



Accessible Boundary Points in the Shift Locus of a Family of Meromorphic Functions with Two Finite Asymptotic Values

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Abstract

In this paper, we continue the study, began in Chen et al. (Slices of parameter space for meromorphic maps with two asymptotic values, [arXiv:1908.06028](https://arxiv.org/abs/1908.06028), 2019), of the bifurcation locus of a family of meromorphic functions with two asymptotic values, no critical values, and an attracting fixed point. If we fix the multiplier of the fixed point, either of the two asymptotic values determines a one-dimensional parameter slice for this family. We proved that the bifurcation locus divides this parameter slice into three regions, two of them analogous to the Mandelbrot set and one, the shift locus, analogous to the complement of the Mandelbrot set. In Fagella and Keen (Stable components in the parameter plane of meromorphic functions of finite type, [arXiv:1702.06563](https://arxiv.org/abs/1702.06563), 2017) and Chen and Keen (Discrete and Continuous Dynamical Systems 39(10):5659–5681, 2019), it was proved that the points in the bifurcation locus corresponding to functions with a parabolic cycle, or those for which some iterate of one of the asymptotic values lands on a pole are accessible boundary points of the hyperbolic components of the Mandelbrot-like sets. Here, we prove these points, as well as the points where some iterate of the asymptotic value lands on a repelling periodic cycle are also accessible from the shift locus.

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1 Introduction

The investigation of the bifurcation locus in the parameter plane of quadratic polynomials where the dynamics is unstable has led to a lot of interesting mathematics and is still not completely understood. In an early paper, [17], part of the parameter space for the dynamics of the family \mathcal{R}_2 of rational maps with one attractive fixed point and two critical values was shown to be similar to the parameter space of quadratic polynomials, although the existence of two varying singular values and poles made its structure more complicated. In particular, in a one-dimensional slice formed by fixing the multiplier of the fixed point, the bifurcation locus separates the parameter plane into three regions, one like the complement of the Mandelbrot set, and called the shift locus, where both critical values are attracted to the same cycle and two complementary regions that are Mandelbrot sets containing stable, or hyperbolic components where the critical values are attracted to different cycles.

In this paper, we look at how the situation differs for the family:

$$\mathcal{F}_2 = \left\{ f_{\lambda, \rho}(z) = \frac{e^z - e^{-z}}{\frac{e^z}{\lambda} - \frac{e^{-z}}{\mu}}, \frac{1}{\lambda} - \frac{1}{\mu} = \frac{2}{\rho} \right\}$$

of meromorphic functions with two asymptotic values λ and μ , no critical values, and a fixed point at the origin whose multiplier ρ lies in the punctured unit disk. Some things are the same, of course. In particular, stable dynamical behavior is always eventually periodic and controlled by the singular values. There are, though, significant differences due to the maps in \mathcal{F}_2 being infinite to one and to their branching over the singular values being logarithmic rather than algebraic.

In [9], we studied the family \mathcal{F}_2 using the holomorphic dependence of the functions on two parameters, the multiplier ρ and the asymptotic value λ , the other asymptotic value μ being a simple function of ρ and λ . We proved that, like \mathcal{R}_2 , if we take a slice by fixing ρ in the punctured unit disk, the bifurcation locus in the resulting parameter plane again divides it into three distinct regions, one, a shift locus like the complement of the Mandelbrot set, where both asymptotic values are attracted to the fixed point at the origin, and two complementary regions that are Mandelbrot-like. They each contain infinitely many hyperbolic components where the asymptotic values are attracted to different periodic cycles. It had already been shown, see [10, 16, 19], that each hyperbolic component of the Mandelbrot-like sets is a universal cover of \mathbb{D}^* and that the covering map extends continuously to the boundary. Like the hyperbolic components of Mandelbrot set, the boundary contains points where the map has a parabolic cycle. Unlike the Mandelbrot set, however, the hyperbolic components do not contain a “center” where the periodic cycle contains the critical value and has

multiplier zero. Instead, they contain a distinguished boundary point with the property that as the parameter approaches this point, the limit of the multiplier of the periodic cycle attracting the asymptotic value is zero. It is thus called a “virtual center”. Virtual centers are also characterized by the property that one of the asymptotic values is a prepole, that is: some iterate lands on infinity and its orbit are finite.

In this paper, we are interested in the bifurcation locus in the slice of \mathcal{F}_2 with ρ fixed. In particular, we characterize two subsets of points that are accessible from inside the hyperbolic components in the sense that there is a curve in the hyperbolic component whose accumulation set on the boundary consists only of that point. In addition, we prove that points in the bifurcation locus where the asymptotic value lands on a repelling periodic cycle are accessible from the shift locus. In Sect. 5, we prove our main result:

Main Theorem *The parameters in the bifurcation locus that are virtual centers and the parameters for which the function $f_{\lambda,\rho}$ has a parabolic cycle or for which an asymptotic value is mapped onto a repelling cycle by some iterate of $f_{\lambda,\rho}$ are accessible from inside the shift locus.*

In other words, the points with parabolic cycles and the virtual centers in the bifurcation locus are accessible both from inside the hyperbolic components of the Mandelbrot-like sets and from the inside of the shift locus.

The first step in proving our results is to put a “coordinate structure” on the shift locus. We showed in [9] that the shift locus is an annulus. We summarize that argument in Sect. 3. The discussion is similar to that for polynomials and rational maps. It uses quasiconformal mappings together with the dynamics of a fixed “model function” to characterize the shift locus by defining a “Green’s function” for the model. This function pulls back from the dynamic space of the model to the shift locus where it measures the relative rates of attraction of the asymptotic values to zero. The inverse of the Green’s function defines level and gradient curves for those rates in the shift locus.

The transcendental qualities of $f_{\lambda,\rho}$ impart a much more complicated structure near the boundary of the shift locus than one has for rational maps. We describe this structure first in the model. As we did for rational maps in [17], we start with a fixed level curve of Green’s function and apply the dynamics of the model map. In that case, there were two preimages of the curve, but now there are infinitely many.

To understand the structure, we need first to identify each of the infinitely many inverse branches of the function $f_{\lambda,\rho}$ with an integer. The n^{th} backward orbit of a point can then be assigned to a sequence of n of these integers. Applying $f_{\lambda,\rho}$ to the map acts as a shift map on the sequence. For the Julia set of a rational map, the periodic points are assigned infinite periodic sequences. Poles, which are now preimages of the essential singularity, correspond to finite sequences. Thus, assigning the “integer” infinity to the point at infinity and taking the closure in the space of finite and infinite sequences of integers, we obtain a representation of the Julia set of the model map with its dynamics by a sequence space that is compatible with the shift map.

We use this identification of the Julia set with the sequence space to construct paths in our model space, and in Sect. 4, we transfer these paths from the model to the shift locus. The heart of the proof of the main theorem is to show that the paths in the shift locus have unique end points.

2 Notation and Basics

Here, we briefly recall the basic definitions, concepts, and notation we will use. We refer the reader to standard sources on meromorphic dynamics for details and proofs. See, e.g., [2–7, 13, 19].

We denote the complex plane by \mathbb{C} , the Riemann sphere by $\widehat{\mathbb{C}}$ and the unit disk by \mathbb{D} . We denote the punctured plane by $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and the punctured disk by $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$.

To study the dynamics of a family of meromorphic functions, $\{f_\lambda(z)\}$, we look at the orbits of points formed by iterating the function $f(z) = f_\lambda(z)$. If $f^k(z) = \infty$ for some $k > 0$, z is called a prepole of order k —a pole is a prepole of order 1. For meromorphic functions, the poles and prepoles have finite orbits that end at infinity. The *Fatou set or Stable set*, F_f , consists of those points at which the iterates form a normal family. The Julia set J_f is the complement of the Fatou set and contains all the poles and prepoles.

If there exists a minimal n , such that $f^n(z) = z$, then z is called *periodic*. Periodic points are classified by their multipliers, $\rho(z) = (f^n)'(z)$ where n is the period: they are repelling if $|\rho(z)| > 1$, attracting if $0 < |\rho(z)| < 1$, super-attracting if $\rho = 0$ and neutral otherwise. A neutral periodic point is *parabolic* if $\rho(z) = e^{2\pi i p/q}$ for some rational p/q . The Julia set is the closure of the set of repelling periodic points and is also the closure of the prepoles, (see, e.g., [5]).

A point a is a *singular value* of f if f is not a regular covering map over a .

- a is a *critical value* if, for some z , $f'(z) = 0$ and $f(z) = a$.
- a is an *asymptotic value* for f if there is a path $\gamma(t)$, such that $\lim_{t \rightarrow \infty} \gamma(t) = \infty$ and $\lim_{t \rightarrow \infty} f(\gamma(t)) = a$; $\gamma(t)$ is called an *asymptotic curve or an asymptotic path* for a .
- The *set of singular values* S_f consists of the closure of the critical values and the asymptotic values. The *post-singular set* is

$$P_f = \overline{\cup_{a \in S_f} \cup_{k=0}^{\infty} f^k(a) \cup \{\infty\}}.$$

For notational simplicity, if a prepole p_n of order n is a singular value, $\cup_{k=0}^n f^k(p_n)$ is a finite set with $f^n(p_n) = \infty$.

