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RESEARCH CONTRIBUTION

The Structure of the Conjugate Locus of a General Point on Ellipsoids and Certain Liouville Manifolds

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Abstract
It is well known since Jacobi that the geodesic flow of the ellipsoid is “completely integrable”, which means that the geodesic orbits are described in a certain explicit way. However, it does not directly indicate that any global behavior of the geodesics becomes easy to see. In fact, it happened quite recently that a proof for the statement “The conjugate locus of a general point in two-dimensional ellipsoid has just four cusps” in Jacobi’s Vorlesungen über dynamik appeared in the literature. In this paper, we consider Liouville manifolds, a certain class of Riemannian manifolds which contains ellipsoids. We solve the geodesic equations; investigate the behavior of the Jacobi fields, especially the positions of the zeros; and clarify the structure of the conjugate locus of a general point. In particular, we show that the singularities arising in the conjugate loci are only cuspidal edges and D_4^+ Lagrangian singularities, which would be the higher dimensional counterpart of Jacobi’s statement.

Keywords Conjugate locus · Ellipsoid · Singularity

Mathematics Subject Classification Primary 53C22 ; Secondary 53A07 · 58E10

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1 Introduction

Let M be a Riemannian manifold ($\dim M = n$) and let $\gamma(t)$ be a geodesic with $\gamma(0) = p \in M$. Then $\gamma(t_1)$ ($t_1 > 0$) is called the *first conjugate point* of p along $\gamma(t)$ if $t = t_1$ is the largest value such that $\gamma|_{[0,t]}$ is the shortest one among the curves which join $\gamma(0)$ and $\gamma(t)$ and which are "infinitesimally close to $\gamma|_{[0,t]}$". More precisely and more generally, the point $\gamma(T)$ ($T > 0$) is called a *conjugate point* of p along the geodesic $\gamma(t)$ if there is a non-zero Jacobi field $Y(t)$ along $\gamma(t)$ such that $Y(0) = 0$, $Y(T) = 0$. Conjugate points of p along $\gamma(t)$ are discrete in t; $\gamma(t_1)$, $\gamma(t_2)$, $\ldots (0 < t_1 \leq t_2 \leq \ldots)$, called the first conjugate point, the second conjugate point, etc. The multiplicity is less than or equal to $n - 1$.

The ith *conjugate locus* of $p \in M$ is the set of all ith conjugate point of p along the geodesics emanating from p. The term "conjugate locus" is usually used with the meaning of the first conjugate locus. For the generality of conjugate points and conjugate loci, one can refer to [10,11]. The simplest example of the conjugate locus is that of the sphere of constant curvature S^n. In this case, the first conjugate point of each $x \in S^n$ along any geodesic is the antipodal point $-x$ and its multiplicity is $n-1$; thus, the ith conjugate locus ($1 \leq i \leq n-1$) of x is equal to $\{-x\}$, whereas the jth conjugate locus ($n \leq j \leq 2n-2$) is equal to $\{x\}$.

To understand the global behavior of the geodesics, it is crucial to know the structure of conjugate loci and cut loci of points. In general, however, it is quite difficult to determine conjugate loci and cut loci explicitly, except for symmetric spaces and other few examples (see [4, Introduction]). In the previous papers [3,5], we determined the structure of conjugate loci and cut loci of points for the tri-axial ellipsoid and certain Liouville surfaces. Also we determined in [4], the structure of cut loci of points for the ellipsoid and certain Liouville manifolds of dimension greater than two. In this paper, we clarify the structure of the conjugate locus of a general point on the ellipsoid and certain Liouville manifolds of dimension greater than two. In particular, we give a detailed description for the singular points on the conjugate locus. This would be a higher dimensional counterpart of "the last geometric statement of Jacobi", which says that the conjugate locus of a non-umbilic point of the two-dimensional ellipsoid has exactly four cusps ([6,7]; see also [3,5,12,13]).

Now, let us illustrate our results in detail by taking the ellipsoid $M : \sum_{i=0}^{n} u_i^2/a_i = 1$ ($0 < a_n < \cdots < a_0$) as an example. *The elliptic coordinate system* $(\lambda_1, \ldots, \lambda_n)$ on M ($\lambda_n \leq \cdots \leq \lambda_1$) is defined by the following identity in λ:

$$\sum_{i=0}^{n} \frac{u_i^2}{a_i - \lambda} - 1 = \frac{\lambda \prod_{k=1}^{n} (\lambda_k - \lambda)}{\prod_i (a_i - \lambda)}.$$

For a fixed $u \in M$, λ_k are determined by n "confocal quadrics" passing through u. From λ_k's, u_i are explicitly described as

$$u_i^2 = \frac{a_i \prod_{k=1}^{n} (\lambda_k - a_i)}{\prod_{j \neq i} (a_j - a_i)},$$

and the range of λ_k is $a_k \leq \lambda_k \leq a_{k-1}$. Also the metric g is described as

$$g = \sum_{i=1}^{n} \frac{(-1)^n \lambda_i \prod_{l \neq i}(\lambda_l - \lambda_i)}{4 \prod_{j=0}^{n}(\lambda_i - a_j)} \, d\lambda_i^2. \tag{1.1}$$

Let N_k be the ellipsoid of codimension one in M defined by

$$N_k = \{u = (u_0, \ldots, u_n) \in M \mid u_k = 0\} \qquad (0 \leq k \leq n),$$

which is a totally geodesic submanifold of M. By the elliptic coordinates the submanifold N_k is expressed as

$$N_k = \{\lambda_k = a_k \quad \text{or} \quad \lambda_{k+1} = a_k\}.$$

We shall say that $p \in M$ is a *general point* if $p \notin N_k$ for any k.

Now, let $p \in M$ be a general point. Each element v of the tangent space T_pM at p is expressed as

$$v = \sum_{i=1}^{n} v_i \frac{\partial}{\partial \lambda_i}.$$

Then putting

$$\tilde{v}_i = \sqrt{\frac{(-1)^{n-i} \lambda_i \prod_{l \neq i}(\lambda_l - \lambda_i)}{(-1)^i \, 4 \prod_{j=0}^{n}(\lambda_i - a_j)}} \, v_i,$$

we have an Euclidean coordinate system $(\tilde{v}_1, \ldots, \tilde{v}_n)$ on T_pM, i.e.,

$$g(v, v) = 1 \quad \text{if and only if} \quad \sum_{i=1}^{n} \tilde{v}_i^2 = 1.$$

We now define an elliptic coordinate system $(\mu_1, \ldots, \mu_{n-1})$ on the unit tangent space $U_pM \subset T_pM$ by the following identity in μ:

$$\sum_{i=1}^{n} \frac{\tilde{v}_i^2}{\mu - \lambda_i(p)} = \frac{\prod_{k=1}^{n-1}(\mu - \mu_k)}{\prod_{j=1}^{n}(\mu - \lambda_j(p))}, \quad \lambda_{i+1}(p) \leq \mu_i \leq \lambda_i(p).$$

Then

$$\tilde{v}_i^2 = \frac{\prod_{k=1}^{n-1}(\lambda_i(p) - \mu_k)}{\prod_{j \neq i}(\lambda_i(p) - \lambda_j(p))}, \qquad \sum_{i=1}^{n} \tilde{v}_i^2 = 1.$$

Define the submanifolds (with boundary) $C_i^{\pm}$ $(1 \le i \le n-1)$ of U_pM by

$$C_i^- = \{v \in U_pM \mid \mu_i(v) = \lambda_{i+1}(p)\}, \quad C_i^+ = \{v \in U_pM \mid \mu_i(v) = \lambda_i(p)\}.$$

It is seen that $C_{i-1}^- \cup C_i^+$ is equal to the great sphere $\tilde{v}_i = 0$ and they are diffeomorphic to

$$C_i^- \simeq S^{i-1} \times \bar{D}^{n-1-i}, \quad C_i^+ \simeq \bar{D}^{i-1} \times S^{n-1-i},$$

where S^k and $\bar{D}^k$ stand for the k-sphere and the closed k-disk (S^0 and $\bar{D}^0$ stand for the set of two points and that of one point), respectively. Also for the boundary $\partial C_i^{\pm}$ of $C_i^{\pm}$,

$$\begin{aligned} \partial C_i^+ = \partial C_{i-1}^- = C_i^+ \cap C_{i-1}^- \simeq S^{i-2} \times S^{n-1-i} \quad (2 \le i \le n-1), \\ \partial C_{n-1}^- = \emptyset = \partial C_1^+. \end{aligned}$$

Put $V_i = \pm(\partial/\partial\mu_i)/\|\partial/\partial\mu_i\|$ $(1 \le i \le n-1)$. One can see that at each point $v \in U_pM - \partial C_i^{\pm}$, the vector field V_i is smoothly defined on a neighborhood of the point by taking the appropriate sign. Let $\gamma_v(t)$ be the geodesic on M with the initial vector $\dot{\gamma}_v(0) = v \in U_pM$ and let $Y_i(t, v)$ $(1 \le i \le n-1)$ be the Jacobi field along the geodesic $\gamma_v(t)$ defined by the initial data $Y_i(0, v) = 0$, $Y_i'(0, v) = V_i(v)$ ("prime" represents the covariant derivative in t). Assume first that $v \notin \partial C_j^{\pm}$ for any j. Then as was already shown in [4, Proposition 5.1], the Jacobi field $Y_i(t, v)$ is of the form

$$Y_i(t, v) = y_i(t, v)\tilde{V}_i(t, v),$$

where $y_i(t, v)$ is a function and $\tilde{V}_i(t, v)$ is the parallel vector field along the geodesic $\gamma_v(t)$ such that $\tilde{V}_i(0, v) = V_i(v)$. (Actually, we may say $\tilde{V}_i(t, v) = V_i(\dot{\gamma}_v(t))$.) Let $t = r_i(v)$ be the first zero of the function $t \mapsto y_i(t, v)$ for $t > 0$. It turns out that the function $r_i(v)$ can be continuously extended to all over U_pM and is of C^∞ outside $\partial C_i^{\pm}$. Then our first result is the following:

A (Proposition 6.1)

(1) $r_{n-1}(v) \le r_{n-2}(v) \le \cdots \le r_1(v)$ *for any* $v \in U_pM$.
(2) $r_{i-1}(v) = r_i(v)$ *if and only if* $v \in \partial C_{i-1}^- = \partial C_i^+$ $(2 \le i \le n-1)$.

Put

$$\tilde{K}_i(p) = \{r_i(v)v \mid v \in U_pM\}, \quad K_i(p) = \{\gamma_v(r_i(v)) \mid v \in U_pM\}.$$

As a consequence of the above proposition, we have

B (Theorem 6.2)

(1) $K_{n-1}(p)$ *is the (first) conjugate locus of* p.

3-dimensional case

$$U_pM \quad \ni \quad v \quad \longmapsto \quad \mathrm{Exp}_p(r_2(v)v) \quad \in \quad K_2(p) \subset M$$

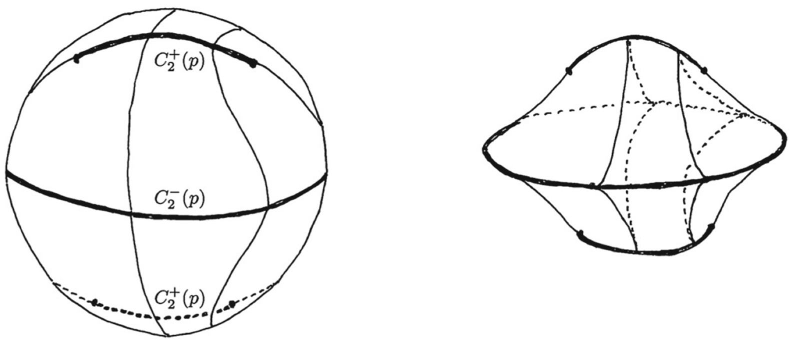

Fig. 1 3D case

(2) If M is close to the round sphere in an appropriate sense, then $K_{n-i}(p)$ is the ith conjugate locus of p for $2 \leq i \leq n-1$.

In this case, $\tilde{K}_{n-1}(p)$ is called the *tangential conjugate locus* of p, and under the situation of (2) $\tilde{K}_{n-i}(p)$ is called the ith tangential conjugate locus. The assumption in (2) of the above theorem is actually given as follows: "if the second zero, say $r_{n-1}^2(v)$, of $y_{n-1}(t, v)$ is greater than $r_1(v)$ for any $v \in U_pM$".

Next, let us explain our results on the "singular points" of the conjugate locus. Define the map $\Phi : U_pM \to M$ by

$$\Phi(v) = \mathrm{Exp}_p(r_{n-1}(v)v) = \gamma_v(r_{n-1}(v)),$$

whose image is the conjugate locus $K_{n-1}(p)$ of p. Then we have

C (Theorem 6.7)

(1) Φ is an immersion outside $C_{n-1}^- \cup C_{n-1}^+$.
(2) The germ of Φ is a cuspidal edge at each point of C_{n-1}^- and each interior point of C_{n-1}^+; the restriction of Φ to (the interior of) $C_{n-1}^\pm$ are immersions to the edges of the vertices.

See Fig. 1 for the three-dimensional case.

We note that, as was shown in [4, Theorem 7.1], the restriction of Φ to C_{n-1}^- is actually an embedding and the image bounds the cut locus of p.

As for the singularities arising on the boundary ∂C_{n-1}^+, we need to treat them as the singularities of the map $\mathrm{Exp}_p : T_pM \to M$ since the map Φ is not differentiable at points on ∂C_{n-1}^+ (and the tangential conjugate locus $\tilde{K}_{n-1}(p)$ is not smooth at $r_{n-1}(v)v$, $v \in \partial C_{n-1}^+$). However, it should be noted that the function r_{n-1} restricted

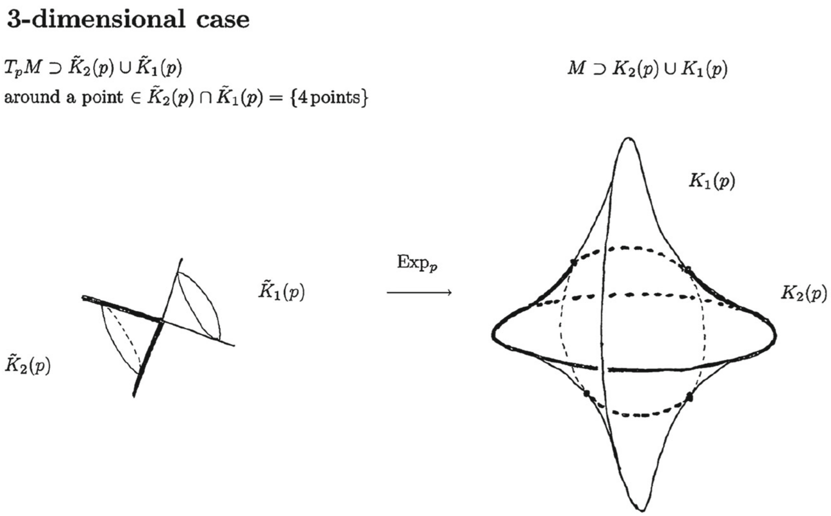

Fig. 2 The first and the second conjugate locus

to ∂C^+_{n-1} is smooth. Thus, $\tilde{S} = \{r_{n-1}(v)v \mid v \in \partial C^+_{n-1}\}$ is a submanifold of T_pM diffeomorphic to $S^{n-3} \times S^0$.

D (Corollary 7.13) *The germ of the map* $\mathrm{Exp}_p : T_pM \to M$ *at each point* $w \in \tilde{S}$ *is a* D_4^+ *Lagrangian singularity.*

The notion of D_4^+ Lagrangian singularity first appeared in the work of Arnold [1], where he classified the "simple Lagrangian singularities" (see also [2]). We shall give its precise description in §7. Here we only note one consequence of the above result: The singularity at each point of $\tilde{S} \subset \tilde{K}_{n-1}(p)$ is a cone-edge, i.e., there is a coordinate system $(w_1, \dots, w_n)$ on T_pM around the point (represented by the origin) such that $\tilde{K}_{n-1}(p)$ and $\tilde{S}$ are described as

$$\begin{aligned} &\tilde{K}_{n-1}(p): \quad w_1^2 + w_2^2 = w_3^2, \quad w_3 \le 0, \\ &\tilde{S}: \quad w_1 = w_2 = w_3 = 0 \end{aligned}$$

and this cone-edge is connected to $\tilde{K}_{n-2}(p)$ as

$$\tilde{K}_{n-2}(p): \quad w_1^2 + w_2^2 = w_3^2, \quad w_3 \ge 0.$$

This illustrates the zeros of the smooth function $\det d(\mathrm{Exp}_p)$ near $r_{n-1}(v)v$, where $v \in \partial C^+_{n-1}$. See Fig. 2 for the three-dimensional case.

Under the situation of the statement **B** (2), we have the similar result for $K_{n-i}(p)$.

E (Theorem 6.7, Corollary 7.14) *Suppose* $r^2_{n-1}(v)$ *is greater than* $r_1(v)$ *for any* $v \in U_pM$. *Then defining the map* $\Phi_{n-i} : U_pM \to M$ *by*

$$\Phi_{n-i}(v) = Exp_p(r_{n-i}(v)v),$$

we have

(1) Φ_{n-i} *is an immersion outside* $C^-_{n-i} \cup C^+_{n-i}$.
(2) The germ of Φ_{n-i} *is a cuspidal edge at each interior point of* C^-_{n-i} *and* C^+_{n-i}*; the restriction of* Φ_{n-i} *to the interior of* $C^{\pm}_{n-i}$ *is immersions to the edges of the vertices.*
(3) The germ of the map $\mathrm{Exp}_p : T_pM \to M$ *at each point* $r_{n-i}(v)v$, $v \in \partial C^{\pm}_{n-i}$, *is a* D_4^+ *Lagrangian singularity and the restriction Exp* $_p|_{\partial C^{\pm}_{n-i}}$ *is an immersion to the edge of vertices.*

The present paper is partly a continuation of our previous paper [4], where we studied the cut loci of points on certain Liouville manifolds diffeomorphic to the sphere. Each Liouville manifold which is considered here is, as in [4], defined with $n+1$ constants $a_0 > \cdots > a_n > 0$ and a positive function $A(\lambda)$ on $[a_n, a_0]$ ($A(\lambda) = \sqrt{\lambda}$ in the case of the ellipsoid). This function $A(\lambda)$ is assumed to satisfy a certain monotonicity condition, which is a bit stronger than condition (4.1) posed in [4]. We shall explain this condition in §4. In §2 and §3, we give a brief summary of Liouville manifolds and the behavior of geodesics on them. Since they are almost the same as those in [4], we omit the proofs there.

Under the condition given in §4, we shall investigate the positions of zeros of Jacobi fields in detail in §5 and describe the structure of the conjugate locus of a general point in §6. In particular, we shall show in this section that the major part of the singularities of the conjugate locus of a general point are cuspidal edges. Also, as an application of the results obtained in §5, we shall illustrate there an interesting asymptotic nature of the distribution of zeros of Jacobi fields.

In §7, we investigate the remaining singularities, which appear as the end points of the cuspidal edges and which are also points of double conjugacy. We shall show there that those are D_4^+ Lagrangian singularities. In the final section, §8, we mention some problems related to our results. We added two figures in this introduction at the request of the referee, which are hand-written and numerical correctness is not expected, though.

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Preliminary Remarks and Notations

In this paper, the geodesics will be described in the Hamiltonian formalism. Therefore, the geodesic flow is described in the cotangent bundle. Let M be a Riemannian manifold and g its Riemannian metric. By $\flat : TM \to T^*M$, we denote the bundle isomorphism determined by g (Legendre transformation). We also use the symbol $\sharp = \flat^{-1}$. The canonical 1-form on the cotangent bundle T^*M is denoted by α. For a canonical coordinate system (x, ξ) on T^*M (x being a coordinate system on M), α is expressed as $\sum_i \xi_i dx_i$. Then the 2-form $d\alpha$ represents the standard symplectic structure on T^*M.

Let E be the function on T^*M defined by

$$E(\lambda) = \frac{1}{2} g(\sharp(\lambda), \sharp(\lambda)) = \frac{1}{2} \sum_{i,j} g^{ij}(x) \xi_i \xi_j, \quad \lambda = (x, \xi) \in T^*M.$$

We call it the (kinetic) energy function of M. For a function F, H on T^*M, the Hamiltonian vector field X_F and the Poisson bracket $\{F, H\}$ are defined by

$$X_F = \sum_i \left(\frac{\partial F}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i} \frac{\partial}{\partial \xi_i} \right), \qquad \{F, H\} = X_F H.$$

Then X_E generates the geodesic flow $\{\zeta_t\}_{t \in \mathbb{R}}$, i.e., each curve $\gamma(t) = \pi(\zeta_t \lambda)$ ($\lambda \in T^*M$) is a geodesic of the Riemannian manifold M, where $\pi : T^*M \to M$ is the bundle projection. In this case, we have $\flat(\dot{\gamma}(t)) = \zeta_t \lambda$.

2 Liouville Manifolds

Liouville manifold is, roughly speaking, a class of Riemannian manifold whose geodesic equations are "integrated in the same way as those of ellipsoids". The precise definition is as follows. Let M be a Riemannian manifold of dimension n and let $\mathcal{F}$ be an n-dimensional vector space of functions on the cotangent bundle T^*M. Then the pair $(M, \mathcal{F})$ is called a Liouville manifold if (1) each $F \in \mathcal{F}$ is fiberwise a homogeneous quadratic polynomial; (2) those quadratic forms are simultaneously normalizable on each fiber; (3) $\mathcal{F}$ is commutative with respect to the Poisson bracket; (4) $\mathcal{F}$ contains the energy function E (the Hamiltonian of the geodesic flow); and (5) $\{F|_{T_p^*M} \mid F \in \mathcal{F}\}$ is n-dimensional at some point $p \in M$. For the generality of Liouville manifolds, we refer to [9].

As in [4], we treat in this paper a subclass of "compact Liouville manifolds of rank one and type (A) (cf. [9])". An explanation of this subclass and the geodesic equations on it were already given in [4]. We shall briefly illustrate it in this and the next sections (without proof) for the sake of convenience.

