Arnold Mathematical Journal (2021) 7:267–311
https://doi.org/10.1007/s40598-020-00168-2

RESEARCH CONTRIBUTION

Hypergeometric Integrals Modulo p and Hasse–Witt Matrices

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Received: 14 May 2020 / Revised: 13 October 2020 / Accepted: 27 October 2020 /
Published online: 21 November 2020

Abstract

We consider the KZ differential equations over $\mathbb{C}$ in the case, when the hypergeometric solutions are one-dimensional integrals. We also consider the same differential equations over a finite field $\mathbb{F}_p$. We study the space of polynomial solutions of these differential equations over $\mathbb{F}_p$, constructed in a previous work by Schechtman and the second author. Using Hasse–Witt matrices, we identify the space of these polynomial solutions over $\mathbb{F}_p$ with the space dual to a certain subspace of regular differentials on an associated curve. We also relate these polynomial solutions over $\mathbb{F}_p$ and the hypergeometric solutions over $\mathbb{C}$.

Keywords KZ equations · Hypergeometric integrals · Hasse–Witt matrix · Reduction to characteristic p

Mathematics Subject Classification 13A35 (33C60 , 32G20)

Contents

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1 Introduction

The KZ equations were discovered by Vadim Knizhnik and Alexander Zamolodchikov [5] to describe the differential equations for conformal blocks on sphere in the Wess–Zumino–Witten model of conformal field theory. The hypergeometric solutions of the

KZ equations were constructed more than 30 years ago, see [10,11]. The polynomial solutions of the KZ equations over the finite field $\mathbb{F}_p$ with a prime number p of elements were constructed recently in [12]. We call these solutions over $\mathbb{F}_p$ the $\mathbb{F}_p$-*hypergeometric solutions*. The general problem is to find the dimension of the space of $\mathbb{F}_p$-hypergeometric solutions and to understand relations between the hypergeometric solutions of the KZ equations over $\mathbb{C}$ and the $\mathbb{F}_p$-hypergeometric solutions.

In this paper, we consider an example of the KZ differential equations, whose hypergeometric solutions over $\mathbb{C}$ are n-vectors of the integrals

$$I^{(\gamma)}(z_1,\dots,z_n)=\left(\int_\gamma \frac{1}{x-z_1}\frac{\mathrm{d}x}{y},\ \dots,\ \int_\gamma \frac{1}{x-z_n}\frac{\mathrm{d}x}{y}\right), \tag{1.1}$$

where

$$y^q=(x-z_1)\dots(x-z_n) \tag{1.2}$$

and γ is a suitable 1-cycle. It is well known that the space of such n-vectors is $n-1$-dimensional. We consider the same differential KZ equations over the field $\mathbb{F}_p$ under the assumption that q is also a prime number and $p>q$, $p>n$, $n=kq+1$ for some positive integer k. We show that the dimension of the space of $\mathbb{F}_p$-hypergeometric solutions equals only a fraction of n. Namely, let a_1 be the unique positive integer such that $1\le a_1\le q-1$ and $a_1p\equiv 1 \pmod q$. This a_1 is the inverse of p modulo q. It turns out that the dimension of the space of $\mathbb{F}_p$-hypergeometric solutions equals a_1k. More precisely, the dimension of the space of $\mathbb{F}_p$-hypergeometric solutions can be defined as follows. Consider the curve X defined by the affine equation (1.2). The cyclic group $\mathbb{Z}_q$ of qth roots of unity acts on X by multiplication on the coordinate y. The space $\Omega^1(X)$ of regular differentials on X splits into eigenspaces of the $\mathbb{Z}_q$-action, $\Omega^1(X)=\bigoplus_{a=1}^{q-1}\Omega^1_a(X)$, where $\Omega^1_a(X)$ consists of differentials of the form $u(x)\mathrm{d}x/y^a$. We show that the dimension of the space of $\mathbb{F}_p$-hypergeometric solutions equals the dimension of $\Omega^1_{a_1}(X)$. Moreover, we establish an isomorphism of the space of $\mathbb{F}_p$-hypergeometric solutions and the space dual to $\Omega^1_{a_1}(X)$. That isomorphism is constructed with the help of the map adjoint to the corresponding Cartier map, and more precisely, with the help of the corresponding Hasse–Witt matrix. This is our first main result.

We also choose one solution of the KZ equations over $\mathbb{C}$ and call it *distinguished*. We expand the distinguished solutions into the Taylor series at some point, reduce the coefficients of the Taylor expansion modulo p and present this reduced Taylor series as an infinite formal sum of $\mathbb{F}_p$-hypergeometric solutions, with coefficients being matrix elements of iterates of the associated Hasse–Witt matrix. Moreover, this presentation allows one to recover a basis of $\mathbb{F}_p$-hypergeometric solutions in terms of the reduced Taylor expansion. This statement is our second main result.

Our comparison of the reduced Taylor expansion of a solution over $\mathbb{C}$ and $\mathbb{F}_p$-hypergeometric solutions is analogous to Manin's considerations of the elliptic integral in his classical paper [7] in 1961, see also "Manin's Result: The Unity of Mathematics" in the book [2] by Clemens.

For $q = 2$, the results of this paper have been obtained in [17].

The paper is organized as follows. In Sect. 2, we describe the differential KZ equations considered in this paper. We construct the hypergeometric solutions of these equations over $\mathbb{C}$ and the $\mathbb{F}_p$-hypergeometric solutions. In Sect. 3, we define the module of $\mathbb{F}_p$-hypergeometric solutions, show that it is free, and calculate the rank of the module. In Sect. 3.3, we describe the fusion procedure for modules of $\mathbb{F}_p$-hypergeometric solutions.

In Sect. 4, we prove a generalization of the classical Lucas theorem, which allows us to reduce modulo p the coefficient of the Taylor expansion of the distinguished solution.

In Sect. 5, we discuss different bases in the module of $\mathbb{F}_p$-hypergeometric solutions. One of the bases naturally appears in the defining construction of $\mathbb{F}_p$-hypergeometric solutions and the other is convenient to relate the $\mathbb{F}_p$-hypergeometric solutions to the Taylor expansion of the distinguished solution.

In Sects. 6 and 7, we study the Cartier map related to our $\mathbb{F}_p$-hypergeometric solutions and prove our first main result by identifying the module of $\mathbb{F}_p$-hypergeometric solutions with the space dual to $\Omega^1_{a_1}(X)$, see Theorems 6.2 and 7.4. In Sect. 8, we change variables in the Hasse–Witt matrix preparing it for application to the study of the distinguished solution.

In Sect. 9, we compare the reduced Taylor series of the distinguished solution and $\mathbb{F}_p$-hypergeometric solutions, see Theorem 9.8.

The authors thank Etingof for useful discussions.

2 KZ Equations

2.1 Description of Equations

Let $\mathfrak{g}$ be a simple Lie algebra over the field $\mathbb{C}$, $\Omega \in \mathfrak{g}^{\otimes 2}$ the Casimir element corresponding to an invariant scalar product on $\mathfrak{g}$, $V_1, \dots, V_n$ finite-dimensional irreducible $\mathfrak{g}$-modules.

The system of KZ equations with parameter $\kappa \in \mathbb{C}^\times$ on a tensor $\otimes_{i=1}^n V_i$ valued function $I(z_1, \dots, z_n)$ is the system of the differential equations

$$\frac{\partial I}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega^{(i,j)}}{z_i - z_j} I, \quad i = 1, \dots, n, \tag{2.1}$$

where $\Omega^{(i,j)}$ is the Casimir element acting in the ith and jth factors, see [3,5]. The KZ differential equations commute with the action of $\mathfrak{g}$ on $\otimes_{i=1}^n V_i$, in particular, they preserve the subspaces of singular vectors of a given weight.

In [10,11], the KZ equations restricted to the subspace of singular vectors of a given weight were identified with a suitable Gauss–Manin differential equations and the corresponding solutions of the KZ equations were presented as multidimensional hypergeometric integrals.

Let p be a prime number and $\mathbb{F}_p$ the field with p elements. Let $\mathfrak{g}^p$ be the same Lie algebra considered over $\mathbb{F}_p$. Let $V_1^p, \dots, V_n^p$ be the $\mathfrak{g}^p$-modules which are reductions modulo p of $V_1, \dots, V_n$, respectively. If κ is an integer and p large enough with respect to κ, then one can look for solutions $I(z_1, \dots, z_n)$ of the KZ equations in $\otimes_{i=1}^n V_i^p \otimes \mathbb{F}_p[z_1, \dots, z_n]$. Such solutions were constructed in [12].

In this paper, we address two questions:

A. What is the number of independent solutions constructed in [12] for a given $\mathbb{F}_p$?
B. How are those solutions related to the solutions over $\mathbb{C}$ that are given by hypergeometric integrals?

We answer these questions in an example in which the hypergeometric solutions are presented by one-dimensional integrals. The case of hyperelliptic integrals was considered in [17]. The object of our study is the following joint system of differential and algebraic equations.

2.1.1 Assumptions in This Paper

We fix prime numbers p, q, $p > q$, a positive integer n, a vector $\Lambda = (\Lambda_1, \dots, \Lambda_n) \in \mathbb{Z}^n_{>0}$, such that $\Lambda_i < q$ for all $i = 1, \dots, n$.

For $z = (z_1, \dots, z_n)$, we study the column vectors $I(z) = (I_1(z), \dots, I_n(z))$ satisfying the system of differential and algebraic linear equations:

$$\frac{\partial I}{\partial z_i} = \frac{1}{q}\sum_{j\neq i}\frac{\Omega_{ij}}{z_i - z_j}I, \quad i = 1, \dots, n, \quad \Lambda_1 I_1(z) + \dots + \Lambda_n I_n(z) = 0, \tag{2.2}$$

where

$$\Omega_{ij} = \begin{pmatrix} & \overset{i}{\vdots} & & \overset{j}{\vdots} & \\ i \cdots & -\Lambda_j & \cdots & \Lambda_j & \cdots \\ & \vdots & & \vdots & \\ j \cdots & \Lambda_i & \cdots & -\Lambda_i & \cdots \\ & \vdots & & \vdots & \end{pmatrix}, \tag{2.3}$$

and all other entries are zero.

In this paper, this joint system of *differential and algebraic equations* will be called the *KZ differential equations*.

We will construct solutions of these KZ differential equations over $\mathbb{C}$ and over $\mathbb{F}_p$ and compare the properties of solutions.

Remark The system of equations (2.2) is the system of the true KZ differential equations (2.1) with parameter $\kappa = q$, associated with the Lie algebra $\mathfrak{sl}_2$ and the subspace of singular vectors of weight $\sum_{i=1}^n \Lambda_i - 2$ of the tensor product $V_{\Lambda_1} \otimes \dots \otimes V_{\Lambda_n}$, where V_{Λ_i} is the irreducible $\Lambda_i + 1$-dimensional $\mathfrak{sl}_2$-module, up to a gauge transformation, see this example in [14, Section 1.1].

Notice also that the assumption $\Lambda_i < q$ for $i = 1, \ldots, n$ appears when one is interested in differential equations for $\mathfrak{sl}_2$ conformal blocks with central charge $q-2$.

2.2 Solutions over $\mathbb{C}$

Consider the *master function*

$$\Phi(t, z_1, \ldots, z_n) = \prod_{a=1}^{n} (t - z_a)^{-\Lambda_a/q} \tag{2.4}$$

and the n-vector of hypergeometric integrals

$$I^{(\gamma)}(z) = (I_1(z), \ldots, I_n(z)), \tag{2.5}$$

where

$$I_j = \int \Phi(t, z_1, \ldots, z_n)\, \frac{dt}{t - z_j}, \quad j = 1, \ldots, n. \tag{2.6}$$

The integrals I_j, $j = 1, \ldots, n$ are over an element γ of the first homology group of the algebraic curve with affine equation

$$y^q = (t - z_1)^{\Lambda_1} \ldots (t - z_n)^{\Lambda_n}.$$

Starting from such γ, chosen for given $\{z_1, \ldots, z_n\}$, the vector $I^{(\gamma)}(z)$ can be analytically continued as a multivalued holomorphic function of z to the complement in $\mathbb{C}^n$ to the union of the diagonal hyperplanes $z_i = z_j$.

Theorem 2.1 *The vector $I^{(\gamma)}(z)$ satisfies the KZ differential equations* (2.2).

Theorem 2.1 is a classical statement. Much more general algebraic and differential equations satisfied by analogous multidimensional hypergeometric integrals were considered in [10,11]. Theorem 2.1 is discussed as an example in [14, Section 1.1].

Theorem 2.2 [13, Formula (1.3)] *All solutions of equations* (2.2) *have this form, namely the complex vector space of solutions of the form* (2.5)–(2.6) *is $n-1$-dimensional.*

This theorem follows from the determinant formula for multidimensional hypergeometric integrals in [13], in particular, from [13, Formula (1.3)].

2.3 Solutions over $\mathbb{F}_p$

Polynomial solutions of system (2.2), considered over the field $\mathbb{F}_p$, were constructed in [12].

For $i = 1, \dots, n$, choose positive integers M_i such that

$$M_i \equiv -\frac{\Lambda_i}{q} \pmod{p}, \tag{2.7}$$

that is, project Λ_i, q to $\mathbb{F}_p$, calculate $-\frac{\Lambda_i}{q}$ in $\mathbb{F}_p$ and then choose positive integers M_i satisfying these equations. Denote $M = (M_1, \dots, M_n)$. Consider the *master polynomial*

$$\Phi_p(t, z, M) := \prod_{i=1}^{n} (t - z_i)^{M_i}, \tag{2.8}$$

and the Taylor expansion with respect to the variable t of the vector of polynomials

$$P(t, z, M) := \Phi_p(t, z, M) \left(\frac{1}{t - z_1}, \dots, \frac{1}{t - z_n} \right) = \sum_i P^i(z, M)\, t^i,$$

where $P^i(z, M)$ are n-vectors of polynomials in $z_1, \dots, z_n$ with coefficients in $\mathbb{F}_p$.

Theorem 2.3 [12, Theorem 1.2] *For any positive integer l, the vector of polynomials $P^{lp-1}(z, M)$ satisfies the KZ differential equations* (2.2).

Theorem 2.3 is a particular case of [12, Theorem 2.4]. Cf. Theorem 2.3 in [4]. See also [15–17].

The solutions $P^{lp-1}(z, M)$ given by this construction will be called the $\mathbb{F}_p$-*hypergeometric solutions* of the KZ differential equations (2.2).