A map f is *hyperbolic* if $J_f \cap P_f = \emptyset$.

In [13], it is proved that every component of the Fatou set of a function with two asymptotic values and no critical values is eventually periodic: that is, $f^n(D) \subseteq f^m(D)$ for some integers n, m . In addition, the periodic cycles of stable domains for these functions are classified there as follows:

- **Attracting** If the periodic cycle of domains contains an attracting cycle in its interior.
- **Parabolic** If there is a parabolic periodic cycle on its boundary.
- **Rotation** If $f^n : D \rightarrow D$ is holomorphically conjugate to a rotation map. It follows from arguments in [19] that for maps with only two asymptotic values and no critical values, rotation domains are always simply connected. These domains are called *Siegel disks*.

A standard result in dynamics is that each attracting or parabolic cycle of domains contains a singular value. The boundary of each rotation domain is contained in the accumulation set of the forward orbit of a singular value. (See, e.g., [21], chap 8–11 or [2], Sect. 4.3.)

By a theorem of Nevanlinna [24], any meromorphic function with only two asymptotic values and no critical values can be explicitly written as a linear transformation of the exponential map. Therefore, putting the essential singularity at infinity and conjugating by an affine map, we may assume that the origin is a fixed point with multiplier ρ , and we may write \mathcal{F}_2 as a family of functions of the form:

$$\mathcal{F}_2 = \left\{ f_{\lambda,\rho}(z) = \frac{e^z - e^{-z}}{\frac{e^z}{\lambda} - \frac{e^{-z}}{\mu}}, \frac{1}{\lambda} - \frac{1}{\mu} = \frac{2}{\rho} \right\},$$

so that λ and μ are the asymptotic values. Note that $f_{\lambda,\rho}(z)$ is not defined for $\lambda = 0, \rho/2$.

The family \mathcal{F}_2 depends on two complex parameters. We form a *dynamically natural slice* of \mathcal{F}_2 , in the sense of [16], by fixing $\rho, |\rho| < 1$, and taking the asymptotic value $\lambda \in \mathbb{C} \setminus \{0, \rho/2\}$ as the parameter. The other asymptotic value μ is then a simple function of ρ and λ . We write the functions in the slice as $f_\lambda = f_{\lambda,\rho}$.

Since the origin is an attracting fixed point, for every $\lambda \in \mathbb{C} \setminus \{0, \rho/2\}$, either λ or $\mu = \mu(\lambda, \rho)$ is attracted by 0.

Definition 1 Let

$$\begin{aligned} \mathcal{M}_\lambda &= \{\lambda \in \mathbb{C} \setminus \{0, \rho/2\} \mid \lambda \text{ is NOT attracted to the origin}\}, \\ \mathcal{M}_\mu &= \{\lambda \in \mathbb{C} \setminus \{0, \rho/2\} \mid \mu \text{ is NOT attracted to the origin}\} \end{aligned}$$

and

$$\mathcal{S} = \{\lambda \in \mathbb{C} \setminus \{0, \rho/2\} \mid \lambda, \mu \text{ are both attracted to the origin}\}.$$

We focus the discussion below on \mathcal{M}_λ . It is a summary of results in [9,16]. We refer the reader to those papers for proofs. There is a completely analogous discussion for \mathcal{M}_μ . See Figs. 1 and 2.

Recall that a function f_λ is *hyperbolic* if the orbits of the asymptotic values remain bounded away from its Julia set. The interior of \mathcal{M}_λ contains all the hyperbolic components Ω_p in which the orbit of λ tends to an attracting periodic cycle of period p . These are called *Shell components* because of their shape. In [16], it is proved that each Ω_p is a universal covering of the punctured disk \mathbb{D}^* where the covering map is defined by the multiplier of the cycle. This map extends continuously to the boundary and there is a standard bifurcation at each rational boundary point where the multiplier is of the form $e^{2p\pi i/q}$. There is a unique point $\lambda^* \in \partial\Omega_p$, such that as $\lambda \rightarrow \lambda^*$ along a path in Ω_p , the multiplier of the cycle tends to zero. This boundary point is called the *virtual center*, since it plays the role for Ω_p played by the center of a hyperbolic component of the Mandelbrot set for $z^2 + c$.

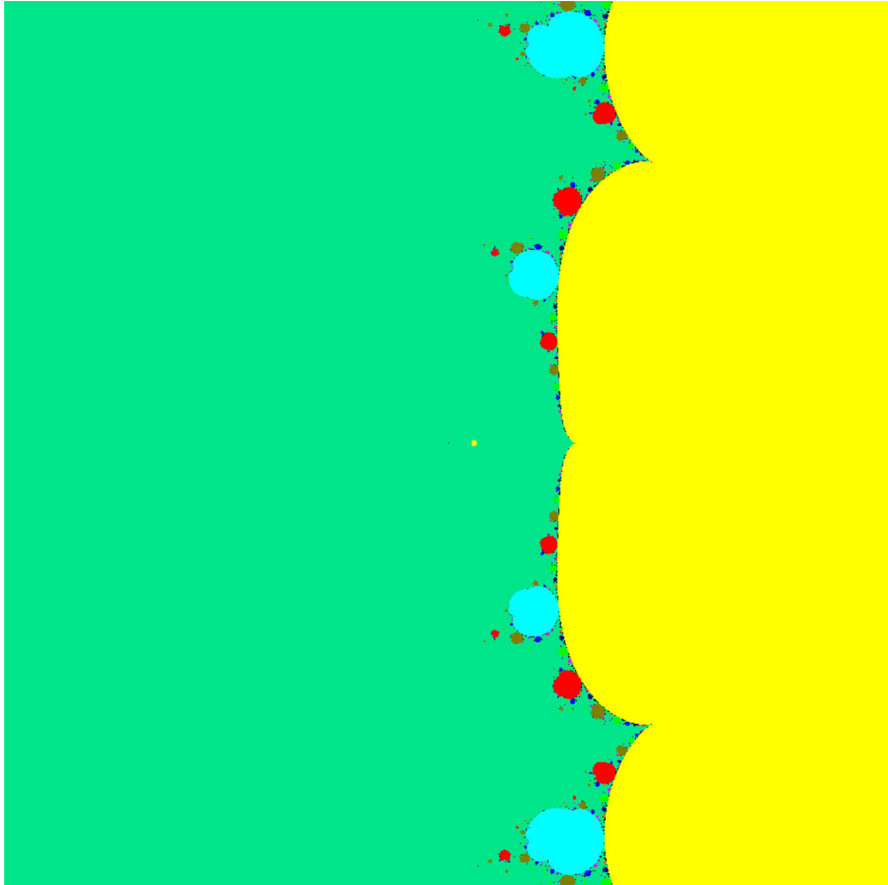


Fig. 1 The λ plane is divided into the shift locus and shell components. The shift locus is shown in green. In the yellow shell component, λ is attracted to a fixed point; in the cyan components, it is attracted to a period two cycle; in the red components, a period three cycle and in the khaki components a period four cycle (color figure online)

For $p \neq 1$, λ^* is finite and has the additional property that $f_{\lambda^*}^{p-1}(\lambda^*) = \infty$; that is, λ^* is a prepole. Since the limit along a path approaching infinity from inside the asymptotic tract of an asymptotic value is the asymptotic value, λ^* , its iterates, where the p^{th} iterate is defined by this limit, form a *virtual cycle*. A parameter with this property is called a *virtual cycle parameter*, and in [16], it is proved that every virtual center parameter on the boundary of a shell component is a virtual cycle parameter.

Remark 1 We note that all the rational boundary points and the virtual center of a shell component Ω_p are accessible boundary points in the sense that the accumulation set of any path inside Ω_p tending to the point consists of a single point.

There is a unique component Ω_1 in which λ is attracted to a fixed point $q_\lambda (\neq 0)$. It is unbounded and, by abuse of language, we say that its virtual center is infinity.

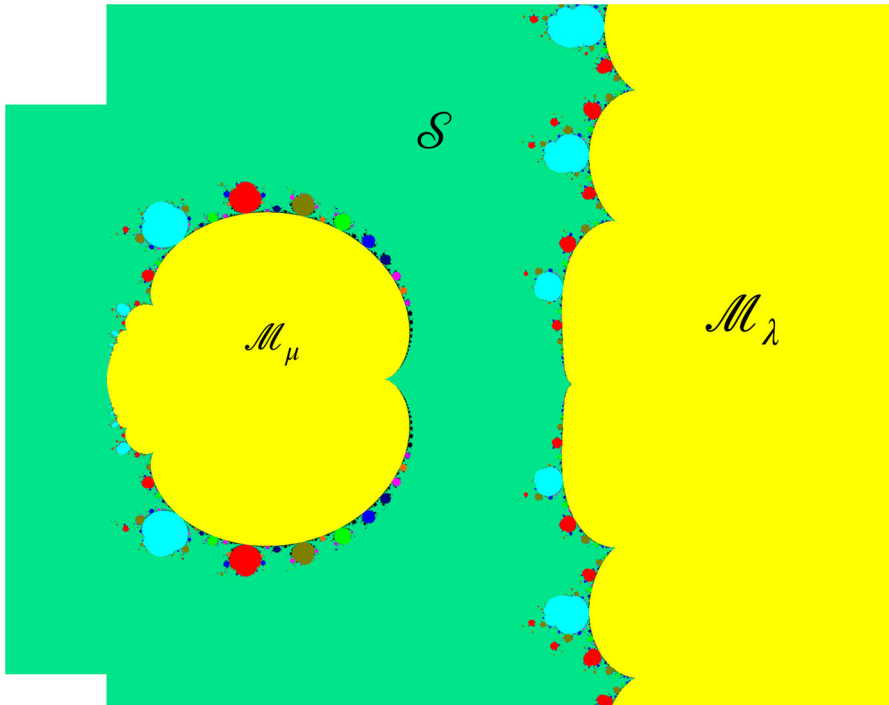


Fig. 2 The λ -plane with a blow-up of the \mathcal{M}_μ region. The color coding is the same as in Fig. 1

3 The Model Map

We fix $\rho = \rho_0, |\rho_0| < 1$, once and for all for this paper and write f_λ for f_{λ, ρ_0} . The figures in the paper were made by taking $\rho_0 = 2/3$.