Each Liouville manifold treated here is constructed from $n + 1$ constants $a_0 > \cdots > a_n > 0$ and a positive C^∞ function $A(\lambda)$ on the closed interval $a_n \le \lambda \le a_0$. Let $\alpha_1, \dots, \alpha_n$ be positive numbers defined by

$$\alpha_i = 2 \int_{a_i}^{a_{i-1}} \frac{A(\lambda)\, d\lambda}{\sqrt{(-1)^i \prod_{j=0}^n (\lambda - a_j)}} \qquad (i = 1, \dots, n).$$

Define the C^∞ function f_i on the circle $\mathbb{R}/\alpha_i \mathbb{Z} = \{x_i\}$ ($1 \le i \le n$) by the conditions:

$$\left(\frac{df_i}{dx_i} \right)^2 = \frac{(-1)^i 4 \prod_{j=0}^n (f_i - a_j)}{A(f_i)^2} \tag{2.1}$$

$$f_i(0) = a_i, \; f_i(\frac{\alpha_i}{4}) = a_{i-1}, \quad f_i(-x_i) = f_i(x_i) = f_i\left(\frac{\alpha_i}{2} - x_i \right). \tag{2.2}$$

Then the range of f_i is $[a_i, a_{i-1}]$, and f_i actually has the period $\alpha_i/2$.

Put

$$R = \prod_{i=1}^{n} (\mathbb{R}/\alpha_i \mathbb{Z}).$$

Let τ_i $(1 \leq i \leq n-1)$ be the involutions on the torus R defined by

$$\tau_i(x_1, \ldots, x_n) = \left(x_1, \ldots, x_{i-1}, -x_i, \frac{\alpha_{i+1}}{2} - x_{i+1}, x_{i+2}, \ldots, x_n\right),$$

and let G $(\simeq (\mathbb{Z}/2\mathbb{Z})^{n-1})$ be the group of transformations generated by $\tau_1, \ldots, \tau_{n-1}$. Then the quotient space $M = R/G$ is homeomorphic to the n-sphere, and moreover, M has a unique differentiable structure so that the quotient map $R \to M$ is of C^∞ and the symmetric 2-form g given by

$$g = \sum_i (-1)^{n-i} \left(\prod_{l \neq i} (f_l(x_l) - f_i(x_i)) \right) dx_i^2, \tag{2.3}$$

represents a C^∞ Riemannian metric on M. We regard M as a Riemannian manifold with this metric g. As a result, M is diffeomorphic to the n-sphere S^n.

Now, put

$$b_{ij}(x_i) = \begin{cases} (-1)^i \prod_{\substack{1 \leq k \leq n-1 \\ k \neq j}} (f_i(x_i) - a_k) & (1 \leq j \leq n-1) \\ (-1)^{i+1} \prod_{k=1}^{n-1} (f_i(x_i) - a_k) & (j = n) \end{cases},$$

and define functions $F_1, \ldots, F_{n-1}, F_n = 2E$ on the cotangent bundle by

$$\sum_{j=1}^{n} b_{ij}(x_i) F_j(x, \xi) = \xi_i^2 \quad (1 \leq i \leq n), \tag{2.4}$$

where ξ_i are the fiber coordinates with respect to the base coordinates $(x_1, \ldots, x_n)$. Then F_i represent well-defined C^∞ functions on T^*M.

Computing the inverse matrix of (b_{ij}) explicitly, we have

$$2E = \sum_{i=1}^{n} \frac{(-1)^{n-i} \xi_i^2}{\prod_{\substack{1 \leq l \leq n \\ l \neq i}} (f_l(x_l) - f_i(x_i))}$$

$$F_j = \frac{1}{\prod_{\substack{1 \leq k \leq n-1 \\ k \neq j}} (a_k - a_j)} \sum_{i=1}^{n} \frac{(-1)^{n-i} \prod_{\substack{1 \leq l \leq n \\ l \neq i}} (f_l(x_l) - a_j)}{\prod_{\substack{1 \leq l \leq n \\ l \neq i}} (f_l(x_l) - f_i(x_i))} \xi_i^2$$

$$(1 \leq j \leq n-1).$$

Therefore, E is the energy function, i.e., the Hamiltonian of the associated geodesic flow of M. From formula (2.4), one can easily see that

$$\{F_i, F_j\} = 0 \qquad (1 \le i, j \le n),$$

where $\{,\}$ denotes the Poisson bracket (see [9, Prop. 1.1.3]). Thus, denoted by $\mathcal{F}$ the vector space spanned by $F_1, \dots, F_n$, the pair $(M, \mathcal{F})$ becomes a Liouville manifold.

The following proposition is obvious.

Proposition 2.1 *For each i, the map*

$$x \mapsto (\dots, x_{i-1}, -x_i, x_{i+1}, \dots) \quad or \quad x \mapsto \left(\dots, x_i, \frac{\alpha_{i+1}}{2} - x_{i+1}, x_{i+2}, \dots\right)$$

defines an isometry of M which preserves F_j for any j. This map is the symmetry with respect to N_i defined below.

As examples, if $A(\lambda)$ is a constant function, then M is the sphere of constant curvature. This case is explained in detail in [9, pp.71–74]. If $A(\lambda) = \sqrt{\lambda}$, then M is isometric to the ellipsoid $\sum_{i=0}^{n} u_i^2/a_i = 1$. In this case, the system of functions $(f_1(x_1), \dots, f_n(x_n))$ is nothing but the elliptic coordinate system (see [4, p.261]).

The manifold M has some special submanifolds: put

$$N_k = \{x \in M \mid f_k(x_k) = a_k \quad \text{or} \quad f_{k+1}(x_{k+1}) = a_k\} \quad (0 \le k \le n),$$
$$J_k = \{x \in M \mid f_k(x_k) = f_{k+1}(x_{k+1}) = a_k\} \quad (1 \le k \le n-1).$$

Then we have, putting $(F_k)_p = F_k|_{T_p^*M}$,

Proposition 2.2 *(1) $J_k = \{p \in M \mid (F_k)_p = 0\}$.*
(2) $N_k = \{p \in M \mid$ rank $(F_k)_p \le 1\} \quad (1 \le k \le n-1)$.
(3) $\bigcup_k J_k$ is identical with the branch locus of the covering $R \to M = R/G$.
(4) N_k is a totally geodesic submanifold of codimension one $(0 \le k \le n)$.
(5) $J_k \subset N_k$, and J_k is diffeomorphic to $S^{k-1} \times S^{n-k-1}$.

In the case of the ellipsoid, N_k is equal to the intersection with the Cartesian hyperplane $u_k = 0$, and the set $\bigcup_k J_k$ is identical with the locus where some principal curvature has multiplicity two.

3 Geodesic Equations

Suppose that $c = (c_1, \dots, c_{n-1}, 1)$ is a regular value of the map

$$\mathbf{F} = (F_1, \dots, F_{n-1}, 2E) : T^*M \to \mathbb{R}^n,$$

then its inverse image is a disjoint union of tori, and the vector fields X_{F_j}, X_E on it are mutually commutative and linearly independent everywhere. Here X_f denotes the

Hamiltonian vector field determined by a function f;

$$X_f = \sum_i \left(\frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial}{\partial \xi_i} \right).$$

Let ω_j $(1 \leq j \leq n)$ be the dual 1-forms of $\{\pi_* X_{F_j}\}$, where $\pi : T^*M \to M$ is the bundle projection. Then by (2.4), we have

$$\omega_l = \sum_i \frac{b_{il}}{2\xi_i}\, dx_i \qquad (1 \leq l \leq n).$$

They are closed 1-forms, and the geodesic orbits are determined by

$$\omega_l = 0 \qquad (1 \leq l \leq n-1), \tag{3.1}$$

and the length parameter t on an orbit is given by

$$dt = 2\omega_n. \tag{3.2}$$

Thus, the geodesics are described with the integration of closed 1-forms which contains c_is as parameters.

To observe the behavior of geodesics, it is more convenient to use the constants $b_1, \dots, b_{n-1}$ defined below than using c_is as parameters. Put

$$\Theta(\lambda) = \sum_{j=1}^{n-1} \left(\prod_{\substack{1 \leq k \leq n-1 \\ k \neq j}} (\lambda - a_k) \right) c_j - \prod_{k=1}^{n-1} (\lambda - a_k).$$

If a unit covector $(x, \xi) \in U^*M$ with every $\xi_i \neq 0$ lies on $\mathbf{F}^{-1}(c)$, then we have, by (2.4),

$$(-1)^i \Theta(f_i(x_i)) = \xi_i^2 > 0.$$

Therefore, the algebraic equation $\Theta(\lambda) = 0$ has $n-1$ real roots in this case. It thus follows by continuity that for each $c \in \mathbf{F}(U^*M)$, there are constants $b_1 \geq \cdots \geq b_{n-1}$ such that

$$\Theta(\lambda) = -\prod_{i=1}^{n-1} (\lambda - b_i),$$
$$f_i(x_i) \geq b_i \geq f_{i+1}(x_{i+1}) \qquad (1 \leq i \leq n-1).$$

Note that the range of b_is are given by

$$a_{i+1} \leq b_i \leq a_{i-1}, \qquad b_{i+1} \leq b_i. \tag{3.3}$$

In fact, it can be verified that for each b_is satisfying (3.3) there is a unit covector $\mu \in U^*M$ such that $F_i(\mu) = c_i$ $(1 \leq i \leq n-1)$, where

$$c_i = \frac{-\prod_{l=1}^{n-1}(a_i - b_l)}{\prod_{\substack{1\leq k\leq n-1\\ k\neq i}}(a_i - a_k)} \qquad (1 \leq i \leq n-1). \tag{3.4}$$

Note also that if $b_1, \ldots, b_{n-1}$ satisfy

$$a_{i+1} < b_i < a_{i-1}, \quad b_i \neq a_i, \quad b_{i+1} < b_i \qquad \text{for any } i, \tag{3.5}$$

then the corresponding $c = (c_1, \ldots, c_{n-1}, 1)$ is a regular value of $\mathbf{F}$.

Now, put

$$a_i^+ = \max\{a_i, b_i\} \quad (1 \leq i \leq n-1), \quad a_n^+ = a_n$$
$$a_i^- = \min\{a_i, b_i\} \quad (1 \leq i \leq n-1), \quad a_0^- = a_0.$$

If $b_1, \ldots, b_{n-1}$ satisfy condition (3.5), then the π-image of a connected component of $\mathbf{F}^{-1}(c)$ (a Lagrange torus) is of the form

$$L_1 \times \cdots \times L_n \subset M,$$

where each L_i is a connected component of the inverse image of $[a_i^+, a_{i-1}^-]$ by the map

$$f_i : \mathbb{R}/\alpha_i\mathbb{Z} \to [a_i, a_{i-1}].$$

(Precisely speaking, $L_1 \times \cdots \times L_n \subset R$; but it is injectively mapped into M by the branched covering $R \to M$.) Along a corresponding geodesic, the coordinate function $x_i(t)$ moves on L_i and $f_i(x_i(t)) \in [a_i^+, a_{i-1}^-]$.

After all, the equations of geodesic orbits

$$\omega_l = 0 \qquad (1 \leq l \leq n-1),$$

are described as

$$\sum_{i=1}^{n} \frac{\epsilon_i(-1)^i \prod_{\substack{1\leq k\leq n-1\\ k\neq l}}(f_i(x_i) - a_k)\, dx_i}{\sqrt{(-1)^{i-1}\prod_{k=1}^{n-1}(f_i(x_i) - b_k)}} = 0 \qquad (1 \leq l \leq n-1),$$

where $\epsilon_i = \operatorname{sign}(\xi_i) = \operatorname{sign}(dx_i/dt)$. This system of equations is equivalent to

$$\sum_{i=1}^{n} \frac{\epsilon_i(-1)^i G(f_i)\, dx_i}{\sqrt{(-1)^{i-1}\prod_{k=1}^{n-1}(f_i - b_k)}} = 0, \tag{3.6}$$

for any polynomial $G(\lambda)$ of degree $\le n-2$. Thus, by (3.6), we have

$$\sum_{i=1}^{n} \int_s^t \frac{(-1)^i G(f_i)}{\sqrt{(-1)^{i-1} \prod_{k=1}^{n-1}(f_i - b_k)}} \left| \frac{dx_i(t)}{dt} \right| dt = 0, \tag{3.7}$$

for any period $[s,t]$, where $f_i = f_i(x_i(t))$. By (2.1), those equations are also described as

$$\sum_{i=1}^{n} \int_s^t \frac{(-1)^i G(f_i) A(f_i)}{\sqrt{-\prod_{k=1}^{n-1}(f_i - b_k) \cdot \prod_{k=0}^{n}(f_i - a_k)}} \left| \frac{df_i(x_i(t))}{dt} \right| dt = 0. \tag{3.8}$$

Also, integrating $dt = 2\omega_n = \sum_i (b_{in}/\xi_i) dx_i$, we have

$$\sum_{i=1}^{n} \int_s^t \frac{(-1)^{i+1} \tilde{G}(f_i)}{\sqrt{(-1)^{i-1} \prod_{k=1}^{n-1}(f_i - b_k)}} \left| \frac{dx_i(t)}{dt} \right| dt = t - s, \tag{3.9}$$

where $\tilde{G}(\lambda)$ is any monic polynomial in λ of degree $n-1$.

Finally, let us illustrate the behavior of each coordinate function $x_i(t)$ along a geodesic $\gamma(t) = (x_1(t), \dots, x_n(t))$. Since

$$x_i'(t) = \frac{\partial E}{\partial \xi_i} = \frac{\pm\sqrt{(-1)^{i-1} \prod_{k=1}^{n-1}(f_i(x_i) - b_k)}}{(-1)^{n-i} \prod_{l \ne i}(f_l(x_l) - f_i(x_i))}, \tag{3.10}$$

and since

$$f_i - b_{i+1} \ge f_i - f_{i+1} \ge 0, \quad b_{i-2} - f_i \ge f_{i-1} - f_i \ge 0,$$

we have

$$\left| x_i'(t) \right| \ge c\sqrt{(f_i(x_i)) - b_i)(b_{i-1} - f_i(x_i))},$$

for some constant $c > 0$ (at least, outside the branch locus of the covering $R \to M$). Therefore, if $a_i^+ < f_i(x_i(t)) < a_{i-1}^-$, then it reaches the boundary at a finite time $t = t_0$. And if $f_i(x_i(t_0))$ is equal to $b_i (\ne a_i)$ or $b_{i-1} (\ne a_{i-1})$, then $x_i'(t_0) = 0$, $x_i''(t_0) \ne 0$, and if $f_i(x_i(t_0))$ is equal to $a_i (\ne b_i)$ or $a_{i-1} (\ne b_{i-1})$, then $x_i'(t_0) \ne 0$; and $(d/dt) f_i(x_i(t))$ changes the sign when t passes t_0 in each case. Thus, if $a_i^- < a_i^+ < a_{i-1}^- < a_{i-1}^+$, then $x_i(t)$ oscillates on L_i if L_i is an interval, or $x_i(t)$ moves monotonously if L_i is the whole circle, and the function $f_i(x_i(t))$ oscillates on the interval $[a_i^+, a_{i-1}^-]$.

If $a_{i+1}^+ < a_i^- = a_i^+ < a_{i-1}^-$ and if $x_i(t)$ and $x_{i+1}(t)$ satisfy

$$a_{i+1}^+ < f_{i+1}(x_{i+1}(t)) < a_i^- = a_i^+ < f_i(x_i(t)) < a_{i-1}^-, \quad \frac{d}{dt} f_i(x_i(t)) < 0,$$

at some t, then $x_i(t)$ reaches the boundary point $\in f_i^{-1}(a_i^+)$ at a finite time t_0 by the same reason as above. Moreover, at that time,

$$f_{i+1}(x_{i+1}(t_0)) = a_i^- = a_i^+ = f_i(x_i(t_0)),$$

and this point $\gamma(t_0)$ is a branch point of the covering $R \to M$. In fact, observe formula (3.8) for $G(\lambda) = \prod_{k\neq i}(\lambda - b_k)$ and take the limit $t \to t_0 - 0$ there. Since the ith summand tends to ∞, if other b_j and a_k are all distinct, then the $(i+1)$-st summand must tend to ∞ and thus $f_{i+1}(x_{i+1}(t_0)) = a_i^\pm$. In this case, the geodesic passes through a point on J_i and intersects N_i transversally at the point. For the general case, we take a sequence of geodesics satisfying the above condition and obtain the same result. Since $\gamma(t_0)$ is a branch point, there are two possible ways of description for $x_i(t)$ after passing the time t_0; but anyway, the function $f_i(x_i(t))$ turns the direction at $t = t_0$ and $(d/dt)(f_i(x_i(t)) > 0$ for $t > t_0$ near t_0.

If $a_i^+ = a_{i-1}^-$, then there are several possible cases. If $a_i < b_i = b_{i-1} < a_{i-1}$, then $x_i(t)$ is constant along the geodesic. This case will be investigated in detail in §7. If $a_i = b_{i-1}$ (resp. if $b_i = a_{i-1}$), then the function $x_i(t)$ is again constant, but in this case the geodesic is totally contained in the totally geodesic submanifold N_i (resp. N_{i-1}), and such type of geodesic is not considered in this paper.

4 A Monotonicity Condition for Liouville Manifolds

In this section, we first introduce a monotonicity condition on the positive function $A(\lambda)$, under which the structures of the conjugate loci on the corresponding Liouville manifolds become simple:

$$\begin{aligned}&\text{The function } \tilde{A}(\lambda) = (\lambda - a_n)A(\lambda) \text{ satisfies}\\ &\quad (-1)^k \tilde{A}^{(k)}(\lambda) > 0 \quad \text{on } [a_n, a_0] \qquad (2 \le k \le n),\end{aligned} \tag{4.1}$$

where $\tilde{A}^{(k)}(\lambda)$ denotes the kth derivative of $\tilde{A}(\lambda)$ in λ. The following proposition indicates that this condition is stronger than the condition (4.1) in [4], which is

$$(-1)^k A^{(k)}(\lambda) < 0 \quad \text{on } [a_n, a_0] \quad (1 \le k \le n-1). \tag{4.2}$$

Proposition 4.1 *If a positive function $A(\lambda)$ on $[a_n, a_0]$ satisfies the condition (4.1), then it also satisfies the condition (4.2).*

Proof Since $A(\lambda)$ is described in the form

$$A(\lambda) = \frac{\tilde{A}(\lambda) - \tilde{A}(a_n)}{\lambda - a_n} = \int_0^1 \tilde{A}'(t\lambda + (1-t)a_n)\, dt,$$

we have

$$A^{(k)}(\lambda) = \int_0^1 t^k\, \tilde{A}^{(k+1)}(t\lambda + (1-t)a_n)\, dt \qquad (1 \le k \le n-1).$$

The lemma immediately follows from this formula. □

Remark 4.2 If a positive function $A(\lambda)$ satisfies the condition (4.1), then $\tilde{A}'(\lambda)$ is positive on $[a_n, a_0]$. In fact, $\tilde{A}'(\lambda) = A(\lambda) + (\lambda - a_n)A'(\lambda)$ and $A'(\lambda) > 0$ by Proposition 4.1.

It is easily seen that $A(\lambda) = \sqrt{\lambda}$, i.e., the case of the ellipsoid $\sum_i u_i^2/a_i = 1$, satisfies the condition (4.1). From now on, we shall always assume that the condition (4.1) is satisfied. Added to Proposition 4.1 in [4], we shall prove a similar proposition below. To do so, we need two lemmas; the first one being the same as [4, Lemmas 4.2], we shall omit the proof. For the two lemmas we assume $b_1, \dots, b_{n-1}$ and $a_0, \dots, a_n$ are all distinct.

Lemma 4.3

$$\sum_{i=1}^{n} \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^i G(\lambda)\, d\lambda}{\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k) \cdot \prod_{k=0}^{n}(\lambda - a_k)}} = 0,$$

for any polynomial $G(\lambda)$ *of degree* $\leq n - 2$.

Lemma 4.4 *Let* J *be any subset of* $\{1, \dots, n-1\}$*, and let* $B(\lambda)$ *be the function defined by*

$$\frac{A(\lambda)(\lambda - a_n)}{\prod_{j \in J}(\lambda - b_j)} = \sum_{j \in J} \frac{e_j}{\lambda - b_j} + B(\lambda), \quad e_j = \frac{A(b_j)(b_j - a_n)}{\prod_{\substack{l \in J \\ l \neq j}}(b_j - b_l)}. \tag{4.3}$$

Then

$$B(\lambda) = \int_{D_k} \tilde{A}^{(k)}\left(\left(1 - \sum_{l=1}^{k} s_l\right)\lambda + s_k b_{i_k} + \dots + s_1 b_{i_1}\right) ds_1 \dots ds_k, \tag{4.4}$$

where $J = \{i_1, \dots, i_k\}$ *and*

$$D_k = \{(s_1, \dots, s_k) \in \mathbb{R}^k \mid s_i \geq 0\,(1 \leq i \leq k), \sum_{i=1}^{k} s_i \leq 1\}.$$

In particular, $B(\lambda)$ *satisfies* $(-1)^{\#J} B(\lambda) > 0$ *for* $a_n \leq \lambda \leq a_0$ *if* $\#J \geq 2$.

Proof We prove this by induction in $k = \#J$. When $k = 0$, the assertion is trivial. Let $k \geq 0$ and assume that the assertion is true for J with $\#J \leq k$. Suppose $J = \{i_1, \dots, i_{k+1}\}$ and put $J_0 = J - \{i_{k+1}\}$. Define $B_0(\lambda)$ as the function $B(\lambda)$ in formula (4.3) for J_0. By the induction assumption we have formula (4.4) for B_0.

By the defining formula (4.3), the functions $B(\lambda)$ for J and $B_0(\lambda)$ for J_0 are related as

$$B(\lambda) = \frac{B_0(\lambda) - B_0(b_{i_{k+1}})}{\lambda - b_{i_{k+1}}} = \int_0^1 B_0'(t\lambda + (1-t)b_{i_{k+1}})\, dt.$$

By the induction assumption the right-hand side is equal to

$$\int_0^1 \int_{D_k} \Big(1-\sum_{l=1}^k s_l\Big) \tilde{A}^{(k+1)}\Big(\Big(1-\sum_{l=1}^k s_l\Big)(t\lambda+(1-t)b_{i_{k+1}})+\sum_{l=1}^k s_l b_{i_l}\Big)\, ds_1 \ldots ds_k dt.$$

Therefore, changing the variable $t \to s_{k+1} = (1 - \sum_{l=1}^k s_l)(1 - t)$, we obtain formula (4.4) for J. □

Proposition 4.5 *If $b_1, \ldots, b_{n-1}$ and $a_0, \ldots, a_n$ are all distinct, then the following inequalities hold:*

(1)

$$\sum_{l=1}^{n} \int_{a_l^+}^{a_{l-1}^-} \frac{(-1)^{n-l+\#I} A(\lambda)\,(\lambda - a_n) \prod_{j \in I}(\lambda - b_j)}{\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k) \cdot \prod_{k=0}^{n}(\lambda - a_k)}}\, d\lambda > 0,$$

where I is any (possibly empty) subset of $\{1, \ldots, n-1\}$ such that $\#I \le n-3$.