3 Module of $\mathbb{F}_p$-Hypergeometric Solutions

3.1 Definition of the Module

Denote $\mathbb{F}_p[z^p] := \mathbb{F}_p[z_1^p, \dots, z_n^p]$. The set of all polynomial solutions of (2.2) with coefficients in $\mathbb{F}_p$ is a module over the ring $\mathbb{F}_p[z^p]$ since equations (2.2) are linear and $\frac{\partial z_i^p}{\partial z_j} = 0$ in $\mathbb{F}_p[z]$ for all i, j.

The $\mathbb{F}_p$-hypergeometric solutions $P^{lp-1}(z, M)$ of equations (2.2) depend on the choice of the positive integers $M = (M_1, \dots, M_n)$ in congruences (2.7). Given M satisfying (2.7), denote by

$$\mathcal{M}_M = \left\{ \sum_l c_l(z) P^{lp-1}(z, M) \mid c_l(z) \in \mathbb{F}_p[z^p] \right\} \tag{3.1}$$

the $\mathbb{F}_p[z^p]$-module generated by the solutions $P^{lp-1}(z, M)$.

Let $M = (M_1, \dots, M_n)$ and $M' = (M'_1, \dots, M'_n)$ be two vectors of positive integers, each satisfying congruences (2.7). We say that $M' > M$ if $M'_i \geq M_i$ for $i = 1, \dots, n$ and there exists i such that $M'_i > M_i$.

Assume that $M' > M$. Then $M'_i = M_i + pN_i$ for some $N_i \in \mathbb{Z}_{\geq 0}$. We have

$$P(t, z, M') = \left(\prod_{i=1}^{n}(t - z_i)^{pN_i}\right) P(t, z, M) = \left(\prod_{i=1}^{n}(t^p - z_i^p)^{N_i}\right) P(t, z, M). \tag{3.2}$$

This identity defines an embedding of modules,

$$\varphi_{M',M} : \mathcal{M}_{M'} \hookrightarrow \mathcal{M}_M. \tag{3.3}$$

Namely, formula (3.2) allows us to present any solution $P^{l'p-1}(z, M')$, coming from the Taylor expansion of the left-hand side, as a linear combination of the solutions $P^{lp-1}(z, M)$, coming from the Taylor expansion of the right-hand side, with coefficients in $\mathbb{F}_p[z^p]$.

Clearly, if $M'' > M'$ and $M' > M$, then $M'' > M$ and

$$\varphi_{M',M}\, \varphi_{M'',M'} = \varphi_{M'',M}. \tag{3.4}$$

Theorem 3.1 *For any $M' > M$, the embedding $\varphi_{M',M} : \mathcal{M}_{M'} \hookrightarrow \mathcal{M}_M$ is an isomorphism.*

Proof Let v' and v be the greatest integers such that $v'p - 1 \leq \deg_t P(t, z, M')$ and $vp - 1 \leq \deg_t P(t, z, M)$. Comparing the coefficients in (3.2), we observe that

$$\begin{bmatrix} P^{v'p-1}(z, M') \\ \vdots \\ \vdots \\ P^{(v'-v+1)p-1}(z, M') \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & 1 \end{bmatrix} \cdot \begin{bmatrix} P^{vp-1}(z, M) \\ \vdots \\ \vdots \\ P^{p-1}(z, M) \end{bmatrix}, \tag{3.5}$$

where all the diagonal entries are 1s and stars denote some polynomials in $\mathbb{F}_p[z^p]$. Hence, this matrix is invertible and therefore $\mathcal{M}_M \subset \mathcal{M}_{M'}$. The theorem is proved. □

The set of tuples $M = (M_1, \dots, M_n)$ satisfying (2.7) has the minimal element $\bar{M} = (\bar{M}_1, \dots, \bar{M}_n)$, where $\bar{M}_i$ is the minimal positive integer satisfying (2.7). Hence, for any $M = (M_1, \dots, M_n)$ satisfying (2.7), we have an isomorphism

$$\varphi_{M,\bar{M}} : \mathcal{M}_M \hookrightarrow \mathcal{M}_{\bar{M}}. \tag{3.6}$$

The module $\mathcal{M}_M$, which does not depend on the choice of M, will be called the *module of $\mathbb{F}_p$-hypergeometric solutions* and denoted by $\mathcal{M}_\Lambda$, where Λ can be seen in Sect. 2.1.1.

3.2 The Module Is Free

Recall $\Lambda = (\Lambda_1, \dots, \Lambda_n) \in \mathbb{Z}^n_{>0}$ with $\Lambda_i < q$ for $i = 1, \dots, n$. Recall $\bar M = (\bar M_1, \dots, \bar M_n)$ defined in Sect. 3.1. Denote

$$d_\Lambda := \left[\sum_{i=1}^n \bar M_i / p \right], \tag{3.7}$$

the integer part of the number $\sum_{i=1}^n \bar M_i / p$.

Consider the module $\mathcal{M}_{\bar M}$ spanned over $\mathbb{F}_p[z^p]$ by the solutions $P^{lp-1}(z, \bar M)$, corresponding to $\bar M$. The range for the index l is defined by the inequalities $0 < lp - 1 \le \sum_{i=1}^n \bar M_i - 1$. This means that $l = 1, \dots, d_\Lambda$.

Theorem 3.2 *The solutions $P^{lp-1}(z, \bar M)$, $l = 1, \dots, d_\Lambda$ are linearly independent over the ring $\mathbb{F}_p[z^p]$, that is, if $\sum_{l=1}^{d_\Lambda} c_l(z) P^{lp-1}(z, \bar M) = 0$ for some $c_l(z) \in \mathbb{F}_p[z^p]$, then $c_l(z) = 0$ for all l.*

Corollary 3.3 *The module $\mathcal{M}_\Lambda$ of $\mathbb{F}_p$-hypergeometric solutions is free of rank d_Λ.*

For $q = 2$, this theorem is Theorem 3.1 in [17]. The proof of Theorem 3.2 below is the same as the proof of [17, Theorem 3.1].

Proof For $l = 1, \dots, d_\Lambda$, the coordinates of the vector

$$P^{lp-1}(z, \bar M) = (P_1^{lp-1}(z, \bar M), \dots, P_n^{lp-1}(z, \bar M))$$

are homogeneous polynomials in $z_1, \dots, z_n$ of degree $\sum_{i=1}^n \bar M_i - lp$ and

$$P_j^{lp-1}(z, \bar M) = \sum P^{lp-1}_{j;\ell_1,\dots,\ell_n} z_1^{\ell_1} \dots z_n^{\ell_n},$$

where the sum is over the elements of the set

$$\Gamma_j^l = \left\{ (\ell_1, \dots, \ell_n) \in \mathbb{Z}^n_{\ge 0} \mid \sum_{s=1}^n \ell_s = \sum_{i=1}^n \bar M_i - lp,\ 0 \le \ell_j \le \bar M_j - 1,\ 0 \le \ell_i \le \bar M_i \text{ for } i \ne j \right\}$$

and

$$P^{lp-1}_{j;\ell_1,\dots,\ell_n} = (-1)^{\sum_{i=1}^n \bar M_i - lp} \binom{\bar M_j - 1}{\ell_j} \prod_{i \ne j} \binom{\bar M_i}{\ell_i} \in \mathbb{F}_p.$$

Notice that all coefficients $P^{lp-1}_{j;\ell_1,\dots,\ell_n}$ are nonzero. Hence, each solution $P^{lp-1}(z, \bar M)$ is nonzero.

We show that the first coordinates $P_1^{lp-1}(z, \bar M)$, $l = 1, \dots, d_\Lambda$ are linearly independent over the ring $\mathbb{F}_p[z^p]$.

Let $\bar{\Gamma}_1^l \subset \mathbb{F}_p^n$ be the image of the set Γ_1^l under the natural projection $\mathbb{Z}^n \to \mathbb{F}_p^n$. The points of $\bar{\Gamma}_1^l$ are in bijective correspondence with the points of Γ_1^l since $\bar{M}_i < p$ for all i. Any two sets $\bar{\Gamma}_1^l$ and $\bar{\Gamma}_1^{l'}$ do not intersect, if $l \neq l'$.

For any l and any nonzero polynomial $c_l(z) \in \mathbb{F}_p[z^p]$, consider the nonzero polynomial $c_l(z) P_1^{lp-1}(z, \bar{M}) \in \mathbb{F}_p[z_1, \dots, z_n]$ and the set Γ_{1,c_l}^l of vectors $\vec{\ell} \in \mathbb{Z}^n$ such that the monomial $z_1^{\ell_1} \dots z_n^{\ell_n}$ enters $c_l(z) P_1^{lp-1}(z, \bar{M})$ with nonzero coefficient. Then the natural projection of Γ_{1,c_l}^l to $\mathbb{F}_p^n$ coincides with $\bar{\Gamma}_1^l$. Hence, the polynomials $P_1^l(z, \bar{M})$, $l = 1, \dots, d_\Lambda$ are linearly independent over the ring $\mathbb{F}_p[z^p]$. □

3.3 Fusion of $\mathbb{F}_p$-Hypergeometric Solutions

Consider solutions $I(z_1, \dots, z_n)$ of the KZ differential equations over $\mathbb{C}$ with values in some tensor product $V_1 \otimes \dots \otimes V_n$. Assume that $z_1, \dots, z_n$ tend to some limit $\tilde{z}_1, \dots, \tilde{z}_{\tilde{n}}$, in which some groups of the points $z_1, \dots, z_n$ collide. It is well known that under this limit the leading term of asymptotics of solutions satisfies the KZ equations with respect to $\tilde{z}_1, \dots, \tilde{z}_{\tilde{n}}$ with values in the tensor product $\tilde{V}_1 \otimes \dots \otimes \tilde{V}_{\tilde{n}}$, where each $\tilde{V}_j$ is the tensor product of some of $V_1, \dots, V_n$.

In this section, we show that the $\mathbb{F}_p$-hypergeometric solutions of the KZ equations have a similar functorial property but even simpler.

Namely, let $\Lambda = (\Lambda_1, \dots, \Lambda_n) \in \mathbb{Z}_{>0}^n$ with $\Lambda_i < q$ for $i = 1, \dots, n$, and $\tilde{\Lambda} = (\tilde{\Lambda}_1, \dots, \tilde{\Lambda}_{\tilde{n}}) \in \mathbb{Z}_{>0}^{\tilde{n}}$ with $\tilde{\Lambda}_j < q$ for $j = 1, \dots, \tilde{n}$.

Assume that there is a partition $\{1, \dots, n\} = I_1 \cup \dots \cup I_{\tilde{n}}$ such that

$$\sum_{i \in I_j} \Lambda_i = \tilde{\Lambda}_j, \quad j = 1, \dots, \tilde{n}. \tag{3.8}$$

We say that $\tilde{\Lambda}$ is a *fusion* of Λ.

Recall the modules of $\mathbb{F}_p$-hypergeometric solutions $\mathcal{M}_\Lambda$ and $\mathcal{M}_{\tilde{\Lambda}}$. We define an epimorphism of modules,

$$\psi_{\Lambda, \tilde{\Lambda}} : \mathcal{M}_\Lambda \to \mathcal{M}_{\tilde{\Lambda}}, \tag{3.9}$$

as follows.

Let $M = (M_1, \dots, M_n)$ be a vector of positive integers such that $M_i \equiv -\Lambda_i/q \mod p$ for $i = 1, \dots, n$. Define $\tilde{M} = (\tilde{M}_1, \dots, \tilde{M}_{\tilde{n}})$ by the formula

$$\tilde{M}_j = \sum_{i \in I_j} M_i, \quad j = 1, \dots, \tilde{n}.$$

Then $\tilde{M}_j \equiv -\tilde{\Lambda}_j/q \pmod p$ for $j = 1, \dots, \tilde{n}$.

Consider the module $\mathcal{M}_M$ generated by the solutions $(P^{lp-1}(z, M))_l$, which are n-vectors of polynomials in $z_1, \dots, z_n$. Consider the module $\mathcal{M}_{\tilde{M}}$ generated by the solutions $(P^{lp-1}(\tilde{z}, \tilde{M}))_l$, which are $\tilde{n}$-vectors of polynomials in $\tilde{z}_1, \dots, \tilde{z}_{\tilde{n}}$.

Define the module homomorphism

$$\psi_{\Lambda,\tilde{\Lambda}} : \mathcal{M}_\Lambda = \mathcal{M}_M \to \mathcal{M}_{\tilde{M}} = \mathcal{M}_{\tilde{\Lambda}},$$

as the map which sends a generator $P^{lp-1}(z, M)$ to the generator $P^{lp-1}(\tilde{z}, \tilde{M})$.

On the level of coordinates of these vectors, the vectors $P^{lp-1}(z, M)$ and $P^{lp-1}(\tilde{z}, \tilde{M})$ have the following relation. Choose a coordinate $P_a^{lp-1}(z, M)$ of $P^{lp-1}(z, M)$, replace in it every z_i with $\tilde{z}_j$ if $i \in I_j$, then the resulting polynomial $P_a^{lp-1}(z(\tilde{z}), M)$ equals the bth coordinate $P_b^{lp-1}(\tilde{z}, \tilde{M})$ of $P^{lp-1}(\tilde{z}, \tilde{M})$ if $a \in I_b$.

It is easy to see that the homomorphism $\mathcal{M}_\Lambda \to \mathcal{M}_{\tilde{\Lambda}}$ does not depend of the choice of M solving congruences (2.7).

Also it is easy to see that if $\tilde{\Lambda}$ is a fusion of Λ and $\hat{\Lambda}$ is a fusion of $\tilde{\Lambda}$, then $\hat{\Lambda}$ is a fusion of Λ and

$$\psi_{\tilde{\Lambda},\hat{\Lambda}}\, \psi_{\Lambda,\tilde{\Lambda}} = \psi_{\Lambda,\hat{\Lambda}}. \tag{3.10}$$

4 Binomial Coefficients Modulo *p*

4.1 Lucas' Theorem

Theorem 4.1 [6] *For nonnegative integers m and n and a prime p, the following congruence relation holds:*

$$\binom{n}{m} \equiv \prod_{i=0}^{k} \binom{n_i}{m_i} \pmod{p}, \tag{4.1}$$

where $m = m_k p^k + m_{k-1} p^{k-1} + \cdots + m_1 p + m_0$ *and* $n = n_k p^k + n_{k-1} p^{k-1} + \cdots + n_1 p + n_0$ *are the base p expansions of m and n, respectively. This uses the convention that* $\binom{n}{m} = 0$ *if* $n < m$.

On Lucas' theorem see, for example, [8].

4.2 Factorization of $\binom{-1/q}{m}$ Modulo *p*

In the next sections, we will use the binomial coefficients $\binom{-1/q}{m}$ modulo p. Recall that $p > q$ are prime numbers. Denote by d the order of p modulo q, that is the least integer k such that $p^k \equiv 1 \pmod{q}$.

