of this theorem takes the rest of this section.

Since the spaces $\mathcal{P}^{\mathbb{C}}$ and $\mathcal{P}_2^{\mathbb{C}}$ are simply connected, these fundamental groups are generated by "pinches", i.e. the loops going first from the distinguished point X to a smooth piece of the *discriminant set* $\Sigma \equiv \mathcal{P}^{\mathbb{C}} \backslash \mathcal{R}eg^{\mathbb{C}}$ of non-generic hyperplanes, then running a small circle around this piece, and coming back to X along the first part of the loop.

The discriminant variety Σ consists of four irreducible components Σ_j, $j = 1, \dots, 4$, formed, respectively, by hyperplanes

(Σ_1) tangent to the variety A at its non-singular finite points,

(Σ_2) tangent to the variety $\operatorname{sing} A$ at its finite points,

(Σ_3) defined by equation (4) with

$$\alpha_1^2 + \cdots + \alpha_n^2 + \gamma_1^2 + \cdots + \gamma_m^2 = 0 \tag{20}$$

(i.e. asymptotic to the smooth part of A: the closures in $\mathbb{CP}^{n+m}$ of these hyperplanes are tangent to the intersection of the closure of A and the "infinitely distant" hyperplane $\mathbb{CP}_\infty^{n+m-1} \equiv \mathbb{CP}^{n+m} \backslash \mathbb{C}^{n+m}$), and

(Σ_4) defined by equation (4) with

$$\gamma_1^2 + \cdots + \gamma_m^2 = 0 \tag{21}$$

(i.e. asymptotic with respect to $\operatorname{sing} A$).

Let us study the action of pinches embracing these components on the group $rH_{n+m-1}(A, A \cap X)$.

3.1 Σ_1: Standard Picard–Lefschetz Operator

Proposition 2 *The pinches in $\mathcal{R}eg^{\mathbb{C}}$ going around smooth pieces of the component Σ_1 act trivially on the group $rH_{n+m-1}(A, A \cap X)$.*

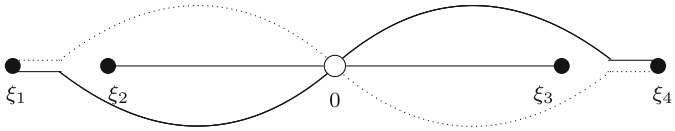


Fig. 4 Vanishing cycles and their complex conjugates

Proof. Let s be a regular point of A, at which the second fundamental form of A is non-degenerate; let $Y \in \Sigma_1$ be the hyperplane tangent to A at this point. We assume that Y is not tangent to A at any other points and does not belong to any other components of Σ (which is true for almost all points $s \in A$). Then Σ is smooth at the point Y. Let $L : \mathbb{C}^{n+m} \to \mathbb{C}$ be a linear (generally non-homogeneous) function such that Y is defined by the equation $L = 0$. Let B be a small ball centered at s. Consider the one-parametric family of parallel hyperplanes X_λ, $\lambda \in \mathbb{C}^1$, defined by equations $L(x, y) = \lambda$. For all λ with sufficiently small $|\lambda| \neq 0$, these hyperplanes belong to $\mathcal{R}eg^{\mathbb{C}}$. Let $\lambda_0 \neq 0$ be such a value of the parameter λ with very small $|\lambda_0|$. We need to calculate the monodromy action on $rH_{n+m-1}(A, A \cap X_{\lambda_0})$ defined by the circle in $\mathcal{R}eg^{\mathbb{C}}$ consisting of all hyperplanes X_λ with $\lambda = e^{it}\lambda_0$, $t \in [0, 2\pi]$. This action is described by the Picard–Lefschetz formula (22) formulated in the following terms.