Let λ_0 be a point in $\Omega_1 \subset \mathcal{M}_\lambda$, such that at the fixed point $q_0 = q(\lambda_0)$ of f_{λ_0} , $f'_{\lambda_0}(q_0) = \rho_0$. We remark that λ_0 is not uniquely defined, because there is a discrete set of preimages of ρ_0 under the multiplier map. Choose one and set $Q(z) = f_{\lambda_0}(z)$; it is our *Model map*.¹ Let K_0 be the immediate attracting basin of q_0 for Q .

Since the orbits of the asymptotic values either tend to q_0 or 0, the map Q is hyperbolic. Its Julia set J_0 is the common boundary of the attracting basins of 0 and q_0 ; it is well known to be equal to the closure of both the set of repelling periodic points of Q and the set of prepoles of Q . It will be convenient to identify both these sets with a space of sequences. To that end, we define the following.

Definition 2 Let $\mathbf{j}_n = j_1 j_2 \dots j_i \dots j_n, j_i \in \mathbb{Z}$ be a sequence of length n whose entries are integers and set $\mathbf{J}_n = \{\mathbf{j}_n\}$; let $\mathbf{j}_\infty = j_1 j_2 \dots j_i \dots j_n \dots, j_i \in \mathbb{Z}$ be a sequence of infinite length whose entries are integers and set $\mathbf{J}_\infty = \{\mathbf{j}_\infty\}$. Let \mathbf{j} denote an element

¹ Note that conjugating by the affine map $w = z - q_0/2$, we obtain a map of the form $\alpha \tan w$ for some $\alpha, |\alpha| > 1$, with fixed points at $\pm q_0/2$. In particular, if ρ_0 is real, λ_0 can be chosen real, and then the attracting basin of q_0 is a half plane and the Julia set is a line.

of either \mathbf{J}_n or \mathbf{J}_∞ . Define the sequence space:

$$\Sigma = \left\{ \mathbf{j} \in \bigcup_n \mathbf{J}_n \cup \mathbf{J}_\infty \cup \{\infty\} \right\}$$

and give it the standard sequence topology. The shift map σ defined by dropping j_1 defines a continuous self-map on Σ .

We will use Σ to label the inverse branches of Q . (See Fig. 4.)

Let l be a curve that joins λ_0 to ∞ in K_0 ; it is an asymptotic curve for λ_0 . Set $l_* = \bigcup_{n \geq 1} Q^n(l)$; this union is a continuous curve with endpoints q_0 and λ_0 . Although it does not matter in this section, we will see in the next section that we can choose l , so that it depends holomorphically on ρ_0 .

Since $\mu_0 \notin \overline{K_0}$, we can define the inverse branches R_j of Q , $j \in \mathbb{Z}$, on $K^* = \overline{K_0} \setminus l$. Note that Q is periodic with period πi , and so, if $q_j = q_0 + j\pi i$, then $Q(q_j) = q_0$; it will be convenient to choose l so that each R_j is defined in a full neighborhood of q_0 and is labeled so that $R_j(q_0) = q_j$. Our figures are computed with $\rho_0 = 2/3$, so that λ_0 is real and l is contained in the real axis.

We can do this by setting:

$$R_j(w) = \frac{1}{2} \text{Log} \left(\frac{w/\mu_0 - 1}{w/\lambda_0 - 1} \right) + \pi i j,$$

where Log stands for the principal branch of the logarithm.

Since Q is periodic, we can speak of adjacent poles in J_0 . Let l^+ and l^- mark the upper and lower edges of the curve l . Then, R_j maps $K_0 \setminus l$ to a strip of width π in K_0 bounded by the curves $R_j(l^-)$ and $R_j(l^+)$, each of which joins a pole in J_0 to infinity; moreover, the poles are adjacent. We label these poles p_j and p_{j+1} , respectively. Specifically, for $w \in K^*$, we set:

$$p_j = \lim_{w \rightarrow \infty \text{ asymptotic to } l^-} R_j(w) = \frac{1}{2} \text{Log} \frac{\lambda_0}{\mu_0} + \pi i j$$

and

$$p_{j+1} = \lim_{w \rightarrow \infty \text{ asymptotic to } l^+} R_j(w) = \frac{1}{2} \text{Log} \frac{\lambda_0}{\mu_0} + \pi i (j + 1).$$

Note that although each pole is defined by two limits, the index of the pole is well defined. (See Fig. 4.)

Taking one-sided limits, we define the images of l under R_j to be the lines:

$$l_j = \{R_j(w) \mid w \in l^-\}.$$

1. Set:

$$R_{\mathbf{j}_n} = R_{j_1 j_2 \dots j_n} = R_{j_1} \circ R_{j_2} \circ \dots \circ R_{j_n}.$$

2. We can enumerate the preimages of the fixed points in the same fashion. We set:

$$q_{\mathbf{j}_n} = q_{j_1 j_2 \dots j_n} = R_{\mathbf{j}_n}(q_0).$$

3. We can enumerate the prepoles. Set:

$$p_{\mathbf{j}_n} = p_{j_1 j_2 \dots j_n} = R_{\mathbf{j}_n}(\infty).$$

This scheme enumerates all the prepoles by assigning each a unique element of Σ . Note that:

$$Q(p_{j_1 j_2 \dots j_n}) = p_{j_2 \dots j_n},$$

so that Q acts as a shift map σ on the sequence.

4. We can also enumerate the repelling periodic points in J_0 . If $z \in J_0$ and $Q^n(z) = z$, we can find branches R_{j_k} , such that:

$$z = R_{j_1 j_2 \dots j_n}(z) = R_{j_1 j_2 \dots j_n} R_{j_1 j_2 \dots j_n}(z) = \dots R_{j_1 j_2 \dots j_n}(z)$$

or

$$z = R_{\mathbf{j}_n}(z) = R_{j_n} R_{\mathbf{j}_n}(z) = \dots R_{j_n} R_{\mathbf{j}_n}(z).$$

That is, we can associate the infinite repeating sequence $\mathbf{j}_\infty = \mathbf{j}_n \mathbf{j}_n \dots \mathbf{j}_n \dots$ to the point and set:

$$z = z_{\mathbf{j}_\infty} = R_{\mathbf{j}_n}(z_{\mathbf{j}_\infty}) = Q^n(z_{\mathbf{j}_\infty}).$$

Again, Q acts as a shift map and Q^n leaves the infinite sequence invariant.

Proposition 1 *There is a homeomorphism from Σ to J_0 , such that for $\mathbf{j} \in \Sigma$ and $z \in J_0$, $Q(z_{\mathbf{j}}) = z_{\sigma(\mathbf{j})}$.*

In [9], using results in [16], we proved that the finite sequence defining the prepole that is the virtual cycle parameter of a component Ω_p of \mathcal{M}_λ uniquely characterizes that component. Thus, the finite sequences in Σ are in one-to-one correspondence with these boundary points of \mathcal{M}_λ .

3.1 A Structure for K_0

There is a linearizing homeomorphism ϕ_0 , defined in a largest neighborhood $\Delta = O_{\lambda_0}$ of q_0 to a disk \mathbb{D}_0 centered at the origin of radius r_0 , such that $\phi_0(q_0) = 0, \phi'_0(q_0) = 1$. Moreover, for $z \in \Delta$, $\phi_0(Q(z)) = \rho_0 \phi_0(z)$, $\lambda_0 \in \partial \Delta$, ϕ_0 extends continuously to the boundary and $r_0 = |\phi_0(\lambda_0)|$. The function $\log |\phi_0|$ is like a Green's function for Δ . The preimages of the circles $|\zeta| = r$ in \mathbb{D}_0 are the *level curves* and the preimages of the radii are the *gradient curves* in Δ . Thus, the level of $\partial \Delta$ is r_0 .

The map ϕ_0 depends holomorphically on ρ_0 . Therefore, a canonical choice for the curve l used to define the branches R_j above can be made using the polar coordinates of \mathbb{D}_0 . For example, if ρ is real, we define $l_* = \phi_0^{-1}(t)$, $t \in [0, r_0) \in \mathbb{D}_0$ and let $l = R_0(l_*) \setminus l_*$. If it is not, we adjust appropriately.