Let

$$\frac{p^d - 1}{q} = A_0 + A_1 p + A_2 p^2 + \cdots + A_{d-1} p^{d-1} \tag{4.2}$$

be the base p expansion of $(p^d - 1)/q$.

Theorem 4.2 *Let m be a positive integer with the base p expansion given by*

$$m = m_b p^b + m_{b-1} p^{b-1} + \cdots + m_1 p + m_0. \tag{4.3}$$

Then the binomial coefficient $\binom{-1/q}{m}$ *is a well-defined modulo p and the following congruence holds:*

$$\binom{-1/q}{m} \equiv \prod_{j \ge 0} \prod_{i=0}^{d-1} \binom{A_i}{m_{dj+i}} \pmod{p}. \tag{4.4}$$

The case $q = 2$ was considered in [17].

Proof The proof of this theorem is based on Lucas' theorem and the following three lemmas. □

Lemma 4.3 *For any positive integer c*

$$\frac{p^{dc} - 1}{q} = \sum_{j=0}^{c-1} \left(\sum_{i=0}^{d-1} A_i p^i \right) p^{jd}. \tag{4.5}$$

Proof We have

$$\frac{p^{dc} - 1}{q} = \frac{(p^d - 1)(p^{d(c-1)} + \cdots + p + 1)}{q} = \sum_{j=0}^{c-1} \left(\sum_{i=0}^{d-1} A_i p^i \right) p^{jd},$$

where we use (4.2) to obtain the last equality. □

Lemma 4.4 *Let m be a positive integer with the base p expansion given by* (4.3)*. If c is a positive integer such that $dc > b$, then*

$$\binom{(p^{dc} - 1)/q}{m} \equiv \prod_{j=0}^{c-1} \prod_{i=0}^{d-1} \binom{A_i}{m_{dj+i}} \pmod{p}. \tag{4.6}$$

Proof The lemma follows from Theorem 4.1 and Lemma 4.3. □

Given a prime p, define the *p-adic norm* on $\mathbb{Q}$ as follows. Any nonzero rational number x can be represented uniquely by $x = p^\ell (r/s)$, where r and s are integers not divisible by p. Set $|x|_p = p^{-\ell}$. Also define the p-adic value $|0|_p = 0$. We call $p^\ell (r/s)$ the *p-reduced presentation* of x.

Lemma 4.5 *Let m be a positive integer with the base p expansion given by* (4.3)*. Then the following statements hold.*

(i) *The binomial coefficient* $\binom{-1/q}{m}$ *is well-defined modulo p.*

(ii) *For any positive integer c such that $dc > b+1$, we have*

$$\binom{-1/q}{m} \equiv \binom{(p^{dc}-1)/q}{m} \pmod{p}. \tag{4.7}$$

Proof We have

$$\binom{-1/q}{m} = (-1/q)^m \prod_{\ell=0}^{m-1} \frac{q\ell+1}{\ell+1}. \tag{4.8}$$

We also have

$$\binom{(p^{dc}-1)/q}{m} = (-1/q)^m \prod_{\ell=0}^{m-1} \frac{q\ell+1-p^{dc}}{\ell+1} = (-1/q)^m \prod_{\ell=0}^{m-1} \frac{q\ell+1}{\ell+1} + \dots \tag{4.9}$$

For each $\ell = 0, \dots, m-1$ we have $\left|\frac{q\ell+1-p^{dc}}{\ell+1}\right|_p = \left|\frac{q\ell+1}{\ell+1}\right|_p$, since $q\ell+1 < qm < qp^{b+1} < p^{b+2} \le p^{dc}$.

By the same reasoning for every ℓ the power of p in the p-reduced presentation of the number $\frac{p^{dc}}{\ell+1}$ is greater than the power of p in the p-reduced presentation of $\frac{q\ell+1}{\ell+1}$. This observation implies that on the right-hand side of (4.9), the power of p in the p-reduced presentation of terms denoted by '…' is greater than the corresponding power in the p-reduced presentation of $(-1/q)^m \prod_{\ell=0}^{m-1} \frac{q\ell+1}{\ell+1}$. Since the left-hand side of (4.9) is an integer, we conclude that in the p-reduced presentation of the binomial coefficient $\binom{-1/q}{m}$ the power of p is non-negative, i.e. $\binom{-1/q}{m}$ is well-defined modulo p. This gives part (i). Moreover, the following congruence holds:

$$\binom{(p^{dc}-1)/q}{m} \equiv (-1/q)^m \prod_{\ell=0}^{m-1} \frac{q\ell+1}{\ell+1} \pmod{p}. \tag{4.10}$$

Formulas (4.8) and (4.10) give (4.7). □

Proof Theorem 4.2 follows from Lemmas 4.4 and 4.5. □

4.3 Factorization of $\binom{u/v}{m}$ modulo p

Here are some generalizations of Theorem 4.2. Let p be prime. Let u, v be relatively prime integers with $0 < u, v < p$.

Assume that there exists a positive integer ℓ such that

$$p^\ell + u \equiv 0 \pmod{v}. \tag{4.11}$$

Let

$$\frac{p^\ell + u}{v} = B_0 + B_1 p + B_2 p^2 + \dots + B_{\ell-1} p^{\ell-1} \tag{4.12}$$

be the base p expansion of $(p^\ell + u)/v$.

Let $\varphi(v)$ be the number of positive divisors of v and

$$\frac{p^{\varphi(v)} - 1}{v} = C_0 + C_1 p + C_2 p^2 + \cdots + C_{\varphi(v)-1} p^{\varphi(v)-1} \tag{4.13}$$

the base p expansion of $(p^{\varphi(v)} - 1)/v$.

Let m be a positive integer with the base p expansion

$$m = m_b p^b + m_{b-1} p^{b-1} + \cdots + m_1 p + m_0.$$

Theorem 4.6 *Under these assumptions the binomial coefficient $\binom{u/v}{m}$ is well-defined modulo p and the following congruence holds:*

$$\binom{u/v}{m} \equiv \left[\prod_{i=0}^{\ell-1} \binom{B_i}{m_i}\right] \cdot \left[\prod_{j\ge 0} \prod_{i=0}^{\varphi(v)-1} \binom{C_i}{m_{\ell+j\varphi(v)+i}}\right] \pmod{p}. \tag{4.14}$$

Proof The proof of Theorem 4.6 is parallel to the proof of Theorem 4.2 and follows from the analysis of the base p expansion of the integers of the form

$$\frac{p^{k\varphi(v)+\ell} + u}{v} = p^\ell \, \frac{p^{k\varphi(v)} - 1}{v} + \frac{p^\ell + u}{v}$$

for $k \in \mathbb{Z}_{\ge 0}$. □

In the same way, we prove the following statement.

Theorem 4.7 *Let p be a prime. Let u, v be integers such that p, u, v are pairwise relatively prime. Let*

$$\frac{u}{v} = A_0 + A_1 p + A_2 p^2 + \dots, \quad 0 \le A_i < p \quad \text{for } i \in \mathbb{Z}_{\ge 0} \tag{4.15}$$

be the p-adic presentation of u/v. Let m be a positive integer with the base p expansion

$$m = m_b p^b + m_{b-1} p^{b-1} + \cdots + m_1 p + m_0.$$

Then

$$\binom{u/v}{m} \equiv \prod_{i\ge 0} \binom{A_i}{m_i} \pmod{p}. \tag{4.16}$$

4.4 Map η

Recall that p and q are prime numbers, $p > q$, and d the order of p modulo q. Define the map

$$\eta \;:\; \{1, \dots, q-1\} \;\to\; \{1, \dots, q-1\}, \quad a \;\mapsto\; \eta(a), \tag{4.17}$$

the division by p modulo q. More precisely, $\eta(a)$ is defined by the conditions

$$\eta(a)p \;\equiv\; a \pmod{q}, \quad 1 \le \eta(a) < q. \tag{4.18}$$

Denote $\eta^{(s)} = \eta \circ \eta \circ \dots \circ \eta$ the sth iteration of η. We have $\eta^{(s+d)} = \eta^{(s)}$. Define

$$a_s \;:=\; \eta^{(s)}(1). \tag{4.19}$$

We have $a_{s+d} = a_s$ and $a_0 = 1$.

The integer a_1 will play a special role in the next sections. It is the unique integer such that

$$1 \le a_1 < q \quad \text{and} \quad q \mid (a_1 p - 1). \tag{4.20}$$

Example 4.8 Let $p = 5,\ q = 3$. The order d of 5 modulo 3 is 2. The map $\eta : \{1, 2\} \to \{2, 1\}$ is the transposition. We have $a_0 = 1$, $a_1 = 2$, $a_2 = 1$.

4.5 Formulas for A_s

Recall the base p expansion

$$\frac{p^d - 1}{q} = A_0 + A_1 p + A_2 p^2 + \dots + A_{d-1} p^{d-1}. \tag{4.21}$$

Lemma 4.9 *The integers A_s are given by the formula*

$$A_s = \frac{a_{s+1} p - a_s}{q}, \quad s = 0, \dots, d-1. \tag{4.22}$$

Moreover, $A_s > 0$ for $s = 0, \dots, d-1$.

Proof We have

$$0 < \frac{a_{s+1} p - a_s}{q} < p \tag{4.23}$$

for every s. Indeed, since $1 \le a_s \le q-1$ and $p > q$, we have

$$0 < \frac{p - a_s}{q} \le \frac{a_{s+1} p - a_s}{q} < \frac{a_{s+1} p}{q} < p.$$

We show that (4.21) holds for A_s given by (4.22). Indeed

$$q(A_0 + A_1 p + \cdots + A_{d-1} p^{d-1}) = q\left(\frac{a_1 p - 1}{q} + \frac{a_2 p - a_1}{q} p + \cdots + \frac{p - a_{d-1}}{q} p^{d-1}\right)$$
$$= p^d - 1.$$

□

5 Solutions for the Case $\Lambda_i = 1, i = 1, \ldots, n$

In Sects. 5, 6, and 9, we study the $\mathbb{F}_p$-hypergeometric solutions of the KZ equations in the case $\Lambda_i = 1, i = 1, \ldots, n$.

5.1 Basis of $\mathbb{F}_p$-Hypergeometric Solutions

Recall that p, q are prime numbers, $p > q$. Assume that

$$n = kq + 1 \tag{5.1}$$

for some positive integer k and

$$\Lambda = (1, \ldots, 1) \in \mathbb{Z}^n. \tag{5.2}$$

The minimal positive solution of the system of the congruences

$$M_i \equiv -1/q \pmod{p}, \quad i = 1, \ldots, n, \tag{5.3}$$

is the vector

$$\bar{M} = (\bar{M}_1, \ldots, \bar{M}_n) := ((a_1 p - 1)/q, \ldots, (a_1 p - 1)/q), \tag{5.4}$$

where a_1 is introduced in (4.20).

Recall that the rank of the module $\mathcal{M}_\Lambda$ of $\mathbb{F}_p$-hypergeometric solutions in this case equals

$$d_\Lambda = [n(a_1 p - 1)/qp], \tag{5.5}$$

see Corollary 3.3.

Lemma 5.1 *If $p > n$, then $d_\Lambda = a_1 k$.*

Proof We have

$$d_\Lambda = \left[n\frac{a_1 p - 1}{qp}\right] = \left[\frac{na_1}{q} - \frac{n}{qp}\right] = \left[ka_1 + \frac{a_1}{q} - \frac{n}{pq}\right].$$

Notice that $1/q \le a_1/q < 1$ and $n/qp < 1/q$. Hence, $d_\Lambda = a_1 k$. □

The master polynomial is

$$\Phi_p(x,z,\bar{M}) = \prod_{i=1}^{n} (x-z_i)^{(a_1p-1)/q}. \tag{5.6}$$

The n-vector of polynomials $P(x,z,\bar{M})$ is

$$P(x,z,\bar{M}) = (P_1(x,z,\bar{M}),\dots,P_n(x,z,\bar{M})), \quad P_j(x,z,\bar{M}) = \frac{\Phi_p(x,z,\bar{M})}{x-z_j}, \tag{5.7}$$

with the Taylor expansion

$$P(x,z,\bar{M}) = \sum_{i=0}^{n(a_1p-1)/q-1} P^i(z,\bar{M})\,x^i, \quad P^i(z,\bar{M}) = (P^i_1(z,\bar{M}),\dots,P^i_n(z,\bar{M})). \tag{5.8}$$

The coefficients $P^i(z,\bar{M})$ with $i = lp-1$ are solutions of (2.2). There are ka_1 of them. They are linearly independent by Theorem 3.2. Denote them

$$I^m(z) = (I^m_1(z),\dots,I^m_n(z)), \quad I^m(z) := P^{(a_1k-m)p+p-1}(z,\bar{M}), \tag{5.9}$$

for $m = 1,\dots,a_1k$. The coordinates of the n-vector of polynomials $I^m(z)$ are *homogeneous polynomials in z of degree*

$$(a_1p-1)/q + (m-1)p - k.$$

For example, $I^1 = P^{a_1kp-1}(z,\bar{M})$ is a homogeneous polynomial in z of degree $(a_1p-1)/q-k$, and $I^{a_1k} = P^{p-1}(z,\bar{M})$ is a homogeneous polynomial in z of degree $(a_1p-1)/q-k+(a_1k-1)p$.

5.2 Change of the Basis of $\mathcal{M}_\Lambda$

For future use, we introduce a new basis of $\mathcal{M}_\Lambda$. For $m = 1,\dots,a_1k$, define

$$J^m(z) = \sum_{l=1}^{m} I^{m+1-l}(z)\,z_1^{(l-1)p}\binom{a_1k-m-1+l}{a_1k-m}, \tag{5.10}$$

that is,

$$\begin{aligned} J^1(z) &= I^1(z), \\ J^2(z) &= I^1(z)\,z_1^p\binom{a_1k-1}{a_1k-2} + I^2(z), \end{aligned}$$

$$J^3(z) = I^1(z)\, z_1^{2p} \binom{a_1k-1}{a_1k-3} + I^2(z)\, z_1^{p} \binom{a_1k-2}{a_1k-3} + I^3(z),$$

and so on. The coordinates of the n-vector of polynomials $J^m(z)$ are *homogeneous polynomials in z of degree*

$$(a_1p-1)/p + (m-1)p - k.$$

Lemma 5.2 *The n-vectors of polynomials $J^m(z)$ belong to the module $\mathcal{M}_\Lambda$ of $\mathbb{F}_p$-hypergeometric solutions and form a basis of $\mathcal{M}_\Lambda$.*

The solutions $J^m(z)$ of the KZ equations (2.2) can be defined as coefficients of the Taylor expansion of a suitable n-vector of polynomials similar to the definition of solutions $I^m(z)$ in (5.9).