The group $H_{n+m-1}(A \cap B, A \cap B \cap X_{\lambda_0})$ is isomorphic to $\mathbb{Z}$ and is generated by the *vanishing relative cycle* $\Delta(\lambda_0)$, see, e.g. [11]. The boundary map $H_{n+m-1}(A \cap B, A \cap B \cap X_{\lambda_0}) \to H_{n+m-2}(A \cap B \cap X_{\lambda_0})$ is an isomorphism, and the latter group is generated by the absolute cycle $\partial\Delta(\lambda_0)$.

Further, let Δ be any element of the group $H_{n+m-1}(A, A \cap X_{\lambda_0})$, and $\partial\Delta$ be its boundary in $H_{n+m-2}(A \cap X_{\lambda_0})$. Then the monodromy operator in question sends our class Δ to

$$\Delta + (-1)^{(n+m)(n+m-1)/2}\langle \partial\Delta, \partial\Delta(\lambda_0)\rangle \Delta(\lambda_0) \ , \tag{22}$$

where $\langle \cdot, \cdot \rangle$ is the intersection index in $\mathring{A} \cap X_{\lambda_0}$. Therefore, it remains to prove the following lemma.

Lemma 6 *For $X \in \mathcal{R}eg^{\mathbb{C}}$, any two elements of $rH_{n+m-1}(A, A \cap X)$ can be represented by relative cycles $\Delta, \Delta' \subset \mathring{A}$ such that $\langle \partial\Delta, \partial\Delta' \rangle = 0$.*

Proof. It is sufficient to prove this for hyperplane X used in the proof of Lemma 4, and for basic Lefschetz thimbles issuing from the four critical points of the function $x_1 + y_1 : \mathring{A} \to \mathbb{C}$, see Figs. 3 and 4. All these critical points are real, and Morse indices of their restrictions to the real part ∂W of A are even. Let ξ_j, $j \in \{1, \ldots, 4\}$, be critical values $\pm 1 \pm \sqrt{2}\varepsilon$ of this restriction ordered by their increase. Consider some such value ξ_j, and let ξ' be a non-critical real value of this restriction which is very close to ξ_j. Let $X_{\xi'}$ be the hyperplane in $\mathbb{C}^{n+m}$ defined by equation $x_1 + y_1 = \xi'$. The vanishing relative cycle in $(\mathring{A}, \mathring{A} \cap X_{\xi'})$ corresponding to the critical point with value ξ_j is located in a small ball B around this critical point and generates the group $H_{n+m-1}(\mathring{A} \cap B, \mathring{A} \cap X_\xi \cap B)$. This vanishing cycle $\Delta_j(\xi')$ can be chosen to be invariant under the complex conjugation; moreover, since the Morse index of the function $(x_1 + y_1)|_{\partial W}$ is even, this conjugation preserves the orientation of this cycle

and homology class of its boundary in $H_{n+m-2}(\mathring{A} \cap X_{\xi'})$. By isotopy considerations, this invariance holds also for all cycles $\Delta_j(\tilde{\xi})$ in similar groups $H_{n+m-2}(\mathring{A} \cap X_{\tilde{\xi}})$, where $\tilde{\xi}$ belong to the same interval of real non-critical values as ξ', and $\Delta_j(\tilde{\xi})$ are obtained from our vanishing cycle $\Delta_j(\xi')$ by the Gauss–Manin connection over this interval.

On the other hand, the complex conjugation changes the orientation of the $2(n+m-2)$-dimensional manifold $\mathring{A} \cap X_{\tilde{\xi}}$; therefore, the intersection index of any two invariant cycles is opposite to itself, and hence is equal to 0. In particular, $\langle \partial\Delta_i(\xi), \partial\Delta_{i+1}(\xi)\rangle = 0$, if ξ is a non-critical value in the interval bounded by the critical values ξ_i and ξ_{i+1}, and $\Delta_j(\xi)$, $j = i$ or $i+1$, is the Lefschetz thimble in $(\mathring{A}, X_\xi)$ defined by the segment connecting ξ with the corresponding endpoint of this interval.