(2)

$$\frac{\partial}{\partial b_i} \sum_{l=1}^{n} \int_{a_l^+}^{a_{l-1}^-} \frac{(-1)^l A(\lambda)\,(\lambda - a_n)\, G(\lambda)\, d\lambda}{\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k) \cdot \prod_{k=0}^{n}(\lambda - a_k)}},$$

is negative for $G(\lambda) = \prod_{k \ne i}(\lambda - b_k)$ and is positive for $G = \prod_{k \ne i,j}(\lambda - b_k)$, $(j \ne i)$.

(3)

$$\frac{\partial^2}{\partial b_i^2} \sum_{l=1}^{n} \int_{a_l^+}^{a_{l-1}^-} \frac{(-1)^l A(\lambda)\,(\lambda - a_n)\, G(\lambda)\, d\lambda}{\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k) \cdot \prod_{k=0}^{n}(\lambda - a_k)}}$$

is positive for $G(\lambda) = \prod_{k \ne i}(\lambda - b_k)$.

Proof Put $J = \{1, \ldots, n-1\} - I$, $\tilde{A}(\lambda) = (\lambda - a_n)A(\lambda)$ and define $B(\lambda)$ by

$$\frac{\tilde{A}(\lambda)}{\prod_{j \in J}(\lambda - b_j)} = \sum_{j \in J} \frac{1}{\lambda - b_j} \frac{\tilde{A}(b_j)}{\prod_{\substack{k \in J \\ k \ne j}}(b_j - b_k)} + B(\lambda). \tag{4.5}$$

Then by Lemma 4.4 we have $(-1)^{\#J} B(\lambda) > 0$ on the interval $[a_n, a_0]$. Since the sum in (1) is equal to

$$\sum_{l=1}^{n} \int_{a_l^+}^{a_{l-1}^-} \frac{(-1)^{l-1+\#J} B(\lambda) \prod_{k=1}^{n-1}(\lambda - b_k)}{\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k) \cdot \prod_{k=0}^{n}(\lambda - a_k)}}\, d\lambda, \tag{4.6}$$

by Lemma 4.3, and since $(-1)^{l-1} \prod_{k=1}^{n-1}(\lambda - b_k)$ is positive on every interval (a_l^+, a_{l-1}^-), we have the inequality (1).

To prove (2) for $G(\lambda) = \prod_{k \neq i,j}(\lambda - b_k)$, we use formula (4.5) with $J = \{i, j\}$. In this case,

$$B(\lambda) = \int_0^1 \int_0^{1-t} \tilde{A}''((1 - t - s)\lambda + tb_i + sb_j)\, dsdt,$$

and the formula in (2) is written as

$$\frac{\partial}{\partial b_i} \sum_{l=1}^{n} \int_{a_l^+}^{a_{l-1}^-} \frac{(-1)^l B(\lambda) \prod_{k=1}^{n-1}(\lambda - b_k)}{\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k) \cdot \prod_{k=0}^{n}(\lambda - a_k)}}\, d\lambda. \tag{4.7}$$

Then in the same way as the proof of Proposition 4.1 (2) in [4], we see that the above formula is equal to

$$\frac{1}{2} \sum_{l=1}^{n} \int_{a_l^+}^{a_{l-1}^-} \frac{(-1)^l \left(\frac{\partial}{\partial b_i} B(\lambda)\right) \prod_{k=1}^{n-1}(\lambda - b_k)}{\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k) \cdot \prod_{k=0}^{n}(\lambda - a_k)}}\, d\lambda, \tag{4.8}$$

which is positive, since $\frac{\partial}{\partial b_i} B(\lambda) < 0$.

In the case where $G(\lambda) = \prod_{k \neq i}(\lambda - b_k)$, we also have the same formula as above with

$$B(\lambda) = \int_0^1 \tilde{A}'(t\lambda + (1 - t)b_i)dt. \tag{4.9}$$

Since $\frac{\partial}{\partial b_i} B(\lambda) > 0$ in this case, the assertion follows.

(3) Differentiating formula (4.8) by b_i under the equality (4.9), we have

$$\begin{aligned} &\frac{1}{2} \sum_{l=1}^{n} \int_{a_l^+}^{a_{l-1}^-} \frac{(-1)^l \left(\frac{\partial^2}{\partial b_i^2} B(\lambda)\right) \prod_{k=1}^{n-1}(\lambda - b_k)}{\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k) \cdot \prod_{k=0}^{n}(\lambda - a_k)}}\, d\lambda \\ &\quad - \frac{1}{4} \sum_{l=1}^{n} \int_{a_l^+}^{a_{l-1}^-} \frac{(-1)^l \left(\frac{\partial}{\partial b_i} B(\lambda)\right) \prod_{k \neq i}(\lambda - b_k)}{\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k) \cdot \prod_{k=0}^{n}(\lambda - a_k)}}\, d\lambda. \end{aligned} \tag{4.10}$$

By Lemma 4.3 in [4], the second line of this formula is equal to

$$-\frac{1}{4} \sum_{l=1}^{n} \int_{a_l^+}^{a_{l-1}^-} \frac{(-1)^l \tilde{B}(\lambda) \prod_{k=1}^{n-1}(\lambda - b_k)}{\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k) \cdot \prod_{k=0}^{n}(\lambda - a_k)}}\, d\lambda, \tag{4.11}$$

where

$$\tilde{B}(\lambda) = \frac{\frac{\partial}{\partial b_i} B(\lambda) - \frac{1}{2}\tilde{A}''(b_i)}{\lambda - b_i} = \frac{1}{2}\frac{\partial^2}{\partial b_i^2} B(\lambda).$$

Note that $(\partial/\partial b_i)B(\lambda)|_{\lambda=b_i} = (1/2)\tilde{A}''(b_i)$.

Therefore, formula (4.10) is equal to

$$\frac{3}{8}\sum_{l=1}^{n}\int_{a_l^+}^{a_{l-1}^-} \frac{(-1)^l \left(\frac{\partial^2}{\partial b_i^2} B(\lambda)\right) \prod_{k=1}^{n-1}(\lambda - b_k)}{\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k)\cdot \prod_{k=0}^{n}(\lambda - a_k)}}\, d\lambda. \tag{4.12}$$

Since

$$\frac{\partial^2}{\partial b_i^2} B(\lambda) = \int_0^1 (1-t)^2 \tilde{A}'''(t\lambda + (1-t)b_i)dt, \tag{4.13}$$

is negative, the assertion follows. □

In the later applications, we also need certain limit cases of the above proposition, which may be stated as follows.

Proposition 4.6 *Let $b^k = (b_1^k, \ldots, b_{n-1}^k)$ $(k = 1, 2, \ldots)$ be a sequence such that*

$$a_{i+1} < b_i^k < a_{i-1},\ b_i^k \neq a_i,\ b_i^k < b_{i-1}^k \ \text{ for any } k, i,$$

and such that the ordering of a_i and b_i^k does not change when k varies for each i. Suppose that b^k converges to $b^\infty = (b_1, \ldots, b_{n-1})$ as $k \to \infty$. Then when $k \to \infty$, each formula in (1), (2), (3) in Proposition 4.5 for b^k converges to a nonzero value, namely those inequalities are still valid in the limit case.

Proof We first consider case (1) in Proposition 4.5. Let us observe formula (4.6) for b^k and take the limit $k \to \infty$. We regard the sum of the integrals as the integral over $[a_n, a_0]$ of the single function $E^k(\lambda)$, where

$$E^k(\lambda) = \begin{cases} \frac{(-1)^{l-1+\#J} B(\lambda) \prod_{i=1}^{n-1}(\lambda - b_i^k)}{\sqrt{-\prod_{i=1}^{n-1}(\lambda - b_i^k)\cdot\prod_{i=0}^{n}(\lambda - a_i)}} & (\lambda \in [a_l^+, a_{l-1}^-]) \\ 0 & (\lambda \notin \cup_{i=1}^{n}[a_i^+, a_{i-1}^-]) \end{cases}.$$

In view of formula (4.4) we see that there is a constant c which does not depend on k such that

$$|E^k(\lambda)| \leq \frac{c}{\sqrt{\prod_{i=0}^{n}|\lambda - a_i|}} \qquad (a_n \leq \lambda \leq a_0),$$

for any k. Therefore, by Lebesgue's convergence theorem, we have

$$\lim_{k\to\infty}\int_{a_n}^{a_0} E^k(\lambda)d\lambda = \int_{a_n}^{a_0} E^\infty(\lambda)d\lambda. \quad E^\infty(\lambda) = \lim_{k\to\infty} E^k(\lambda).$$

Since there is at least one index i such that $a_i^+ = a_i$ and $a_{i-1}^- = a_{i-1}$ for each k and since this index i does not depend on k by the assumption, it follows that $a_i^+ = a_i$ and $a_{i-1}^- = a_{i-1}$ for $k = \infty$. Therefore, $E^\infty(\lambda)$ is positive on the open interval (a_i, a_{i-1}) and nonnegative on the whole interval $[a_n, a_0]$. Thus, the assertion follows.

For (2) and (3), we use (4.8) and (4.12) instead of (4.6). Since the proof goes in completely the same way as above, we omit the detail. □

5 Zeros of Jacobi Fields

Let $\gamma(t) = (x_1(t), \ldots, x_n(t))$ be a geodesic which is not totally contained in the totally geodesic submanifolds N_i $(1 \le i \le n-1)$. Let us denote by $\sigma_i(t)$ the total variation of $f_i(x_i(t))$:

$$\sigma_i(t) = \int_0^t \left|\frac{df_i(x_i(s))}{ds}\right| ds \qquad (1 \le i \le n).$$

When $a_i^+ < a_{i-1}^-$, this function is strictly increasing (cf. §3), and we then define the time $t = t_i$ by the equality

$$\sigma_i(t_i) = 2(a_{i-1}^- - a_i^+), \tag{5.1}$$

which represents a half of the period in some sense. Note that t_n is the same one as t_0 defined in [4, §6]. Note also that, in view of (2.1), the following equalities hold:

$$\begin{aligned}
&\int_0^{t_i} \frac{(-1)^i G(f_i)}{\sqrt{(-1)^{i-1}\prod_{k=1}^{n-1}(f_i - b_k)}} \left|\frac{dx_i(t)}{dt}\right| dt \\
&= \frac{1}{2}\int_0^{t_i} \frac{(-1)^i G(f_i)A(f_i)}{\sqrt{-\prod_{k=1}^{n-1}(f_i - b_k)\cdot\prod_{k=0}^{n}(f_i - a_k)}} \left|\frac{df_i(x_i(t))}{dt}\right| dt \\
&= \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^i G(\lambda)A(\lambda)\, d\lambda}{\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k)\cdot\prod_{k=0}^{n}(\lambda - a_k)}}.
\end{aligned} \tag{5.2}$$

Those equalities will be frequently used below.

In the rest of this section, we shall assume that the corresponding $n-1$ constants b_i and the $n+1$ constants a_j are all distinct unless otherwise stated. We have already seen in [4, Proposition 6.5] that $t_n < t_i$ for any $i \le n-1$. Here one can obtain a stronger result.

Proposition 5.1 $t_n < t_{n-1} < \cdots < t_1$.

Proof Fix k such that $2 \leq k \leq n-1$ and assume that $t_i \leq t_k$ for some $i \leq k-1$. Put

$$I = \{i \mid 1 \leq i \leq n,\ t_i \leq t_k,\ i \neq k\}.$$

Let J be the set of j such that $1 \leq j \leq n-1$ and either $j \in I$ and $j+1 \in I$, or $j \notin I$ and $j+1 \notin I$. Since there is some $i \in I$ such that $i < k$ by the assumption, and since $n \in I$ as remarked above and $k \notin I$, it follows that $\#J \leq n-3$.

We then consider the equality (the geodesic equation)

$$\begin{aligned}&\sum_{l=1}^{n}\int_{t_l}^{t_k}\frac{(-1)^l G(f_l)}{\sqrt{(-1)^{l-1}\prod_{k=1}^{n-1}(f_l-b_k)}}\left|\frac{dx_l(t)}{dt}\right|dt\\&\quad+\sum_{l=1}^{n}\int_{a_l^+}^{a_{l-1}^-}\frac{(-1)^l G(\lambda)A(\lambda)\,d\lambda}{\sqrt{-\prod_{k=1}^{n-1}(\lambda-b_k)\cdot\prod_{k=0}^{n}(\lambda-a_k)}}=0,\end{aligned}\tag{5.3}$$

where $G(\lambda) = (\lambda - a_n)\prod_{j\in J}(\lambda - b_j)$. Since the sign of $(-1)^l G(f_l)$ are the same for any $l \in I$, and since $n \in I$, it follows that

$$(-1)^{n-\#J+l}G(f_l)\begin{cases}\geq 0 & (l \in I)\\ \leq 0 & (l \notin I)\end{cases}.$$

Also, we have

$$t_k \geq t_l \quad (l \in I), \quad t_k \leq t_l \quad (l \notin I).$$

Therefore, the sign of the first line of formula (5.3) is $(-1)^{n-\#J}$. On the other hand, the second line of (5.3) is nonzero and its sign is, by Proposition 4.5 (1), equal to $(-1)^{n-\#J}$, which is a contradiction. Therefore, we have $t_k < t_i$ for any $1 \leq i \leq k-1$, and the proposition thus follows. □

Let H_i $(1 \leq i \leq n-1)$ denote the first integral of the geodesic flow whose value is expressed by b_i, i.e., H_i are functions on the unit cotangent bundle U^*M defined by the following identity in λ;

$$\begin{aligned}&\sum_{j=1}^{n-1}\left(\prod_{\substack{1\leq k\leq n-1\\ k\neq j}}(\lambda-a_k)\right)F_j(\mu)-\prod_{k=1}^{n-1}(\lambda-a_k)=-\prod_{i=1}^{n-1}(\lambda-H_i(\mu)),\\&\qquad H_1(\mu)\geq H_2(\mu)\geq\cdots\geq H_{n-1}(\mu),\quad \mu\in U^*M.\end{aligned}$$

We extend H_i to $T^*M - \{0 - \text{section}\}$ as a function of degree 0, i.e.,

$$H_i(t\lambda) = H_i(\lambda), \qquad t > 0, \lambda \in T^*M, \lambda \neq 0.$$

Then the π_*-image of the vector X_{H_i} at $\flat(\dot{\gamma}(t))$ is perpendicular to $\dot{\gamma}(t)$.

In [4, Proposition 5.1], we proved that the Jacobi fields on the manifolds we are considering possess a remarkable property, which may be stated as follows.

Proposition 5.2 *There are smooth vector fields $V_i(t)$ $(1 \leq i \leq n-1)$ along the geodesic $\gamma(t)$ satisfying*

$$\pi_*\big((X_{H_i})_{\flat(\dot{\gamma}(t))}\big) = h_i(t)V_i(t), \quad |V_i(t)| = 1,$$

for some functions $h_i(t)$ and they have the following properties:

(1) Each $V_i(t)$ is parallel along $\gamma(t)$.
(2) $V_1(t), \ldots, V_{n-1}(t)$ are mutually orthogonal for any $t \in \mathbb{R}$.
(3) Any Jacobi field $Z(t)$ satisfying $Z(0), Z'(0) \in \mathbb{R}V_i(0)$ is of the form $z(t)V_i(t)$ with some function $z(t)$ for any $t \in \mathbb{R}$ and any i.

We prove here the following proposition, which will be necessary in later sections.

Proposition 5.3 *The 1-form*

$$\tilde{\omega}_i = \sum_{k=1}^{n} \frac{\epsilon_k(-1)^k G_i(f_k)\,dx_k}{\sqrt{(-1)^{k-1}\prod_{l=1}^{n-1}(f_k - b_l)}}, \quad G_i(\lambda) = \prod_{\substack{1\leq l\leq n-1\\ l\neq i}} (\lambda - b_l),$$

satisfies

$$\tilde{\omega}_i(V_k(t)) = 0 \quad (k \neq i), \quad \tilde{\omega}(\dot{\gamma}(t)) = 0,$$

at $\gamma(t)$ for any $t \in \mathbb{R}$ such that $(f_{i+1}(x_{i+1}(t)) - b_i)(f_i(x_i(t)) - b_i) \neq 0$ $(1 \leq i \leq n-1)$. Here $\epsilon_k =$ sign of $\xi_k =$ sign of $x_k'(t)$. At $t \in \mathbb{R}$ with $f_{i+1}(x_{i+1}(t)) = b_i$ (resp. $f_i(x_i(t)) = b_i$) the one-form dx_{i+1} (resp. dx_i) has the same property.

Proof By the identity

$$\prod_{l=1}^{n-1}(f_k(x_k) - H_l)\cdot 2E = (-1)^{k+1}\xi_k^2, \tag{5.4}$$

one obtains

$$\sum_{i=1}^{n-1}\prod_{l\neq i}(f_k - b_l)\cdot\pi_*(X_{H_i}) - 2\prod_{l=1}^{n-1}(f_k - b_l)\cdot\pi_*(X_E) = (-1)^k 2\xi_k\frac{\partial}{\partial x_k},$$

for $(x, \xi) \in U^*M$ at which $H_l = b_l$ (and $2E = 1$). Then taking the dual one-forms $\eta_1, \ldots, \eta_n$ of $\pi_*(X_{H_1}), \ldots, \pi_*(X_{H_{n-1}}), \pi_*(X_E)$, we have

$$\eta_i = \sum_{k=1}^{n}\frac{(-1)^k\prod_{l\neq i}(f_k - b_l)}{2\xi_k}\,dx_k \quad (1 \leq i \leq n-1).$$

Then taking (5.4) into account, we have the proposition for the points where $\pi_*(X_{H_i}) \neq 0$ for any i. For the point where $\pi_*(X_{H_i}) = 0$, i.e., $f_{i+1}(x_{i+1}) = b_i$ or $f_i(x_i) = b_i$, we can take the limit:

$$\lim_{f_{i+1}(x_{i+1}) \to b_i} \epsilon_{i+1}\sqrt{b_i - f_{i+1}(x_{i+1})}\,\tilde{\omega}_i = \sqrt{G_i(f_{i+1})}\,dx_{i+1},$$
$$\lim_{f_i(x_i) \to b_i} \epsilon_i\sqrt{f_i(x_i) - b_i}\,\tilde{\omega}_i = \sqrt{G_i(f_i)}\,dx_i.$$

Thus, the proposition follows. □

Let us define the Jacobi field $Y_i(t)$ by the initial condition:

$$Y_i(0) = 0, \quad Y_i'(0) = V_i(0) \quad (1 \le i \le n-1).$$

Then $Y_i(t)$ is of the form $y_i(t)V_i(t)$ for some function $y_i(t)$. Let $t = r_i$ be the first zero of $Y_i(t)$ for $t > 0$. We have already seen that $r_i \ge t_n$ for any i ([4, Proposition 5.3]). Moreover, let S_i be the discrete subset of $\mathbb{R}$ such that

$$t \in S_i \iff \begin{cases} f_i(x_i(t)) = b_i & \text{if } b_i = a_i^+ \\ f_{i+1}(x_{i+1}(t)) = b_i & \text{if } b_i = a_i^- \end{cases},$$

as given in [4, §5]. Let s_i^1 and s_i^2 be the first and the second positive time in S_i, respectively. We then have, by the definition and Proposition 5.1 in [4],

$$s_i^1 < r_i,\ t_i < s_i^2 \quad (b_i = a_i^+), \quad s_i^1 < r_i,\ t_{i+1} < s_i^2 \quad (b_i = a_i^-) \quad \text{if} \quad 0 \notin S_i, \tag{5.5}$$

$$r_i = s_i^1 = t_i \quad (b_i = a_i^+), \quad r_i = s_i^1 = t_{i+1} \quad (b_i = a_i^-) \quad \text{if} \quad 0 \in S_i. \tag{5.6}$$

Now we prove the following proposition about the ordering of r_i and t_j.