Namely, change x to $x+z_1$ in (5.6) to obtain the polynomial

$$\tilde{\Phi}_p(x,z) = x^{(a_1p-1)/q} \prod_{i=2}^{n} (x+z_1-z_i)^{(a_1p-1)/q} \in \mathbb{F}_p[x,z]. \tag{5.11}$$

Consider the associated n-vector of polynomials

$$\tilde{P}(x,z) = \tilde{\Phi}_p(x,z) \left(\frac{1}{x}, \frac{1}{x+z_1-z_2}, \dots, \frac{1}{x+z_1-z_n} \right) \tag{5.12}$$

and its Taylor expansion

$$\tilde{P}(x,z) = \sum_{i=0}^{n(a_1p-1)/q-1} \tilde{P}^i(z)x^i, \quad \tilde{P}^i(z) \in \mathbb{F}_p[z]^n. \tag{5.13}$$

Lemma 5.3 [17, Lemma 5.2] *For $m = 1, \dots, a_1k$, we have*

$$J^m(z) = \tilde{P}^{(a_1k-m)p+p-1}(z). \tag{5.14}$$

5.3 Change of Variables in $J^m(z)$

Lemma 5.4 *For $m = 1, \dots, a_1k$, the homogeneous polynomial $J^m(z)$ can be written as a homogeneous polynomial in variables z_2-z_1, z_3-z_1, ..., z_n-z_1, see* (5.11) *and* (5.12).

Hence, we may pull out from the polynomial $J^m(z)$ the factor $(z_2-z_1)^{(a_1p-1)/q+(m-1)p-k}$ and present $J^m(z)$ in the form

$$J^m(z) = (z_2-z_1)^{(a_1p-1)/q+(m-1)p-k}\, K^m(\lambda), \tag{5.15}$$

where $\lambda := (\lambda_3, \dots, \lambda_n)$,

$$\lambda_i \ := \frac{z_i - z_1}{z_2 - z_1}, \quad i = 3, \dots, n, \tag{5.16}$$

and $K^m(\lambda)$ is a suitable n-vector of polynomials in λ.

Another way to define the n-vector $K^m(\lambda)$ is as follows.

Consider the polynomial

$$\hat{\Phi}_p(x, \lambda) = x^{(a_1 p-1)/q} (x-1)^{(a_1 p-1)/q} \prod_{i=3}^{n} (x - \lambda_i)^{(a_1 p-1)/q} \ \in \ \mathbb{F}_p[x, \lambda] \tag{5.17}$$

and the associated n-vector of polynomials

$$\hat{P}(x, \lambda) = \hat{\Phi}_p(x, \lambda) \left(\frac{1}{x}, \frac{1}{x-1}, \frac{1}{x-\lambda_3}, \dots, \frac{1}{x-\lambda_n} \right) \tag{5.18}$$

with Taylor expansion

$$\hat{P}(x, \lambda) = \sum_{i=0}^{n(a_1 p-1)/q-1} \hat{P}^i(\lambda) x^i, \quad \hat{P}^i(\lambda) \in \mathbb{F}_p[\lambda]^n. \tag{5.19}$$

Lemma 5.5 *For $m = 1, \dots, a_1 k$, we have*

$$K^m(\lambda) = \hat{P}^{(a_1 k - m)p + p - 1}(\lambda), \tag{5.20}$$

cf. formulas (5.9) *and* (5.14).

5.4 Formula for $K^m(\lambda)$

For $m = 1, \dots, a_1 k$, denote

$$\Delta^m = \Bigg\{ (\ell_3, \dots, \ell_n) \in \mathbb{Z}_{\geq 0}^{n-2} \mid 0 \leq \sum_{i=3}^{n} \ell_i + k - (m-1)p \leq (a_1 p - 1)/q, \\ \ell_j \leq (a_1 p - 1)/q \ \text{ for } \ j = 3, \dots, n \Bigg\}. \tag{5.21}$$

Theorem 5.6 *For $m = 1, \dots, a_1 k$, we have*

$$K^m(\lambda) = \sum_{(\ell_3, \dots, \ell_n) \in \Delta^m} K^m_{\ell_3, \dots, \ell_n}(\lambda), \tag{5.22}$$

where

$$K^m_{\ell_3,\dots,\ell_n}(\lambda) = (-1)^{(a_1p-1)/q+(m-1)p-k}\binom{(a_1p-1)/q}{\sum_{i=3}^n \ell_i + k - (m-1)p}$$
$$\times \prod_{i=3}^n \binom{(a_1p-1)/q}{\ell_i} \lambda_3^{\ell_3}\dots\lambda_n^{\ell_n} \left(1, -q\left(\sum_{i=3}^n \ell_i + k\right), q\ell_3+1, \dots, q\ell_n+1\right) \tag{5.23}$$

modulo p. In particular, all coefficients $K^m_{\ell_3,\dots,\ell_n}(\lambda)$ are nonzero.

Proof We have

$$K^m_1(\lambda) = (-1)^{(a_1p-1)/q+(m-1)p-k}\sum_{\Delta}\binom{(a_1p-1)/q}{\ell_2}\dots\binom{(a_1p-1)/q}{\ell_n}\lambda_3^{\ell_3}\dots\lambda_n^{\ell_n},$$

where

$$\Delta = \left\{(\ell_2,\dots,\ell_n) \in \mathbb{Z}^{n-1}_{\geq 0} \mid \sum_{i=2}^n \ell_i = (a_1p-1)/q + (m-1)p - k,\ \ \ell_i \leq (a_1p-1)/q\right\}.$$

Expressing ℓ_2 from the conditions defining Δ, we write

$$K^m_1(\lambda) = (-1)^{(a_1p-1)/q+(m-1)p-k}$$
$$\times \sum_{\Delta^m}\binom{(a_1p-1)/q}{\sum_{i=3}^n \ell_i - (m-1)p + k}$$
$$\times \prod_{i=3}^n \binom{(a_1p-1)/q}{\ell_i}\lambda_3^{\ell_3}\dots\lambda_n^{\ell_n}.$$

Similarly, we have

$$K^m_2(\lambda) = (-1)^{(a_1p-1)/q+(m-1)p-k}\sum_{\Delta'}\binom{(a_1p-1)/q-1}{\ell_2}$$
$$\times \prod_{i=3}^n \binom{(a_1p-1)/q}{\ell_i}\lambda_3^{\ell_3}\dots\lambda_n^{\ell_n},$$

where

$$\Delta' = \left\{(\ell_2,\dots,\ell_n) \in \mathbb{Z}^{n-1}_{\geq 0} \mid \sum_{i=2}^n \ell_i = (a_1p-1)/q + (m-1)p - k,\right.$$
$$\left.\ell_2 \leq (a_1p-1)/q - 1 \text{ and } \ell_i \leq (a_1p-1)/q \text{ for } i > 2\right\}.$$

Expressing ℓ_2 from the conditions defining Δ', we write

$$\begin{aligned} K_2^m(\lambda) &= (-1)^{(a_1p-1)/q+(m-1)p-k} \\ &\quad\times \sum_{\Delta^m} \binom{(a_1p-1)/q-1}{\sum_{i=3}^n \ell_i-(m-1)p+k-1} \prod_{i=3}^n \binom{(a_1p-1)/q}{\ell_i} \lambda_3^{\ell_3}\dots\lambda_n^{\ell_n} \\ &= (-1)^{(a_1p-1)/q+(m-1)p-k}\, \frac{\sum_{i=3}^n \ell_i-(m-1)p+k}{(a_1p-1)/q} \\ &\quad\times \sum_{\Delta^m} \binom{(a_1p-1)/q}{\sum_{i=3}^n \ell_i-(m-1)p+k} \prod_{i=3}^n \binom{(a_1p-1)/q}{\ell_i} \lambda_3^{\ell_3}\dots\lambda_n^{\ell_n}. \end{aligned}$$

For $j=3,\dots,n$, we have

$$K_j^m(\lambda) = (-1)^{(a_1p-1)/q+(m-1)p-k} \sum_{\Delta''} \binom{(a_1p-1)/q-1}{\ell_j} \prod_{\substack{i=2\\ i\neq j}}^n \binom{(a_1p-1)/q}{\ell_i} \lambda_3^{\ell_3}\dots\lambda_n^{\ell_n},$$

where

$$\Delta'' = \Big\{ (\ell_2,\dots,\ell_n)\in\mathbb{Z}_{\geq 0}^{n-1} \mid \sum_{i=2}^n \ell_i = (a_1p-1)/q+(m-1)p-k, \\ \ell_j \leq (a_1p-1)/q-1 \ \text{ and } \ \ell_i \leq (a_1p-1)/q \ \text{ for } \ i\neq j \Big\}.$$

Expressing ℓ_2 from the conditions defining Δ'', we write

$$\begin{aligned} K_j^m(\lambda) &= (-1)^{(a_1p-1)/q+(m-1)p-k} \\ &\quad\times \sum_{\Delta^m} \binom{(a_1p-1)/q}{\sum_{i=3}^n \ell_i-(m-1)p+k} \binom{(a_1p-1)/q-1}{\ell_j} \prod_{\substack{i=3\\ i\neq j}}^n \binom{(a_1p-1)/q}{\ell_i} \lambda_3^{\ell_3}\dots\lambda_n^{\ell_n} \\ &= (-1)^{(a_1p-1)/q+(m-1)p-k} \left(1-\frac{\ell_j q}{a_1p-1}\right) \\ &\quad\times \sum_{\Delta^m} \binom{(a_1p-1)/q}{\sum_{i=3}^n \ell_i-(m-1)p+k} \prod_{i=3}^n \binom{(a_1p-1)/q}{\ell_i} \lambda_3^{\ell_3}\dots\lambda_n^{\ell_n}. \end{aligned}$$

The congruences

$$\frac{\sum_{i=3}^n \ell_i-(m-1)p+k}{(a_1p-1)/q} \equiv -q\left(\sum_{i=3}^n \ell_i+k\right) \pmod{p},$$
$$1-\frac{\ell_j q}{a_1p-1} \equiv q\ell_j+1 \pmod{p}$$

allow us to rewrite $K_j^m(\lambda)$, $j=2,\dots,n$, in the form indicated in the theorem. □

5.5 Example

For $p = 5$, $q = 3$, $n = 4$, the module $\mathcal{M}_\Lambda$ of $\mathbb{F}_p$-hypergeometric solutions is two-dimensional and is generated by solutions J^1 and J^2 which are 4-vectors, whose coefficients are homogeneous polynomials in z_1, z_2, z_3, z_4 of degrees 2 and 7, respectively.

The corresponding 4-vectors K^1 and K^2 are given by the formulas

$$\begin{aligned} K^1(\lambda_3, \lambda_4) &= (3, 1, 3, 3) + (4, 1, 1, 4)\lambda_3 + (4, 1, 4, 1)\lambda_4 \\ &\quad + (3, 3, 1, 3)\lambda_3^2 + (4, 4, 1, 1)\lambda_3\lambda_4 + (3, 3, 3, 1)\lambda_4^2, \\ K^2(\lambda_3, \lambda_4) &= (2, 0, 0, 3)\lambda_3^2\lambda_4 + (1, 0, 2, 2)\lambda_3^2\lambda_4^2 + (2, 0, 3, 0)\lambda_3\lambda_4^3 \\ &\quad + (1, 2, 0, 2)\lambda_3^3\lambda_4^2 + (1, 2, 2, 0)\lambda_3^2\lambda_4^3 + (2, 3, 0, 0)\lambda_3^3\lambda_4^3. \end{aligned}$$

6 Cartier Map

6.1 Matrices and Semilinear Algebra, [1]

6.1.1 Bases, Matrices, and Linear Operators

Let W be a vector space over a field $\mathbb{K}$ with basis $\mathcal{C} = \{w_1, \dots, w_n\}$. Any $w \in W$ is expressible as $w = \sum c_i w_i$. Let $[w]_\mathcal{C}$ denote the column vector $[w]_\mathcal{C} = (c_1, \dots, c_n)^T$.

Let V be a vector space with basis $\mathcal{B} = \{v_1, \dots, v_m\}$, and $f : W \to V$ a linear map. The matrix of f relative to the bases $\mathcal{C}$ and $\mathcal{B}$ is $[f]_{\mathcal{B}\leftarrow\mathcal{C}} = (a_{ij}) \in \mathrm{Mat}_{m\times n}(\mathbb{K})$, where $f(w_j) = \sum_{i=1}^m a_{ij} v_i$. Matrix multiplication gives

$$[f(w)]_\mathcal{B} = [f]_{\mathcal{B}\leftarrow\mathcal{C}}\, [w]_\mathcal{C}.$$

Let $f : V \to V$ be an endomorphism, and $\mathcal{B}$ and $\mathcal{D}$ two bases for V. Then

$$[f]_{\mathcal{D}\leftarrow\mathcal{D}} = [\mathrm{id}]_{\mathcal{D}\leftarrow\mathcal{B}}\, [f]_{\mathcal{B}\leftarrow\mathcal{B}}\, [\mathrm{id}]_{\mathcal{B}\leftarrow\mathcal{D}}.$$

If $S = [\mathrm{id}]_{\mathcal{B}\leftarrow\mathcal{D}}$, then

$$[f]_{\mathcal{D}\leftarrow\mathcal{D}} = S^{-1}\, [f]_{\mathcal{B}\leftarrow\mathcal{B}}\, S.$$

6.1.2 Semilinear Algebra

Let τ be an automorphism of $\mathbb{K}$ and $\sigma = \tau^{-1}$. Then $f : V \to V$ is called τ*-linear*, if for $a \in \mathbb{K}$ and $v \in V$,

$$f(av) = a^\tau f(v),$$

where $a^\tau = \tau(a)$. Let $f(v_j) = \sum_i a_{ij} v_i$. If $v = \sum_j c_j v_j$, then

$$f(v) = \sum_j f(c_j v_j) = \sum_j c_j^\tau f(v_j) = \sum_j \left(\sum_i a_{ij} v_i \right) c_j^\tau$$

and so

$$[f(v)]_{\mathcal{B}} = [f]_{\mathcal{B}\leftarrow\mathcal{B}} \, [v]_{\mathcal{B}}^\tau,$$

where B^τ is the matrix obtained by applying τ to each entry of B.