Since $\mu_0 \notin K_0$, the map ϕ_0 can be extended by analytic continuation to all of K_0 . The curves $R_0^{-1}(\partial\Delta)$ have level r_0/ρ_0 and end at infinity. We use the level and gradient curves to define a coordinate structure on K_0 as follows. The coordinates are locally defined on preimages of $A_0 = R_0(\Delta) \setminus (\Delta \cup l)$ and $\mathbf{B}_0 = R_0(\overline{\Delta} \setminus \{\lambda_0\})$.

3.1.1 Fundamental Domains

Definition 3 We say that a region $D \subset K_0$, with interior D_0 , is a *fundamental domain* for the action of Q if:

- for any pair $(z_1, z_2) \in D_0$, $z_1 \neq z_2$, $Q(z_1) \neq Q(z_2)$ and if
- for some integer $n \in \mathbb{Z}$, $\bigcup_{k \geq n} Q^k(D_0) = \overline{\Delta}$.

We now inductively define a set of fundamental domains that defines a partition of K_0 into fundamental domains. Each fundamental domain D_0 will have boundary curves that are identified by Q . In the process, we give an enumeration scheme for these domains. The curves referred to in the description are shown in Fig. 3.

1. Let $\gamma_0^* = \gamma^* = \partial\Delta$. Then, $Q(\Delta) \subset \Delta$ and Δ contains the positive orbit of λ_0 . Set $\gamma_{-n}^* = Q^n(\gamma^*)$, $n = 1, \dots, \infty$. Note that $R_0^n(\gamma_{-n}^*) = \gamma^*$. These curves are nested about q_0 in Δ . Any annulus A_{-n}^* in Δ bounded by γ_{-n}^* and γ_{-n-1}^* is a fundamental domain for Q . In Fig. 3, γ^* is drawn in black and γ_{-1}^* is drawn as a dotted red curve. The annulus A_0^* between them is a fundamental domain.
2. For all $j \in \mathbb{Z}$, set $\mathbf{B}_j = R_j(\overline{\Delta} \setminus \{\lambda_0\})$. In Fig. 3, we see that the domains \mathbf{B}_j , $j \neq 0$, are simply connected and are bounded by dotted doubly infinite black curves $\gamma_j = R_j(\gamma^* \setminus \{\lambda_0\})$; the dotted red curves inside the \mathbf{B}_j are $R_j(\gamma_{-1}^*)$; each \mathbf{B}_j contains the preimage of the fixed point q_j . We single out the boundary curve of the domain \mathbf{B}_0 , $\gamma_0 = R_0(\gamma^* \setminus \{\lambda_0\})$ and color it red, because, as we will see, its preimages are somewhat different from the preimages of the other γ_j 's. Note that \mathbf{B}_0 has a puncture at λ_0 . The annulus $A_0 = \mathbf{B}_0 \setminus (\overline{\Delta} \setminus \{\lambda_0\})$ between γ_0 and γ^* is a fundamental domain, and we will be particularly interested in its preimages. Note that it contains the line l .
3. Next, we denote the preimages of A_0 by $A_{j0} = R_j(A_0)$, $j \in \mathbb{Z}$. To see what they look like, we look at the preimages of its boundary curves. First, consider the boundary curve γ^* : its preimages are the curves $\gamma_j = R_j(\gamma^*)$. The other boundary curve is γ_0 : its preimages are the curves $\gamma_{j0} = R_j(\gamma_0)$; each of these curves, drawn in red in Fig. 3, joins the pair of poles (p_j, p_{j+1}) . Now, consider the preimages of the line l inside A_0 : these are the lines $l_j = R_j(l^-)$ and $l_{j+1} = R_j(l^+) = R_{j+1}(l^-)$ that join the poles p_j and p_{j+1} to infinity; they are drawn in green in Fig. 3. Thus, we see that each A_{j0} is bounded by four curves, the red curves γ_j , γ_{j0} , and the green curves l_j and l_{j+1} .

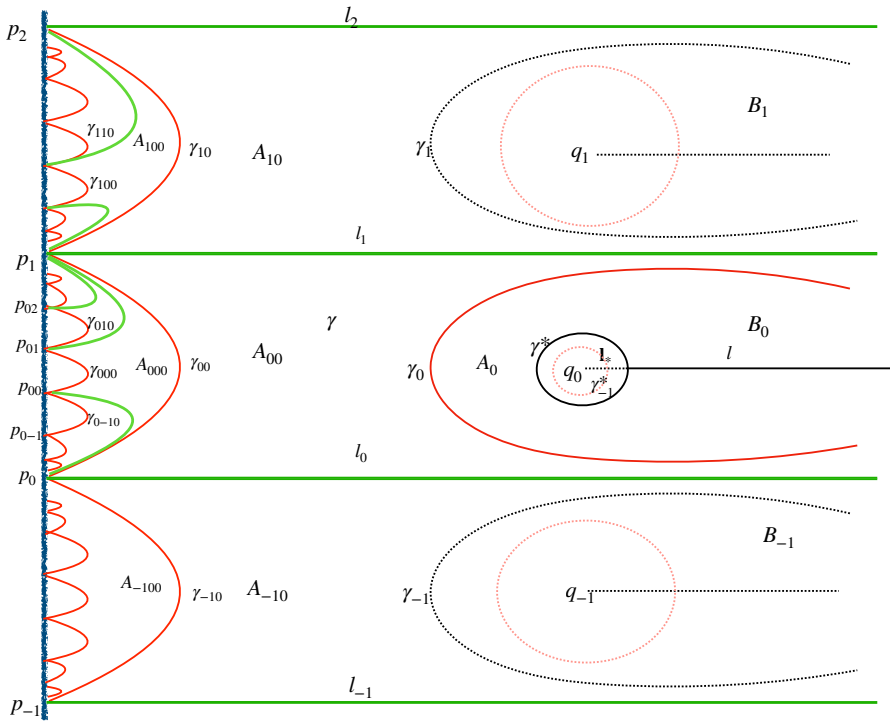


Fig. 3 The model space shown with level (red) and gradient (green) curves. The dotted black curves are also level curves

Each A_{j_0} has three vertices on ∂K_0 , the poles p_j and p_{j+1} where the lines l_j and l_{j+1} meet the ends of γ_{j_0} , and infinity, the common endpoint of the doubly infinite γ_j and an endpoint of each of the lines l_j and l_{j+1} .

Note that if we consider a pole as a prepole of order one, and infinity as a “prepole” of order zero, the red curves labeled by γ_i join a prepole to a prepole of the same order, whereas the green curves labeled by l_i join a prepole of order 0 to a prepole of order 1.

For later use, we set $\mathbf{A}_0 = \overline{\cup_j A_{j_0}}$; it is a simply connected domain.

4. We inductively define the domains $A_{j_1 \dots j_{n-1} 0} = R_{j_1}(A_{j_2 \dots j_{n-1} 0})$. Their boundaries contain two pairs of boundary curves. The pair drawn in red consists of $\gamma_{j_1 \dots j_{n-1} 0}$ which joins adjacent prepoles of order $n - 1$, and $\gamma_{j_2 \dots j_{n-1} 0}$ which joins a prepole of order $n - 2$ to itself if $j_{n-1} \neq 0$ and joins two adjacent prepoles of order $n - 2$ if $j_{n-1} = 0$. The other pair, drawn in green, consists of $l_{j_1 \dots j_{n-2} j_{n-1} 0}$ and $l_{(j_1+1) j_2 \dots j_{n-2} j_{n-1} 0}$, each joining a prepole of order $n - 2$ to a prepole of order $n - 1$. If $j_{n-1} = 0$, all four boundary prepoles are distinct; if not, the prepoles of order $n - 1$ are the same.

The domains A_{000} and A_{100} are shown in Fig. 3. Although not all the curves are labeled, the red curves are preimages of γ_0 and the green curves are preimages of l .

Again, for later use, we define the simply connected domains:

$$\mathbf{A}_{j_1 \dots j_{n-1} 0} = R_{j_1}(\mathbf{A}_{j_2 \dots j_{n-1} 0})$$

for all $n > 1$.

5. Unlike the preimages of γ_0 which join two adjacent poles, the curves $\gamma_{ji} = R_j(\gamma_i)$, $i \neq 0$ are curves that join a pole to itself. This is because of the way we have defined the R_i . The pole p_j is one endpoint of each the curves $\gamma_{(j-1)0}$ and γ_{j0} . The curves γ_{ji} , $i > 0$, are loops that come in to p_j , tangent to, and under γ_{j0} , while the curves $\gamma_{(j-1)i}$, $i < 0$, are loops that come in to p_j , tangent to, and under $\gamma_{(j-1)0}$. Thus, in the preimage, $R_i(K_0)$, the curves γ_{ji} are tangent to the pole p_i for $j \geq 0$ and tangent to the pole p_{i+1} for $j \leq 0$.

Therefore, the loops γ_{ji} , $j \neq 0$, bound simply connected domains $\mathbf{B}_{ji} = R_j(\mathbf{B}_i)$. They are tessellated by the fundamental annuli $B_{ji}^k = R_{ji}(A_{-k}^*)$, $k \geq 0$ and each \mathbf{B}_{ji} contains a curve that is a preimage of the line l . The disjoint domains \mathbf{B}_{ji} form an infinite cluster at each pole. For considerations of space and clarity, we have not included them in the figure.