Proposition 5.4 *If $0 \notin S_j$, then $t_{j+1} < r_j < t_j$ for each $1 \le j \le n-1$.*

Proof We shall first prove that $r_j < t_1$ for any j. Suppose that $r_j \ge t_1$ for some j. Let $G(\lambda) = \prod_{k \ne j}(\lambda - b_k)$ and observe the following formula (a part of geodesic equations):

$$\begin{aligned} t = &\sum_{l=1}^{n} \int_{t_l}^{t} \frac{(-1)^l G(f_l)\,(f_l - a_n)}{\sqrt{(-1)^{l-1}\prod_{k=1}^{n-1}(f_l - b_k)}} \left|\frac{dx_l(t)}{dt}\right| dt \\ &+ \sum_{l=1}^{n} \int_{a_l^+}^{a_{l-1}^-} \frac{(-1)^l G(\lambda)\,(\lambda - a_n) A(\lambda)\,d\lambda}{\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k)\cdot\prod_{k=0}^{n}(\lambda - a_k)}}. \end{aligned} \tag{5.7}$$

We differentiate the above formula in terms of the deformation parameter defining the Jacobi field cY_j, c being $\pm$ (the norm of $\partial/\partial b_j$ at $\gamma(0)$), and put $t = r_j$. Then we claim that the resulting formula is

$$\begin{aligned}&\frac{1}{2}\sum_{l=1}^{n}\int_{t_l}^{r_j}\frac{(-1)^l G(f_l)\,(f_l-a_n)}{(f_l-b_j)\sqrt{(-1)^{l-1}\prod_{k=1}^{n-1}(f_l-b_k)}}\left|\frac{dx_l(t)}{dt}\right|dt\\&+\frac{\partial}{\partial b_j}\sum_{l=1}^{n}\int_{a_l^+}^{a_{l-1}^-}\frac{(-1)^l G(\lambda)\,(\lambda-a_n)A(\lambda)\,d\lambda}{\sqrt{-\prod_{k=1}^{n-1}(\lambda-b_k)\cdot\prod_{k=0}^{n}(\lambda-a_k)}}=0.\end{aligned}\tag{5.8}$$

To prove this, we first assume that $b_j = a_j^+$. Let

$$\gamma(t,u) = (x_1(t,u),\dots,x_n(t,u)) \quad (|u|<\epsilon),$$

be a variation of the geodesic $\gamma(t)=\gamma(t,0)$ such that

$$\frac{\partial\gamma}{\partial u}(t,0) = cY_j(t),$$

and that the value of each first integral H_k $(k\neq j)$ for the geodesics $t\to\gamma(t,u)$ remains to be b_k (constant) for any u. (In this case, b_j is a function of u such that $db_j/du = 1$ at $u=0$.) For $l\neq j$ and for $t>t_l$ satisfying

$$f_l(x_l(t,0)) \neq a_l^+,\, a_{l-1}^-,$$

we define times t^* and $\hat{t}$ such that $t_l < t^* \le \hat{t} < t$ and that

$$\begin{aligned}f_l(x_l(t^*,u)),\ f_l(x_l(\hat{t},u)) =\ & a_l^+ \text{ or } a_{l-1}^-,\\ & a_l^+ < f_l(x_l(s,u)) < a_{l-1}^- \quad \text{for } s\in[\,t_l,t^*)\cup(\hat{t},t\,].\end{aligned}$$

Then one obtains the following expression for sufficiently small $|u|$:

$$\begin{aligned}&\int_{t_l}^{t}\frac{(-1)^l G(f_l)\,(f_l-a_n)}{\sqrt{(-1)^{l-1}\prod_{k=1}^{n-1}(f_l-b_k)}}\left|\frac{\partial x_l(s,u)}{\partial s}\right|ds\\&=\frac{\epsilon}{2}\int_{f_l(x_l(t_l,u))}^{f_l(x_l(t^*,u))}\frac{(-1)^l G(\lambda)\,(\lambda-a_n)A(\lambda)\,d\lambda}{\sqrt{-\prod_{k=1}^{n-1}(\lambda-b_k)\cdot\prod_{k=0}^{n}(\lambda-a_k)}}\\&\quad+\frac{k}{2}\int_{a_l^+}^{a_{l-1}^-}\frac{(-1)^l G(\lambda)\,(\lambda-a_n)A(\lambda)\,d\lambda}{\sqrt{-\prod_{k=1}^{n-1}(\lambda-b_k)\cdot\prod_{k=0}^{n}(\lambda-a_k)}}\\&\quad+\frac{\epsilon'}{2}\int_{f_l(x_l(\hat{t},u))}^{f_l(x_l(t,u))}\frac{(-1)^l G(\lambda)\,(\lambda-a_n)A(\lambda)\,d\lambda}{\sqrt{-\prod_{k=1}^{n-1}(\lambda-b_k)\cdot\prod_{k=0}^{n}(\lambda-a_k)}},\end{aligned}$$

where ϵ, ϵ' $(=\pm1)$ are the sign of $\partial f_j(x_l(s,u))/\partial s$ at $s=t_l$ and $s=t$, respectively, and k is a certain nonnegative integer.

We differentiate the above formula in u and put $u = 0$. Since $f_l(x_l(t_l, u))$, $f_l(x_l(t^*, u))$, and $f_l(x_l(\hat{t}, u))$ do not depend on u, the right-hand side becomes

$$\begin{aligned}
&\frac{\epsilon}{4}\int_{f_l(x_l(t_l,u))}^{f_l(x_l(t^*,u))} \frac{(-1)^l G(\lambda)\,(\lambda - a_n) A(\lambda)\, d\lambda}{(\lambda - b_j)\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k)\cdot\prod_{k=0}^{n}(\lambda - a_k)}} \\
&\quad + \frac{k}{4}\int_{a_l^+}^{a_{l-1}^-} \frac{(-1)^l G(\lambda)\,(\lambda - a_n) A(\lambda)\, d\lambda}{(\lambda - b_j)\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k)\cdot\prod_{k=0}^{n}(\lambda - a_k)}} \\
&\quad + \frac{\epsilon'}{4}\int_{f_l(x_l(\hat{t},u))}^{f_l(x_l(t,u))} \frac{(-1)^l G(\lambda)\,(\lambda - a_n) A(\lambda)\, d\lambda}{(\lambda - b_j)\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k)\cdot\prod_{k=0}^{n}(\lambda - a_k)}} \\
&\quad + \frac{\epsilon'}{2}\frac{(-1)^l G(f_l)\,(f_l - a_n) A(f_l)}{\sqrt{-\prod_{k=1}^{n-1}(f_l - b_k)\cdot\prod_{k=0}^{n}(f_l - a_k)}}\, f_l'(x_l)\, dx_l(cY_j(t)). \qquad (5.9)
\end{aligned}$$

In the same way, it turns out that the sum of the first three lines of formula (5.9) is equal to

$$\frac{1}{2}\int_{t_l}^{t} \frac{(-1)^l G(f_l)\,(f_l - a_n)}{(f_l - b_j)\sqrt{(-1)^{l-1}\prod_{k=1}^{n-1}(f_l - b_k)}}\left|\frac{dx_l(t)}{dt}\right| dt. \qquad (5.10)$$

Thus, if $f_l(x_l(r_j)) \neq a_l^+, a_{l-1}^-$, then putting $t = r_j$ in (5.9), we have the desired formula for l. If $f_l(x_l(r_j)) = a_l^+, a_{l-1}^-$, then taking $t < r_j$ and taking the limit $t \to r_j$, one obtains the same formula.

In case there are no such times t^* and $\hat{t}$, i.e., if

$$a_l^+ < f_l(x_l(s, u)) < a_{l-1}^- \quad \text{for any } s \in [\,t_l, t\,],$$

then instead of (5.9) one has

$$\begin{aligned}
&\frac{\epsilon}{4}\int_{f_l(x_l(t_l,u))}^{f_l(x_l(t,u))} \frac{(-1)^l G(\lambda)\,(\lambda - a_n) A(\lambda)\, d\lambda}{(\lambda - b_j)\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k)\cdot\prod_{k=0}^{n}(\lambda - a_k)}} \\
&\quad + \frac{\epsilon'}{2}\frac{(-1)^l G(f_l)\,(f_l - a_n) A(f_l)}{\sqrt{-\prod_{k=1}^{n-1}(f_l - b_k)\cdot\prod_{k=0}^{n}(\lambda - a_k)}}\, f_l'(x_l)\, dx_l(cY_j(t)), \qquad (5.11)
\end{aligned}$$

the first line of which is again equal to (5.10). Thus, we have the same result in this case.

Next, we consider the remaining term in (5.7):

$$\int_{t_j}^{t} \frac{(-1)^j G(f_j)\,(f_j - a_n)}{\sqrt{(-1)^{j-1}\prod_{k=1}^{n-1}(f_j - b_k)}}\left|\frac{\partial x_j(t, u)}{\partial t}\right| dt. \qquad (5.12)$$

Let us differentiate (5.12) in u at $u = 0$ and put $t = r_j$. When t is close to r_j, the inequalities (5.5) imply that $f_j(x_j(s)) - b_j$ does not vanish on the interval $t_j \le s \le t$. Thus, the derivative of (5.12) in u at $u = 0$ is described as (5.9) with $k = 0$ or as (5.11). Therefore, putting $t = r_j$, one obtains

$$\frac{1}{2}\int_{t_j}^{r_j} \frac{(-1)^j G(f_j)\,(f_j - a_n)}{(f_j - b_j)\sqrt{(-1)^{j-1}\prod_{k=1}^{n-1}(f_j - b_k)}}\left|\frac{dx_j(t)}{dt}\right| dt.$$

Hence, formula (5.8) follows when $b_j = a_j^+$. The case where $b_j = a_j^-$ is similar and we omit the detail.

Let us come back to the situation before the claim. Since $r_j \ge t_1 \ge t_l$, the first line of formula (5.8) is nonpositive. However, since the second line is negative by Proposition 4.5 (2), it is a contradiction. Thus, $r_j < t_1$ for any j.

Now, we have proved that $t_n < r_j < t_1$. Assume that $t_{m+1} \le r_j \le t_m$ for some $m \ne j$ and put $G(\lambda) = \prod_{l \ne m, j}(\lambda - b_l)$. Then differentiating the formula

$$\begin{aligned}&\sum_{l=1}^{n}\int_{t_l}^{t} \frac{(-1)^l G(f_l)\,(f_l - a_n)}{\sqrt{(-)^{l-1}\prod_{k=1}^{n-1}(f_l - b_k)}}\left|\frac{dx_l(t)}{dt}\right| dt \\ &+\sum_{l=1}^{n}\int_{a_l^+}^{a_{l-1}^-} \frac{(-1)^l G(\lambda)\,(\lambda - a_n)A(\lambda)\,d\lambda}{\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k)\cdot\prod_{k=0}^{n}(\lambda - a_k)}} = 0,\end{aligned} \tag{5.13}$$

by cY_j as above and putting $t = r_j$, we have the same formula as (5.8) with $G(\lambda) = \prod_{l \ne m, j}(\lambda - b_l)$. In this case, each summand of the first line of (5.8) is nonnegative, whereas the second line is positive by Proposition 4.5; again a contradiction. Thus, we have $t_{j+1} < r_j < t_j$ for any j. □

As an application of Proposition 5.4, we shall show that the distribution of conjugate points have some curious asymptotic property. Let $\gamma(t) = (x_1(t), \ldots, x_n(t))$ be a geodesic such that the corresponding $n-1$ values b_i of the first integrals H_i and the $n+1$ constants a_j are all distinct. Let $L \subset M$ be the π-image of the Lagrange torus in T^*M determined by $\cap_i H_i^{-1}(b_i)$ and containing the geodesic orbit $\{\flat(\dot\gamma(t))\}$, where $\pi : T^*M \to M$ is the bundle projection. As stated in §3, L is diffeomorphic to the product $L_1 \times \cdots \times L_n$, where each L_j is either the whole circle $\mathbb{R}/\alpha_j\mathbb{Z}$ or an arc in it. Let $t = r_i^k$ be the kth zero of the Jacobi field $Y_i(t)$ $(1 \le i \le n-1)$; $0 < r_i^1 < r_i^2 < \cdots$.

Theorem 5.5 *The sequence $\{\gamma(r_i^k)\}_{k\in\mathbb{N}}$ of conjugate points of $\gamma(0)$ approaches the boundary ∂L of L when $k \to \infty$, i.e., for any $\epsilon > 0$, there is $N > 0$ such that the distance of $\gamma(r_i^k)$ from ∂L is less than ϵ, if $k \ge N$.*

Proof We shall consider the case where $b_i = a_i^+$. The case where $b_i = a_i^-$ will be similar. Let $\{s_i^k\}$, $0 < s_i^1 < s_i^2 < \cdots$, be the set of times such that $f_i(x_i(s_i^k)) = b_i$. Then by Corollary 5.2 in [4] and Proposition 5.4, we have $s_i^k < r_i^k < s_i^{k+1} < r_i^{k+1}$

and

$$|x_i(r_i^k) - x_i(s_i^k)| > |x_i(r_i^{k+1}) - x_i(s_i^{k+1})|.$$

Note that $\gamma(s_i^k) \in L_1 \times \cdots \times \partial L_i \times \cdots \times L_n \subset \partial L$.

We shall show that

$$\lim_{k\to\infty} |x_i(r_i^k) - x_i(s_i^k)| = 0,$$

which will indicate the theorem. Assume that this is not the case. Then there is a subsequence $r_i^{k(l)}$ $(l = 1, 2, \dots)$ such that

$$\lim_{l\to\infty} x_i(r_i^{k(l)}) = a, \quad f_i(a) > b_i, \quad \lim_{l\to\infty} \gamma(r_i^{k(l)}) = p \in L.$$

Then taking a subsequence if necessary, the sequence of geodesics $\gamma(t + r_i^{k(l)})$ converges to a geodesic $\tilde{\gamma}(t) = (\tilde{x}_1(t), \dots, \tilde{x}_n(t))$ and the Jacobi fields $Y_i(t + r_i^{k(l)})$ converges to a Jacobi field $\tilde{Y}_i(t)$ such that $\tilde{Y}_i(0) = 0$ and $\tilde{Y}_i'(0)$ is a multiple of $\flat(\partial/\partial H_i)$. Let $t = T > 0$ be the first zero of $\tilde{Y}_i(t)$. Then $\tilde{x}_i(T) = \lim_{l\to\infty} x_i(r_i^{k(l)+1})$ and

$$\lim_{l\to\infty} \left|x_i(r_i^{k(l)}) - x_i(s_i^{k(l)})\right| = \lim_{l\to\infty} \left|x_i(r_i^{k(l)+1}) - x_i(s_i^{k(l)+1})\right|,$$
$$|f_i(\tilde{x}_i(0)) - b_i| = |f_i(\tilde{x}_i(T)) - b_i|.$$

Therefore, we have

$$\int_0^T \left|\frac{df_i(\tilde{x}_i(t))}{dt}\right| = 2(a_{i-1}^- - a_i^+),$$

which contradicts Proposition 5.4. □

6 Conjugate Locus

Let $p_0 = (x_{1,0}, \dots, x_{n,0}) \in M$ be a general point, i.e., a point which is not contained in any hypersurfaces N_i $(0 \le i \le n)$. We shall determine the shape of the conjugate locus of p_0. In view of Proposition 2.1, we may assume $0 < x_{i,0} < \alpha_i/4$ for any i without loss of generality. Although the first conjugate locus is our primary concern, we shall also consider the kth conjugate locus for $1 \le k \le n-1$ as well (see Sect. 1). The reason of doing so is that the first $n-1$ conjugate loci can be viewed as a scattered image of the first conjugate locus of a point of the sphere of constant curvature, which is a one point with multiplicity $n-1$, provided M is sufficiently close to the standard sphere in some sense.

We shall parametrize the unit cotangent space $U^*_{p_0}M$ by $(n-1)$-torus as follows: putting $f_{i,0} = f_i(x_{i,0})$,

$$\xi_i = \epsilon_i \sqrt{(-1)^{i-1} \prod_{k=1}^{n-1} (f_{i,0} - b_k(u_k))},$$
$$b_k(u_k) = f_{k+1,0}(\cos u_k)^2 + f_{k,0}(\sin u_k)^2,$$

$u = (u_1, \dots, u_{n-1}) \in (\mathbb{R}/2\pi\mathbb{Z})^{n-1}$, where the sign ϵ_i is chosen to be equal to that of $\cos u_i \sin u_{i-1}$ if $2 \le i \le n-1$, that of $\cos u_1$ if $i = 1$, and that of $\sin u_{n-1}$ if $i = n$. We denote by $[u] \in U^*_{p_0}M$ the corresponding covector. Observe that $(b_1, \dots, b_{n-1})$ is, in the case of the ellipsoid, essentially the same as the elliptic coordinates $(\mu_1, \dots, \mu_{n-1})$ on U_pM described in Introduction. Accordingly, we define submanifolds (with boundary) $C_k^{\pm}$ $(1 \le k \le n-1)$ of $U^*_{p_0}M$ by

$$C_k^- = \{[u] \mid u_k = 0, \pi\}, \qquad C_k^+ = \left\{[u] \mid u_k = \pm\frac{\pi}{2}\right\}.$$

Then $C_{k-1}^- \cup C_k^+$ is equal to the great sphere $\xi_k = 0$, they are diffeomorphic to

$$C_k^- \simeq S^{k-1} \times \bar{D}^{n-1-k}, \quad C_k^+ \simeq \bar{D}^{k-1} \times S^{n-1-k},$$

and the boundaries $\partial C_k^{\pm}$ satisfy

$$\partial C_k^+ = \partial C_{k-1}^- = C_k^+ \cap C_{k-1}^- \simeq S^{k-2} \times S^{n-1-k} \quad (2 \le k \le n-1),$$
$$\partial C_{n-1}^- = \emptyset = \partial C_1^+.$$

We shall denote by

$$t \mapsto \gamma(t, u) = (x_1(t, u), \dots, x_n(t, u)),$$

the geodesic such that $\gamma(0, u) = p_0$ and $\flat((\partial\gamma/\partial t)(0, u)) = [u] \in U^*_{p_0}M$. Put

$$Y_i(t, u) = \frac{\partial \gamma}{\partial u_i}(t, u) \qquad (1 \le i \le n-1),$$

and let $t = r_i(u)$ be the first zero of the Jacobi field $t \mapsto Y_i(t, u)$ for $t > 0$. Note that the Jacobi fields Y_i are identical with constant multiple of the ones defined in the previous section. When $[u] \in C_{i-1}^- \cap C_i^+$, $Y_{i-1}(t, u)$ and $Y_i(t, u)$ vanish identically. So, in this case, we use the Jacobi fields $Z_{i-1}(t)$ and $Z_i(t)$ defined in [4, §5] (see also §7.2 in this paper), and define $t = r_k(u)$ as the first zero of $Z_k(t)$ for $t > 0$ $(k = i, i-1)$. Actually, $Z_{i-1}(t)$ and $Z_i(t)$ are also defined for $[u]$ near the points in $C_{i-1}^- \cap C_i^+$ and they are linear combinations of $Y_{i-1}(t, u)$ and $Y_i(t, u)$ there. Thus, the functions $r_i(t)$ are continuous at any $[u] \in U^*_{p_0}M$.

In view of Proposition 5.4 and [4, Proposition 5.5], we obtain the following proposition.

Proposition 6.1 *$r_i(u) \le r_{i-1}(u)$ for any $u \in (\mathbb{R}/2\pi\mathbb{Z})^{n-1}$, and the equality holds if and only if $[u] \in C^-_{i-1} \cap C^+_i$, i.e., $b_i(u_i) = b_{i-1}(u_{i-1})$.*

Proof Let $t_i = t_i(u)$ be the value defined by formula (5.1) for the geodesic $\gamma(t, u)$. We proved in Proposition 5.4 and remarked in (5.6) that

$$r_i(u) \le t_i(u) \le r_{i-1}(u),$$

and $r_i(u) = t_i(u)$ if and only if $u_i = \pm\pi/2$ and $r_{i-1}(u) = t_i(u)$ if and only if $u_{i-1} = 0, \pi$, provided $b_1(u_1), \dots, b_{n-1}(u_{n-1})$ and $a_0, \dots, a_n$ are all distinct. We shall prove that $r_i(u) \ne t_i(u)$ if $u_i \ne \pm\pi/2$ and $r_{i-1}(u) \ne t_i(u)$ if $u_{i-1} \ne 0, \pi$, not necessarily assuming $b_1, \dots, b_{n-1}$ and $a_1, \dots, a_n$ are distinct, which will indicate the proposition.

Let us observe formula (5.8) for j being replaced by i:

$$\begin{aligned} &\frac{1}{2}\sum_{l=1}^{n}\int_{t_l}^{r_i} \frac{(-1)^l G(f_l)\,(f_l - a_n)}{(f_l - b_i)\sqrt{(-1)^{l-1}\prod_{m=1}^{n-1}(f_l - b_m)}}\left|\frac{\partial x_l(t,u)}{\partial t}\right| dt \\ &\quad + \frac{\partial}{\partial b_i}\sum_{l=1}^{n}\int_{a_l^+}^{a_{l-1}^-} \frac{(-1)^l G(\lambda)\,(\lambda - a_n)A(\lambda)\,d\lambda}{\sqrt{-\prod_{m=1}^{n-1}(\lambda - b_m)\cdot\prod_{m=0}^{n}(\lambda - a_m)}} = 0, \end{aligned} \tag{6.1}$$

where $G(\lambda) = \prod_{m \ne i, i-1}(\lambda - b_m)$ and $b_m = b_m(u_m)$. This formula is effective for u such that $b_1(u_1), \dots, b_{n-1}(u_{n-1})$ and $a_1, \dots, a_n$ are all distinct. We now take any $u \in (\mathbb{R}/2\pi\mathbb{Z})^{n-1}$ such that $u_i \ne \pm\pi/2$ (i.e., $b_i(u_i) < f_{i,0}$) and take a sequence $u^k = (u_1^k, \dots, u_{n-1}^k) \in (\mathbb{R}/2\pi\mathbb{Z})^{n-1}$ such that $u^k \to u$ as $k \to \infty$ and such that $b_1(u_1^k), \dots, b_{n-1}(u_{n-1}^k)$ and $a_1, \dots, a_n$ are all distinct for any k. We also assume that the ordering of a_j and $b_j(u_j^k)$ does not change when k varies for each j. By Proposition 4.6 we see that the second line of formula (6.1) for $(b_m) = (b_m(u_m^k))$ has a limit value as $k \to \infty$ and the value is still positive. Also, each summand in the first line of formula (6.1) is positive if $l \ne i$ and is negative if $l = i$ for any k by Proposition 4.5. Therefore, there is a constant $c > 0$ such that

$$\int_{r_i(u^k)}^{t_i(u^k)} \frac{(-1)^i G(f_i)\,(f_i - a_n)}{(f_i - b_i^k)\sqrt{(-1)^{i-1}\prod_{m=1}^{n-1}(f_i - b_m^k)}}\left|\frac{\partial x_i(t,u^k)}{\partial t}\right| dt \ge c, \tag{6.2}$$

for sufficiently large k, where $b_m^k = b_m(u_m^k)$. We note that

$$f_i(x_i(t_i(u^k), u^k)) = f_{i,0}, \quad b_i^k < f_{i,0}, \quad a_i < f_{i,0} < a_{i-1}.$$

We now consider the following two cases separately: $f_{i,0} < b_{i-1}(u_{i-1})$; and $f_{i,0} = b_{i-1}(u_{i-1})$. Let us first assume that $f_{i,0} < b_{i-1}(u_{i-1})$. By formula (3.10) we see that there are (sufficiently small) constant $\delta > 0$ and positive constants c_1, c_2 not depending on k such that

$$c_1 \leq \left| \frac{\partial x_i(t, u^k)}{\partial t} \right| \leq c_2 \quad \text{if} \quad |f_i(x_i(t, u^k)) - f_{i,0}| \leq \delta, \tag{6.3}$$

for any (sufficiently large) k. Since $|df_i/dx_i|$ is bounded both above and below if $(a_i - f_i)(f_i - a_{i-1})$ is bounded away from 0 in view of (2.1), we also have

$$c_1' \leq \left| \frac{\partial f_i(x_i(t, u^k))}{\partial t} \right| \leq c_2' \quad \text{if} \quad |f_i(x_i(t, u^k)) - f_{i,0}| \leq \delta, \tag{6.4}$$

for some positive constants c_1' and c_2' not depending on k. Thus the map $t \mapsto f_i(x_i(t, u^k))$ and its inverse are "equicontinuous" in k on a neighborhood of $t = t_i(u)$. Now, suppose that $r_i(u) = t_i(u)$. Then for large k we have

$$|f_i(x_i(t, u^k)) - f_{i,0}| \leq \delta \quad \text{for any} \quad t \in [r_i(u^k), t_i(u^k)],$$

and by (6.2),

$$c'(t_i(u^k) - r_i(u^k)) \geq c, \tag{6.5}$$

for some constant $c' > 0$ not depending on k. Thus, taking a limit $k \to \infty$, we have a contradiction.