Change of basis is accomplished with τ-twisted conjugacy:

$$[f]_{\mathcal{D}\leftarrow\mathcal{D}} = [\mathrm{id}]_{\mathcal{D}\leftarrow\mathcal{B}} \, [f]_{\mathcal{B}\leftarrow\mathcal{B}} \, [\mathrm{id}]_{\mathcal{B}\leftarrow\mathcal{D}}^\tau = S^{-1} \, [f]_{\mathcal{B}\leftarrow\mathcal{B}} \, S^\tau.$$

The iterates of f are represented by

$$[f^{\circ r}] = [f] \, [f]^\tau \, [f]^{\tau^2} \ldots [f]^{\tau^{r-1}}. \tag{6.1}$$

6.1.3 Adjoint Map

Let V^* be the dual vector space of V and $(\cdot, \cdot)_V : V \times V^* \to \mathbb{K}$ the natural pairing, linear with respect to each argument. Similarly, let W^* be the dual vector space of W and $(\cdot, \cdot)_W : W \times W^* \to \mathbb{K}$ the natural pairing.

Let $f : W \to V$ be τ-linear. Define the *adjoint map* $f^* : V^* \to W^*$ by the formula

$$(w, f^*(\varphi))_W = (f(w), \varphi)_V^\sigma, \quad w \in W, \ \varphi \in V^*.$$

The map f^* is σ-linear, $f^*(a\varphi) = a^\sigma f^*(\varphi)$ for $a \in \mathbb{K}$. Indeed,

$$(w, f^*(a\varphi))_W = (f(w), a\varphi)_V^\sigma = a^\sigma (f(w), \varphi)_V^\sigma = a^\sigma (w, f^*(\varphi))_W = (w, a^\sigma f^*(\varphi))_W.$$

Let $\mathcal{C} = \{w_1, \ldots, w_n\}$ be a basis of W and $\mathcal{B} = \{v_1, \ldots, v_m\}$ a basis of V. Let $\mathcal{C}^* = \{\varphi_1, \ldots, \varphi_n\}$ be the dual basis of W^* and $\mathcal{B}^* = \{\psi_1, \ldots, \psi_m\}$ the dual basis of V^*.

If $[f]_{\mathcal{B}\leftarrow\mathcal{C}} = (a_{ij})$, then

$$[f^*]_{\mathcal{C}^*\leftarrow\mathcal{B}^*} = ([f]_{\mathcal{B}\leftarrow\mathcal{C}}^\sigma)^T. \tag{6.2}$$

6.2 Field $\mathbb{K}(u)$

In this paper, we consider some particular fields $\mathbb{K}(u)$.

Let $u = (u_1, \dots, u_r)$ be variables. Let $\mathbb{K}(u)$ be the field of rational functions in variables

$$u_i^{1/p^s}, \quad i = 1, \dots, r, \quad s \in \mathbb{Z}_{>0}, \tag{6.3}$$

with coefficients in $\mathbb{F}_p$. Thus, an element of $\mathbb{K}(u)$ is the ratio of two polynomials in variables u_i^{1/p^s} with coefficients in $\mathbb{F}_p$.

For any $f(u_1, \dots, u_r) \in \mathbb{K}(u)$, we have

$$f(u_1, \dots, u_r)^{1/p} = f(u_1^{1/p}, \dots, u_r^{1/p}), \tag{6.4}$$

cf. [9].

The field $\mathbb{K}(u)$ has the Frobenius automorphism

$$\sigma \, : \, \mathbb{K}(u) \to \mathbb{K}(u), \quad f(u) \mapsto f(u)^p, \tag{6.5}$$

and its inverse

$$\tau \, : \, \mathbb{K}(u) \to \mathbb{K}(u), \quad f(u) \mapsto f(u)^{1/p}. \tag{6.6}$$

6.3 Curve *X*

Recall that $n = qk + 1$, see Sect. 5.1. Consider the field $\mathbb{K}(z)$, $z = (z_1, \dots, z_n)$. Consider the algebraic curve X over $\mathbb{K}(z)$ defined by the affine equation

$$y^q = F(x, z) := (x - z_1)(x - z_2) \dots (x - z_n). \tag{6.7}$$

The curve has genus

$$g \, := \, k \, \frac{q(q-1)}{2}. \tag{6.8}$$

The space $\Omega^1(X)$ of regular 1-forms on X is the direct sum

$$\Omega^1(X) = \bigoplus_{a=1}^{q-1} \Omega^1_a(X), \tag{6.9}$$

where $\dim \, \Omega^1_a(X) = ak$, and $\Omega^1_a(X)$ has basis :

$$x^{i-1} \frac{\mathrm{d}x}{y^a}, \quad i = 1, \dots, ak. \tag{6.10}$$

6.4 Cartier Operator

Following [1], we introduce the *Cartier operator*

$$C\ :\ \Omega^1(X)\ \to\ \Omega^1(X), \tag{6.11}$$

which is τ-linear. It has block structure. For $a = 1, \dots, q-1$, we have

$$C\left(\Omega^1_a(X)\right)\ \subset\ \Omega^1_{\eta(a)}(X), \tag{6.12}$$

where $\eta : \{1, \dots, q-1\} \to \{1, \dots, q-1\}$ is the division by p modulo q defined in (4.17).

We define the Cartier operator by the action on the basis vectors as follows.

For $a = 1, \dots, q-1$, formula (4.18) implies $q \mid (\eta(a)p - a)$. Hence, for $f = 1, \dots, ak$, we have

$$\begin{aligned} x^{ak-f}\frac{\mathrm{d}x}{y^a} &= x^{ak-f}\,y^{\eta(a)p-a}\,\frac{\mathrm{d}x}{y^a\,y^{\eta(a)p-a}} = x^{ak-f}\,(y^q)^{(\eta(a)p-a)/q}\,\frac{\mathrm{d}x}{y^{\eta(a)p}} \\ &= x^{ak-f}\,F(x,z)^{(\eta(a)p-a)/q}\,\frac{\mathrm{d}x}{y^{\eta(a)p}} = \sum_{w\geq 0}{}^{a}F^{w}_{f}(z)\,x^{w}\,\frac{\mathrm{d}x}{y^{\eta(a)p}}, \end{aligned} \tag{6.13}$$

where ${}^aF^w_f(z) \in \mathbb{F}_p[z]$.

The Cartier operator is defined by the formula

$$x^{ak-f}\frac{\mathrm{d}x}{y^a}\ \ \mapsto\ \ \sum_{h=1}^{\eta(a)k}\left({}^{a}F_f^{(\eta(a)k-h)p+p-1}(z)\right)^{1/p}\ x^{\eta(a)k-h}\,\frac{\mathrm{d}x}{y^{\eta(a)}}. \tag{6.14}$$

This formula has the following meaning. If w in (6.13) is not of the form $lp + p - 1$ for some l, then the summand ${}^aF^w_f(z)\,x^w\,\frac{\mathrm{d}x}{y^{\eta(a)p}}$ in (6.13) is ignored in the definition (6.14), and if $w = lp + p - 1$ for some l, then the term ${}^aF^{lp+p-1}_f(z)\,x^{lp+p-1}\,\frac{\mathrm{d}x}{y^{\eta(a)p}}$ produces the summand

$$\left({}^{a}F_f^{lp+p-1}(z)\right)^{1/p}\ x^{l}\,\frac{\mathrm{d}x}{y^{\eta(a)}}$$

in the definition (6.14). It turns out that there are exactly $\eta(a)k$ such values $w = lp + p - 1$ and that explains the upper index $\eta(a)k$ in the sum in (6.14).

The coefficients $\left({}^{a}F_f^{(\eta(a)k-h)p+p-1}(z)\right)^{1/p}$ form the $g \times g$-matrix of the Cartier operator with respect to the basis $x^{i-1}\mathrm{d}x/y^a$ in (7.4). The matrix is called the *Cartier–Manin matrix*.

Let $\Omega^1(X)^*$ be the space dual to $\Omega^1(X)$. Let

$$\varphi_a^i, \quad a = 1, \dots, q-1, \quad i = 1, \dots, ka, \tag{6.15}$$

denote the basis of $\Omega^1(X)^*$ dual to the basis $(x^{i-1}\mathrm{d}x/y^a)$ of $\Omega^1(X)$.

The map

$$C^* : \Omega^1(X)^* \to \Omega^1(X)^*$$

adjoint to the Cartier operator is σ-linear. Following Serre, the matrix of the map C^* is called the *Hasse–Witt matrix* with respect to the basis (φ_a^i), see [1]. The entries of the Hasse–Witt matrix are the *polynomials*

$${}^aF_f^{(\eta(a)k-h)p+p-1}(z) \in \mathbb{F}_p[z],$$

see (6.2).

Remark The map C^* is identified with the Frobenius map

$$H^1(X, \mathcal{O}(X)) \to H^1(X, \mathcal{O}(X)),$$

see, for example, [1].

For each $a = 1, \dots, q-1$, consider the columns of the Hasse–Witt matrix, corresponding to the basis vectors of $\Omega^1_{\eta(a)}(X)^*$ and the rows corresponding to the basis vectors of $\Omega^1_a(X)^*$. The respective block of the Hasse–Witt matrix of size $ak \times \eta(a)k$ is denoted by ${}^a\mathcal{I}$. Its entries are denoted by

$${}^a\mathcal{I}_f^h(z) := {}^aF_f^{(\eta(a)k-h)p+p-1}(z), \quad f = 1, \dots, ak, \quad h = 1, \dots, \eta(a)k. \tag{6.16}$$

Lemma 6.1 *The entry ${}^a\mathcal{I}_f^h(z)$ is a homogeneous polynomial in $z_1, \dots, z_n$ of degree*

$$(\eta(a)p - a)/q - (f-1) + (h-1)p.$$

6.5 Cartier Map and $\mathbb{F}_p$-Hypergeometric Solutions

Let $W(X)$ be the n-dimensional $\mathbb{K}(z)$-vector space spanned by the following differential 1-forms on X:

$$\frac{1}{x - z_i}\frac{\mathrm{d}x}{y}, \quad i = 1, \dots, n. \tag{6.17}$$

Notice that these are the differential 1-forms on X, which appear in the construction of the solutions of the KZ equations over $\mathbb{C}$. Notice also that they are not regular on X.

Let $W(X)^*$ be the space dual to $W(X)$ and ψ_j, $j = 1, \ldots, n$, the basis of $W(X)^*$ dual to the basis $(\mathrm{d}x/y(x - z_i))$.

Define the τ-linear Cartier map

$$\hat{C} \,:\, W(X) \,\to\, \Omega^1_{a_1}(X) \tag{6.18}$$

in the standard way. Namely we have

$$\begin{aligned}\frac{1}{x - z_j}\,\frac{\mathrm{d}x}{y} &= \frac{F(x,z)^{(a_1 p - 1)/q}}{x - z_j}\,\frac{\mathrm{d}x}{y^{a_1 p}} = \frac{\Phi_p(x, z, \bar{M})}{x - z_j}\,\frac{\mathrm{d}x}{y^{a_1 p}} \\ &= P_j(z, \bar{M})\,\frac{\mathrm{d}x}{y^{a_1 p}} = \sum_i P^i_j(z, \bar{M})\, x^i\,\frac{\mathrm{d}x}{y^{a_1 p}},\end{aligned}$$

where Φ_p, P_j, P^i_j see in (5.6), (5.7), (5.8). Define $\hat{C}$ by the formula

$$\frac{1}{x - z_j}\,\frac{\mathrm{d}x}{y} \quad\mapsto\quad \sum_{h=1}^{a_1 k}\Big(P_j^{(a_1 k - h)p + p - 1}(z, \bar{M})\Big)^{1/p}\, x^{a_1 k - h}\,\frac{\mathrm{d}x}{y^{a_1}}, \tag{6.19}$$

cf. (6.14).

The map

$$\hat{C}^* \,:\, \Omega^1_{a_1}(X)^* \,\to\, W(X)^* \tag{6.20}$$

adjoint to the Cartier map $\hat{C}$ is σ-linear. The matrix of the map $\hat{C}^*$ will be called the *Hasse–Witt matrix* with respect to the bases $(\varphi^i_{a_1})$ and (ψ_j). The entries of the Hasse–Witt matrix are the *polynomials* $P_j^{(a_1 k - h)p + p - 1}(z, \bar{M})$.

For any $m = 1, \ldots, a_1 k$, we have

$$\hat{C}^*(\varphi^m_{a_1}) \,=\, \sum_{j=1}^{n} P_j^{(a_1 k - m)p + p - 1}(z, \bar{M})\,\psi_j. \tag{6.21}$$

This formula shows that the coordinate vector

$$(P_1^{(a_1 k - m)p + p - 1}(z, \bar{M}), \ldots, P_n^{(a_1 k - m)p + p - 1}(z, \bar{M})) \tag{6.22}$$

of $\hat{C}^*(\varphi^h_{a_1})$ is exactly the $\mathbb{F}_p$-hypergeometric solution $I^m(z)$ defined in (5.9).

This construction gives us the following theorem.

Theorem 6.2 *The Hasse–Witt matrix of the map $\hat{C}^*$ defines an isomorphism*

$$\iota_\Lambda \,:\, \Omega_{a_1}(X)^* \,\to\, \mathcal{M}_\Lambda, \quad \varphi^m_{a_1} \,\mapsto\, I^m(z), \tag{6.23}$$

of the vector space $\Omega_{a_1}(X)^*$ and the module $\mathcal{M}_\Lambda$ of $\mathbb{F}_p$-hypergeometric solutions for $\Lambda = (1, \ldots, 1)$.

Proof Clearly the map ι_Λ is an epimorphism. The fact that ι_Λ is an isomorphism follows from the fact that $\dim \mathcal{M}_\Lambda = \dim \Omega_{a_1}(X)^* = a_1 k$ by Lemma 5.1. □

6.6 $\mathbb{F}_p$-Hypergeometric Solutions from Iterates

Consider an iterate of the Cartier map,

$$C^{\circ b} \circ \hat{C} := C \circ C \cdots \circ C \circ \hat{C} \;:\; W(X) \;\to\; \Omega^1(X).$$

The image lies in $\Omega^1_{a_b}(X)$. Choose any element $\varphi^m_{a_b}$ of the basis of $\Omega^1_{a_b}(X)^*$ dual to the basis $(x^{i-1}\mathrm{d}x/y^{a_b})$ and express the vector

$$(C^{\circ b} \circ \hat{C})^*(\varphi^m_{a_b})$$

in terms of the basis (ψ_j) of $W(X)^*$. By Theorem 6.2, the coordinates of that vector is an $\mathbb{F}_p$-hypergeometric solution, namely it is the solution

$${}^bI^m(z) := \sum_{m_1,\dots,m_b} {}^{a_b}\mathcal{I}^m_{m_b}\left(z_1^{p^b},\dots,z_n^{p^b}\right)\cdots {}^{a_1}\mathcal{I}^{m_2}_{m_1}\left(z_1^p,\dots,z_n^p\right)\cdot I^{m_1}(z_1,\dots,z_n). \tag{6.24}$$

Here $m = 1,\dots,a_b k$. We will call these $\mathbb{F}_p$-hypergeometric solutions the *iterated* *$\mathbb{F}_p$-hypergeometric solutions*.