Note that the fundamental domains B_{ji}^k are analogous to the fundamental domains A_{j0} , whereas the unions of fundamental domains \mathbf{B}_{ji} are analogous to the unions of fundamental domains $\mathbf{A}_{j_{n-1}0}$. We will preserve this analogy and notation in the inductive definition below.

6. We inductively define the domains $\mathbf{B}_{j_n} = R_j(\mathbf{B}_{j_{n-1}})$. Thus, \mathbf{j}_n has the form $j j_{n-2} i$, $i \neq 0$. For each j , if $i > 0$, the \mathbf{B}_{j_n} cluster at the prepole $p_{j_{n-2}i}$, and if $i < 0$, they cluster at the prepole $p_{j_{n-2}(i+1)}$. Each has an outer biinfinite boundary curve, γ_{j_n} , both of whose endpoints are at the same pole and an interior curve l_{j_n} , joining the pole $p_{j_{n-2}i}$ to q_{j_n} . Each of the domains \mathbf{B}_{j_n} is a union of fundamental domains $B_{j_n}^k$.
7. We remark that the admissibility condition for the \mathbf{A}_{j_n} is that the rightmost entry in the sequence \mathbf{j}_n is always zero, while the condition for the \mathbf{B}_{j_n} is that the rightmost entry is never zero. The geometry of these regions is different. The \mathbf{A}_{j_n} have infinitely many vertices at infinitely many distinct prepoles. Each interior fundamental domain has vertices at three or four distinct prepoles. The \mathbf{B}_{j_n} , on the other hand, have only one vertex at a single prepole, or infinity if $n = 1$, where all the boundary curves meet. The fundamental domains contained in them are annuli, the outermost of which has a prepole on its boundary.

3.1.2 The Coordinates

We extend the map ϕ_0 defined above to all of K_0 by analytic continuation. Continuing across the boundary of Δ , we have:

$$\phi_0^{-1}(\{\zeta \mid r_0 \leq |\zeta| \leq r_0/\rho\}) = \overline{A_0}.$$

We use the closure here to include the boundary curves. We extend the map to all of $K_0 \setminus \{\lambda_0\}$ in the obvious way: if z is in a fundamental domain $Q^{-n}(A_0)$ for any n inverse branches, we set $\rho^n \phi_0(z) = \phi_0(Q^n(z))$.

We define coordinates for K_0 in terms of local coordinates in the fundamental domains described above.

1. The curves $\phi_0^{-1}(|\zeta| = r)$ are the *level curves of level r* in K_0 . The level of γ^* is r_0 and the level of γ_j is r_0/ρ .

The outer boundary of each $B_{\mathbf{j}_n}$ has level r_0/ρ^n ; passing through the interior fundamental domains, the levels decrease to zero by powers of ρ^k . In each $A_{\mathbf{j}_n}$, the levels go from r_0/ρ^{n-1} to r_0/ρ^n and the same is true in the interior fundamental domains.

2. The *gradient curves* are preimages of the radii $\theta = \theta_0$ for a fixed $\theta_0 \in \mathbb{R}$ under the map $\phi_0^{-1}(re^{i\theta})$. For example, each of the fundamental domains $A_{\mathbf{j}_n}$ has four boundary curves; one pair of opposite curves, labeled with γ 's and shown in red in Fig. 3, are level curves and the other pair of opposite curves, labeled with l 's and shown in green in Fig. 3, are gradient curves along which the level rises from r_0/ρ^{n-1} to level r_0/ρ^n .
3. The level and gradient curves define a set of *local coordinates in K_0* : the preimages of the circles and radii in \mathbb{D}_0 under the extension of ϕ_0 pull back to each fundamental domain. We denote the coordinates of the point $z \in K_0$ by:

$$z = (X_{\mathbf{j}_n}, r, \theta + \pi(n - 1)),$$

where $X_{\mathbf{j}_n} = \overline{A_{\mathbf{j}_n}}$ if $\mathbf{j}_n = j_1 \dots j_{n-1}0$ and $X_{\mathbf{j}_n} = \overline{B_{\mathbf{j}_n}^k}$ if $\mathbf{j}_n = j_1 \dots j_{n-1}j$, $j \neq 0$. Because k can be read off from $r \in [0, \infty)$, it is enough to write $X_{\mathbf{j}_n}$. We let $\theta \in [-\pi, \pi)$ and note that because the inverse branches are defined as one-sided limits on the boundaries of the fundamental domains, the θ coordinate varies continuously across common boundaries.

Theorem 2 *Every point $z \in K_0$, $z \neq \lambda_0$, is in either a unique $A_{\mathbf{j}_n}$ or a unique $B_{\mathbf{j}_n}^k$ or on the boundary of two such domains: either the common boundary of some $A_{\mathbf{j}_n}$ and some $A_{\mathbf{j}_{n+1}}$ where $\mathbf{j}_{n+1} = \mathbf{j}_n0$, or the common boundary of an $A_{\mathbf{j}_n}$ and a $B_{\mathbf{j}_{n-1}}^0$ where $\mathbf{j}_n = \mathbf{j}_{n-1}j_n$, $j_n \neq 0$ or the common boundary of a $B_{\mathbf{j}_{n-1}}^k$ and a $B_{\mathbf{j}_{n-1}}^{k+1}$.*

Proof Let $z \in K_0$, $z \neq \lambda_0$. Since every such z is attracted to q_0 and has infinitely many preimages, there is an m , such that $Q^m(z) \in \Delta \cup (\partial\Delta \setminus \{\lambda_0\})$. If $\zeta = Q^m(z) \in \Delta$, there is a unique set of preimages R_j , such that $z = R_{\mathbf{j}_m}(\zeta)$. If $\zeta \in (\partial\Delta \setminus \{\lambda_0\})$, the θ

coordinate is defined as a one-sided limit for each preimage, and the limits agree on the boundary curves. □

3.1.3 The Tree in K_0

In this section, we use the domains $\mathbf{A}_{\mathbf{j}_n}$ which are unions of the fundamental domains $A_{\mathbf{j}_n 0}$ to define a tree in K_0 . For readability, we will say a level curve of level r_0/ρ^n has level n .

Among the boundary curves of the union $\mathbf{A}_{\mathbf{j}_n}$, there is a distinguished boundary curve of level n ; it is a preimage of γ_0 under the map $R_{\mathbf{j}_n}$. The remaining infinitely many boundary curves of the union have level $n + 1$ and are images under the maps $R_{\mathbf{j}_n j}$. Note that the non-distinguished boundary curves of $\mathbf{A}_{\mathbf{j}_n}$ of level n are also boundaries of some $\mathbf{B}_{\mathbf{j}_n}$. At level $n = 1$, we will fix a root node on γ_0 . On each $\gamma_{\mathbf{j}_n 0}$ of level n , $n > 1$, we will define the preimage of the root to be a node of the tree. We call these interior nodes. We will also put nodes at every preimage of every order on the tree.

The children of the interior node of level n are the interior nodes of level $n + 1$, a prepole of level n , and infinitely many prepoles of level $n + 1$. We will define branches of the tree that connect a node to each of its children. The nodes that are prepoles have no children and are called leaves of the tree. Each interior node has only one parent and each prepole node has two parents. Paths through the tree start at the root and consist of a connected set of branches joining nodes in the tree. Some will be finite, ending at leaves, and others will be infinite.

For the explicit construction of the tree (see Fig. 4), fix a point x_0^* on the boundary curve γ_0 of the domain A_0 . It has level 1 and is the first node, or root of the tree. For each $n > 1$, and each $\mathbf{j}_n = \mathbf{j}_{n-1}0$, the interior nodes of level n are defined as the points $x_{\mathbf{j}_n 0}^* = R_{\mathbf{j}_n}(x_0^*)$. The prepoles of all orders are leaves of the tree.

The first step of the construction is to define a tree T^* . Join the root x_0^* to each of the nodes $x_{j_0}^*$, $j \in \mathbb{Z}$, by a branch s_j , and join it to the pole p_j by a branch t_j . Also define the segment r_0 of γ_0 from x_0^* to infinity, asymptotic to the line l_0 as a branch joining the root to the “prepole” infinity. The root with these branches connected to the leaves is a small tree T^* contained in \mathbf{A}_0 . Since the domain \mathbf{A}_0 admits a hyperbolic metric, we may choose s_j as geodesics in the hyperbolic metric and take the t_j and r_0 along the level curves. The hyperbolic lengths of the s_j go to infinity with $|j|$, while the lengths of the t_j and r_0 are always infinite.

Now, define a small tree at each node $x_{\mathbf{j}_n 0}^*$, and contained in $\mathbf{A}_{\mathbf{j}_n 0}$, by $T_{\mathbf{j}_n}^* = R_{\mathbf{j}_n}(T^*)$. Note that as often happens, the spatial relationship of the nodes in the tree is dual to the dynamic relationship. The nodes $x_{j_1 k_0}^*$, $k \in \mathbb{Z}$, are children of the node $x_{j_1 0}^*$, whereas the nodes $x_{k j_1 0}^*$ which are preimages of $x_{j_1 0}^*$ are not. Thus, the children of the parent node $x_{\mathbf{j}_n 0}^*$ are the nodes $x_{\mathbf{j}_n k_0}^*$ and not the nodes $x_{k \mathbf{j}_n 0}^*$. The tree has branches $s_{\mathbf{j}_n}$ joining the parent node $x_{\mathbf{j}_n 0}^*$ to its interior children $x_{\mathbf{j}_n k_0}^*$, a branch $r_{\mathbf{j}_n 0}$ joining it to the prepole $p_{\mathbf{j}_n}$ and branches $t_{\mathbf{j}_n k}$ joining it to the prepoles $p_{\mathbf{j}_n k}$. Because the R_j are biholomorphic, the hyperbolic lengths of the branches are preserved.

