



Near Parabolic Renormalization for Unicritical Holomorphic Maps

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Abstract

Inou and Shishikura provided a class of maps that is invariant by near-parabolic renormalization, and that has proved extremely useful in the study of the dynamics of quadratic polynomials. We provide here another construction, using more general arguments. This will allow to extend the range of applications to unicritical polynomials of all degrees.

Summary of Notations

$\cdots[f]$	Used to emphasize the dependence on f of a given object
A	Immediate basin of the parabolic point of $f \in \mathcal{F}$
B_d	A unicritical Blaschke product with a parabolic point at $z = 1$
\widehat{B}_d	Another normalization of B_d
b_*	The U_1 -box that contains a punctured neighborhood of the origin
β_t	Constant so that $\Phi_t(z) = \Phi_{\text{attr}}[f_t](z) + \beta_t$ has a critical value independent of t ; $\beta_t = \sigma_d - \Phi_{\text{attr}}[f_t](v_t)$
\mathfrak{B}	The main dynamical upper chessboard box of A
C	A curve through the orbit of the critical value, Proposition 36
C	Main object of study of Sect. 3.7
C_d	A semiconjugate of B_d , so that the parabolic point has only one attracting petal
\mathbb{D}	The open unit disk in \mathbb{C}
d_1	Infimum over \mathcal{F} of some hyperbolic distance
$\text{Dom}(f)$	Domain of definition of the map f
d_U	Hyperbolic distance w.r.t. U
E	$E(z) = e^{2i\pi z}$

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f_0	An element of \mathcal{F}
\mathcal{F}	Shishikura's invariant class
\mathcal{F}_ε	A class of maps with slightly less structure
f_t	A deformation of f_0 , element of \mathcal{F}_t
\mathbb{H}	The upper half plane in \mathbb{C}
$\mathcal{H}[f]$	The horn map of f , semi-conjugated by E
$h[f]$	Normalized extended horn maps, $h[f] = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$
$\ell(x)$	The hyperbolic distance from 0 to x in \mathbb{D}
$\lambda[f_0](z)$	Some contraction factor in W_0
\odot	$U \odot r$ is the set of points $z \in U$ with $d_U(0, z) < d_{\mathbb{D}}(0, r)$
\Vdash	$V \Vdash r$ is the set of points $z \in V$ with $E(z) \in E(V) \odot r$
$PC(f)$	The post critical set of f
$\mathcal{R}[f]$	The upper parabolic renormalization of f
r_2	Defined in Proposition 79
ρ_U	Element of hyperbolic metric w.r.t. U
\mathcal{S}	Parabolic renormalization invariant class with the full structure
σ_d	Complex number chosen so that $E(\sigma_d)$ is the critical value of $\mathcal{R}[B_d]$
σ_t	A motion appearing in the decomposition $f_t = f_0 \circ \sigma_t$
\mathcal{SL}	The class of Schlicht maps
T_0	For $f \in \mathcal{F}_{[0, T_0[}$ have a (unique) critical value
T_1	$z \mapsto z + 1$
T'_1	For $f \in \mathcal{F}_{[0, T'_1]}$, the critical value is attracted to 0
T_3	Some parameter in Lemma 67, later chosen to be $= T'_1/2$
T_5	$\exists! T_5 \in]0, 1[$ s.t. $\ell(1 - T_5) = d_1(\delta) + \ell(1 - \varepsilon')$
U_1	Domain of $f \in \mathcal{F}$
U_u	Upper component of $\Psi_{\text{rep}}^{-1}(A)$, also of $\text{Dom}(h[f])$
$V_\delta[f]$	The δ -neighborhood of $PC(f)$
$\hat{V}_\eta[f]$	Some domain used in the proofs
W_0	The complement in \mathbb{C} of the closure of the post critical set of f_0
W'_0	$W'_0 = f_0^{-1}(W_0)$
Φ_{attr}	Attracting Fatou coordinates; normalized and extended except at the beginning of Appendix A; normalized by the expansion at infinity in Sect. 3
Ψ_{rep}	Repelling inverse Fatou coordinates; same remarks as for Φ_{attr} apply
Φ_t	$\Phi_t = \Phi_{\text{attr}}[f_t] + \beta_t$ with β_t a constant so that the critical value of Φ_t is independent of t
Ψ_t	$\Psi_t(z) = \Psi_{\text{rep}}[f_t](z - \beta'_t)$ for $\beta'_t = \beta_t - i\pi\gamma[f_t]$ with γ the iterative residue
$\Omega_\theta(R)$	Some domain in the coordinates $u = -1/c_f z$, extending the half plane on which we control the Fatou coordinates

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We give here a brief summary of frequently used notations: \mathbb{D} refers to the unit disk in the complex plane and \mathbb{H} to the upper half plane. The translation by 1 in \mathbb{C} is denoted by $T_1 : z \mapsto z + 1$. By convention, \mathbb{N} includes 0 and \mathbb{N}^* denotes the set of positive integers. We will often make use of the map $E(z) = e^{2i\pi z}$. We adopt the following convention for open and semi-open intervals: $]a, b[$, $[a, b[$, $]a, b]$. The notation \mathcal{SL} refers to the class of Schlicht maps, i.e. holomorphic injective maps $\phi : \mathbb{D} \rightarrow \mathbb{C}$ with $\phi(0) = 0$ and $\phi'(0) = 1$. There are a lot of more specific notations in this article, and a (partial) summary of symbols has been included before the table of contents.