Next, suppose that $f_{i,0} = b_{i-1}(u_{i-1})$. In this case $b_{i-1}(u_{i-1})$ is not equal to $b_{i-2}(u_{i-2})$, because

$$b_{i-1}(u_{i-1}) = f_{i,0} < a_{i-1} \leq b_{i-2}(u_{i-2}).$$

Therefore, we may assume that u^k are chosen so that $u_{i-1}^k = u_{i-1}$ for any k. Then $t = t_i(u^k)$ is the turning point of the functions $x_i(t, u^k)$ and $f_i(x_i(t, u^k))$ (including the case $k = \infty$), and by the same reason as in the previous case, the functions

$$\frac{|\partial x_i(t, u^k)/\partial t|}{\sqrt{b_{i-1}(u_{i-1}) - f_i(x_i(t, u^k))}}, \qquad \frac{|\partial f_i(x_i(t, u^k))/\partial t|}{\sqrt{b_{i-1}(u_{i-1}) - f_i(x_i(t, u^k))}},$$

are bounded both above and below by positive constants not depending on k if

$$|f_i(x_i(t, u^k)) - f_{i,0}| \leq \delta, \tag{6.6}$$

where δ is a constant not depending on k. Thus, as in the previous case, we see that there is a constant $\delta' > 0$ not depending on k such that inequality (6.6) holds for t with

$$|t - t_i(u^k)| \leq \delta', \tag{6.7}$$

for any (sufficiently large) k. Therefore, if we assume $r_i(u) = t_i(u)$, then we again have inequality (6.5) for large k, which is a contradiction.

We have thus proved that $r_i(u) < t_i(u)$ if $u_i \neq \pm\pi/2$. The implication of $t_i(u) < r_{i-1}(u)$ for u with $u_{i-1} \neq 0, \pi$ is similar to the above, and we omit the detail. □

Theorem 6.2 *(1) $t = r_{n-1}(u)$ represents the first conjugate point of p_0 along the geodesic $\gamma(t, u)$.*

(2) If the Riemannian manifold M is close to the round sphere so that the second zero $t = r^2_{n-1}(u)$ of the Jacobi field $Y_{n-1}(t, u)$ is greater than $r_1(u)$, then $t = r_i(u)$ represents the $(n - i)$th conjugate point of p_0 along the geodesic $\gamma(t, u)$ for $2 \leq i \leq n - 1$.

Proof The assertions are immediate from Proposition 6.1. For (2), we remark that for the round sphere the first zeros of $Y_i(t, u)$ $(1 \leq i \leq n - 1)$ coincide, and the second zero of $Y_{n-1}(t, u)$ is greater than them. Therefore, if M is "close to the round sphere", then the second zero of $Y_{n-1}(t, u)$ appears after the first zero of $Y_1(t, u)$.

We put

$$\tilde{K}_i(p_0) = \{r_i(u)\sharp[u] \in T_{p_0}M \mid u \in (\mathbb{R}/2\pi\mathbb{Z})^{n-1}\}$$
$$K_i(p_0) = \{\gamma(r_i(u), u) \mid u \in (\mathbb{R}/2\pi\mathbb{Z})^{n-1}\}.$$

Then $K_{n-1}(p_0)$ ($\tilde{K}_{n-1}(p_0)$) represents the first (tangential) conjugate locus of p_0, and if M is close to the round sphere, then $K_{n-j}(p_0)$ ($\tilde{K}_{n-j}(p_0)$) represents the jth (tangential) conjugate locus of p_0 for $2 \leq j \leq n - 1$. For the smoothness of the functions $r_i(u)$, we have the following.

Lemma 6.3 *$u \mapsto r_i(u)$ is of C^∞ for $[u] \notin \partial C_i^\pm$, i.e., for u with $b_i(u_i) \neq b_{i-1}(u_{i-1})$ and $b_i(u_i) \neq b_{i+1}(u_{i+1})$.*

Proof Under the given assumption, $Y_i(t, u)$ is written as $y_i(t, u)V_i(t, u)$, as described in the previous section. Since $(\partial/\partial t)y_i(t, u) \neq 0$ at $t = r_i(u)$, the lemma follows from the implicit function theorem. □

Proposition 6.4 *(1) $\dfrac{\partial}{\partial u_i} r_i(u) \neq 0$ if $[u] \notin C_i^\pm$.*

(2) $\dfrac{\partial}{\partial u_i} r_i(u) = 0$ and $\dfrac{\partial^2}{\partial u_i^2} r_i(u) \neq 0$ for $[u] \in C_i^\pm$, $[u] \notin \partial C_i^\pm$.

Proof (1) First we assume that $b_1(u), \ldots, b_{n-1}(u)$ and $a_0, \ldots, a_n$ are all distinct. Differentiating formula (5.8) (j being i here) in the proof of Proposition 5.4 in terms of the deformation parameter defining cY_j once again, we have

$$\frac{3}{4}\sum_{l=1}^{n}\int_{t_l}^{r_i}\frac{(-1)^l G(f_l)}{(f_l-b_i)^2\sqrt{(-1)^{l-1}\prod_{k=1}^{n-1}(f_l-b_k)}}\left|\frac{\partial x_l(t,u)}{\partial t}\right| dt$$
$$+2\frac{\partial^2}{\partial b_i^2}\sum_{l=1}^{n}\int_{a_l^+}^{a_{l-1}^-}\frac{(-1)^l G(\lambda)A(\lambda)\,d\lambda}{\sqrt{-\prod_{k=1}^{n-1}(\lambda-b_k)\cdot\prod_{k=0}^{n}(\lambda-a_k)}}$$
$$+\frac{c}{2}\frac{\partial r_i}{\partial u_i}\sum_{l=1}^{n}\frac{(-1)^l G(f_l)}{(f_l-b_i)\sqrt{(-1)^{l-1}\prod_{k=1}^{n-1}(f_l-b_k)}}\left|\frac{\partial x_l(t,u)}{\partial t}\right|_{t=r_i(u)}=0, \quad (6.8)$$

where $G(\lambda)=(\lambda-a_n)\prod_{k\neq i}(\lambda-b_k)$,

$$c=\left(\frac{db_i}{du_i}\right)^{-1}=\frac{1}{2\sin u_i\cos u_i(f_{i,0}-f_{i+1,0})},$$

and f_l in the third line is equal to $f_l(x_l(r_i(u),u))$. Since $r_i<t_l$, $f_l-b_i>0$ for $l\le i$, and $r_i>t_l$, $f_l<b_i$ for $l\ge i+1$, the first line of the above formula is positive; while the second line is also positive by Proposition 4.5 (3). Therefore it follows that $\partial r_i/\partial u_i\neq 0$.

Next we consider the general case. As before, we take a sequence $u^k\in(\mathbb{R}/2\pi\mathbb{Z})^{n-1}$ such that $u^k\to u$ as $k\to\infty$ and that $b_j(u_j^k)$ $(1\le j\le n-1)$ and a_l $(0\le l\le n)$ are all distinct for each k. Let us consider formula (6.8) for u^k and take a limit $k\to\infty$. The second line then converges to a positive value by Proposition 4.6 and the first line is positive for each k. For the third line, we note that

$$f_l(x_l(r_i(u),u))\neq b_i(u_i)\quad(l=i,i+1),$$

by the proof of Proposition 6.1 and

$$\sqrt{\prod_{m=1}^{n-1}|f_l(x_l(r_i(u^k),u^k))-b_m(u_m^k)|}\left|\frac{\partial x_l}{\partial t}(r_i(u^k),u^k)\right|\le 1,$$

since

$$\sqrt{\prod_{\substack{1\le m\le n\\ m\neq l}}|f_m-f_l|}\left|\frac{\partial x_l}{\partial t}(r_i(u^k),u^k)\right|\le 1,$$

by the expression of metric (2.3) ($f_m=f_m(x_m(r_i(u^k),u^k))$) and

$$|f_l-b_m|\le\begin{cases}|f_l-f_m| & (1\le m\le l-1)\\ |f_l-f_{m+1}| & (l\le m\le n-1)\end{cases}.$$

Therefore, the integral in the third line of (6.8) for u^k remains finite as $k \to \infty$. Those facts indicate that $(\partial r_i/\partial u_i)(u)$ does not vanish.

(2) Let us consider the case where $u_i = 0$ ($u_{i+1} \neq \pm\pi/2$). We describe (5.8) for u with $u_i < 0$ and u_i being close to 0 in the form:

$$\frac{1}{2}\sum_{\substack{1\le l\le n\\ l\neq i+1}}\int_{t_l}^{r_i}\frac{(-1)^l G(f_l)}{(f_l-b_i)\sqrt{(-)^{l-1}\prod_{k=1}^{n-1}(f_l-b_k)}}\left|\frac{dx_l(t)}{dt}\right|dt$$
$$+\frac{1}{4}\int_{f_{i+1}(x_{i+1}(r_i(u),u))}^{f_{i+1}(x_{i+1,0})}\frac{(-1)^{i+1}G(\lambda)A(\lambda)\,d\lambda}{(\lambda-b_i)\sqrt{-\prod_{k=1}^{n-1}(\lambda-b_k)\cdot\prod_{k=0}^{n}(\lambda-a_k)}}$$
$$+\frac{\partial}{\partial b_i}\sum_{l=1}^{n}\int_{a_l^+}^{a_{l-1}^-}\frac{(-1)^l G(\lambda)A(\lambda)\,d\lambda}{\sqrt{-\prod_{k=1}^{n-1}(\lambda-b_k)\cdot\prod_{k=0}^{n}(\lambda-a_k)}}=0, \tag{6.9}$$

where $G = (\lambda - a_n)\prod_{k\neq i,i+1}(\lambda - b_k)$. Note that this formula is effective for general u, i.e., for u not necessarily satisfying that $b_l(u_l)$ $(1 \le l \le n-1)$ and a_m $(0 \le m \le n)$ are all distinct. In fact, since the second and the third lines have definite values at such u, so is the first line. Note also that in this case $b_i = a_i^-$. Since we are assuming $0 < x_{i+1,0} < \alpha_{i+1}/4$, we have

$$f'_{i+1}(x_{i+1}(t_{i+1}(u),u)) = f'_{i+1}(x_{i+1,0}) > 0,$$

and since $u_i < 0$ and u_i is close to 0, we also have

$$t_{i+1}(u) < r_i(u), \quad \frac{\partial x_{i+1}}{\partial t}(t,u) < 0.$$

Therefore,

$$f_{i+1}(x_{i+1}(r_i(u),u)) < f_{i+1}(x_{i+1,0}) < b_i(u_i),$$

when $u_i < 0$, and they all coincide when $u_i = 0$, by (5.5), (5.6), and Proposition 5.4.

We denote by $\Phi(\lambda)$ the integrand of the second line in formula (6.9):

$$\Phi(\lambda) = \frac{(-1)^{i+1}G(\lambda)A(\lambda)}{(\lambda-b_i)\sqrt{-\prod_{k=1}^{n-1}(\lambda-b_k)\cdot\prod_{k=0}^{n}(\lambda-a_k)}}.$$

When λ is in the interval of the integration;

$$f_{i+1}(x_{i+1}(r_i(u),u)) \le \lambda \le f_{i+1}(x_{i+1,0}),$$

then $\Phi(\lambda) < 0$ and

$$-\Phi(\lambda) \le c\left(b_i(u_i) - f_{i+1}(x_{i+1,0})\right)^{-\frac{3}{2}} = \frac{c'}{|\sin u_i|^3},$$

for some positive constants c, c'. Thus, one obtains

$$0 < -\int_{f_{i+1}(x_{i+1}(r_i(u),u))}^{f_{i+1}(x_{i+1,0})} \Phi(\lambda)\, d\lambda \le \frac{c\left|x_{i+1,0} - x_{i+1}(r_i(u), u)\right|}{|\sin u_i|^3}, \tag{6.10}$$

for some (another) constant $c > 0$.

Now we need the following lemma, which is essentially the same as [5, Lemma 8.2].

Lemma 6.5 *Regarded as a function of u_i (other u_js being fixed),*

$$x_{i+1}(r_i(u), u) - x_{i+1,0} = c\, u_i^3 + O(u_i^4),$$
$$c = \frac{1}{3}\left(\frac{\partial^2 x_{i+1}}{\partial t \partial u_i}(r_i(u), u)\frac{\partial^2 r_i}{\partial u_i^2}(u)\right)\bigg|_{u_i=0}.$$

Proof We have

$$\frac{\partial}{\partial u_i} x_{i+1}(r_i(u), u) = \frac{\partial x_{i+1}}{\partial t}(r_i(u), u)\,\frac{\partial r_i}{\partial u_i}(u).$$

Since $(\partial r_i/\partial u_i)(u) = (\partial x_{i+1}/\partial t)(r_i(u), u) = 0$ when $u_i = 0$, it, therefore, follows that

$$\frac{\partial^3}{\partial u_i^3} x_{i+1}(r_i(u), u)\Big|_{u_i=0} = 2\,\frac{\partial^2 x_{i+1}}{\partial t \partial u_i}(r_i(u), u)\,\frac{\partial^2 r_i}{\partial u_i^2}(u)\Big|_{u_i=0}, \tag{6.11}$$

which indicates the lemma.

We continue the proof of Proposition 6.4. Assume that

$$\frac{\partial^2 r_i}{\partial u_i^2}(u)\Big|_{u_i=0} = 0.$$

Then by (6.10) and Lemma 6.5, the second line of formula (6.9) tends to 0 when $u_i \to 0$. However, the first line of formula (6.9) is nonnegative and the third line us positive by Proposition 4.5 (2) and Proposition 4.6, which is a contradiction. Thus, we have

$$\frac{\partial^2 r_i}{\partial u_i^2}(u)\Big|_{u_i=0} \ne 0,$$

in this case. The case where $u_i = \pi$ is similar.

For the cases where $u_i = \pm\pi/2$, $b_i = a_i^+$ and one should consider the integral concerning the variable x_i in formula (6.9) instead of that concerning the variable x_{i+1} as above. Then the argument is parallel as above and we shall omit the detail. This finishes the proof of Proposition 6.4. □

We remark that in the above proof, we have actually proved the following fact.

Corollary 6.6 *The constant c which appeared in Lemma 6.5 does not vanish.*

Thus, as a consequence of Proposition 6.4 and Corollary 6.6, we have the following theorem.

Theorem 6.7 *The following statements hold for each i $(1 \leq i \leq n-1)$. For $i \neq n-1$, we assume that the second zero $t = r^2_{n-1}(u)$ of the Jacobi field $Y_{n-1}(t, u)$ is greater than $r_1(u)$ for any $u \in (\mathbb{R}/2\pi\mathbb{Z})^{n-1}$.*

(1) The map $u \mapsto \gamma(r_i(u), u)$ is an immersion at $[u]$ with $[u] \notin C^{\pm}_i$. In particular, $K_i(p_0)$ is a smooth hypersurface around such points $\gamma(r_i(u), u)$.
(2) For each $p = \gamma(r_i(u), u)$ with $[u] \in C^{\pm}_i$, $[u] \notin \partial C^{\pm}_i$, there is a neighborhood U of p and a function x, y on U such that $dx \wedge dy \neq 0$ and $U \cap K_i(p_0)$ is given by $x^3 = y^2$ and such that the edge of vertices $x = y = 0$ corresponds to $\{\gamma(r_i(u), u) \mid [u] \in C^{\pm}_i\}$. Namely, $K_i(p_0)$ is diffeomorphic to a cuspidal edge around p.

Proof (1) By the assumption, we see that $n-1$ vectors

$$\frac{\partial}{\partial u_i}\gamma(r_i(u), u) = \frac{\partial \gamma}{\partial t}(r_i(u), u)\frac{\partial r_i}{\partial u_i}(u),$$

and

$$\frac{\partial}{\partial u_k}\gamma(r_i(u), u) = \frac{\partial \gamma}{\partial t}(r_i(u), u)\frac{\partial r_i}{\partial u_k}(u) + \frac{\partial \gamma}{\partial u_k}(r_i(u), u) \quad (k \neq i),$$

are linearly independent. Therefore, the map $u \to \gamma(r_i(u), u)$ is an immersion.

(2) We fix $u_0 = (u_{1,0}, \ldots, u_{n-1,0})$ such that $[u_0] \in C^{\pm}_i$, $[u_0] \notin \partial C^{\pm}_i$. We consider the case where $u_{i,0} = 0$. Other cases $(u_{i,0} = \pi, \pm\pi/2)$ will be similar. From the assumption, the $n-1$ vectors

$$\frac{\partial \gamma}{\partial t}(r_i(u), u), \quad \frac{\partial \gamma}{\partial u_k}(r_i(u), u) \quad (k \neq i),$$

are linearly independent at $u = u_0$ and, by Proposition 5.3, their dx_{i+1}-components vanish. Therefore, we can take a coordinate system $(z_1, \ldots, z_n)$ around the point $p = \gamma(r_i(u_0), u_0)$ such that

$$z_1 = x_{i+1}, \quad dz_2\left(\frac{\partial \gamma}{\partial t}(r_i(u_0), u_0)\right) \neq 0, \quad z_k(p) = 0 \text{ for any } k,$$

and the Jacobian of the map

$$u \mapsto (z_3, \ldots, z_n, u_i),$$

given by $z_k = z_k(\gamma(r_i(u), u))$ does not vanish at $u = u_0$. Then putting

$$u' = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{n-1}),$$

we have

$$z_1(\gamma(r_i(u), u)) = c_1(u_i, u')u_i^3, \quad z_2(\gamma(r_i(u), u)) = c_2(u_i, u')u_i^2,$$

where c_1 and c_2 are functions of $u = (u_i, u')$ which do not vanish at $u_0 = (0, u'_0)$. We may assume that c_2 is positive at u_0. Thus, we can replace the coordinate function u_i with

$$v_i = \sqrt{c_2(u_i, u')}u_i,$$

so that (v_i, u') is the new coordinate system on $U^*_{p_0}M$, and we have, putting $z_k(v_i, u') = z_k(\gamma(r_i(u), u))$,

$$z_1(v_i, u') = c_3(v_i, u')v_i^3, \quad z_2(v_i, u') = v_i^2,$$

and $c_3(0, u'_0) \neq 0$.

Since the map

$$(v_i, u') \mapsto (z_3(v_i, u'), \ldots, z_n(v_i, u'), v_i),$$

is a local diffeomorphism around the point $(v_i, u') = (0, u'_0)$, we can take the inverse function so that (v_i, u') is a function of $(z_3, \ldots, z_n, v_i)$. Therefore, the map $u \mapsto \gamma(r_i(u), u)$ is described as a map

$$(z_3, \ldots, z_n, v_i) \mapsto (z_1, \ldots, z_n),$$

such that

$$z_1 = c_4(z_3, \ldots, z_n, v_i)v_i^3, \quad z_2 = v_i^2.$$

and $c_4(0, \ldots, 0) \neq 0$.

We put $z' = (z_3, \ldots, z_n)$ and

$$c_{4,\pm}(z', v_i) = \frac{1}{2}(c_4(z', v_i) \pm c_4(z', -v_i)).$$

Then since $c_{4,+}$ is a even function in v_i, there is a C^∞ function c_5 such that

$$c_{4,+}(z', v_i) = c_5(z', v_i^2).$$

Similarly, we have

$$c_{4,-}(z', v_i) = v_i\, c_6(z', v_i^2),$$

for some C^∞ function c_6. Thus, we have

$$\begin{aligned} z_1 &= (c_5(z', v_i^2) + v_i\, c_6(z', v_i^2))\, v_i^3 \\ &= c_5(z', z_2)\, v_i^3 + c_6(z', z_2)\, z_2^2. \end{aligned}$$

Therefore, replacing the coordinate function z_1 with

$$\bar{z}_1 = \frac{z_1 - c_6(z', z_2)\, z_2^2}{c_5(z', z_2)},$$

we see that the map $u \mapsto \gamma(r_i(u), u)$ is expressed as the map

$$(z', v_i) \mapsto (\bar{z}_1, z_2, z'),$$

such that

$$\bar{z}_1 = v_i^3, \quad z_2 = v_i^2.$$

Thus the theorem has been proved. □

7 Singularities Arising at Points with Double Conjugacy

7.1 Definition of D_4^+ Lagrangian Singularity

We first review the notion of Lagrangian singularity and that of generating family which describes a Lagrangian submanifold. After that, we state the definition of D_4^+ Lagrangian singularity. For the statements of this subsection, we refer to [2] for Lagrangian singularities and [14] for versal deformations.

Lagrangian Singularity

Let N be a manifold and let L be a Lagrangian submanifold of T^*N. A *Lagrangian singularity* is a singularity of the map $\pi \circ i : L \to N$, where i and π denote the inclusion $L \to T^*N$ and the bundle projection $T^*N \to N$, respectively. More precisely, for points $\lambda_0 \in L$ and $q_0 = \pi(\lambda_0) \in N$, we consider the "Lagrangian equivalence class" of the map-germ $(\pi \circ i) : (L, \lambda_0) \to (N, q_0)$. Two such map-germs $(L, \lambda_0) \to (N, q_0)$ and $(\pi' \circ i') : (L', \lambda_0') \to (N', q_0')$ are said to be Lagrangian equivalent if there is a diffeomorphism $\phi : (N, q_0) \to (N', q_0')$ and a symplectic diffeomorphism $\Phi : (T^*N, \lambda_0) \to (T^*N', \lambda_0')$ such that the diagram

$$\begin{array}{ccc} (T^*N, \lambda_0) & \xrightarrow{\Phi} & (T^*N', \lambda_0') \\ \pi \downarrow & & \downarrow \pi' \\ (N, q_0) & \xrightarrow[\phi]{} & (N', q_0') \end{array}$$

is commutative and such that $\Phi(L, \lambda_0) = (L', \lambda_0')$. Actually, Φ is described as

$$\Phi(\lambda) = (\phi^*)^{-1}(\lambda) + dh_{\phi(\pi(\lambda))}, \quad \lambda \in T^*N,$$

for some function h on N' in this case.