Finally, joining all these small trees together, we obtain the full tree:

$$T_\infty^* = \bigcup_n \bigcup_{\mathbf{j}_n} (T_{\mathbf{j}_n}^*).$$

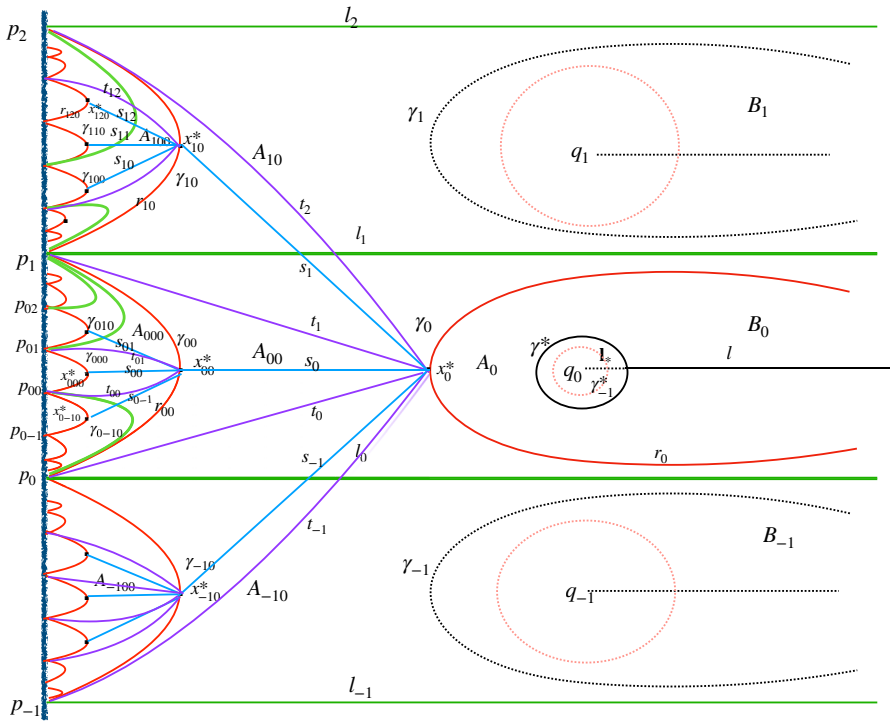


Fig. 4 The fundamental domains of the model space with the tree

If \mathbf{j}_n is periodic with period n , there is a sequence $\mathbf{j}_n = j_1 \dots j_n$, such that $\mathbf{j}_\infty = \mathbf{j}_n \mathbf{j}_n \mathbf{j}_n \dots$. By the periodicity, $j_{1+n} = j_1$, so that the node $x_{\mathbf{j}_n \mathbf{j}_n 0}^*$ is a direct descendant of the node $x_{j_1 0}^*$. This means that there is a path $\tau_{\mathbf{j}_n} = s_{j_1} \cup s_{j_1 j_2} \cup \dots \cup s_{j_1 \dots j_{n-1}}$ from the root to the node. It has finite hyperbolic length. Note, however, as we remarked above, applying R_{j_i} to successively obtain the nodes $x_{j_n 0}^*, x_{j_{n-1} j_n 0}^*, \dots, x_{j_2 \dots j_{n-1} j_n 0}^*$, yields a collection of nodes that are not direct descendants of $x_{j_1 0}^*$.

The path

$$\tau_{\mathbf{j}_n \mathbf{j}_n} = R_{\mathbf{j}_n}(\tau_{\mathbf{j}_n}) = R_{\mathbf{j}_n}(s_{j_1}) \cup R_{\mathbf{j}_n}(s_{j_1 j_2}) \cup \dots \cup R_{\mathbf{j}_n}(s_{j_1 \dots j_{n-1}})$$

joins $x_{j_1 0}^*$ to $x_{\mathbf{j}_n \mathbf{j}_n 0}^*$ and has the same hyperbolic length as $\tau_{\mathbf{j}_n}$. Iterating, we obtain a path $\tau_{\mathbf{j}_\infty}$ of infinite length that is invariant under $R_{\mathbf{j}_n}$.

All but the first segment of this path are separated from the root by the level curve $\gamma_{j_1 0}$, so any accumulation point of the path lies in the Julia set between the poles p_{j_1} and p_{j_1+1} . Thus, the path remains inside a compact domain bounded by $\gamma_{j_1 0}$ and the Julia set boundary of K_0 . This implies that the Euclidean lengths of subpaths making up $\tau_{\mathbf{j}_n}$ tend to zero, and since Q is hyperbolic, that the full path has a unique endpoint.

Thus, if \mathbf{j}_n is periodic of period n , the path in the tree with unique endpoint $z_{\mathbf{j}_\infty}$ is invariant under Q^n and corresponds to a repelling periodic point of Q in J_0 whose combinatorics agree with those defined in Proposition 1.

If \mathbf{j}_∞ is preperiodic, $\mathbf{j} = \mathbf{j}_m \mathbf{j}_n \mathbf{j}_m \mathbf{j}_n \dots$, we can construct a periodic infinite subpath of $\tau_{\mathbf{j}_\infty}$ beginning at $x_{\mathbf{j}_m}^* = R_{\mathbf{j}_m}(x_0^*)$, instead of the root, so that it is invariant under $R_{\mathbf{j}_n}$. The argument above shows that it also has a unique preperiodic endpoint.

4 The Shift Locus

In the shift locus \mathcal{S} , both asymptotic values are attracted to the origin. If $\lambda \in \mathcal{S}$, we can define a linearizing map ϕ_λ from the attractive basin A_λ of the origin to the disk \mathbb{D}_0 that is injective on a neighborhood O_λ of the origin. Neither λ nor μ lies in O_λ and one or both lie on ∂O_λ .

We divide \mathcal{S} into disjoint subsets as follows:

$$\begin{aligned} \mathcal{S}_\lambda^0 &= \{\lambda \in \mathcal{S} \mid \mu \in \partial O_\lambda, \lambda \notin \partial O_\lambda\}, \\ \mathcal{S}_\mu^0 &= \{\lambda \in \mathcal{S} \mid \lambda \in \partial O_\lambda, \mu \notin \partial O_\lambda\}, \quad \text{and} \\ \mathcal{S}_* &= \{\lambda \in \mathcal{S} \mid \lambda \in \partial O_\lambda, \mu \in \partial O_\lambda\}, \end{aligned}$$

We normalize the map, so that if z_0 equal to the asymptotic value on the boundary and $z \in O_\lambda$, then:

$$\phi_\lambda(0) = 0, \phi_\lambda(z_0) = \phi_0(\lambda_0) = r_0, \text{ and } \phi_\lambda(f_\lambda(z)) = \rho\phi_\lambda(z).$$

Note that this normalization agrees with our normalization of ϕ_0 , the linearizing map for the model Q ; that is, both map the asymptotic value to the point r_0 on the real axis.

We restrict our discussion here to \mathcal{S}_λ^0 , but there is a comparable discussion for \mathcal{S}_μ^0 .

4.1 Coordinates in the Dynamic Plane A_λ

The scheme we defined above for tessellating the attracting basin K_0 of $Q = f_{\lambda_0}$ works equally well in a subdomain of the attractive basin of zero, A_λ , for $\lambda \in \mathcal{S}_\lambda^0$.

Theorem 3 *Given $\lambda \in \mathcal{S}_\lambda^0$, there is a coordinate structure defined on a subdomain \widehat{A}_λ of A_λ . More precisely, there is an integer N , such that the basin of the origin of f_λ , A_λ , contains a subdomain \widehat{A}_λ tessellated by fundamental domains $\alpha_{\lambda, \mathbf{j}_{n-1}0}$ and $\beta_{\lambda, \mathbf{j}_{n-1}i}^k$, $i \neq 0, k \geq 0$ and $n \leq N$. The boundary curves of these regions are level and gradient curves defined using a normalized linearizing function ϕ_λ near the origin, and pulling back a radius and circle containing $\phi_\lambda(\mu)$ to A_λ . The geometric properties of the $\alpha_{\lambda, \mathbf{j}_{n-1}0}$ and $\beta_{\lambda, \mathbf{j}_{n-1}i}^k$ are analogous to those of the fundamental domains $A_{\mathbf{j}_{n-1}0, \lambda}$ and $B_{\mathbf{j}_{n-1}i, \lambda}^k$ in K_0 . The coordinates in \widehat{A}_λ are $(\sigma_{\mathbf{j}_n}, r, \theta + \pi(n-1))$ where $r \in [0, \infty)$, $\theta \in [-\pi, \pi)$ and $\sigma_{\mathbf{j}_n} = \sigma_{\mathbf{j}_{n-1}i}$ stands for $\alpha_{\lambda, \mathbf{j}_{n-1}0}$ if $i = 0$ and $\beta_{\lambda, \mathbf{j}_{n-1}i}^k$ for some k depending on r for $i \neq 0$.*

Proof For $\lambda \in \mathcal{S}_\lambda^0$, the attractive basin of zero, A_λ , contains both asymptotic values. By definition, the linearizing map ϕ_λ is a homeomorphism from O_λ , an open neighborhood of the origin with μ on its boundary, onto the disk \mathbb{D}_0 of radius r_0 . It is normalized, so

that $\phi_\lambda(0) = 0$ and $\phi_\lambda(\mu) = \phi_0(\lambda_0) = r_0$. Extending ϕ_λ by analytic continuation, the analogues of the domains $A_{\mathbf{j}_n}$ and $B_{\mathbf{j}_n}^k$ can be defined as in Sect. 3.1.2 until, for some $n = N$, one of them contains λ . That is, $r_0/\rho^{N-1} < |\phi_\lambda(\lambda)| \leq r_0/\rho^N$. The level and gradient curves are well defined in these domains by the branches of ϕ_λ^{-1} .