We obtain immediately the equality $\langle \partial\Delta_2, \partial\Delta_3\rangle = 0$, where Δ_2, Δ_3 are the Lefschetz thimbles in $(\mathring{A}, \mathring{A} \cap X)$ defined by segments in Fig. 4 connecting 0 and critical values $\pm(1-\sqrt{2}\varepsilon)$.

Moreover, the intersection index does not change if we deform continuously in $\mathbb{C}^1$ the pattern consisting of the non-critical value ξ' and two paths connecting it to the neighbouring (fixed) critical values in such a way that all this construction has no additional meetings with the set of critical values during this movement. For instance, the union of two solid paths in Fig. 4 connecting the critical values ξ_1 and ξ_2 to the non-critical value 0 can be obtained by such a deformation from two parts of the segment $[\xi_1, \xi_2] \subset \mathbb{R}^1$ connecting the same critical values to some point of this segment. This implies equality $\langle \partial\Delta_1, \partial\Delta_2\rangle = 0$ and, analogously, $\langle \partial\Delta_3, \partial\Delta_4\rangle = 0$.

Further, denote by $\tilde{\Delta}_1$ and $\tilde{\Delta}_4$ Lefschetz thimbles in $(\mathring{A}, \mathring{A} \cap X)$ which are complex conjugate to Δ_1 and Δ_4, respectively, and therefore, can be defined by the pointed paths in Fig. 4. The cycles $\partial\Delta_1 + \partial\tilde{\Delta}_1$ and $\partial\Delta_4 + \partial\tilde{\Delta}_4$ are invariant under the complex conjugation; therefore, we have

$$\langle \partial\Delta_1 + \partial\tilde{\Delta}_1, \partial\Delta_3\rangle = 0, \quad \langle \partial\Delta_2, \partial\Delta_4 + \partial\tilde{\Delta}_4\rangle = 0,$$
$$\langle \partial\Delta_1 + \partial\tilde{\Delta}_1, \partial\Delta_4 + \partial\tilde{\Delta}_4\rangle = 0.$$

By Picard–Lefschetz formula, $\partial\tilde{\Delta}_1$ is equal to $\partial\Delta_1 \pm \langle \partial\Delta_1, \partial\Delta_2\rangle \partial\Delta_2$, and hence (by the equality $\langle \partial\Delta_1, \partial\Delta_2\rangle = 0$ proved in the previous paragraph) to $\partial\Delta_1$. In a similar way, $\partial\tilde{\Delta}_4 = \partial\Delta_4$. Therefore, the previous three equalities reduce to $\langle \partial\Delta_1, \partial\Delta_3\rangle = 0$, $\langle \partial\Delta_2, \partial\Delta_4\rangle = 0$, and $\langle \partial\Delta_1, \partial\Delta_4\rangle = 0$.

Finally, all self-intersection indices $\langle \partial\Delta_j, \partial\Delta_j\rangle$ of odd-dimensional cycles are trivial. □

3.2 Σ_2: Tangents to the Singular Locus

Let Y be a generic hyperplane in $\mathbb{C}^{n+m}$ tangent to $\operatorname{sing} A$ at some point s, so that its linear (generally, non-homogeneous) equation $L = 0$ defines a Morse function on the stratified variety A in the sense of [8], and Y does not belong to other local branches of Σ. Let Y_ξ, $\xi \in \mathbb{C}$, be the family of hyperplanes parallel to $Y \equiv Y_0$, which are defined by the equations $L(x, y) = \xi$. Let B be a small ball in $\mathbb{C}^{n+m}$ centered at s. Let $\xi_0 \neq 0$ be a number such that $|\xi_0|$ is much smaller than the radius of the ball B. We consider

the monodromy operator acting on the group

$$H_{n+m-1}(A, A \cap Y_{\xi_0}) \tag{23}$$

and defined by the Gauss–Manin connection over the circle in $\mathcal{R}eg^{\mathbb{C}}$ consisting of hyperplanes Y_ξ, where ξ runs the circle $e^{it}\xi_0$, $t \in [0, 2\pi]$.