We will see these solutions in Sects. 9.5 and 9.6.

7 Cartier Map and Module $\mathcal{M}_\Lambda$ for More General Λ

7.1 Curve $\tilde{X}$

Recall that $n = qk + 1$, see Sect. 5.1. Let $\tilde{\Lambda} = (\tilde{\Lambda}_1,\dots,\tilde{\Lambda}_{\tilde{n}}) \in \mathbb{Z}^{\tilde{n}}_{>0}$ be such that

$$\sum_{j=1}^{\tilde{n}} \tilde{\Lambda}_j = n, \quad \tilde{\Lambda}_j < q, \quad j = 1,\dots,\tilde{n}.$$

This means that $\tilde{\Lambda}$ is a fusion of $\Lambda = (1,\dots,1) \in \mathbb{Z}^n_{>0}$, see Sect. 3.3.

For $j = 1,\dots,\tilde{n}$, $a = 1,\dots,q-1$, denote

$$e_j(a) := \left\lceil \frac{\tilde{\Lambda}_j a + 1}{q} - 1 \right\rceil, \quad e(a) = \sum_{j=1}^{\tilde{n}} e_j(a), \tag{7.1}$$

where $\lceil x \rceil$ is the smallest integer greater than x or equal to x.

Consider the field $\mathbb{K}(\tilde{z})$, $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_{\tilde{n}})$. Consider the algebraic curve $\tilde{X}$ over $\mathbb{K}(\tilde{z})$ defined by the affine equation

$$y^q = \tilde{F}(x, \tilde{z}) := (x - \tilde{z}_1)^{\tilde{\Lambda}_1}(x - \tilde{z}_2)^{\tilde{\Lambda}_2} \dots (x - \tilde{z}_{\tilde{n}})^{\tilde{\Lambda}_{\tilde{n}}}. \tag{7.2}$$

The space $\Omega^1(\tilde{X})$ of regular 1-forms on $\tilde{X}$ is the direct sum

$$\Omega^1(\tilde{X}) = \bigoplus_{a=1}^{q-1} \Omega^1_a(\tilde{X}), \tag{7.3}$$

where $\Omega^1_a(\tilde{X})$ consists of 1-forms

$$u(x)\,\frac{\mathrm{d}x}{y^a}, \tag{7.4}$$

such that $u(x)$ is a polynomial in x of degree $< ak$, and for any $j = 1, \dots, \tilde{n}$, the polynomial $u(z)$ has zero at $\tilde{z}_j$ of multiplicity at least $e_j(a)$.

This fact is checked by writing $u(x)\mathrm{d}x/y^a$ in local coordinates on $\tilde{X}$ at $\infty, \tilde{z}_1, \dots, \tilde{z}_{\tilde{n}}$. Namely, at $x = \infty$, we have

$$x = t^{-q}, \quad y = t^{-n}(1 + \mathcal{O}(t)),$$

where t is a local coordinate. If $d = \deg_x u(x)$, then $u(x)\mathrm{d}x/y^a = -qt^{an-dq-q-1}(1+ \mathcal{O}(t))dt$. Hence, $u(x)\mathrm{d}x/y^a$ is regular at infinity if $an - dq - q - 1 \geq 0$, which is equivalent to $d < ak$. At $x = \tilde{z}_j$, we have

$$x - \tilde{z}_j = t^q, \quad y = \text{const}\, t^{\tilde{\Lambda}_j}(1 + \mathcal{O}(t)),$$

where t is a local coordinate. If d is the multiplicity of $u(x)$ at $x = \tilde{z}_j$, then $u(x)\mathrm{d}x/y^a = \text{const}\, t^{-a\tilde{\Lambda}_j+dq+q-1}(1 + \mathcal{O}(t))dt$. Hence $u(x)\mathrm{d}x/y^a$ is regular at $x = \tilde{z}_j$ if $-a\tilde{\Lambda}_j + dq + q - 1 \geq 0$, which is equivalent to $d \geq e_j(a)$.

Therefore,

$$\dim \Omega^1_a(\tilde{X}) = ak - e(a), \tag{7.5}$$

and $\Omega^1_a(\tilde{X})$ consists of elements

$$\tilde{u}(x)\,\frac{\mathrm{d}x}{y^a}\,\prod_{j=1}^{\tilde{n}}(x - \tilde{z}_j)^{e_j(a)}, \tag{7.6}$$

where $\tilde{u}(x)$ is an arbitrary polynomial of degree less than $ak - e(a)$.

7.2 Rank of $\mathcal{M}_{\tilde\Lambda}$

Consider the module $\mathcal{M}_{\tilde\Lambda}$ of $\mathbb{F}_p$-hypergeometric solutions of the KZ equations associated with $\tilde\Lambda$.

Theorem 7.1 *Let $p > n = kq + 1$, then the rank $d_{\tilde\Lambda}$ of $\mathcal{M}_{\tilde\Lambda}$ equals $a_1 k - e(a_1)$.*

This theorem is a generalization of Lemma 5.1.

Proof We have $d_\Lambda = \left[\sum_{j=1}^{\tilde n} \bar M_j / p\right]$ by formula (3.7). Here $\bar M_j$ is the minimal positive integer solution of the congruence

$$M_j \equiv -\frac{\bar\Lambda_j}{q} \pmod{p}. \tag{7.7}$$

Recall the integer a_1 defined by

$$1 \le a_1 < q \quad \text{and} \quad q \mid (a_1 p - 1). \tag{7.8}$$

Then

$$\tilde M_j = \tilde\Lambda_j \, \frac{a_1 p - 1}{q} \tag{7.9}$$

is another solution of the congruence in (7.7). Denote

$$\tilde M = (\tilde M_1, \dots, \tilde M_{\tilde n}). \tag{7.10}$$

□

Lemma 7.2 *We have*

$$\tilde M_j = \bar M_j + e_j(a_1) p. \tag{7.11}$$

Proof Clearly, we have $\tilde M_j = \bar M_j + lp$, where

$$l = \left[\frac{\tilde\Lambda_j (a_1 p - 1)/q}{p}\right] = \left[\frac{\tilde\Lambda_j a_1}{q} - \frac{\tilde\Lambda_j}{qp}\right].$$

Since $\tilde\Lambda_j < q < p$, we have

$$\frac{\tilde\Lambda_j}{qp} < \frac{q}{qp} < \frac{1}{q}.$$

Furthermore, since $\tilde{\Lambda}_j, a_1 < q$ and q is prime, we conclude that $\tilde{\Lambda}_j a_1/q$ is not an integer. These two observations imply that

$$l = \left[\frac{\tilde{\Lambda}_j a_1}{q} - \frac{\tilde{\Lambda}_j}{qp}\right] = \left[\frac{\tilde{\Lambda}_j a_1}{q}\right].$$

On the other hand,

$$e_j(a_1) = \left\lceil \frac{\tilde{\Lambda}_j a_1 + 1}{q} - 1 \right\rceil = \left\lceil \frac{\tilde{\Lambda}_j a_1}{q} + \frac{1}{q} \right\rceil - 1 = \left\lceil \frac{\tilde{\Lambda}_j a_1}{q} \right\rceil - 1 = \left[\frac{\tilde{\Lambda}_j a_1}{q}\right],$$

where the last two equalities follow from the fact that $\tilde{\Lambda}_j a_1/q$ is not an integer. The lemma is proved. □

To finish the proof of Theorem 7.1, we observe that

$$\begin{aligned} d_{\tilde{\Lambda}} &= \left[\sum_{j=1}^{\tilde{n}} \frac{\bar{M}_j}{p}\right] = \left[\sum_{j=1}^{\tilde{n}} \left(\tilde{\Lambda}_j \frac{(a_1 p - 1)/q}{p} - e_j(a_1)\right)\right] \\ &= \left[n \frac{(a_1 p - 1)/q}{p}\right] - e(a_1) = a_1 k - e(a_1), \end{aligned}$$

where we use Lemmas 7.2 and 5.1. □

7.3 Cartier Map for $\tilde{X}$

Let $W(\tilde{X})$ be the $\tilde{n}$-dimensional $\mathbb{K}(\tilde{z})$-vector space spanned by the following differential 1-forms on $\tilde{X}$:

$$\frac{1}{x - \tilde{z}_i} \frac{\mathrm{d}x}{y}, \quad i = 1, \ldots, \tilde{n}. \tag{7.12}$$

Let $W(\tilde{X})^*$ be the space dual to $W(\tilde{X})$ and ψ_j, $j = 1, \ldots, \tilde{n}$, be the basis of $W(\tilde{X})^*$ dual to the basis $(\mathrm{d}x/y(x - \tilde{z}_i))$.

Define the τ-linear Cartier map $\hat{C}$ of the space $W(\tilde{X})$ to the space of differential 1-forms on $\tilde{X}$ in the standard way. Namely we have

$$\frac{1}{x - \tilde{z}_j} \frac{\mathrm{d}x}{y} = \frac{\tilde{F}(x, \tilde{z})^{(a_1 p - 1)/q}}{x - \tilde{z}_j} \frac{\mathrm{d}x}{y^{a_1 p}} = P_j(\tilde{z}, \tilde{M}) \frac{\mathrm{d}x}{y^{a_1 p}} = \sum_l P_j^l(\tilde{z}, \tilde{M})\, x^l \frac{\mathrm{d}x}{y^{a_1 p}},$$

where $\tilde{M}$, P_j, P_j^l see in (7.10), (5.7), (5.8). Define $\hat{C}$ by the formula

$$\frac{1}{x - \tilde{z}_j} \frac{\mathrm{d}x}{y} \quad \mapsto \quad \sum_{h=1}^{a_1 k} \left(P_j^{(a_1 k - h)p + p - 1}(\tilde{z}, \tilde{M})\right)^{1/p} x^{a_1 k - h} \frac{\mathrm{d}x}{y^{a_1}}, \tag{7.13}$$

cf. (6.14), (7.13).

Theorem 7.3 *For $j = 1, \dots, \tilde n$, the 1-form $\hat C(\mathrm{d}x/y(x-\tilde z_j))$ lies in $\Omega^1_{a_1}(\tilde X)$, and hence the Cartier map maps $W(\tilde X)$ to $\Omega^1_{a_1}(\tilde X)$.*

Proof The polynomial

$$u(x) := \sum_{h=1}^{a_1 k}\Big(P_j^{(a_1k-h)p+p-1}(\tilde z, \tilde M)\Big)^{1/p}\, x^{a_1k-h}\,\frac{\mathrm{d}x}{y^{a_1}}$$

has degree $< a_1 k$. We need to check that for any $i = 1, \dots, \tilde n$, the polynomial $u(x)$ has zero at $x = \tilde z_i$ of multiplicity at least $e_i(a_1)$.

Indeed, on the one hand, we have

$$\tilde F(x+\tilde z_i, \tilde z)^{(a_1p-1)/q} = x^{\bar M_i + e_i(a_1)p}\prod_{l\neq i}(x+\tilde z_i - \tilde z_l)^{\tilde\Lambda_i(a_1p-1)/q},$$

by Lemma 7.2. Hence, in the Taylor expansion

$$\frac{\tilde F(x+\tilde z_i, \tilde z)^{(a_1p-1)/q}}{x-\tilde z_j} =: \sum_l \tilde P_j^l(\tilde z)\, x^l, \tag{7.14}$$

we have

$$\tilde P_j^{lp+p-1}(\tilde z) = 0, \quad l = 0, \dots, e_i(a_1)-1. \tag{7.15}$$

On the other hand, we have

$$\frac{\tilde F(x+\tilde z_i, \tilde z)^{(a_1p-1)/q}}{x-\tilde z_j} = \sum_l P_j^l(\tilde z, \tilde M)\,(x+\tilde z_i)^l. \tag{7.16}$$

By Lucas' Theorem 4.1, we have

$$\tilde P^{lp+p-1}(\tilde z) = \sum_{h\geq 0}\binom{l+h}{h}\tilde z_i^{(l+h)p}\, P_j^{(l+h)p+p-1}(\tilde z, \tilde M), \tag{7.17}$$

for any l. Formulas (7.15) and (7.17) show that the polynomial $u(x)$ has zero at $x = \tilde z_i$ of multiplicity at least $e_i(a_1)$. The theorem is proved. □

7.4 Cartier Map and $\mathbb{F}_p$-Hypergeometric Solutions

The map

$$\hat C^* : \Omega^1_a(\tilde X)^* \to W(\tilde X)^*, \tag{7.18}$$

adjoint to the Cartier operator $\hat{C}$ is σ-linear.

Elements of $\Omega^1_a(\tilde{X})$ have the form

$$u(x)\frac{\mathrm{d}x}{y^{a_1}} = \sum_{i=0}^{a_1k-1} u^i x^i \frac{\mathrm{d}x}{y^{a_1}}$$

with suitable coefficients u^i, see Sect. 7.1. For $m = 0, \dots, a_1k - 1$ define an element $\varphi^m \in W(\tilde{X})^*$ by the formula

$$\varphi^m \ : \ u(x)\frac{\mathrm{d}x}{y^{a_1}} \ \mapsto \ u^m. \tag{7.19}$$

For any $m = 1, \dots, a_1k$, we have

$$\hat{C}^*(\varphi^m) = \sum_{j=1}^{\tilde{n}} P_j^{mp+p-1}(\tilde{z}, \tilde{M})\,\psi_j. \tag{7.20}$$

This formula shows that the coordinate vector

$$(P_1^{mp+p-1}(\tilde{z}, \tilde{M}), \dots, P_{\tilde{n}}^{mp+p-1}(\tilde{z}, \tilde{M})) \tag{7.21}$$

of $\hat{C}^*(\varphi^m)$ is an $\mathbb{F}_p$-hypergeometric solution constructed in Theorem 2.3 and all solutions constructed in Theorem 2.3 are of this form.

This construction gives us the following theorem.

Theorem 7.4 *The map $\hat{C}^*$ adjoint to the Cartier map $\hat{C} : W(\tilde{X}) \to \Omega^1_{a_1}(\tilde{X})$ defines an isomorphism*

$$\iota_{\tilde{\Lambda}} \ : \ \Omega_{a_1}(\tilde{X})^* \to \mathcal{M}_{\tilde{\Lambda}}, \quad \varphi^m \mapsto (P_1^{mp+p-1}(\tilde{z}, \tilde{M}), \dots, P_{\tilde{n}}^{mp+p-1}(\tilde{z}, \tilde{M})), \tag{7.22}$$

of the vector space $\Omega_{a_1}(\tilde{X})^*$ and the module $\mathcal{M}_{\tilde{\Lambda}}$ of $\mathbb{F}_p$-hypergeometric solutions for $\tilde{\Lambda}$.