To this end, we use the map $\xi_\lambda = \phi_0^{-1} \circ \phi_\lambda : O_\lambda \rightarrow O_{\lambda_0}$ that we defined in [9]. The inverse map, ξ_λ^{-1} , extends, as a homeomorphism from a subset, $K_0(\lambda)$ to a largest subset \widehat{A}_λ of A_λ that contains λ . Therefore, ξ_λ^{-1} is defined on the fundamental domains $A_{\mathbf{j}_n}$ and $B_{\mathbf{j}_n}^k$ tessellating K_0 , $k \geq 0$, $n \leq N$, where N is the largest integer, such that $|\phi_\lambda(\lambda)| \leq r_0/\rho^N$.

Set $\alpha_{\lambda, \mathbf{j}_n} = \xi_\lambda^{-1}(A_{\mathbf{j}_n})$ and $\beta_{\lambda, \mathbf{j}_n}^k = \xi_\lambda^{-1}(B_{\mathbf{j}_n}^k)$. The boundary curves of these domains are, by definition, level and gradient curves for A_λ and the relative levels correspond, via the map ξ_λ^{-1} , to the levels of the corresponding curves in K_0 . Moreover, since we can define inverse branches $R_{\lambda, j}$ of f_λ on \widehat{A}_λ using the relation $R_{\lambda, j} = \xi_\lambda \circ R_j \circ \xi_\lambda^{-1}$, the indexing is consistent with the model. Thus, we obtain a coordinate $(\sigma_{\mathbf{j}_n}, r, \theta + \pi(n - 1))$ for \widehat{A}_λ . □

We can also use the map ξ_λ^{-1} to obtain a tree in \widehat{A}_λ , $T_\lambda^* = \xi_\lambda^{-1}(T_\infty \cap K_0^\lambda)$. The root of this tree is $x_\lambda^* = \xi_\lambda^{-1}(x_0^*)$. Its nodes are defined similarly. Note that some of images of infinite paths in T_∞ are truncated and so are finite in T_λ^* .

4.2 Coordinates in \mathcal{S}_λ^0

In [9], we proved.

Theorem 4 *There is a homeomorphism $E : \mathcal{S}_\lambda^0 \rightarrow K_0 \setminus \overline{\Delta}$. Thus, \mathcal{S}_λ^0 is homeomorphic to an annulus \mathbb{A} . If I is the inner boundary of \mathbb{A} , $I = \partial\Delta$ and E^{-1} extends continuously to all points on I except λ_0 . The point $E^{-1}(\lambda_0)$ corresponds to the parameter singularity $\lambda = 0$ on the inner boundary of \mathcal{S}_λ^0 where the function f_λ is not defined. The outer boundary of $E^{-1}(\mathbb{A})$ is contained in $\partial\mathcal{M}_\lambda$ and contains all the virtual centers.*

To define the map E , we use the maps ϕ_0 , ϕ_λ , and $\xi_\lambda = \phi_0^{-1} \circ \phi_\lambda$, and set $E(\lambda) = \xi_\lambda(\lambda)$. It is not difficult to prove the map is injective. We then prove that the map is a homeomorphism by the following construction: to each $\zeta \in K_0 \setminus \Delta$, $\zeta \neq \lambda_0$, we inductively construct a sequence of covering spaces of $K_0 \setminus \{\lambda_0, \zeta\}$ and corresponding covering maps. Using quasiconformal surgery, we prove that the direct limit of this process is a map in \mathcal{S}_λ^0 .

The inverse holomorphic homeomorphism E^{-1} can be used to define a tessellation and coordinates in \mathcal{S}_λ^0 . For each sequence \mathbf{j}_n , define $\mathcal{A}_{\mathbf{j}_n} = E^{-1}(A_{\mathbf{j}_{n-10}})$ and $\mathcal{B}_{\mathbf{j}_n}^k = E^{-1}(B_{\mathbf{j}_{n-1}^k})$, $i \neq 0$, $k \geq 0$. This identification immediately gives us (see Fig. 5),

Theorem 5 *Each point $\lambda \in \mathcal{S}_\lambda^0$ has a unique coordinate $\lambda = (\mathcal{X}_{\mathbf{j}_n}, r, \theta)$ where $\mathcal{X}_{\mathbf{j}_n}$ is either $\mathcal{A}_{\mathbf{j}_n}$ or $\mathcal{B}_{\mathbf{j}_n}^k$, $r \in [0, \infty)$, $\theta = t + (n - 1)\pi \in \mathbb{R}$, $0 \leq t < \pi$.*

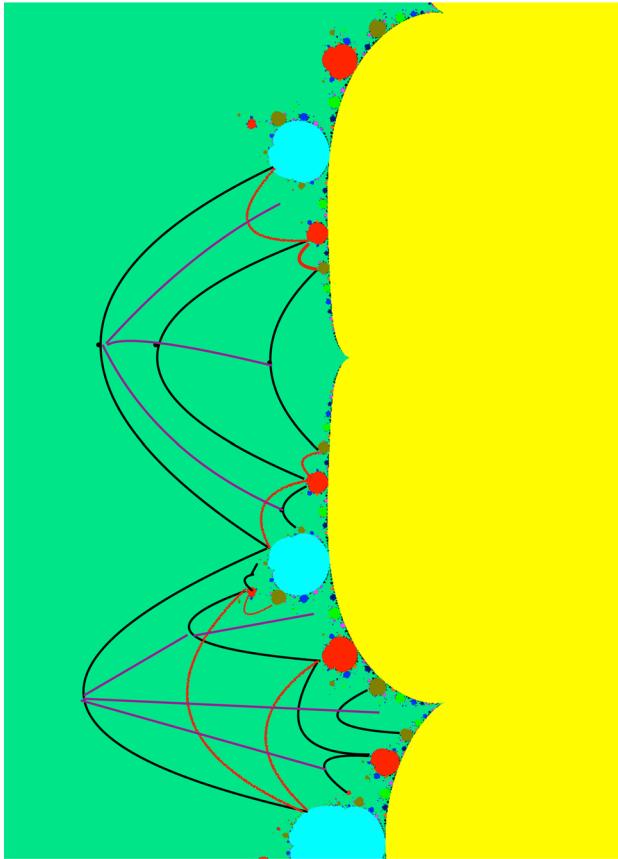


Fig. 5 The parameter space with some of the level curves in black, and gradient curves in red and paths from the tree in purple

5 The Boundaries of K_0 and \mathcal{M}_λ

We are now ready to prove our main result.

Theorem 6 *The injective holomorphic map $E : \mathcal{S}_\lambda^0 \rightarrow K_0 \setminus \overline{\Delta}$ extends continuously to the virtual centers of $\partial \mathcal{S}_\lambda^0$ and maps them to prepoles of Q with the same itinerary.*

Proof Fix a finite sequence \mathbf{j}_n , and let $\tau(t), t \in [0, 1]$ be a path in the tree T_∞ that ends at the prepole $p_{\mathbf{j}_n}$. That is, it passes from the root to the node $x_{\mathbf{j}_n,0}^*$ and its last branch $r_{\mathbf{j}_n,0}$ goes from $x_{\mathbf{j}_n,0}^*$ to the prepole $p_{\mathbf{j}_n}$ along the level curve $\gamma_{\mathbf{j}_n,0}$. The map E^{-1} then maps $\tau(t), t \in [0, 1)$, to a path $\lambda(t) \in \mathcal{S}_\lambda^0$.

We claim that the accumulation set of $\lambda(t)$ as t goes to 1 is a single point and that this point is a virtual cycle parameter.

Let $\lambda_\infty \in \partial \mathcal{S}_\lambda^0$ be an accumulation point of $\lambda(t)$ as t goes to 1 and let t_m be sequence tending to 1, such that $\lambda_m = \lambda(t_m)$ has limit λ_∞ . Since we are only interested in $\tau(t)$

for t close to 1, we may assume that all the points $\tau_m = \tau(t_m)$ belong to the last edge $t_{\mathbf{j}_n}$.