Proposition 3 *This operator acts trivially on the subgroup* $r\,H_{n+m-1}(A, A \cap Y_{\xi_0})$.

Proof. The family of varieties $A \cap Y_\xi \setminus B$ forms a trivial fiber bundle over the disc in $\mathbb{C}$ consisting of values ξ with $|\xi| \le |\xi_0|$; therefore, this monodromy operator adds to any element Δ of the group (23) some element (depending linearly on Δ) which can be realized by a chain inside B. Moreover, if $\Delta \in r\,H_{n+m-1}(A, A \cap Y_{\xi_0})$ then this chain can be realized inside $\mathring{A} \cap B$, and hence its homology class in $H_{n+m-1}(A \cap B, A \cap B \cap Y_{\xi_0})$ belongs to the image of the homomorphism

$$H_{n+m-1}(\mathring{A} \cap B, \mathring{A} \cap B \cap Y_{\xi_0}) \to H_{n+m-1}(A \cap B, A \cap B \cap Y_{\xi_0}) \tag{24}$$

of the exact sequence of the triple $(A \cap B, (\mathring{A} \cup Y_{\xi_0}) \cap B, A \cap B \cap Y_{\xi_0})$. This image is trivial by Lemma 3. □

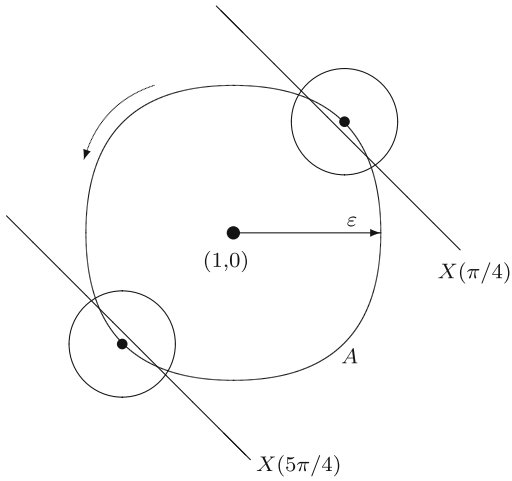
3.3 Σ_3 and Σ_4: Asymptotic Hyperplanes

The space $\mathcal{P}^{\mathbb{C}}$ augmented with the point corresponding to the "infinitely distant" hyperplane in $\mathbb{C}P^{n+m}$ is itself isomorphic to the $(n+m)$-dimensional complex projective space. Any generic two-dimensional projective subspace in it intersects the varieties Σ_3 and Σ_4 (distinguished by conditions (20) and (21)) along two degree 2 complex curves in general position, in particular these curves have exactly four transversal intersection points. (If $m > 2$, then both these curves are non-singular, and in the case $m = 2$ the second of them splits into two lines). By Zariski theorem, the fundamental group of the complement of the union of these curves in this 2-subspace (and hence also of the complement of variety $\Sigma_3 \cup \Sigma_4$ in entire $\mathcal{P}^{\mathbb{C}}$) is isomorphic to $\mathbb{Z}^2$, in particular is commutative. Therefore (and since components Σ_1 and Σ_2 do not contribute to the monodromy action, as is shown in two previous subsections), we can calculate independently the action on $r\,H_{n+m-1}(A, A \cap X)$ of loops embracing these components, and not take care on the choice of the distinguished point in $\mathcal{R}eg^{\mathbb{C}}$. We will do it for some two loops, whose linking numbers with varieties Σ_3 and Σ_4 are equal to $(1, 0)$ and $(1, 1)$.