Generating Family

Let $(L, \lambda_0) \subset T^*N$ and (N, q_0) be as above. Let $x = (x_1, \ldots, x_n)$ be a coordinate system on N so that q_0 corresponds to $a = (a_1, \ldots, a_n)$ $(n = \dim N)$. A function $F(u, x) = F(u_1, \ldots, u_k, x_1, \ldots, x_n)$ defined on a neighborhood of $(b, a) \in \mathbb{R}^k \times \mathbb{R}^n$ is called a "generating family" for L around the reference point $\lambda_0 \in L$ if it satisfies

(1) $0 \in \mathbb{R}^k$ is a regular value of the map

$$d_u F : (u, x) \mapsto (\partial F/\partial u_1, \ldots, \partial F/\partial u_k),$$

and $d_u F(b, a) = 0$. Thus, $C = (d_u F)^{-1}(0)$ is a n-dimensional manifold and $(b, a) \in C$.

(2) The map

$$d_x F : C \ni (u, x) \mapsto \sum_{l=1}^{n} (\partial F/\partial x_l)(u, x)\, dx_l \in T_x^* N \subset T^*N,$$

gives an embedding of C into T^*N whose image is L (a neighborhood of λ_0) and $d_x F(b, a) = \lambda_0$.

It can be seen that

$$k \geq \dim \ker\big((\pi \circ i)_* : T_{\lambda_0} L \to T_{q_0} N\big).$$

If the equality holds, then the generating family is called *minimal*. A way of obtaining a minimal generating family is as follows: Let (x, ξ) be the canonical coordinate system of T^*N associated with a coordinate system $x = (x_1, \ldots, x_n)$ on N. One can choose x so that

$$(\pi \circ i)^*(dx_j) = 0 \quad \text{at } \lambda_0 \quad (1 \leq j \leq k),$$

where $k = \dim \ker((\pi \circ i)_*)_{\lambda_0}$. Then $(\xi_1, \ldots, \xi_k, x_{k+1}, \ldots, x_n)$ form a coordinate system of L around λ_0 and

$$-\sum_{i=1}^{k} x_i d\xi_i + \sum_{j=k+1}^{n} \xi_j dx_j = \sum_{i=1}^{n} \xi_i dx_i - d\left(\sum_{i=1}^{k} \xi_i x_i\right),$$

is a closed form on L. Thus, there is a function

$$\hat{F} = \hat{F}(\xi_1, \ldots, \xi_k, x_{k+1}, \ldots, x_n),$$

on L so that

$$\partial \hat{F}/\partial \xi_i = -x_i|_L, \quad \partial \hat{F}/\partial x_j = \xi_j|_L \qquad (1 \le i \le k,\ k+1 \le j \le n).$$

Then

$$F(\xi_1, \ldots, \xi_k, x_1, \ldots, x_n) = \sum_{i=1}^{k} \xi_i x_i + \hat{F}(\xi_1, \ldots, \xi_k, x_{k+1}, \ldots, x_n)$$

is the desired minimal generating family.

Let $G(v_1, \ldots, v_{k'}, y_1, \ldots, y_n)$ with the base point (b', a') be another minimal generating family for a Lagrangian submanifold $(\tilde{L}, \tilde{\lambda}_0) \subset T^*\tilde{N}$. Then those two minimal generating families are said to be $\mathcal{E}^+$-equivalent if $k' = k$ and there is a diffeomorphism $\Psi : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^n$ $((b, a) \mapsto (b', a'))$ of the form

$$\Psi(u, x) = (\psi(u, x), \phi(x)),$$

and a function $h(x)$ so that $F(u, x) = G(\Psi(u, x)) + h(x)$. The following criterion is crucial (see the theorem in [2, p.304] and its proof).

Theorem 7.1 *Two minimal generating families $F(u, x)$ and $G(v, y)$ are $\mathcal{E}^+$-equivalent if and only if the corresponding Lagrangian submanifolds $(L, \lambda_0) \subset T^*N$ and $(\tilde{L}, \tilde{\lambda}_0) \subset T^*\tilde{N}$ are Lagrangian equivalent.*

Versal Deformation of a Function Germ

Let $F(u, x)$ be a function germ on $\mathbb{R}^k \times \mathbb{R}^n$ at (b, a) and put

$$f(u) = f(u_1, \ldots, u_k) = F(u, a).$$

Such F is called a deformation (or an unfolding) of the function germ $(f(u), b)$. We are interested in the case where $F(u, x)$ is a *versal* deformation of f. We do not explain the original definition of versality here; the following characterization by Mather is enough for our purpose (see [14, §3] for the proof of the next two theorems and a detailed explanation on the theory of versal deformations).

Theorem 7.2 *The function germ $(F(u, x), (b, a))$ is a versal deformation of the function germ $(f(u), b)$ if and only if the quotient space*

$$\mathcal{E}_k \Big/ \left(\frac{\partial f}{\partial u_1}, \ldots, \frac{\partial f}{\partial u_k} \right)$$

is spanned by elements represented by constant functions and

$$\frac{\partial F}{\partial x_j}(u, a) \quad (1 \le j \le n)$$

as a vector space.

Here $\mathcal{E}_k$ denotes the algebra of function germs in $(u_1, \ldots, u_k)$ at $u = b$ and $(\ldots, \partial f/\partial u_j, \ldots)$ stands for its ideal generated by $\partial f/\partial u_j$ $(1 \le j \le k)$.

Theorem 7.3 *Let $(F(u, x), (b, a))$ and $(H(v, y), (b', a'))$ be two deformation germs on $\mathbb{R}^k \times \mathbb{R}^n$ of $f(u) = F(u, a)$ and $h(v) = H(v, a')$ respectively. Suppose F and H are versal deformations. Then the two deformation germs F and H are $\mathcal{E}^+$-equivalent if and only if the function germs $(f(u), b)$ and $(h(v), b')$ are equivalent, i.e., there is a diffeomorphism germ $\phi : (\mathbb{R}^k, b) \to (\mathbb{R}^k, b')$ and a constant $c \in \mathbb{R}$ such that $f = h \circ \phi + c$.*

The $\mathcal{E}^+$-equivalence in the above theorem is the same as that for generating families. If $(F(u, x), (b, a))$ is a versal deformation of $(f(u), b)$, then it is known that the function germ $f(u)$ is *finitely determined*, i.e., there is a positive integer l such that any function germ $(h(u), b)$ whose l-jet is equal to the l-jet of $f(u)$ at b is equivalent to $(f(u), b)$. (In this case $(f(u), b)$ is said to be l-determined.) Therefore, we have the following criterion for Lagrangian equivalence of Lagrangian singularities.

Theorem 7.4 *Let $(F(u, x), (b, a))$, a function germ on $\mathbb{R}^k \times \mathbb{R}^n$, be a minimal generating family for a Lagrangian submanifold $(L, \lambda_0) \subset T^*N$. Suppose F is a versal deformation of $f(u) = F(u, a)$ at b and $f(u)$ is l-determined. Let $H(v, y), (b', a'))$ be another function germ on $\mathbb{R}^k \times \mathbb{R}^n$ and is a minimal generating family of a Lagrangian submanifold $(L', \lambda'_0) \subset T^*N'$. Suppose also that H is a versal deformation o $h(v) = H(v, a')$ at b'. Then the Lagrangian singularity $\pi \circ i : (L, \lambda_0) \to (N, q_0)$ is Lagrangian equivalent to $\pi' \circ i : (L', \lambda'_0) \to (N', q'_0)$ if and only if there is a diffeomorphism germ $\phi : (\mathbb{R}^k, b) \to (\mathbb{R}^k, b')$ and a constant $c \in \mathbb{R}$ such that the l-jets of $h(\phi(u)) + c$ and $f(u)$ at b coincide.*

D_4^+ Singularity

The equivalence class of the function germ $f(u_1, u_2) = u_1^3 + u_1 u_2^2$ at $0 \in \mathbb{R}^2$ is called the D_4^+ singularity. It is 3-determined and the quotient space

$$\mathcal{E}_2 \Big/ \left(\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2} \right)$$

is spanned by $1, u_1, u_2$, and u_2^2. Put

$$F(u_1, u_2, x_1, \ldots, x_n) = u_1^3 + u_1 u_2^2 + x_1 u_1 + x_2 u_2 + x_3 u_2^2 + \sum_{j=4}^{n} c_j x_j,$$

where $c_4, \ldots, c_n \in \mathbb{R}$. Then $(F(u,x), (0,0))$ is a versal deformation of $(f(u), 0)$. Putting

$$C = \{(u, x) \mid \partial F/\partial u_1 = \partial F/\partial u_2 = 0\},$$

we define a germ of a Lagrangian submanifold $(L, \lambda_0) \subset T^*\mathbb{R}^n$ as the image of the map

$$C \ni (u, x) \mapsto \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}(u, x) dx_j \in T^*\mathbb{R}^n, \quad \lambda_0 = \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}(0, 0) dx_j,$$

namely $L \subset T^*\mathbb{R}^n = \{(x, \xi)\}$ is parametrized by $(u_1, u_2, x_3 \ldots, x_n)$ as

$$x_1 = -(3u_1^2 + u_2^2), \quad x_2 = -2(u_1 + x_3)u_2, \quad \xi = (u_1, u_2, u_2^2, c_4, \ldots, c_n).$$

The Lagrangian equivalence class represented by

$$\pi \circ i : (L, \lambda_0) \to (\mathbb{R}^n, 0)$$

is called the D_4^+ Lagrangian singularity.

7.2 Singularities at Points with Double Conjugacy

We now come back to the situation at §6. Let $p_0 = x_0 = (x_{1,0}, \ldots, x_{n,0}) \in M$ be a general point and let $\lambda_0 = (x_0, \xi_0) \in U_{p_0}^* M$ be a covector, where $b_j = b_{j-1}$ for a fixed j $(2 \leq j \leq n-1)$, i.e., $\lambda_0 \in C_j^+ \cap C_{j-1}^-$. We shall denote by $b_{k,0}$ the value of b_k at λ_0 $(1 \leq k \leq n-1)$. Since the coordinate functions u_{j-1} and u_j on $U_{p_0}^* M$ in the previous section are not appropriate at λ_0, we introduce the following functions ν_1, ν_2 instead:

$$2\nu_1 = b_j + b_{j-1} - 2f_{j,0}, \quad \nu_2 = \epsilon\sqrt{(b_{j-1} - f_{j,0})(f_{j,0} - b_j)},$$

where $f_{j,0} = f_j(x_{j,0})$ and $\epsilon = \pm 1$ is chosen so that it is the sign of ξ_j. Thus ν_1 and ν_2 are smooth functions on $U_{p_0}^* M$ around λ_0, $d\nu_1 \wedge d\nu_2 \neq 0$, and

$$\begin{aligned}
\xi_j &= \nu_2 \sqrt{(-1)^j \prod_{k \neq j, j-1} (f_{j,0} - b_k)}, \\
\xi_i &= \epsilon_i \sqrt{(-1)^{i-1} \prod_{k \neq j, j-1} (f_{i,0} - b_k)} \\
&\quad \times \sqrt{(f_{i,0} - f_{j,0})^2 - 2(f_{i,0} - f_{j,0})\nu_1 - \nu_2^2} \quad (i \neq j),
\end{aligned}$$

where ϵ_i is the same one as in §6 and $f_{i,0} = f_i(x_{i,0})$.

Also we take coordinate functions $(\tilde{w}_1, \dots, \tilde{w}_{n-3})$ instead of b_k's $(k \neq j, j-1)$ so that the product structure

$$\{d\nu_1 = d\nu_2 = 0\} \times \{d\tilde{w}_k = 0, 1 \leq k \leq n-3\}$$

coincides with that of

$$\{db_j = db_{j-1} = 0\} \times \{db_k = 0, 1 \leq k \leq n-1, k \neq j, j-1\}.$$

(One can take them as u_ks $(k \neq j, j-1)$ if λ_0 is not contained in any $\partial C_k^{\pm}$ other than $\partial C_j^+ = \partial C_{j-1}^-$.) We put

$$\tilde{S} = \{\lambda \in W \subset U_{p_0}^* M \mid \nu_1(\lambda) = \nu_2(\lambda) = 0\},$$

where W is a neighborhood of λ_0 in $U_{p_0}^* M$. We shall use the abbreviated notations

$$\nu = (\nu_1, \nu_2), \quad \tilde{w} = (\tilde{w}_1, \dots, \tilde{w}_{n-3}), \quad \lambda = (\nu, \tilde{w}) \in U_{p_0}^* M.$$

We take $\tilde{w}$ so that $\lambda_0 = (0, 0)$.

Let $\gamma(t) = \gamma(t, \nu, \tilde{w}) = (x_1(t, \nu, \tilde{w}), \dots, x_n(t, \nu, \tilde{w}))$ be the geodesic such that $\gamma(0) = p_0$ and $\flat(\dot{\gamma}(0)) = (\nu, \tilde{w})$. Let $Z_{j-1}(t)$ and $Z_j(t)$ be the Jacobi fields along the geodesic $\gamma(t)$ defined by the initial conditions

$$Z_{j-1}(0) = 0,\ Z_j(0) = 0, \quad Z'_{j-1}(0) = \frac{1}{2}\sharp\left(\frac{\partial}{\partial \nu_1}\right),\ Z'_j(0) = \frac{1}{2}\sharp\left(\frac{\partial}{\partial \nu_2}\right).$$

They are equal with the Jacobi fields $Z_{j-1,0}(t)$ and $Z_{j,0}(t)$ given in p. 271 of our previous paper [4, §5]. There we proved the following proposition ([4, p. 272]), which we also need here. Let $t = \tau_1 > 0$ be the first zero of $Z_j(t)$ along the geodesic $\gamma(t, 0, 0)$.

Proposition 7.5 *(1)* $Z_{j-1}(\tau_1) = Z_j(\tau_1) = 0$.
(2) $Z_{j-1}(t)$ *and* $Z_j(t)$ *are linearly independent for any* $0 < t < \tau_1$.

We now assume that the following condition is satisfied:

There is no Jacobi field $Y(t) \not\equiv 0$ with $Y(0) = 0, Y(\tau_1) = 0$ along the geodesic $\gamma(t, 0, 0)$ other than linear combinations of $Z_j(t)$ and $Z_{j-1}(t)$. (7.1)

This condition is automatically satisfied when $j = n-1$ by Proposition 6.1. Put

$$\tilde{L} = \{t\lambda \mid |t - \tau_1| < \epsilon,\ \lambda \in W \subset U_{p_0}^* M\} \subset T_{p_0}^* M,$$

for a small constant $\epsilon > 0$ and let $\phi : \tilde{L} \to T^* M$ by

$$\phi(t\lambda) = \zeta_1(t\lambda) = t\zeta_t(\lambda),$$

where $\{\zeta_t\}$ denotes the geodesic flow on T^*M. Put

$$L = \phi(\tilde{L}), \quad \lambda_1 = \phi(\tau_1\lambda_0).$$

Then L is a Lagrangian submanifold of T^*M, and we have the following.

Theorem 7.6 *The map-germ $\pi|_L : (L, \lambda_1) \to (M, p_1)$ is a D_4^+ Lagrangian singularity.*

To prove this theorem we shall prepare good coordinate functions y_0, y_1, y_2, and w_k $(1 \le k \le n-3)$ around the point $p_1 = \gamma(\tau_1, 0, 0) \in M$ so that the criterion in Theorem 7.4 will be easily applicable. First, we define y_0, y_1 and y_2. If the condition

$$f_l(x_l(\tau_1, 0, 0)) \ne b_{k,0} \qquad \text{for any } k \ne j-1, j \text{ and } 1 \le l \le n, \tag{7.2}$$

is satisfied, then we put

$$\begin{aligned} y_0 =& A_1(f_{j,0})(f_j(x_j) - f_{j,0}) \\ y_\alpha =& \sum_{\substack{1\le i\le n\\ i\ne j}} \int_{x_{i,1}}^{x_i} \epsilon_i \sqrt{(-1)^{i-1} \prod_{k\ne j-1,j} (f_i(x_i) - b_{k,0})}\, \frac{(f_i(x_i) - f_{j,0})^\alpha}{|f_i(x_i) - f_{j,0}|}\, dx_i \\ & (\alpha = 1, 2), \end{aligned}$$

where $x_{i,1} = x_i(\tau_1, 0, 0)$, ϵ_i is the sign of $(\partial x_i/\partial t)(\tau_1, 0, 0)$, and

$$A_1(\lambda) = \frac{\sqrt{(-1)^j \prod_{k\ne j,j-1}(\lambda - b_k)}\, A(\lambda)}{2\sqrt{(-1)^j \prod_{l=0}^{n}(\lambda - a_l)}}. \tag{7.3}$$

If (7.2) is not satisfied for some k, then we put

$$I = \{i \mid 1 \le i \le n,\ i \ne j,\ f_i(x_{i,1}) \ne b_{l,0} \text{ for any } l \ne j, j-1\},$$

and

$$\begin{aligned} y_\alpha =& \sum_{i\in I} \int_{x_{i,1}}^{x_i} \epsilon_i \sqrt{(-1)^{i-1} \prod_{k\ne j-1,j} (f_i(x_i) - b_{k,0})}\, \frac{(f_i(x_i) - f_{j,0})^\alpha}{|f_i(x_i) - f_{j,0}|}\, dx_i \\ & (\alpha = 1, 2). \end{aligned}$$

Next, we shall define $(w_1, \ldots, w_{n-3})$. First we define them on the submanifold

$$S = \{\gamma(t, 0, \tilde{w}) \mid (0, \tilde{w}) \in \tilde{S} \subset U_{p_0}^* M,\ |t - \tau_1| < \epsilon\},$$

by $w_k(\gamma(t, 0, \tilde{w})) = \tilde{w}_k$ $(1 \le k \le n-3)$. Note that S is really a submanifold due to the assumption (7.1). Along S we define mutually orthogonal unit vector fields V_1

and V_2 which are normal to S. Then we extend w_k's to a neighborhood of p_1 in M so that they satisfy

$$dw_k(V_i) = 0 \qquad (1 \le k \le n-3,\ i = 1, 2),$$

at each point on S.

Lemma 7.7 *$dy_0 \wedge dy_1 \neq 0$ at p_1, $dy_0 = dy_1 = 0$ on $T_{p_1}S$, and*

$$\frac{d}{dt} y_2(\gamma(t, 0, 0)) \neq 0.$$

In particular, $(y_0, y_1, y_2, w_1, \ldots, w_{n-3})$ form a coordinate system of M around p_1.

Proof $dy_0 \wedge dy_1 \neq 0$ is obvious, since y_1 does not contain the variable x_j. Since $f_j(x_j)$ remains constant $(= f_{j,0})$ on the geodesic $\gamma(t, 0, \tilde{w})$, we have $dy_0|_{TS} = 0$ and $dy_0 \neq 0$ at each point on S. For y_1, we observe Proposition 5.3, which is effective for $\lambda \in U^*_{p_1}M$ such that $b_1, \ldots, b_{n-1}$ and $a_1, \ldots, a_n$ are all distinct. We then have

$$\lim_{\lambda \to \lambda_1} (\tilde{\omega}_j + \tilde{\omega}_{j-1}) = 2dy_1,$$

and it, therefore, follows that $dy_1 = 0$ on TS. Also, for y_2, we observe formula (3.9). Taking a limit as above, we have

$$\frac{d}{dt} y_2(\gamma(t, 0, 0)) \neq 0.$$

□

Let $(\eta_0, \eta_1, \eta_2, v_1, \ldots, v_{n-3})$ be the canonical fiber coordinates of T^*M associated with the coordinate system $(y_0, y_1, y_2, w_1, \ldots, w_{n-3})$ of M. Using the coordinate system $(t, \nu, \tilde{w})$ on $\tilde{L}$, we put

$$y_\alpha \circ \phi(t, \nu, \tilde{w}) = y_\alpha(t, \nu, \tilde{w}) \quad (0 \le \alpha \le 2),$$
$$w_k \circ \phi(t, \nu, \tilde{w}) = w_k(t, \nu, \tilde{w}) \quad (1 \le k \le n-3),$$

and we also define $\eta_\alpha(t, \nu, \tilde{w})$ and $v_k(t, \nu, \tilde{w})$ in the same way. Note that

$$\pi \circ \phi(t, \nu, \tilde{w}) = \gamma(t, \nu, \tilde{w}).$$

Therefore, we have $y_\alpha(t, \nu, \tilde{w}) = y_\alpha(\gamma(t, \nu, \tilde{w}))$, etc. For those functions, we have the following proposition; the proof will be given in the next subsection.

Proposition 7.8 *There are nonzero constants c and c' such that*

$$\begin{aligned} y_0(t,\nu,0) &= 2c\nu_1\nu_2 + c'\nu_2(t-\tau_1) + O((|\nu|+|t-\tau_1|)^3) \\ y_1(t,\nu,0) &= c(3\nu_1^2+\nu_2^2) + c'\nu_1(t-\tau_1) + O((|\nu|+|t-\tau_1|)^3) \\ y_2(t,\nu,0) &= t-\tau_1 + O((|\nu|+|t-\tau_1|)^3) \\ w_k(t,\nu,0) &= O((|\nu|+|t-\tau_1|)^3). \end{aligned}$$

By this proposition, we have the following lemmas.

Lemma 7.9 *(1)* $\dim\ker(\pi|_L)_* = 2$ *at* λ_1.
(2) The system of functions $(\eta_0, \eta_1, y_2, w_1, \ldots, w_{n-3})$ *becomes a coordinate system of* L *around the point* λ_1.

Proof Proposition 7.8 and the fact that $\partial w_k/\partial \tilde{w}_l = \delta_{kl}$ at p_1 imply that $(\pi|_L)^*(dy_0) = (\pi|_L)^*(dy_1) = 0$ at λ_1 and

$$(\pi|_L)^*(dy_2 \wedge dw_1 \wedge \cdots \wedge dw_{n-3}) \neq 0 \text{ at } \lambda_1.$$

Since L is a Lagrangian submanifold, the lemma follows from those facts.

Lemma 7.10

$$\begin{aligned} \eta_0(\tau_1,\nu,0) &= \eta_0(\tau_1,0,0) + e\nu_2 + O(|\nu|^2), \\ \eta_1(\tau_1,\nu,0) &= \eta_1(\tau_1,0,0) + e\nu_1 + O(|\nu|^2), \end{aligned}$$

for some nonzero constant e.