Proof Clearly the map ι_Λ is an epimorphism. The fact that ι_Λ is an isomorphism follows from the fact that $\dim \mathcal{M}_{\tilde{\Lambda}} = \dim \Omega_{a_1}(\tilde{X})^* = a_1k - e(a_1)$ by Theorem 7.1. □

8 Hasse–Witt Matrix for Curve *Y*

In Sect. 6, we introduced the curve X over the field $\mathbb{K}(z)$ and determined its Hasse–Witt matrix. In this section, we consider the same curve over a new field $\mathbb{K}(\lambda)$, where z and λ are related by a fractional linear transformation, and calculate its Hasse–Witt matrix. We will use that new Hasse–Witt matrix to relate the $\mathbb{F}_p$-hypergeometric solutions of the KZ equations over $\mathbb{C}$ and over $\mathbb{F}_p$.

8.1 Curve *Y*

Recall that $n = qk + 1$, see Sect. 5.1. Consider the field $\mathbb{K}(\lambda)$, $\lambda = (\lambda_3, \dots, \lambda_n)$. Consider the algebraic curve Y over $\mathbb{K}(\lambda)$ defined by the affine equation

$$y^q = G(x,\lambda) := x(x-1)(x-\lambda_2)\dots(x-\lambda_n). \tag{8.1}$$

The space $\Omega^1(Y)$ of regular 1-forms on Y is the direct sum $\Omega^1(Y) = \bigoplus_{a=1}^{q-1} \Omega^1_a(Y)$, where dim $\Omega^1_a(Y) = ak$, and $\Omega^1_a(Y)$ has basis :

$$x^{i-1}\frac{\mathrm{d}x}{y^a}, \quad i = 1, \dots, ak. \tag{8.2}$$

We define the Cartier operator

$$C \,:\, \Omega^1(Y) \to \Omega^1(Y) \tag{8.3}$$

in the same way as in Sect. 6.4. The operator has block structure. For $a = 1, \dots, q-1$, we have $C\left(\Omega^1_a(Y)\right) \subset \Omega^1_{\eta(a)}(Y)$.

For $f = 1, \dots, ak$, we have

$$x^{ak-f}\frac{\mathrm{d}x}{y^a} = x^{ak-f}\,G(x,\lambda)^{(\eta(a)p-a)/q}\,\frac{\mathrm{d}x}{y^{\eta(a)p}} = \sum_{w\geq 0} {}^aG^w_f(\lambda)\,x^w\,\frac{\mathrm{d}x}{y^{\eta(a)p}}, \tag{8.4}$$

where ${}^aG^w_f(\lambda) \in \mathbb{F}_p[\lambda]$. The Cartier operator is defined by the formula

$$x^{ak-f}\frac{\mathrm{d}x}{y^a} \quad\mapsto\quad \sum_{h=1}^{\eta(a)k} \left({}^aG_f^{(\eta(a)k-h)p+p-1}(\lambda)\right)^{1/p}\; x^{\eta(a)k-h}\,\frac{\mathrm{d}x}{y^{\eta(a)}}. \tag{8.5}$$

Let $\Omega^1(Y)^*$ be the space dual to $\Omega^1(Y)$. Let

$$\varphi^i_a, \quad a = 1, \dots, q-1, \quad i = 1, \dots, ka, \tag{8.6}$$

denote the basis of $\Omega^1(Y)^*$ dual to the basis $(x^{i-1}\mathrm{d}x/y^a)$ of $\Omega^1_a(Y)$.

The map

$$C^* : \Omega^1(Y)^* \to \Omega^1(Y)^*$$

adjoint to the Cartier operator is σ-linear. The matrix of the map C^* is called the *Hasse–Witt matrix* with respect to the basis (φ^i_a). The entries of the Hasse–Witt matrix are the *polynomials* ${}^aG_f^{(\eta(a)k-h)p+p-1}(\lambda)$, see (6.2).

For each $a = 1, \dots, q-1$, consider the columns of the Hasse–Witt matrix, corresponding to the basis vectors of $\Omega^1_{\eta(a)}(Y)^*$ and the rows corresponding to the basis vectors of $\Omega^1_a(Y)^*$. The respective block of the Hasse–Witt matrix of size $ak \times \eta(a)k$ is denoted by ${}^a\mathcal{K}$. Its entries are denoted by

$$ {}^a\mathcal{K}^h_f(z) := {}^aG_f^{(\eta(a)k-h)p+p-1}(z), \quad f = 1, \dots, ak, \quad h = 1, \dots, \eta(a)k. \quad (8.7) $$

8.2 Example

For $p = 5$, $q = 3$, $n = 4$, we have

$$ \begin{aligned} & y^3 = x(x-1)(x-\lambda_3)(x-\lambda_4), \\ & \Omega^1(Y) = \Omega^1_1(Y) \oplus \Omega^1_2(Y) = \left\langle \frac{\mathrm{d}x}{y} \right\rangle \oplus \left\langle \frac{\mathrm{d}x}{y^2},\ x\frac{\mathrm{d}x}{y^2} \right\rangle, \\ & \frac{\mathrm{d}x}{y} \mapsto \left({}^1\mathcal{K}^1_1\right)^{1/p} \frac{\mathrm{d}x}{y^2} + \left({}^1\mathcal{K}^2_1\right)^{1/p} \frac{x\mathrm{d}x}{y^2}, \\ & \frac{\mathrm{d}x}{y^2} \mapsto \left({}^2\mathcal{K}^1_1\right)^{1/p} \frac{\mathrm{d}x}{y}, \\ & \frac{x\mathrm{d}x}{y^2} \mapsto \left({}^2\mathcal{K}^1_2\right)^{1/p} \frac{\mathrm{d}x}{y}, \end{aligned} $$

where

$$ \begin{aligned} & {}^1\mathcal{K}^1_1(\lambda) = -\lambda_3^3 - \lambda_4^3 - 9\lambda_3^2\lambda_4 - 9\lambda_4^2\lambda_3 - 9\lambda_3^2 - 9\lambda_4^2 - 9\lambda_3 - 9\lambda_4 - 27\lambda_3\lambda_4 - 1, \\ & {}^1\mathcal{K}^2_1(\lambda) = 3\lambda_3^2\lambda_4^2(\lambda_3\lambda_4 + \lambda_3 + \lambda_4), \\ & {}^2\mathcal{K}^1_1(\lambda) = -\lambda_3 - \lambda_4 - 1, \\ & {}^2\mathcal{K}^1_2(\lambda) = 1. \end{aligned} $$

The Hasse–Witt matrix is

$$ \begin{pmatrix} 0 & {}^1\mathcal{K}^1_1(\lambda) & {}^1\mathcal{K}^2_1(\lambda) \\ {}^2\mathcal{K}^1_1(\lambda) & 0 & 0 \\ {}^2\mathcal{K}^1_2(\lambda) & 0 & 0 \end{pmatrix}. $$

8.3 Homogeneous Polynomials ${}^a\mathcal{J}^h_f(z)$

Change variables in the polynomial ${}^a\mathcal{K}^h_f(\lambda)$,

$$ \lambda_j = \frac{z_j - z_1}{z_2 - z_1}, \quad j = 3, \dots, n, $$

and multiply the result by $(z_2 - z_1)^{(\eta(a)p-a)/q+(h-1)p-(f-1)}$.

Lemma 8.1 *The function*

$$ {}^a\mathcal{J}^h_f(z) := (z_2 - z_1)^{(\eta(a)p-a)/q+(h-1)p-(f-1)} \cdot {}^a\mathcal{K}^h_f(\lambda(z)) \quad (8.8) $$

is a homogeneous polynomial in $z_1, \dots, z_n$ *of degree* $(\eta(a)p - a)/q + (h-1)p - (f-1)$.

Remark The polynomials ${}^a\mathcal{J}^h_f(z)$ are entries of the Hasse–Witt matrix of the curve defined by equation

$$y^q = x(x-(z_2-z_1))\dots(x-(z_n-z_1)),$$

and that curve is isomorphic to the curve with equation

$$y^q = (x-z_1)(x-z_2)\dots(x-z_n),$$

which is discussed in Sect. 6.

8.4 Formula for ${}^{a_s}\mathcal{K}^h_f(\lambda)$

Recall the numbers a_s introduced in (4.19), the base p expansion

$$\frac{p^d-1}{q} = A_0 + A_1 p + A_2 p^2 + \dots + A_{d-1}p^{d-1}$$

in (4.21), and the relation

$$A_s = \frac{a_{s+1}p - a_s}{q}, \quad s = 0, \dots, d-1,$$

in Lemma 4.9.

For $s \in \mathbb{Z}_{\ge 0}$, $f = 1, \dots, a_s k$, $h = 1, \dots, a_{s+1}k$, define the sets

$$\begin{aligned} {}^s\Delta^h_f = \Big\{ & (\ell_3, \dots, \ell_n) \in \mathbb{Z}^{n-2}_{\ge 0} \mid 0 \le \sum_{i=3}^n \ell_i + f - 1 - (h-1)p \le A_s, \\ & \ell_j \le A_s \text{ for } j = 3, \dots, n \Big\} \end{aligned} \tag{8.9}$$

Lemma 8.2 *We have*

$${}^{a_s}\mathcal{K}^h_f(\lambda) = \sum_{(\ell_3,\dots,\ell_n)\in {}^s\Delta^h_f} {}^{a_s}\mathcal{K}^h_{f;\ell_3,\dots,\ell_n}(\lambda), \tag{8.10}$$

where

$$\begin{aligned} {}^{a_s}\mathcal{K}^h_{f;\ell_3,\dots,\ell_n}(\lambda_3,\dots,\lambda_n) &= (-1)^{A_s - f + 1 + (h-1)p} \\ &\times \binom{A_s}{\sum_{i=3}^n \ell_i + f - 1 - (h-1)p} \prod_{i=3}^n \binom{A_s}{\ell_i} \lambda_3^{\ell_3} \dots \lambda_n^{\ell_n}. \end{aligned} \tag{8.11}$$

In particular, all terms ${}^{a_s}\mathcal{K}^h_{f;\ell_3,\dots,\ell_n}(\lambda_3,\dots,\lambda_n)$ *are nonzero.*

Proof The lemma is proved by straightforward calculation similar to the proof of Theorem 5.6. □

Similar formulas can be obtained for all entries ${}^a\mathcal{K}^h_f(\lambda)$.

9 Comparison of Solutions over $\mathbb{C}$ and $\mathbb{F}_p$

In this section, we will

(1) distinguish one holomorphic solution of the KZ equations for $\Lambda = (1,\dots,1)$,
(2) expand it into the Taylor series,
(3) reduce this Taylor expansion modulo p,
(4) observe that the reduction mod p of the Taylor expansion of the distinguished solution gives all $\mathbb{F}_p$-hypergeometric solutions mod p, and conversely the $\mathbb{F}_p$-hypergeometric solutions together with matrix coefficients of the Hasse–Witt matrix determine this Taylor expansion.

9.1 Distinguished Holomorphic Solution

Consider the KZ equations (2.2) for $\Lambda = (1,\dots,1)$ over the field $\mathbb{C}$. We assume that $n = qk + 1$ as in Sect. 5.1.

Recall that the solutions have the form $I^{(\gamma)}(z) = (I_1(z),\dots,I_n(z))$, where

$$I_j(z) = \int_\gamma \frac{1}{\sqrt[q]{(t-z_1)\dots(t-z_n)}}\,\frac{dt}{t-z_j}$$

and γ is an oriented loop on the complex algebraic curve with equation

$$y^q = (x-z_1)\dots(x-z_n).$$

Assume that $z_3,\dots,z_n$ are closer to z_1 than to z_2:

$$\left|\frac{z_j - z_1}{z_2 - z_1}\right| < \frac{1}{2}, \quad j = 3,\dots,n.$$

Choose γ to be the circle $\left\{t \in \mathbb{C} \mid \left|\frac{t-z_1}{z_2-z_1}\right| = \frac{1}{2}\right\}$ oriented counterclockwise, and multiply the vector $I(z)$ by the normalization constant $(-1)^{1/q}/2\pi i$.

We will describe the normalization procedure of the solution $I(z)$ more precisely in Sect. 9.2.

We call the solution $I(z)$ the *distinguished* solution.

9.2 Rescaling

Change variables, $t = (z_2 - z_1)x + z_1$, and write

$$I(z_1, \dots, z_n) = (z_2 - z_1)^{-1/q-k} L(\lambda_3, \dots, \lambda_n), \tag{9.1}$$

where

$$\lambda = (\lambda_3, \dots, \lambda_n) = \left(\frac{z_3 - z_1}{z_2 - z_1}, \dots, \frac{z_n - z_1}{z_2 - z_1} \right),$$

$L(\lambda) = (L_1, \dots, L_n)$,

$$L_j = \frac{(-1)^{1/q}}{2\pi i} \int_{|x|=1/2} \frac{\mathrm{d}x}{\sqrt[q]{x(x-1)(x-\lambda_3)\dots(x-\lambda_n)}} \frac{1}{x - \lambda_j}, \tag{9.2}$$

and $\lambda_1 = 0$, $\lambda_2 = 1$.

The integral $L(\lambda)$ is well defined at $(\lambda_3, \dots, \lambda_n) = (0. \dots, 0)$ and

$$L_j(0, \dots, 0) = \frac{(-1)^{1/q}}{2\pi i} \int_{|x|=1/2} \frac{\mathrm{d}x}{x^k \sqrt[q]{x-1}} \frac{1}{x - \lambda_j}.$$

The q-valued function

$$\frac{(-1)^{1/q}}{x^k \sqrt[q]{x-1}}$$

has no monodromy over the circle $|x| = 1/2$. To fix the value of the integrals in (9.2), we choose over the circle $|x| = 1/2$ that branch of the q-valued function

$$\frac{(-1)^{1/q}}{\sqrt[q]{x(x-1)(x-\lambda_3)\dots(x-\lambda_n)}},$$

which is positive at $x = 1/2$ and $(\lambda_3, \dots, \lambda_n) = (0. \dots, 0)$.

The function $L(\lambda)$ is holomorphic at the point $(\lambda_3, \dots, \lambda_n) = (0, \dots, 0)$. Hence,

$$L(\lambda) = \sum_{(k_3, \dots, k_n) \in \mathbb{Z}_{\geq 0}^{n-2}} L_{k_3, \dots, k_n} \lambda_3^{k_3} \dots \lambda_n^{k_n} \tag{9.3}$$

for suitable complex numbers $L_{k_3, \dots, k_n}$ in a neighborhood of the point $(0, \dots, 0)$.