The first of these loops will be realized inside the subspace $\mathcal{P}_2^{\mathbb{C}} \subset \mathcal{P}^{\mathbb{C}}$ (see Sect. 1.2). This subspace meets the discriminant variety Σ_4 at the points of its singular part only; the complement of this variety in this subspace is simply connected, so that all loops in $\mathcal{R}eg_2^{\mathbb{C}}$ have zero linking number with Σ_4. They generate a subgroup isomorphic to $\mathbb{Z}$ in the lattice $\pi_1(\mathcal{P}^{\mathbb{C}} \setminus (\Sigma_3 \cup \Sigma_4)) \simeq \mathbb{Z}^2$, and the linking numbers with Σ_3 separate the points of this subgroup.

This subspace intersects the set Σ_3 along two lines consisting of hyperplanes given by equations $x_1 = \pm i y_1 + \beta$ with arbitrary β. The space $\mathcal{P}_2^{\mathbb{C}} \setminus \Sigma_3$ can be projected to

Fig. 5 A loop of third type



$\mathbb{CP}^1\backslash\{i, -i\}$ by sending any plane with Eq. (3) to the number p/q. A generator of the fundamental group of this space is provided by any loop along which these numbers run the real axis $\mathbb{RP}^1 \subset \mathbb{CP}^1$, or by any loop sufficiently close to it.

To realize such a loop, let us draw a path on the surface ∂W, connecting two critical points of the restriction of function $x_1 + y_1$ to A inside the plane $\mathbb{R}^2$ defined by conditions

$$x_2 = \cdots = x_n = y_2 = \cdots = y_m = 0, \tag{25}$$

see Fig. 5; the coordinates x_1 and y_1 of the points of this path move as $x_1(\tau) = 1 + \varepsilon \cos\tau$, $y_1(\tau) = \varepsilon \sin\tau$, $\tau \in [\pi/4, 5\pi/4]$. The tangent planes of A at all these points can be defined by equations with real coefficients, in particular are distant from the set Σ_3. Let us deform this path slightly inside $A \cap \mathbb{C}^2$ (where $\mathbb{C}^2$ is the complexification of the plane $\mathbb{R}^2$ distinguished by (25)) in such a way that the tangent hyperplanes of A at all points $\eta(\tau)$, $\tau \in [\pi/4, 5\pi/4]$, of the obtained path are non-singular points of Σ; in particular all these points $\eta(\tau)$ are *non-parabolic* points of A, i.e. the second fundamental form of A is non-degenerate at all of them.

(There are exactly three such points of the initial real path in ∂W, where we need to move slightly to the complex domain. One of them corresponds to $\tau = \pi/2$, where the second fundamental form of ∂W degenerates, the second to $\tau = \pi/2 + \arcsin\varepsilon$, where an extra tangency with A at a distant point occurs, and the third one to $\tau = \pi$, where the tangent plane belongs to Σ_4.)

After that, let us pave a path in $\mathcal{R}eg_2^{\mathbb{C}}$ consisting of hyperplanes $X(\tau)$ which are parallel to the tangent hyperplanes of A at the corresponding points $\eta(\tau)$ of our path, and are extremely close to these tangent hyperplanes. For any point $\eta(\tau) \in A \cap \mathbb{C}^2$ of the first path and a small ball B_τ around this point, we have $H_{n+m-1}(A \cap B_\tau, A \cap B_\tau \cap X(\tau)) \simeq \mathbb{Z}$. Denote by $\delta(\tau)$ the vanishing relative cycles generating these groups and supplied with some orientations depending continuously on τ.

Gauss–Manin connection over our path moves these vanishing cycles ones into the others, in particular it moves the vanishing cycle $\delta(\pi/4)$ related with the starting point to the vanishing cycle $\delta(5\pi/4)$ related with the final one.