Proof Since L is Lagrangian, it follows that

$$\phi^*\left(\sum_{\alpha=0}^{2} d\eta_\alpha \wedge dy_\alpha + \sum_{k=1}^{n-3} dv_k \wedge dw_k\right) = 0.$$

Taking the coefficients of $d\nu_1 \wedge d\nu_2$ of the left-hand side, one therefore obtains, by Proposition 7.8,

$$\begin{aligned} &a_1^0(2c\nu_1 + c'(t-\tau_1)) - a_2^0 \cdot 2c\nu_2 + a_1^1 \cdot 2c\nu_2 \\ &\quad - a_2^1(6c\nu_1 + c'(t-\tau_1)) + O((|\nu|+|t-\tau_1|)^2) = 0, \end{aligned}$$

where $a_i^\alpha = \partial\eta_\alpha(t,\nu,0)/\partial\nu_i$. This implies that

$$a_1^0 = 3a_2^1, \quad a_2^0 = a_1^1, \quad a_1^0 = a_2^1 \quad \text{at } \nu = 0,\ t = \tau_1.$$

Therefore, $a_1^0 = a_2^1 = 0$ there and we obtain the desired formula. Since $d\eta_0 \neq 0$ at λ_1, we also have $e \neq 0$. □

We now define the function $\hat{F}(\eta_0, \eta_1, y_2, w_1, \dots, w_{n-3})$ on L as an integral of the closed form

$$-y_0 d\eta_0 - y_1 d\eta_1 + \hat{\eta}_2 dy_2 + \sum_{k=1}^{n-3} \hat{v}_k dw_k = \alpha - d(\eta_0 y_0 + \eta_1 y_1) - dh,$$

where α denotes the canonical 1-form,

$$\hat{\eta}_\alpha = \eta_\alpha - \eta_\alpha(\tau_1, 0, 0), \quad \hat{v}_k = v_k - v_k(\tau_1, 0, 0),$$

for $0 \le \alpha \le 2$ and $1 \le k \le n-3$, and

$$h = h(y_2, w_1, \dots, w_{n-3}) = \eta_2(\tau_1, 0, 0) y_2 + \sum_{k=1}^{n-3} v_k(\tau_1, 0, 0) w_k.$$

We may take $\hat{F}$ so that $\hat{F} = 0$ at $\lambda_1 \in L$. Then as stated in the previous subsection,

$$F(\eta_0, \eta_1, y_0, y_1, y_2, w_1, \dots, w_{n-3}) = \eta_0 y_0 + \eta_1 y_1 + \hat{F} + h$$

becomes a generating family for L. (Note that F contains (y_0, y_1) as independent variables, while $\hat{F}$ does not.)

Lemma 7.11 $F(\eta_0, \eta_1, 0, \dots, 0) = -ce^{-2}(\hat{\eta}_0^2 \hat{\eta}_1 + \hat{\eta}_1^3) + O(|\hat{\eta}|^4)$.

Proof It is enough to show that $\hat{F}(\eta_0, \eta_1, 0, \dots, 0)$ is equal to the right-hand side. First, we have, by Proposition 7.8 and Lemma 7.10,

$$\phi^* d\hat{F}(\tau_1, \nu, 0) = -ce(2\nu_1 \nu_2 d\nu_2 + (3\nu_1^2 + \nu_2^2) d\nu_1) + O(|\nu|^3) d\nu,$$

and, therefore,

$$\phi^* \hat{F}(\tau_1, \nu, 0) = -ce(\nu_1^3 + \nu_1 \nu_2^2) + O(|\nu|^4).$$

We then need to evaluate the difference

$$\hat{F}(\eta_0, \eta_1, y_2, w_1, \dots, w_{n-3}) - \hat{F}(\eta_0, \eta_1, 0, \dots, 0),$$

which is equal to $Ay_2 + \sum_{k=1}^{n-3} B_k w_k$, where

$$A = A(\eta_0, \eta_1, y_2, w_1, \dots, w_{n-3}) = \int_0^1 \hat{\eta}_2(\eta_0, \eta_1, sy_2, sw_1, \dots, sw_{n-3}) ds,$$

$$B_k = B_k(\eta_0, \eta_1, y_2, w_1, \dots, w_{n-3}) = \int_0^1 \hat{v}_k(\eta_0, \eta_1, sy_2, sw_1, \dots, sw_{n-3}) ds.$$

Let us pull back this formula by ϕ at $(\tau_1, \nu, 0)$. Since $y_2(\tau_1, 0, 0) = w_k(\tau_1, 0, 0) = 0$, we have $\phi^* A(\tau_1, 0, 0) = \phi^* B_k(\tau_1, 0, 0) = 0$. Therefore, it follows that

$$\phi^*(Ay_2 + \sum_{k=1}^{n-3} B_k w_k)(\tau_1, \nu, 0) + O(|\nu|^4) = 0,$$

and thus

$$\phi^*(\hat{F}(\eta_0, \eta_1, 0, \ldots, 0))(\tau_1, \nu, 0) = -ce(\nu_1^3 + \nu_1 \nu_2^2) + O(|\nu|^4).$$

The lemma then follows from Lemma 7.10. □

Lemma 7.12 *The function $\hat{\eta}_2$, restricted to the submanifold L':*

$$L' = \{(\eta_0, \eta_1, y_2, w) \in L \mid y_2 = w = 0\},$$

is described as

$$\hat{\eta}_2|_{L'} = c_1(\hat{\eta}_0^2 + \hat{\eta}_1^2) + c_2 \hat{\eta}_0 \hat{\eta}_1 + c_3(\hat{\eta}_0^2 + 3\hat{\eta}_1^2) + O(|\hat{\eta}|^3),$$

where c_1, c_2, c_3 are constants and $c_1 \neq 0$.

Proof We compute the coefficients of $d\nu_i \wedge dt$ in the 2-form

$$\phi^*\left(\sum_{\alpha=0}^{2} d\eta_\alpha \wedge dy_\alpha + \sum_{k=1}^{n-3} dv_k \wedge dw_k\right) = 0,$$

at the points $(\tau_1, \nu, 0)$. By Proposition 7.8 and Lemma 7.10, we have

$$-2c\nu_2 \frac{\partial \eta_0}{\partial t} - 6c\nu_1 \frac{\partial \eta_1}{\partial t} + ec'\nu_1 + \frac{\partial \eta_2}{\partial \nu_1} + O(|\nu|^2) = 0,$$

as the coefficients of $d\nu_1 \wedge dt$, and

$$-2c\nu_1 \frac{\partial \eta_0}{\partial t} - 2c\nu_2 \frac{\partial \eta_1}{\partial t} + ec'\nu_2 + \frac{\partial \eta_2}{\partial \nu_2} + O(|\nu|^2) = 0,$$

as the coefficients of $d\nu_2 \wedge dt$. Therefore, we obtain

$$\hat{\eta}_2(\tau_1, \nu, 0) = -\frac{ec'}{2}(\nu_1^2 + \nu_2^2) + 2c\nu_1\nu_2 \frac{\partial \eta_0}{\partial t} + c(3\nu_1^2 + \nu_2^2)\frac{\partial \eta_1}{\partial t} + O(|\nu|^3).$$

We note that $ec' \neq 0$.

Here we need, as in the previous lemma, to observe the difference $\hat{\eta}_2|_L - \hat{\eta}_2|_{L'}$. Since it is described in the form

$$Ay_2 + \sum_{k=1}^{n-3} B_k w_k,$$

for certain functions A and B_k, as in the proof of the previous lemma, it follows that

$$\phi^*(\hat{\eta}_2|_L - \hat{\eta}_2|_{L'})(\tau_1, \nu, 0) = O(|\nu|^3).$$

Therefore, we have

$$\phi^*(\hat{\eta}_2|_{L'})(\tau_1, \nu, 0) = -\frac{ec'}{2}(\nu_1^2 + \nu_2^2) + 2c\nu_1\nu_2\frac{\partial\eta_0}{\partial t} + c(3\nu_1^2 + \nu_2^2)\frac{\partial\eta_1}{\partial t} + O(|\nu|^3).$$

Thus, the lemma follows by Lemma 7.10. □

We now prove Theorem 7.6. By Lemma 7.11, the function germ $f = F(\eta_0, \eta_1, 0, \ldots, 0)$ is equivalent to the D_4^+ function germ since the latter is 3-determined. Also, since $\partial F/\partial y_2 = \hat{\eta}_2$, we see by Lemma 7.12 that F is a versal deformation of f. Therefore, applying the criterion of Theorem 7.4 to the generating family F for (L, λ_1), we see that the map-germ $\pi|_L : (L, \lambda_1) \to (M, p_1)$ is a D_4^+ Lagrangian singularity. This completes the proof of Theorem 7.6 under the assumption of Proposition 7.8. □

As direct consequences of Theorem 7.6, we have the following corollaries.

Corollary 7.13 *The germ of the map* $\pi \circ \zeta_1 : T_{p_0}^*M \to M$ *at* $\lambda \in \flat(\tilde{K}_{n-1})$ *is a* D_4^+ *Lagrangian singularity if* $\lambda/|\lambda| \in \partial C_{n-1}^+$.

Corollary 7.14 *Suppose that the second zero* $t = r_{n-1}^2(u)$ *of the Jacobi field* $Y_{n-1}(t, u)$ *is greater than* $r_1(u)$ *for any* $[u] \in U_{p_0}^*M$. *Then the germ of the map* $\pi \circ \zeta_1 : T_{p_0}^*M \to M$ *at* $\lambda \in \flat(\tilde{K}_i)$ *is a* D_4^+ *Lagrangian singularity if* $\lambda/|\lambda| \in \partial C_i^+ \cup \partial C_i^-$ $(1 \le i \le n-2)$.

7.3 Proof of Proposition 7.8

In this subsection, we shall always assume $\tilde{w} = 0$, so we shall shortly write (t, ν) instead of writing $(t, \nu, 0)$. Also, the value of b_k $(k \neq j, j-1)$ will be fixed to be $b_{k,0}$ throughout this subsection.

First, we would like to define a function $\theta(t, \nu)$ satisfying

$$f_j(x_j(t, \nu)) - f_{j,0} = \nu_1(1 - \cos\theta(t, \nu)) + \nu_2 \sin\theta(t, \nu). \tag{7.4}$$

Lemma 7.15 *There is a unique* C^∞ *function* $\theta(t, \nu)$ *for small* $|\nu|$ *and* $t \in \mathbb{R}$ *satisfying* (7.4) *and the initial condition* $\theta(0, \nu) = 0$. *Moreover, it satisfies*

$$\frac{\partial\theta}{\partial t}(t, \nu) > 0 \qquad \textit{for any } (t, \nu).$$

Proof Formula (7.4) is equivalent to

$$f_j(x_j(t,\nu)) = b_{j-1}(\cos((\theta+\alpha)/2))^2 + b_j(\sin((\theta+\alpha)/2))^2,$$

where $\theta = \theta(t,\nu)$ and α is defined by

$$-(\nu_1,\nu_2) = \sqrt{\nu_1^2+\nu_2^2}\,(\cos\alpha, \sin\alpha).$$

If $\nu_2 \neq 0$, then $b_{j-1} > f_j(x_j(0,\nu)) > b_j$ and the function $t \mapsto f_j(x_j(t,\nu))$ oscillates between b_j and b_{j-1} and the second derivatives do not vanish at the turning points. Therefore, the assertion easily follows in this case, and we have $\partial\theta/\partial t \neq 0$ for any t. Since

$$\frac{d}{dt} f_j(x_j(t,\nu))|_{t=0} = \nu_2 \frac{\partial\theta}{\partial t}(0,\nu),$$

and since

$$\text{sign of } \nu_2 = \text{ sign of } \xi_j = \text{ sign of } df_j(x_j)/dt \text{ at } t=0,$$

it follows that $\partial\theta/\partial t > 0$ at $t=0$ and so for any t. (Note that $\partial f_j/\partial x_j > 0$ by the assumption posed at the beginning of §6.)

Now let us verify that the function $\theta(t,\nu)$ thus obtained for $\nu_2 \neq 0$ is smoothly extended to points where $\nu_2 = 0$. Putting $G(\lambda) = \prod_{k\neq j,j-1}(\lambda - b_k)$ in formula (3.7), we have

$$\sum_{i=1}^{n} U_i(t,\nu) = 0,$$

where $U_i(t,\nu)$ is given by

$$\int_0^t \frac{(-1)^i \prod_{k\neq j,j-1}(f_i - b_k)\,|\partial x_i(t,\nu)/\partial t|\,dt}{\sqrt{(-1)^{i-1}\prod_{k\neq j,j-1}(f_i-b_k)\big((f_i - f_{j,0})^2 - 2\nu_1(f_i - f_{j,0}) - \nu_2^2\big)}}.$$

When $\nu_2 \neq 0$, U_j is rewritten as

$$U_j(t,\nu) = \int_0^{\theta(t,\nu)} A_1(f_j)\,d\theta,$$

where $f_j = f_{j,0} + \nu_1(1-\cos\theta) + \nu_2\sin\theta$ and $A_1(\lambda)$ is as in (7.3). This formula redefine $\theta(t,\nu)$, which is effective for the case $\nu_2 = 0$ and is of C^∞ anywhere. Here, we again have $\partial\theta/\partial t > 0$ since

$$(-1)^i \prod_{k\neq j,j-1} (f_i(x_i(t,\nu)) - b_k)\,|\partial x_i(t,\nu)/\partial t| \leq 0,$$

for any $i \neq j, j-1$ and is strictly negative for i with L_i being the whole circle.

It should be noted that when $\nu = 0$ the function $f_j(x_j(t,\nu))$ is identically equal to $f_{j,0}$ (constant), but the function $\theta(t,0)$ is strictly increasing in t.

Lemma 7.16 $\theta(\tau_1, 0) = 2\pi$.

Proof When $\nu \neq 0$, we have

$$2(b_{j-1} - b_j) = \sigma_j(t_j) = \int_0^{t_j} \left| \frac{df_j(x_j(s,\nu))}{ds} \right| ds.$$

The right-hand side is equal to

$$\begin{aligned} &\int_0^{t_j} |\nu_1 \sin\theta(s,\nu) + \nu_2 \cos\theta(s,\nu)| \frac{\partial\theta}{\partial s} ds \\ &= \sqrt{\nu_1^2 + \nu_2^2} \int_0^{\theta(t,\nu)} |\sin(\theta+\alpha)| d\theta, \end{aligned}$$

where α is the same one as in the proof of Lemma 7.15. Since $b_{j-1} - b_j = 2\sqrt{\nu_1^2 + \nu_2^2}$, it, therefore, follows that $\theta(t_j, \nu) = 2\pi$.

On the other hand, by Propositions 5.4 and 6.1, we have $r_j \leq t_j \leq r_{j-1}$ when $\nu \neq 0$, and r_{j-1}, r_j tend to τ_1 when ν tends to 0. Therefore we have $\theta(\tau_1, 0) = 2\pi$ by continuity. □

We now consider geodesic equations (3.7) and (3.9) for the following polynomials $G(\lambda)$:

$$G(\lambda) = \prod_{k \neq j, j-1} (\lambda - b_k) \cdot (\lambda - f_{j,0})^\alpha \qquad (\alpha = 0, 1, 2).$$

Let us put, for each $\alpha = 0, 1, 2$:

$$\begin{aligned} u_i^\alpha(x_i, \nu) &= \frac{-\sqrt{(-1)^{i-1} \prod_{k \neq j, j-1} (f_i - b_k)}\,(f_i - f_{j,0})^\alpha}{\sqrt{(f_i - f_{j,0})^2 - 2\nu_1(f_i - f_{j,0}) - \nu_2^2}} \qquad (i \neq j), \\ u_j^\alpha(\theta, \nu) &= A_1(f_j)\,(f_j - f_{j,0})^\alpha, \end{aligned}$$

and

$$\begin{aligned} U_i^\alpha(t,\nu) &= \textstyle\int_0^t u_i^\alpha(x_i(t,\nu), \nu) \left| \frac{\partial x_i}{\partial t}(t,\nu) \right| dt \quad (i \neq j), \\ U_j^\alpha(t,\nu) &= \textstyle\int_0^{\theta(t,\nu)} u_j^\alpha(\theta,\nu)\, d\theta, \end{aligned}$$

where $f_i = f_i(x_i)$ $(i \neq j)$ and

$$f_j = f_{j,0} + \nu_1(1 - \cos\theta) + \nu_2 \sin\theta.$$

We then have

$$\sum_{\substack{1\le i\le n\\ i\neq j}} U_i^\alpha(t,\nu)+U_j^\alpha(t,\nu)=\begin{cases}0 & (\alpha=0,1)\\ t & (\alpha=2)\end{cases}. \tag{7.5}$$

The functions y_0, y_1, and y_2 clearly have the following relations with the above functions:

$$y_0=A_1(f_{j,0})(f_j(x_j)-f_{j,0}), \tag{7.6}$$

$$dy_\alpha=\sum_{i\in I}\epsilon_i u_i^\alpha(x_i,0)dx_i \quad (\alpha=1,2). \tag{7.7}$$

Therefore, for the proof of the proposition it is necessary to calculate the first and second derivatives of the above functions at $(t,\nu)=(\tau_1,0)$.

Let us start with the derivatives in t:

$$\frac{\partial U_i^\alpha}{\partial t}(t,\nu)=\epsilon_i u_i^\alpha(x_i(t,\nu),\nu)\frac{\partial x_i}{\partial t} \quad (i\neq j) \tag{7.8}$$

$$\frac{\partial U_j^\alpha}{\partial t}(t,\nu)=\frac{\partial\theta}{\partial t}(t,\nu)\,u_j^\alpha(\theta(t,\nu),\nu). \tag{7.9}$$

Taking the fact

$$U_j^\alpha(t,0)=0 \quad \text{for any } t\in\mathbb{R} \text{ and } \alpha=1,2, \tag{7.10}$$

into account, we have

$$\sum_{i\neq j}\frac{\partial U_i^\alpha}{\partial t}(t,0)=\begin{cases}0 & (\alpha=1)\\ 1 & (\alpha=2)\end{cases}.$$

Since $u_i^\alpha(x_{i,1},0)=(\partial x_i/\partial t)(\tau_1,0)=0$ if $f_i(x_{i,1})=b_k$ for some $k\neq j, j-1$, it, therefore, follows that

$$\frac{\partial y_\alpha}{\partial t}(\tau_1,0)=\begin{cases}0 & (\alpha=1)\\ 1 & (\alpha=2)\end{cases}, \quad \frac{\partial^2 y_\alpha}{\partial t^2}(\tau_1,0)=0 \quad (\alpha=1,2). \tag{7.11}$$

We also have

$$y_0(t,0)=0 \quad \text{for any } t\in\mathbb{R}. \tag{7.12}$$

Next, let us consider the derivatives $\partial U_i^\alpha/\partial t\partial \nu_k$. We have from (7.8) and (7.9):

$$\frac{\partial U_i^\alpha}{\partial t\partial \nu_k}(t,\nu) = \epsilon_i \frac{\partial u_i^\alpha}{\partial x_i}(x_i(t,\nu),\nu)\frac{\partial x_i}{\partial \nu_k}\frac{\partial x_i}{\partial t} + \epsilon_i \frac{\partial u_i^\alpha}{\partial \nu_k}(x_i(t,\nu),\nu)\frac{\partial x_i}{\partial t} + \epsilon_i u_i^\alpha(x_i(t,\nu),\nu)\frac{\partial^2 x_i}{\partial t\partial \nu_k}$$

and

$$\frac{\partial U_j^\alpha}{\partial t\partial \nu_k}(t,\nu) = \frac{\partial^2\theta}{\partial t\partial \nu_k}(t,\nu)\, u_j^\alpha(\theta,,\nu) + \frac{\partial\theta}{\partial t}\frac{\partial\theta}{\partial \nu_k}\frac{\partial u_j^\alpha}{\partial\theta}(\theta,\nu) + \frac{\partial\theta}{\partial t}\frac{\partial u_j^\alpha}{\partial \nu_k}(\theta,\nu).$$

Since $\partial x_i/\partial \nu_k$ $(k=1,2)$ is the coefficient of $\partial/\partial x_i$ in the Jacobi field Z_j or Z_{j-1}, it vanishes at $(t,\nu)=(\tau_1,0)$. Also, we have

$$u_j^\alpha(2\pi,0) = \frac{\partial u_j^\alpha}{\partial\theta}(2\pi,0) = \frac{\partial u_j^\alpha}{\partial \nu_k}(2\pi,0) = 0,$$

for $\alpha = 1,2$. Therefore, we have

$$\sum_{i\neq j}\epsilon_i \frac{\partial u_i^\alpha}{\partial \nu_k}(x_{i,1},0)\frac{\partial x_i}{\partial t}(\tau_1,0) + \sum_{i\neq j}\epsilon_i u_i^\alpha(x_{i,1},0)\frac{\partial^2 x_i}{\partial t\partial \nu_k}(\tau_1,0) = 0, \tag{7.13}$$

for $\alpha = 1,2$. Note that the second sum in the left-hand side of the above equality is equal to $(\partial y_\alpha/\partial t\partial \nu_k)(\tau_1,0)$ and

$$\frac{\partial u_i^\alpha}{\partial \nu_k}(x_{i,1},0) = \begin{cases} u_i^{\alpha-1}(x_{i,1},0) & (k=1) \\ 0 & (k=2) \end{cases}.$$

Therefore,

$$\frac{\partial^2 y_\alpha}{\partial t\partial \nu_2}(\tau_1,0) = 0 \qquad (\alpha = 1,2), \tag{7.14}$$

and

$$\frac{\partial^2 y_\alpha}{\partial t\partial \nu_1}(\tau_1,0) = -\textstyle\sum_{i\neq j} u_i^{\alpha-1}(x_{i,1},0)\frac{\partial x_i}{\partial t}(\tau_1,0) = -\sum_{i\neq j}\frac{\partial U_i^{\alpha-1}}{\partial t}(\tau_1,0)$$
$$= \frac{\partial U_j^{\alpha-1}}{\partial t}(\tau_1,0) = \begin{cases} A_1(f_{j,0})\frac{\partial\theta}{\partial t}(\tau_1,0) & (\alpha=1) \\ 0 & (\alpha=2) \end{cases}. \tag{7.15}$$

For y_0, we have

$$\frac{\partial y_0}{\partial t}(t,\nu) = A_1(f_{j,0})(\nu_1 \sin\theta + \nu_2 \cos\theta)\frac{\partial \theta}{\partial t}.$$

Thus,

$$\frac{\partial^2 y_0}{\partial t \partial \nu_k}(\tau_1, 0) = \begin{cases} 0 & (k=1) \\ A_1(f_{j,0})\frac{\partial \theta}{\partial t}(\tau_1, 0) & (k=2) \end{cases}. \tag{7.16}$$

Next, we shall consider the derivatives in ν_1 and ν_2. For the first derivatives, we have

Lemma 7.17 $\frac{\partial y_\alpha}{\partial \nu_k}(\tau_1, 0) = 0$ *for any* $\alpha = 0, 1, 2$ *and* $k = 1, 2$.