9.3 Taylor Expansion of $L(\lambda)$

Lemma 9.1 *We have*

$$L_{k_3,\dots,k_n} = (-1)^k \binom{-1/q}{k_3+\dots+k_n+k} \prod_{i=3}^{n} \binom{-1/q}{k_i}$$
$$\times \left(1,\ -q\left(\sum_{i=3}^{n} k_i + k\right),\ qk_3+1,\ \dots,\ qk_n+1\right), \qquad (9.4)$$

where the integer k is defined by the equation $n = qk+1$.

Proof Indeed,

$$\begin{aligned} L_1(\lambda) &= \sum_{k_3,\dots,k_n=0}^{\infty} \frac{\lambda_3^{k_3}\dots\lambda_n^{k_n}}{k_3!\dots k_n!} \frac{\partial^{k_3+\dots+k_n} L_1}{\partial\lambda_3^{k_3}\dots\partial\lambda_n^{k_n}}(0) \\ &= \frac{(-1)^{1/q}}{2\pi i} \sum_{k_3,\dots,k_n=0}^{\infty} \lambda_3^{k_3}\dots\lambda_n^{k_n} (-1)^{\sum_{i=3}^n k_i} \prod_{i=3}^{n} \binom{-1/q}{k_i} \\ &\quad\times \int_{|x|=1/2} x^{-(n-1)/q-\sum_{i=3}^n k_i - 1}(x-1)^{-1/q}\mathrm{d}x \\ &= (-1)^k \sum_{k_3,\dots,k_n=0}^{\infty} \lambda_3^{k_3}\dots\lambda_n^{k_n} \binom{-1/q}{\sum_{i=3}^{n} k_i + k} \prod_{i=3}^{n} \binom{-1/q}{k_i}. \end{aligned}$$

Similarly,

$$L_2(\lambda) = (-1)^k \sum_{k_3,\dots,k_n=0}^{\infty} \lambda_3^{k_3}\dots\lambda_n^{k_n} \binom{-1/q}{\sum_{i=3}^{n} k_i + k} \prod_{i=3}^{n} \binom{-1/q}{k_i} \left(-q\left(\sum_{i=3}^{n} k_i + k\right)\right)$$

and

$$L_j(\lambda) = (-1)^k \sum_{k_3,\dots,k_n=0}^{\infty} \lambda_3^{k_3}\dots\lambda_n^{k_n} \binom{-1/q}{\sum_{i=3}^{n} k_i + k} \prod_{i=3}^{n} \binom{-1/q}{k_i} (qk_j+1)$$

for $j = 3, \dots, n$. The lemma is proved. □

Corollary 9.2 *Each coefficient of the series $L(\lambda)$ is well-defined modulo p.*

Proof The corollary follows from Theorem 4.2. □

9.4 Coefficients $L_{k_3,\dots,k_n}$ Nonzero Modulo p

Let $(k_3, \dots, k_n) \in \mathbb{Z}^{n-2}_{\geq 0}$ with

$$k_i = k_i^0 + k_i^1 p + \dots + k_i^b p^b, \quad 0 \leq k_i^j \leq p-1, \quad i = 3, \dots, n,$$

the base p expansions. Assume that not all numbers k_i^b, $i = 3, \dots, n$ are equal to zero.

Recall the base p expansion

$$\frac{p^d - 1}{q} = A_0 + A_1 p + A_2 p^2 + \dots + A_{d-1} p^{d-1}, \quad A_s = \frac{a_{s+1} p - a_s}{q}.$$

Extend the sequence (A_s) d-periodically,

$$A_{s+d} := A_s. \tag{9.5}$$

Lemma 9.3 *We have* $\prod_{i=3}^n \binom{-1/q}{k_i} \not\equiv 0 \pmod{p}$ *if and only if*

$$k_i^s \leq A_s \tag{9.6}$$

for $i = 3, \dots, n, \ s = 0, \dots, b.$

Proof The lemma follows from Theorem 4.2. □

Lemma 9.4 *Assume that condition (9.6) holds. Then for* $s = 0, \dots, b$, *we have*

$$\sum_{i=3}^n k_i^s + a_s k \ < \ a_{s+1} k p. \tag{9.7}$$

Proof We have

$$\sum_{i=3}^n k_i^s \ \leq \ (n-2) A_s \ = \ (qk - 1) A_s \ = \ qk \, \frac{a_{s+1} p - a_s}{q} - A_s$$
$$= a_{s+1} k p - a_s k - A_s \ < \ a_{s+1} k p - a_s k,$$

where the last inequality follows from the inequality $0 < A_s$, see (4.23). □

Following [17] define the *shift coefficients* $(m_0, \dots, m_{b+1})$ as follows. Define $m_0 = k + 1$. For $s = 0$ formula (9.7) takes the form

$$\sum_{i=3}^n k_i^0 + m_0 - 1 < a_1 k p,$$

since $a_0 = 1$. Hence there exists a unique integer m_1, $1 \le m_1 \le a_1 k$, such that

$$0 \le \sum_{i=3}^{n} k_i^0 + m_0 - 1 - (m_1 - 1)p < p.$$

For $s = 1$ formula (9.7) takes the form

$$\sum_{i=3}^{n} k_i^1 + a_1 k < a_2 k p.$$

By construction $m_1 \le a_1 k$, hence

$$\sum_{i=3}^{n} k_i^1 + m_1 - 1 < a_2 k p.$$

Therefore, there exists a unique integer m_2, $1 \le m_2 \le a_2 k$, such that

$$0 \le \sum_{i=3}^{n} k_i^1 + m_1 - 1 - (m_2 - 1)p < p,$$

and so on.

We will obtain m_s with $1 \le m_s \le a_s k$ for $s = 1, \dots, b$. We define m_{b+1} to be the unique integer such that $1 \le m_{b+1} \le a_{b+1} k$ and

$$0 \le \sum_{i=3}^{n} k_i^b + m_b - 1 - (m_{b+1} - 1)p < p.$$

We say that a tuple $(k_3, \dots, k_n)$ is *admissible* with respect to p if the following inequalities hold

$$k_i^s \le A_s, \quad i = 3, \dots, n, \ s = 0, \dots, b, \tag{9.8}$$

$$\sum_{i=3}^{n} k_i^s + m_s - 1 - (m_{s+1} - 1)p \le A_s, \quad s = 0, \dots, b, \tag{9.9}$$

$$m_{b+1} - 1 \le A_{b+1}. \tag{9.10}$$

Lemma 9.5 *We have*

$$\binom{-1/q}{k_3 + \dots + k_n + k} \not\equiv 0 \pmod{p},$$

if and only if the shift coefficients of the tuple $(k_3, \dots, k_n)$ *satisfy* (9.8) *and* (9.9).

Proof The p-ary expansion of $k_3 + \cdots + k_n + k$ is

$$\left(\sum_{i=3}^{n} k_i^0 + m_0 - 1 - (m_1 - 1)p\right) + \left(\sum_{i=3}^{n} k_i^1 + m_1 - 1 - (m_2 - 1)p\right) p + \dots$$
$$+ \left(\sum_{i=3}^{n} k_i^b + m_b - 1 - (m_{b+1} - 1)p\right) p^b + (m_{b+1} - 1)p^{b+1}.$$

By Theorem 4.2, the following congruence holds modulo p:

$$\binom{-1/q}{k_3 + \cdots + k_n + k} \equiv \binom{A_{b+1}}{m_{b+1} - 1} \cdot \prod_{s=0}^{b} \binom{A_s}{\sum_{i=3}^{n} k_i^s + m_s - 1 - (m_{s+1} - 1)p}.$$

The right-hand side of the congruence above is nonzero, if and only if (9.8) and (9.9) hold. □

Recall the sets ${}^s\Delta_f^h$ defined in (8.9).

Lemma 9.6 *The tuple* $(k_3, \dots, k_n)$ *is admissible if and only if* $m_{b+1} - 1 \le A_{b+1}$ *and*

$$(k_3^s, \dots, k_n^s) \in {}^s\Delta_{m_s}^{m_{s+1}} \quad \text{for } s = 0, \dots, b.$$

Theorem 9.7 *The following statements hold true:*

(i) $L_{k_3,\dots,k_n} \not\equiv 0 \pmod p$ *if and only if the tuple* $(k_3, \dots, k_n)$ *is admissible.*
(ii) *If the tuple* $(k_3, \dots, k_n)$ *is admissible, then*

$$L_{k_3,\dots,k_n} \lambda_3^{k_3} \dots \lambda_n^{k_n} \equiv (-1)^{(a_{b+1}p^{b+1}-1)/q + m_{b+1} - 1} \binom{A_{b+1}}{m_{b+1} - 1}$$
$$\times \left(\prod_{s=1}^{b} {}^{a_s}\mathcal{K}^{m_{s+1}}_{m_s; k_3^s, \dots, k_n^s}\left(\lambda_3^{p^s}, \dots, \lambda_n^{p^s}\right)\right) K^{m_1}_{k_3^0, \dots, k_n^0}(\lambda) \pmod p, \quad (9.11)$$

where ${}^{a_s}\mathcal{K}^h_{f;\vec{k}}(\lambda)$ *are terms of the Hasse–Witt matrix expansion in* (8.10) *and* $K^m_{\vec{k}}(\lambda)$ *are the terms of the expansion of vector polynomial* $K^m(\lambda)$ *in* (5.22).

Proof We have $L_{k_3,\dots,k_n} \not\equiv 0 \pmod p$ if and only if each of the binomial coefficients in (9.4) is not divisible by p. By Lemmas 9.3 and 9.5, this is equivalent to saying that the tuple $(k_3, \dots, k_n)$ is admissible. This gives part (i).

By Lemma 9.1, we have

$$L_{k_3,\dots,k_n} = (-1)^k \binom{-1/q}{k_3 + \cdots + k_n + k} \prod_{i=3}^{n} \binom{-1/q}{k_i}$$
$$\times \left(1,\ -q\left(\sum_{i=3}^{n} k_i + k\right),\ qk_3 + 1,\ \dots,\ qk_n + 1\right). \quad (9.12)$$

Theorem 4.2 allows us to write the binomial coefficients in formula (9.12) as products and then formula (9.11) becomes a straightforward corollary of formulas for ${}^{a_s}\mathcal{K}^h_{f;\vec{k}}$ and $K^m_{\vec{k}}$.

Notice that calculating the power of -1 on the right-hand side of formula (9.11), we use the identity

$$A_0 + A_1 p + \dots + A_b p^b = \frac{a_1 p - 1}{q} + \frac{a_2 p - a_1}{q} p + \dots + \frac{a_{b+1} p - a_b}{q} p^b = \frac{a_{b+1} p^{b+1} - 1}{q}.$$

□

9.5 Decomposition of $L(\lambda)$ as a Sum of $K^m(\lambda)$

Define the set

$$M = \{(m_0, \dots, m_{b+1}) \in \mathbb{Z}^{b+2}_{\geq 1} \mid b \in \mathbb{Z}_{\geq 0},\ m_0 = k+1,\ 1 \leq m_s \leq a_s k \\ \text{for } s = 1, \dots, b,\ 1 \leq m_{b+1} \leq a_{b+1} k\}. \tag{9.13}$$

For any $\vec{m} = (m_0, \dots, m_{b+1}) \in M$, define the n-vector of polynomials in λ:

$$N_{\vec{m}}(\lambda) = (-1)^{(a_{b+1}p^{b+1}-1)/q + m_{b+1} - 1} \binom{A_{b+1}}{m_{b+1} - 1} \\ \times \left(\prod_{s=1}^{b} {}^{a_s}\mathcal{K}^{m_{s+1}}_{m_s} \left(\lambda_3^{p^s}, \dots, \lambda_n^{p^s} \right) \right) K^{m_1}(\lambda_3, \dots, \lambda_n). \tag{9.14}$$

Theorem 9.8 *We have*

$$L(\lambda) \equiv \sum_{\vec{m} \in M} N_{\vec{m}}(\lambda) \pmod{p}. \tag{9.15}$$

Moreover, if a monomial $\lambda_3^{k_3} \dots \lambda_n^{k_n}$ enters one of the vector polynomials $N_{\vec{m}}(\lambda)$ with a nonzero coefficient, then this monomial does not enter with nonzero coefficient any other vector polynomial $N_{\vec{m}'}(\lambda)$.

Proof The theorem is a straightforward corollary of Theorem 5.6, Lemma 8.2, and Theorem 9.7. □

For $q = 2$, this theorem is [17, Corollary 7.4].

9.6 Distinguished Solution over $\mathbb{C}$ and Solutions $J^m(z)$ over $\mathbb{F}_p$

Consider the distinguished solution $I(z_1, \dots, z_n)$ of the KZ equations (2.2) over $\mathbb{C}$, see (9.1), and the $\mathbb{F}_p$-hypergeometric solutions $J^m(z)$ of the same equations over $\mathbb{F}_p$, see (5.10).

We have

$$I(z_1, \dots, z_n) = (z_2 - z_1)^{-1/q-k} \cdot L(\lambda_3, \dots, \lambda_n), \tag{9.16}$$
$$J^m(z) = (z_2 - z_1)^{(a_1 p-1)/q+(m-1)p-k} \cdot K^m(\lambda), \tag{9.17}$$
$${}^a\mathcal{J}^h_f(z) = (z_2 - z_1)^{(\eta(a)p-a)/q+(h-1)p-(f-1)} \cdot {}^a\mathcal{K}^h_f(\lambda(z)), \tag{9.18}$$

see (9.1), (5.15), and (8.8).

Expressing $L(\lambda_3, \dots, \lambda_n)$, $K^m(\lambda)$, $\mathcal{K}^h_f(\lambda(z))$ in terms of $I(z_1, \dots, z_n)$, $J^m(z)$, ${}^a\mathcal{J}^h_f(z)$ from these equations, and using the congruence

$$L(\lambda) \equiv \sum_{\vec{m}\in M} N_{\vec{m}}(\lambda) \pmod p \tag{9.19}$$

of Theorem 9.8, we obtain a relation between the distinguished holomorphic solution $I(z)$ and the $\mathbb{F}_p$-hypergeometric data $J^m(z)$ and ${}^a\mathcal{J}^h_f(z)$.

This relation shows that knowing the distinguished holomorphic solution we may recover the $\mathbb{F}_p$-hypergeometric solutions as well as the entries of the associated Hasse–Witt matrix. Conversely, knowing the $\mathbb{F}_p$-hypergeometric solutions and the entries of the associated Hasse–Witt matrix, we may recover the distinguished holomorphic solution modulo p.

Remark Notice that summands on the right-hand side of (9.15) correspond to the iterated $\mathbb{F}_p$-hypergeometric solutions defined in Sect. 6.6. We may interpret formula (9.15) as a statement that the Taylor expansion of the distinguished holomorphic solution reduced modulo p is the sum of all iterated $\mathbb{F}_p$-hypergeometric solutions.

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