Consider two paths in $\mathcal{R}eg_2^{\mathbb{C}}$ connecting the hyperplanes $X(\pi/4)$ and $X(5\pi/4)$ with the hyperplane $X_0 = \{x_1 + y_1 = 0\}$ and consisting of hyperplanes defined by equations $x_1 + y_1 = \lambda$, where λ runs the solid paths in the right-hand part of Fig. 4. Deforming our two vanishing relative cycles $\delta(\pi/4)$ and $\delta(5\pi/4)$ by the local triviality of the fiber bundle of pairs $(A, A \cap X)$ over these paths, we obtain exactly the Lefschetz thimbles Δ_4 and Δ_3, respectively. Therefore, the loop in $\mathcal{R}eg_2^{\mathbb{C}}$ consisting of these two paths (passed in appropriate directions) and our path $\{X(\tau)\}$, $\tau \in [\pi/4, 5\pi/4]$, moves the class $\Delta_4 \in rH_{n+m-1}(A, A \cap X_0)$ into Δ_3.

This loop is a generator of the group $\pi_1(\mathcal{P}^{\mathbb{C}} \setminus \Sigma_3, X_0)$. A different realization of the same generator is provided by the union of the same two paths (passed in opposite directions) and the family of escorting hyperplanes $X(\tau)$ similar to the one considered above but corresponding to $\tau \in [5\pi/4, 9\pi/4]$. By the same considerations as above the latter family moves the vanishing cycle $\delta(5\pi/4)$ to $\delta(\pi/4)$ or $-\delta(\pi/4)$, and hence the entire loop moves the class Δ_3 to Δ_4 (or, respectively, to $-\Delta_4$).

It remains to prove that the correct choice of the sign is $+$. This property is equivalent to the assertion that the transportation of vanishing cycles $\delta(\cdot)$ along entire circle, composed of two paths $\{\eta(\tau)\}$, $\tau \in [\pi/4, 5\pi/4]$ and $\tau \in [5\pi/4, 9\pi/4]$, moves the homology class of the vanishing cycle $\delta(\tau)$ to itself and not to minus itself. We can start proving this from an arbitrary point of the circle.

To do it, notice that we may simplify our paths $\{\eta(\tau)\}$ and $\{X(\tau)\}$, not moving into the complex domain at the points $\tau = \pi$ and 2π: indeed, in these cases the degeneration of the topological type of the pair $(A, A \cap X(\tau))$ happens far away from the support of our cycles $\delta(\tau)$, and the transportation of the cycles $\delta(\tau)$ along these straightened paths gives the same result. Further, we can realize this family of cycles $\delta(\tau)$, $\tau \in [0, 2\pi]$, in such a way that for any two points τ, τ' with $\tau + \tau' = 2\pi$ the corresponding cycles $\delta(\tau)$ and $\delta(\tau')$ will be symmetric to one another with respect to the involution in $\mathbb{C}^{n+m}$ multiplying the coordinate y_1 by -1. Indeed, we can choose both cycles corresponding to $\tau = \pi$ and $\tau = 0 \equiv 2\pi$ to be symmetric to themselves, then realize arbitrarily the family of cycles $\delta(\tau)$ connecting them over $\tau \in [0, \pi]$, and realize the family for $\tau \in [\pi, 2\pi]$ by cycles symmetric to the corresponding cycles of the previous family. Then the transportation of the cycle $\delta(\pi)$ over the half-circles $\tau \in [0, \pi]$ and $[\pi, 2\pi]$ give the same results, which is equivalent to our assertion.

So, our generator of the group $\pi_1(\mathcal{P}^{\mathbb{C}} \setminus \Sigma_3, X_0)$ permutes basic classes Δ_3 and Δ_4 of the group $rH_{n+m-1}(A, A \cap X_0)$. In absolutely the same way, it permutes the classes Δ_1 and Δ_2; so it realizes the first permutation indicated in Theorem 3.