Note that the attractive basins A_λ of f_λ and the boundary curves defining their tessellations by fundamental domains $\alpha_{\lambda, \mathbf{j}_{n-1}0}$ and $\beta_{\lambda, \mathbf{j}_{n-1}i}^k$, $i \neq 0$, in $\widehat{A}_\lambda \subset A_\lambda$ all move holomorphically with λ .

In particular, the unions of these domains $\alpha_{\lambda, \mathbf{j}_{n-1}0} = \xi_\lambda^{-1}(\mathbf{A}_{\mathbf{j}_{n-1}0})$ and $\beta_{\lambda, \mathbf{j}_{n-1}i} = \xi_\lambda^{-1}(\mathbf{B}_{\mathbf{j}_{n-1}i})$, $i \neq 0$, and their prepole boundary points, including the prepole $p_{\lambda, \mathbf{j}_n}$, move holomorphically. Thus, as m goes to infinity, the functions f_{λ_m} converge to f_{λ_∞} and the prepoles $p_{\lambda_m, \mathbf{j}}$ converge to a prepole $p_{\lambda_\infty, \mathbf{j}_n}$ of f_{λ_∞} . Moreover, τ_m is on a level curve in K_0 , and the images under ξ_λ of the level curves in K_0 are level curves in A_λ , so each $\lambda_m = \xi_{\lambda_m}(\tau_m)$ is on a level curve of the same level. The level curves in A_{λ_m} containing λ_m have endpoints at prepoles, so that $\lim_{m \rightarrow \infty} \xi_{\lambda_m}(\tau_m) = p_{\lambda_\infty, \mathbf{j}_n}$. Therefore, either $\lambda_\infty \in A_{\lambda_\infty}$, so that $\lambda_\infty \in \mathcal{S}_\lambda^0$, or:

$$|\lambda_m - p_{\lambda_\infty, \mathbf{j}_n}| \leq |\lambda_m - p_{\lambda_m, \mathbf{j}_n}| + |p_{\lambda_m, \mathbf{j}_n} - p_{\lambda_\infty, \mathbf{j}_n}| \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

The first possibility cannot happen, since we assumed $\lambda_\infty \notin \mathcal{S}_\lambda^0$. The second says that λ_∞ is a virtual cycle parameter. Since the sequence t_m was arbitrary and the prepoles of any given order form a discrete set, the limit is independent of the sequence and thus unique. □

We turn now to the periodic points in the Julia set of Q and show that the map E^{-1} extends to them. The proof is similar to the above.

Theorem 7 *The injective holomorphic map $E^{-1} : K_0 \setminus \overline{\Delta} \rightarrow \mathcal{S}_\lambda^0$ extends continuously to the repelling periodic points in ∂K_0 and maps them to points in $\lambda \in \partial \mathcal{S}_\lambda^0$ for which f_λ has a parabolic cycle of the same period.*

Proof Let $\mathbf{j}_\infty = \mathbf{j}_n \mathbf{j}_n \dots$ be a periodic infinite sequence and let $z_{\mathbf{j}_\infty} \in \partial K_0$ be the repelling point of order n in the Julia set of Q corresponding to this sequence. Let $\tau_{\mathbf{j}_\infty}(t)$, $t \in [0, 1)$ be the infinite path in T_∞ corresponding to the sequence. It is invariant under Q^n and, since Q is hyperbolic, its endpoint in J_0 is well defined and is the repelling periodic point $z_{\mathbf{j}}$.

Let $\lambda(t) = E^{-1}(\tau_{\mathbf{j}_\infty}(t))$. We claim this path lands on $\partial \mathcal{S}_\lambda^0$ as t goes to 1. Let λ_∞ be any point in the accumulation set of the path and let t_m be a sequence tending to 1, such that $\lambda_m = \lambda(t_m)$ has limit λ_∞ .

For each m , there is an integer $k(m)$, such that if $\mathbf{j}_{k(m)}$ is a truncation of the periodic sequence \mathbf{j}_∞ after $k(m)$ repetitions of \mathbf{j}_n , $\lambda_m \in A_{\mathbf{j}_{k(m)}} \subset \mathcal{S}_\lambda^0$. This means that we also have $\lambda_m \in \alpha_{\lambda_m, \mathbf{j}_{k(m)}} \subset \widehat{A}_{\lambda_m}$ and $\xi_{\lambda_m}(\lambda_m) \in \mathbf{A}_{\mathbf{j}_{k(m)}} \subset K_0$. Let $\widehat{\tau}_{\mathbf{j}_{k(m)}} \in T_\infty$ be the tree $\tau_{\mathbf{j}_\infty}$ in K_0 up to the node $x_{\mathbf{j}_{k(m)}0}^*$, and having as its final branch, $r_{\mathbf{j}_{k(m)}}$, the level curve from the node to the prepole $p_{\mathbf{j}_{k(m)}}$. The last fundamental domain it passes through is $\mathbf{A}_{\mathbf{j}_{k(m)}}$.

Using the map ξ_λ^{-1} , we can pull back $\widehat{\tau}_{\mathbf{j}_{k(m)}}$ to a tree $\widehat{\tau}_{\lambda_m, \mathbf{j}_{k(m)}} \subset \widehat{A}_{\lambda_m}$. The last fundamental domain it passes through is $\alpha_{\mathbf{j}_{k(m)}}$ and this fundamental domain contains λ_m . We can modify the branch of $\widehat{\tau}_{\lambda_m, \mathbf{j}_{k(m)}}$ in $\alpha_{\mathbf{j}_{k(m)}}$, so that it passes through λ_m . We will do this, and by abuse of notation, denote the modified tree by $\widehat{\tau}_{\lambda_m, \mathbf{j}_{k(m)}}$ again.

Everything is holomorphic in λ , and as k goes to infinity, $\mathbf{j}_{k(m)} \rightarrow \mathbf{j}_\infty$, so the prepoles $p_{\mathbf{j}_{k(m)}} \in J_0$ tend to the repelling periodic point $z_{\mathbf{j}_\infty} \in J_0$. It follows from the sequence topology that the prepoles $p_{\mathbf{j}_{k(m)}, \lambda_m}$ tend to the repelling periodic point $z_{\lambda_m, \mathbf{j}_\infty}$ and the repelling periodic points $z_{\lambda_m, \mathbf{j}_\infty}$ tend to $z_{\lambda_\infty, \mathbf{j}_\infty}$. This must be a repelling or parabolic periodic point of f_{λ_∞} . It cannot be the point λ_∞ , because an asymptotic value of f_λ cannot be periodic.

We claim that $z_{\lambda_\infty, \mathbf{j}_\infty}$ must be a parabolic periodic point of f_{λ_∞} . We first show that it must be a neutral periodic point. Suppose $z_{\lambda_\infty, \mathbf{j}_\infty}$ is a repelling periodic point. Then, there is a neighborhood U containing λ_∞ , such that $z_{\lambda, \mathbf{j}_\infty}$ is repelling for all $\lambda \in U \cap S_\lambda^0$. In particular, it contains λ_m for large enough m . Then, for each such m , we modify $\widehat{\tau}_{\lambda_m, \mathbf{j}_{k(m)}}$ by changing its last branch. We do this by replacing $r_{\mathbf{j}_{k(m)}} \in \widehat{\tau}_{\mathbf{j}_{k(m)}}$ with a path in K_0 , monotonic increasing with respect to level, and ending at $z_{\mathbf{j}_\infty}$. We call the result $\widehat{\widehat{\tau}}_{\mathbf{j}_{k(m)}}$. Then, $\xi_{\lambda_m}^{-1}(\widehat{\widehat{\tau}}_{\mathbf{j}_{k(m)}})$ is a path in A_{λ_m} ending at the repelling periodic point $z_{\lambda_m, \mathbf{j}_\infty}$. Again, as m goes to infinity, the $\widehat{\widehat{\tau}}_{\mathbf{j}_{k(m)}}$'s converge to a path in \mathbb{C} with endpoint $z_{\lambda_\infty, \mathbf{j}_\infty}$, the periodic endpoint of f_{λ_∞} . If λ_m were a point on $\xi_{\lambda_m}^{-1}(\widehat{\widehat{\tau}}_{\mathbf{j}_{k(m)}})$, the λ_m 's would either converge to a point in A_{λ_∞} or to a repelling periodic point of f_{λ_∞} . The first case cannot happen, since λ_∞ is not an interior point of S_λ^0 , and the second cannot happen, since λ_∞ cannot be periodic.

Therefore, the fixed point $z_{\lambda_\infty, \mathbf{j}_\infty}$ is neutral. A standard application of the Snail Lemma [22, p.154] shows that it must be parabolic. \square

As a corollary of the proof of this theorem, it follows that the injective homeomorphism $E^{-1} : K_0 \setminus \overline{\Delta} \rightarrow S_\lambda^0$ extends continuously to the eventually periodic points in ∂K_0 and maps them to points in $\lambda \in \partial S_\lambda^0$. Because λ_∞ does not belong to the cycle containing $z_{\mathbf{j}_\infty, \lambda_\infty}$, but maps onto it in finitely many steps, and λ_∞ does belong to the Julia set, the cycle is repelling. This, together with Theorem 6 and Theorem 7, completes the proof of the Main Theorem in the introduction.

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