Proof For the case $\alpha = 0$, the assertion follows from the fact that $\partial f_j / \partial \nu_k = 0$ when $\theta = 2\pi$. For $\alpha = 1, 2$, we have

$$\frac{\partial y_\alpha}{\partial \nu_k}(\tau_1, 0) = \sum_{i \in I} \epsilon_i u_i^\alpha(x_{i,1}, 0)\frac{\partial x_i}{\partial \nu_k}(\tau_1, 0).$$

Since $\partial x_i / \partial \nu_k$ is a component of the Jacobi fields Z_j, Z_{j-1}, it vanishes at $(\tau_1, 0)$. Thus, the assertion follows.

To compute the second derivatives in ν_1, ν_2, we begin with the following lemma.

Lemma 7.18 *For each* $\alpha = 0, 1, 2$ *and* $k = 1, 2$,

$$\frac{\partial U_i^\alpha}{\partial \nu_k}(t,\nu) = \epsilon_i u_i^\alpha(x_i(t,\nu),\nu)\frac{\partial x_i}{\partial \nu_k} + \int_0^t \frac{\partial u_i^\alpha}{\partial \nu_k}\left|\frac{\partial x_i}{\partial t}\right| dt.$$

Proof We have

$$\frac{\partial U_i^\alpha}{\partial \nu_k}(t,\nu) = \int_0^t \epsilon_i(t,\nu)\left[\left(\frac{\partial u_i^\alpha}{\partial x_i}\frac{\partial x_i}{\partial \nu_k} + \frac{\partial u_i^\alpha}{\partial \nu_k}\right)\frac{\partial x_i}{\partial t} + u_i^\alpha \frac{\partial^2 x_i}{\partial t \partial \nu_k}\right] dt.$$

Here $\epsilon_i(t,\nu)$ $(= \pm 1)$ stands for the sign of $(\partial x_i / \partial t)(t,\nu)$, which is locally constant in t for each fixed ν outside the turning point, i.e., the point t where $(\partial x_i / \partial t)(t,\nu) = 0$.

Observe that the integrand is equal to

$$\epsilon_i \frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial \nu_k} u_i^\alpha\right) + \frac{\partial u_i^\alpha}{\partial \nu_k}\left|\frac{\partial x_i}{\partial t}\right|.$$

Since $u_i^\alpha(x_i(t,\nu),\nu)$ vanishes at each turning point and $(\partial x_i / \partial \nu_k)(t,\nu)$ vanishes at $t = 0$, we, therefore, obtain

$$\int_0^t \epsilon_i(t,\nu)\frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial \nu_k} u_i^\alpha\right) dt = \epsilon_i \frac{\partial x_i}{\partial \nu_k}(t,\nu) u_i^\alpha(x_i(t,\nu),\nu).$$

Thus, the lemma follows.

Lemma 7.19

$$\frac{\partial \theta}{\partial \nu_1}(\tau_1, 0) = -A_1(f_{j,0})^{-1}C, \quad \frac{\partial \theta}{\partial \nu_2}(\tau_1, 0) = 0,$$

where

$$C = \sum_{i \neq j} \int_0^{\tau_1} \frac{u_i^0(x_i(t,0),0)}{f_i(x_i(t,0)) - f_{j,0}} \left| \frac{\partial x_i}{\partial t}(t,0) \right| dt + 2\pi A_1'(f_{j,0}).$$

Proof We use the formula

$$\sum_{i \neq j} \frac{\partial U_i^0}{\partial \nu_k}(\tau_1, 0) + \frac{\partial U_j^0}{\partial \nu_k}(\tau_1, 0) = 0.$$

By Lemma 7.18, we have

$$\frac{\partial U_i^0}{\partial \nu_k}(\tau_1, 0) = \begin{cases} \int_0^{\tau_1} \frac{u_i^0(x_i(t,0),0)}{f_i(x_i(t,0)) - f_{j,0}} \left| \frac{\partial x_i}{\partial t} \right| dt & (k=1) \\ 0 & (k=2) \end{cases}.$$

Also, we have

$$\frac{\partial U_j^0}{\partial \nu_k}(\tau_1, 0) = \frac{\partial \theta}{\partial \nu_k}(\tau_1, 0) A_1(f_{j,0}) + \int_0^{2\pi} A_1'(f_{j,0}) \frac{\partial f_j}{\partial \nu_k} \, d\theta,$$

and

$$\frac{\partial f_j}{\partial \nu_k} = \begin{cases} 1 - \cos\theta & (k=1) \\ \sin\theta & (k=2) \end{cases}.$$

Therefore, the lemma follows. □

Corollary 7.20

$$\frac{\partial^2 y_0}{\partial \nu_1^2}(\tau_1, 0) = \frac{\partial^2 y_0}{\partial \nu_2^2}(\tau_1, 0) = 0, \quad \frac{\partial^2 y_0}{\partial \nu_1 \partial \nu_2}(\tau_1, 0) = -C.$$

Proof Since

$$y_0(t, \nu) = A_1(f_{j,0})(\nu_1(1 - \cos\theta(t,\nu)) + \nu_2 \sin\theta(t,\nu)),$$

the assertion easily follows from the previous lemma. □

Finally, we shall consider the second derivatives of y_1 and y_2.

Lemma 7.21

$$\frac{\partial^2 U_i^\alpha}{\partial \nu_k \partial \nu_l}(\tau_1, 0) = \epsilon_i \frac{\partial^2 x_i}{\partial \nu_k \partial \nu_l}(\tau_1, 0)\, u_i^\alpha(x_{i,1}, 0) + \int_0^{\tau_1} \frac{\partial^2 u_i^\alpha}{\partial \nu_k \partial \nu_l}(x_i(t,0), 0)\left|\frac{\partial x_i}{\partial t}\right| dt$$

for $k, l = 1, 2$, $\alpha = 1, 2$, *and* $i \neq j$.

Proof We differentiate the formula in Lemma 7.18 by ν_l and put $(t, \nu) = (\tau_1, 0)$. Then the first term in the right-hand side becomes the first term of the right-hand side of the above formula since $\partial x_i / \partial \nu_k$ vanishes at $(\tau_1, 0)$. Also the second term becomes

$$\int_0^{\tau_1} \frac{\partial}{\partial \nu_l}\left[\frac{\partial u_i^\alpha}{\partial \nu_k}(x_i(t,\nu), \nu)\left|\frac{\partial x_i}{\partial t}(t, \nu)\right|\right]_{\nu=0} dt.$$

This integrand is equal to

$$\frac{\partial^2 u_i^\alpha}{\partial \nu_k \partial \nu_l}(x_i(t,0), 0)\left|\frac{\partial x_i}{\partial t}\right| + \epsilon_i(t, \nu)\frac{\partial}{\partial t}\left[\frac{\partial u_i^\alpha}{\partial \nu_k}\frac{\partial x_i}{\partial \nu_l}\right]_{\nu=0}.$$

By the same reason as in the proof of Lemma 7.18, we have

$$\int_0^{\tau_1} \epsilon_i(t, \nu)\frac{\partial}{\partial t}\left[\frac{\partial u_i^\alpha}{\partial \nu_k}\frac{\partial x_i}{\partial \nu_l}\right]_{\nu=0} dt = \frac{\partial u_i^\alpha}{\partial \nu_k}(x_{i,1} 0)\left|\frac{\partial x_i}{\partial \nu_l}(\tau_1, 0)\right| = 0.$$

Thus, the lemma follows. □

Lemma 7.22

$$\frac{\partial^2 U_j^\alpha}{\partial \nu_1^2}(\tau_1, 0) = \begin{cases} 6\pi A_1'(j_{j,0}) & (\alpha = 1) \\ 6\pi A_1(f_{j,0}) & (\alpha = 2) \end{cases}$$

$$\frac{\partial^2 U_j^\alpha}{\partial \nu_2^2}(\tau_1, 0) = \begin{cases} 2\pi A_1'(j_{j,0}) & (\alpha = 1) \\ 2\pi A_1(f_{j,0}) & (\alpha = 2) \end{cases}$$

$$\frac{\partial^2 U_j^\alpha}{\partial \nu_1 \partial \nu_2}(\tau_1, 0) = 0 \quad (\alpha = 1, 2).$$

Proof A direct computation yields

$$\frac{\partial^2 U_i^\alpha}{\partial \nu_k \partial \nu_l}(t, \nu) = \frac{\partial^2 \theta}{\partial \nu_k \partial \nu_l}(t, \nu)\, u_j^\alpha(\theta, \nu) + \frac{\partial \theta}{\partial \nu_k}(t, \nu)\left(\frac{\partial u_j^\alpha}{\partial \nu_l} + \frac{\partial u_j^\alpha}{\partial \theta}\frac{\partial \theta}{\partial \nu_l}\right) + \frac{\partial \theta}{\partial \nu_l}(t, \nu)\frac{\partial u_j^\alpha}{\partial \nu_k}(\theta, \nu) + \int_0^{\theta(t,\nu)} \frac{\partial^2 u_j^\alpha}{\partial \nu_k \partial \nu_l}(\theta, \nu)\, d\theta.$$

Since

$$u_j^\alpha(\theta, \nu) = A_1(f_j)(f_j - f_{j,0})^\alpha, \quad f_j = f_{j,0} + \nu_1(1 - \cos\theta) + \nu_2 \sin\theta,$$

it is easily seen that, for $\alpha = 1, 2$, the functions

$$u_j^\alpha, \quad \frac{\partial u_j^\alpha}{\partial \nu_k}, \quad \frac{\partial u_j^\alpha}{\partial \nu_l}, \quad \frac{\partial u_j^\alpha}{\partial \theta},$$

vanish at $(\theta, \nu) = (2\pi, 0)$. Since $\theta(\tau_1, 0) = 2\pi$, we, therefore, obtain

$$\frac{\partial^2 U_j^\alpha}{\partial \nu_k \partial \nu_l}(\tau_1, 0) = \int_0^{2\pi} \frac{\partial^2 u_j^\alpha}{\partial \nu_k \partial \nu_l}(\theta, 0)\, d\theta.$$

From this formula, the lemma follows immediately. □

Lemma 7.23

$$\frac{\partial^2 y_\alpha}{\partial \nu_k \partial \nu_l}(\tau_1, 0) = \sum_{i \neq j} \epsilon_i \frac{\partial^2 x_i^\alpha}{\partial \nu_k \partial \nu_l}(\tau_1, 0)\, u_i^\alpha(x_{i,1}, 0) \quad (\alpha = 1, 2).$$

Proof First, note that the sum in the right-hand side is equal to the sum in such i that $i \in I$ since for $i \neq j$ with $i \notin I$ the value $f_i(x_{i,1})$ is equal to some b_k $(k \neq j, j-1)$ and $u_i^\alpha(x_{i,1}, 0) = 0$ in this case. By (7.7), we have

$$\frac{\partial y_\alpha}{\partial \nu_k}(t, \nu) = \sum_{i \in I} \epsilon_i \frac{\partial x_i^\alpha}{\partial \nu_k}(t, \nu)\, u_i^\alpha(x_i(t, \nu), 0).$$

Noting the fact that $\partial x_i / \partial \nu_k$ vanishes at $(\tau_1, 0)$, we obtain the lemma by differentiating this formula with ν_l.

From the above lemmas and the formula

$$\sum_{i \neq j} \frac{\partial^2 U_i^\alpha}{\partial \nu_k \partial \nu_l}(\tau_1, 0) + \frac{\partial^2 U_j^\alpha}{\partial \nu_k \partial \nu_l}(\tau_1, 0) = 0,$$

we have

$$\frac{\partial^2 y_\alpha}{\partial \nu_k \partial \nu_l}(\tau_1, 0) = -\sum_{i \neq j} \int_0^{\tau_1} \frac{\partial^2 u_i^\alpha}{\partial \nu_k \partial \nu_l}(x_i(t, 0), 0) \left| \frac{\partial x_i}{\partial t} \right| dt - \frac{\partial^2 U_j^\alpha}{\partial \nu_k \partial \nu_l}(\tau_1, 0). \quad (7.17)$$

Thus, we need to compute the integrals in the right-hand side of the above formula. The following lemma is straightforward.

Lemma 7.24

$$\frac{\partial^2 u_i^\alpha}{\partial v_1^2}(x_i, 0) = \frac{3u_i^\alpha(x_i, 0)}{(f_i - f_{j,0})^2}, \quad \frac{\partial^2 u_i^\alpha}{\partial v_2^2}(x_i, 0) = \frac{u_i^\alpha(x_i, 0)}{(f_i - f_{j,0})^2},$$
$$\frac{\partial^2 u_i^\alpha}{\partial v_1 \partial v_2}(x_i, 0) = 0 \qquad (\alpha = 1, 2).$$

Corollary 7.25

$$\frac{\partial^2 y_1}{\partial v_1^2}(\tau_1, 0) = -3C, \quad \frac{\partial^2 y_1}{\partial v_2^2}(\tau_1, 0) = -C,$$
$$\frac{\partial^2 y_1}{\partial v_1 \partial v_2}(\tau_1, 0) = 0, \quad \frac{\partial^2 y_2}{\partial v_k \partial v_l}(\tau_1, 0) = 0 \quad (k, l = 1, 2),$$

where C is the constant given in Lemma 7.19.

Proof First, we consider the case where $\alpha = 2$ and $k = l$. By (7.17) and Lemmas 7.22 and 7.24, we have

$$\frac{\partial^2 y_2}{\partial v_k^2}(\tau_1, 0) = -e\left[\sum_{i \neq j} U_i^0(\tau_1, 0) + U_j^0(\tau_1, 0)\right] = 0,$$

where $e = 3$ or 1 according as $k = 1$ or 2, respectively. Similarly, for the case where $\alpha = 1$, we have

$$\begin{aligned}\frac{\partial^2 y_1}{\partial v_k^2}(\tau_1, 0) &= -\, e\left[\sum_{i \neq j} \int_0^{\tau_1} \frac{u_i^0(x_i(t, 0), 0)}{f_i(x_i(t, 0)) - f_{j,0}} \left|\frac{\partial x_i}{\partial t}\right| dt + 2\pi A_1'(f_{j,0})\right] \\ &= -\, eC,\end{aligned}$$

Also, for the case where $k \neq l$, we have

$$\frac{\partial y_\alpha}{\partial u_k \partial u_l}(\tau_1, 0) = 0 \quad (\alpha = 1, 2).$$

Thus, the corollary follows. □

By formulas (7.11), (7.12), (7.14), (7.15), (7.16), and by Lemma 7.17, Corollary 7.20, Corollary 7.25, we obtain the formulas in Proposition 7.8 by putting

$$c = -\frac{C}{2}, \qquad c' = A_1(f_{j,0})\frac{\partial \theta}{\partial t}(\tau_1, 0).$$

It is clear that $c' \neq 0$. Therefore, to complete the proof of the proposition, it is enough to show the following lemma.

Lemma 7.26 $C > 0$.

Proof We put

$$W_i(t,\nu) = U_i^1(t,\nu) + (f_{j,0} - a_n)U_i^0(t,\nu).$$

Then we have

$$\sum_{i=1}^{n} W_i(t,\nu) = 0,$$

$$W_i(t,\nu) = \int_0^t \frac{-(f_i - a_n)\sqrt{(-1)^{i-1}\prod_{k\neq j,j-1}(f_i - b_k)}\,|\frac{\partial x_i(t,\nu)}{\partial t}|\,dt}{\sqrt{(f_i - f_{j,0})^2 - 2\nu_1(f_i - f_{j,0}) - \nu_2^2}},$$

for $i \neq j$ and

$$W_j(t,\nu) = \int_0^{\theta(t,\nu)} A_1(f_j)(f_j - a_n)d\theta.$$

We now take the derivative in ν_1 and put $\nu = 0$, $t = \tau_1$. Since

$$\frac{\partial W_j}{\partial \nu_1}(\tau_1, 0) = 2\pi \frac{\partial}{\partial \lambda}(A_1(\lambda)(\lambda - a_n))|_{\lambda = f_{j,0}} + \frac{\partial \theta}{\partial \nu_1}(\tau_1, 0)A_1(f_{j,0})(f_{j,0} - a_n),$$

and since

$$\frac{\partial \theta}{\partial \nu_1}(\tau_1, 0)A_1(f_{j,0}) = -C,$$

we have

$$C(f_{j,0} - a_n) = \sum_{i\neq j} \frac{\partial W_i}{\partial \nu_1}(\tau_1, 0) + 2\pi \frac{\partial}{\partial \lambda}(A_1(\lambda)(\lambda - a_n))|_{\lambda = f_{j,0}}. \tag{7.18}$$

On the other hand, in view of formula (5.2), we have

$$\frac{\partial W_i}{\partial \nu_1}(t_i, 0) = \int_0^{t_i} \frac{-(f_i - a_n)\sqrt{(-1)^{i-1}\prod_{k\neq j,j-1}(f_i - b_k)}\,|\frac{\partial x_i(t,\nu)}{\partial t}|\,dt}{|f_i - f_{j,0}|(f_i - f_{j,0})} \tag{7.19}$$

$$= \lim_{b_j, b_{j-1}\to f_{j,0}} \frac{\partial}{\partial b_j} \int_{a_i^+}^{a_{i-1}^-} \frac{(-1)^l A(\lambda)\,(\lambda - a_n)\,G(\lambda)\,d\lambda}{\sqrt{-\prod_{k=1}^{n-1}(\lambda - b_k)\cdot\prod_{k=0}^{n}(\lambda - a_k)}} \quad (i \neq j), \tag{7.20}$$

where $G(\lambda) = \prod_{k \neq j, j-1} (\lambda - b_k)$ and t_i is the time defined in §5. Also, when $\nu \neq 0$ ($b_j \neq b_{j-1}$), we have

$$W_j(t_j, \nu) = \int_{b_j}^{b_{j-1}} \frac{A(\lambda)(\lambda - a_n)\sqrt{(-1)^j \prod_{k \neq j, j-1}(\lambda - b_k)}\, d\lambda}{\sqrt{(-1)^j \prod_{l=0}^{n}(\lambda - a_l)}\sqrt{(b_{j-1} - \lambda)(\lambda - b_j)}}$$
$$= 2 \int_0^{\pi} A_1(f_j)(f_j - a_n)\, d\theta,$$

where $f_j = f_{j,0} + \nu_1(1 - \cos\theta) + \nu_2 \sin\theta$. Therefore,

$$2\pi \frac{\partial}{\partial \lambda}(A_1(\lambda)(\lambda - a_n))|_{\lambda = f_{j,0}} = \frac{\partial W_j}{\partial \nu_1}(t_j, 0) =$$
$$\lim_{b_j \to f_{j,0} - 0} \frac{\partial}{\partial b_j} \int_{b_j}^{b_{j-1}} \frac{A(\lambda)(\lambda - a_n)\sqrt{(-1)^j \prod_{k \neq j, j-1}(\lambda - b_k)}\, d\lambda}{\sqrt{(-1)^j \prod_{l=0}^{n}(\lambda - a_l)}\sqrt{(b_{j-1} - \lambda)(\lambda - b_j)}}\Bigg|_{b_{j-1} = f_{j,0}}. \tag{7.21}$$

Then taking the limit $b_{j-1} = f_{j,0}, b_j \to f_{j,0} - 0$ (or $\nu_2 = 0, \nu_1 \to -0$) in the inequality of Proposition 4.5 (2), we have by Proposition 4.6

$$\sum_{i \neq j} \frac{\partial W_i}{\partial \nu_1}(t_i, 0) + \frac{\partial W_j}{\partial \nu_1}(t_j, 0) > 0. \tag{7.22}$$

Since $t = \tau_1$ is the first zero of the Jacobi field $Z_j(t)$ (and $Z_{j-1}(t)$), it follows that $t_j = \tau_1$, $t_i \leq \tau_1$ for $i > j$, and $t_i \geq \tau_1$ for $i < j$ by Proposition 5.4. From formula (7.19), it can be easily seen that the integrand of that formula is positive when $i > j$ and is negative when $i < j$. Therefore, we have

$$\frac{\partial W_i}{\partial \nu_1}(\tau_1, 0) \geq \frac{\partial W_i}{\partial \nu_1}(t_i, 0) \qquad (i \neq j).$$

Thus, the assertion follows from (7.18) and (7.22). □

This finishes the proof of Proposition 7.8.

8 Concluding Remarks

Although Riemannian manifolds considered in this paper are rather special in various meanings, it should be noted that the D_4^+ Lagrangian singularity is "stable". Therefore, one could also observe this type of singularities on conjugate loci for manifolds close to those. Related to this viewpoint, there are two problems that seems to be interesting.

1. Since D_4^+ singularities arise at points of double conjugacy, the presence of those singularities might mean that the tangential conjugate loci, the first one and the second for example, are connected. In the case of Riemannian manifolds treated in this paper, the kth and the $(k+1)$th tangential conjugate loci are connected with each other for every $k = 1, \ldots, n-2$ (n being the dimension of the manifold), if they are close to the round sphere. Therefore, in particular, one could ask whether the same situation holds for any Riemannian manifold close to the round sphere. Or, more generally, we might ask: Does the conjugate locus of any point on any Riemannian manifold diffeomorphic to the sphere necessarily contain points of multiple conjugacy?
2. In this paper, we found only cuspidal edges and D_4^+ Lagrangian singularities as singularities on the conjugate loci. Are there any manifolds whose conjugate loci have D_4^- or D_k ($k \geq 5$) Lagrangian singularities? Since it is quite hard to compute conjugate loci for general manifolds, the possible manifolds to be investigated might be those with indefinite metrics and with integrable geodesic flows (of Liouville type). For other choices of targets, one could investigate focal locus of convex bodies in the Euclidean space, i.e., the envelope of the family of straight lines normal to the give convex body. Those focal loci are similar objects to conjugate loci and may be easier to compute (see [8] for the 3D ellipsoid case).

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