The second loop in $\mathcal{R}eg^{\mathbb{C}}$ is provided by the family of hyperplanes defined by real equations

$$(x_1 + y_1)\cos\gamma + (x_2 + y_2)\sin\gamma = 0\,, \quad \gamma \in [0, \pi]\,.$$

It continuously permutes four critical points of the restrictions of functions $\pm(x_1 + y_1)$ to the hypersurface ∂W, and also permutes the corresponding Lefschetz thimbles

$\Delta_2 \leftrightarrow \Delta_3$ and $\Delta_1 \leftrightarrow \Delta_4$. The linking numbers of this loop with varieties (20) and (21) are equal to 1. □

4 Lefschetz Thimbles and Integration Contours

Lemma 7 *If the restriction of a linear function $\mathbb{R}^{n+m} \to \mathbb{R}$ to ∂W is Morse, then it has exactly four critical points.*

Indeed, the number of its critical points cannot be smaller than four by the Morse inequality, and cannot be bigger because even the restriction of the complexification of a generic linear function to $\mathring{A}$ has no more than four critical points. □

By Lemma 5, for any $X \in \mathcal{R}eg^{\mathbb{C}}$ the subgroup $rH_{n+m-1}(A, A \cap X) \subset H_{n+m-1}(A, A \cap X)$ is isomorphic to $\mathbb{Z}^4$ and is generated by the classes Δ_j of four Lefschetz thimbles associated with the critical points of the restriction to the smooth part of A of the linear function L vanishing on the hyperplane X.

By Lemma 6, the homology classes of these thimbles (defined up to a choice of their orientations) do not depend on the paths connecting the critical values of $L|_{\mathring{A}}$ with 0. Also, we can choose their orientations in concordant way so that they constitute one orbit of the group $\pi_1(\mathcal{R}eg^{\mathbb{C}}, X)$. (This choice of $\pi_1(\mathcal{R}eg^{\mathbb{C}}, X)$-concordant orientations is unique up to a simultaneous change of all of them).

Denote by $\Xi_j \in H_{n+m}(\mathbb{C}^{n+m}, A \cup X)$, the preimages of these basic classes Δ_j under the composite isomorphism (18).

Suppose now that the hyperplane X is real, $X \in \mathcal{R}eg^{\mathbb{C}}$, and the linear function $L : (\mathbb{C}^{n+m}, \mathbb{R}^{n+m}) \to (\mathbb{C}, \mathbb{R})$ vanishing on X is Morse on ∂W. Let us fix some orientation of the space $\mathbb{R}^{n+m}$.

Proposition 4 *(see, e.g. [12]) The homology class in $H_{n+m}(\mathbb{C}^{n+m}, A \cup X)$ of the cycle $W \cap L^{-1}((-\infty, 0])$ (respectively, $W \cap L^{-1}([0, +\infty))$) oriented by the chosen orientation of $\mathbb{R}^{n+m}$ is equal to the sum of cycles Ξ_j, defined by the paths connecting* 0 *with all negative* (*respectively, positive*) *critical values and taken with appropriate coefficients equal to 1 or* -1. □

Proposition 5 *For some $\pi_1(\mathcal{R}eg^{\mathbb{C}}, X)$-concordant choice of orientations of cycles Δ_j, all four coefficients, with which the cycles Ξ_j participate in the two sums mentioned in Proposition 4, are equal to* $+1$.

Indeed, the sum of these two sums is equal to the class of the body ∂W, and hence does not depend on the hyperplane X. Therefore, it is invariant under the action of the group $\pi_1(\mathcal{R}eg^{\mathbb{C}}, X)$. This is possible only if all coefficients are equal to one another. We can make all of them to be equal to $+1$ by the choice of one of two sets of concordant orientations. □

Now define the four-valued analytic function Ψ on the space $\mathcal{P}^{\mathbb{C}}$, whose four values at any point $X \in \mathcal{R}eg^{\mathbb{C}}$ are equal to the integrals of the form (9) along all cycles Ξ_j. This function satisfies conditions 1–5 of Theorem 2, and the last statement of this theorem follows from them and Proposition 4. □

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