1 Introduction

This article has a long introduction and the main theorem appears only on page 9.

1.1 Structure of the Paper

In Sect. 1, we sum up the structure of the paper (the present section); we give a definition of a notion we call *structural equivalence* and which, though not strictly necessary, philosophically underlines this work (Sect. 1.2); for maps with a structure that is complete in some sense ($f \in \mathcal{S}_d$), we recall the classical results of Shishikura, Lanford and Yampolsky: universality and renormalization invariant class and restate renormalization invariance using the language of structural equivalence (Sect. 1.3); we restate the Inou–Shishikura theorem (Sect. 1.4) using this language; finally, we state the main theorem (Sect. 1.5).

In Sect. 2, we explain how to understand and visualize the structure of horn maps and other objects associated with parabolic points, using the chessboard graphs. Some of these chessboards are invariant by the dynamics, and we call them *dynamical chessboards*. The others are called *structural chessboards*. These chessboards are a crucial tool in the proof of the main theorem. By universality, to describe the structure of objects associated with maps $f \in \mathcal{S}_d$ we can focus on a particular map. For this, we choose Blaschke products (Sects. 2.1, 2.2). Chessboards are introduced in Sect. 2.3. In Sect. 2.4, we describe and show a computer-generated image of the structure associated with the Inou–Shishikura class. This section shows many illustrative pictures.

In Sect. 3, we prove the main theorem. The titles of its sub-sections are self explanatory and we refer the reader to Sect. 3.2 for an overview of the proof.

In Appendix A, we fix notations and conventions for the objects associated with parabolic points: Fatou coordinates, horn maps, normalizations, parabolic renormalizations, iterative residue, etc. In Appendix B, we motivate the study of horn maps by giving examples of what they can be used for. In Appendix C, we recall the definition and elementary facts, for singular values.

1.2 Structural Equivalence

In the breakthrough by Inou and Shishikura [19], they make use of a class of maps defined as follows (notations and details may differ): \mathcal{F}_{IS} is the set of maps of the form $f = P \circ \phi^{-1}$, where ϕ varies among the univalent maps on V such that $\phi(z) = z + \mathcal{O}(z^2)$ at the origin. Here, $P(z) = z(1+z)^2$ and V is a specific open subset of \mathbb{C} containing 0 defined in their article. The set \mathcal{F}_{IS} is better thought of as the set of maps that cover the plane in a specific way, and with $f(z) = z + \mathcal{O}(z^2)$. They are not covers because they have ramification points. And they are not even ramified covers, because the cardinality of the preimage of a point is not constant, even when counted with multiplicity. This class \mathcal{F}_{IS} comes in fact from another class of maps, invariant by parabolic renormalization (defined later in this section), with a much richer ramified cover structure, but which was too rigid for their purposes, which was to have a class invariant by *near* parabolic renormalization. They extracted a carefully chosen subset of this structure to define their class \mathcal{F}_{IS} .

We are going to use the same idea, but we will keep more of the original ramified cover structure. Let us formalize the notion of structure:

Definition 1 Let X_1, X_2, Y be dimension one analytic manifolds. Consider an index set I , and two collections of marked points $a_i \in X_1$ and $b_i \in X_2$ indexed by $i \in I$. We denote $a : I \rightarrow X_1$ and $b : I \rightarrow X_2$ defined by $a(i) = a_i$ and $b(i) = b_i$. Consider also two analytic maps which are nowhere locally constant $f_1 : X_1 \rightarrow Y$ and $f_2 : X_2 \rightarrow Y$. We will say that the pairs (a, f_1) and (b, f_2) are structurally equivalent if there exists an analytic isomorphism $\phi : X_1 \rightarrow X_2$ such that $f_1 = f_2 \circ \phi$ and $b = \phi \circ a$, i.e., such that the following diagram commutes

$$\begin{array}{ccc} & I & \\ a \swarrow & & \searrow b \\ X_1 & \xrightarrow{\phi} & X_2 \\ f_1 \searrow & & \swarrow f_2 \\ & Y & \end{array}$$

i.e., such that ϕ sends the marked point a_i to b_i and such that it sends the fiber $f_1^{-1}(y)$ in the fiber $f_2^{-1}(y)$ for all $y \in Y$. Note that this requires that $f_2 \circ b = f_1 \circ a$. Structural equivalence is an equivalence relation, which depends on I and Y . To specify them, we will sometimes use the terminology (I, Y) -structurally equivalent or structurally equivalent over Y with marker I . The equivalence classes will be called *structures* (or (I, Y) -structures).

The restriction on these structures (without losing marked points) induces a preorder on structures as follows:

Definition 2 With the same definition as above, but assuming ϕ analytic injective instead of analytic isomorphism (thus dropping the surjectivity assumption), we will say that the structure of (a, f_1) is a *sub-structure* of that of (b, f_2) : this is indeed independent of the choice of representatives in their equivalence classes. We will also say that (b, f_2) has *at least* the structure of (a, f_1) . It is equivalent to the following: (a, f_1) is structurally equivalent to (b, g_2) , where g_2 is a restriction of f_2 to a set containing the image of b . In other words, sub-structures of (b, f_2) are equivalence classes of restrictions of f_2 to open sets containing the marked points.

This preorder is not always an order: for instance if $I = \emptyset$, and the sets $X_1 \subset \mathbb{C}$ defined by $\operatorname{Re}(z) > 0$ and X_2 defined by $\operatorname{Re}(z) > 1/2$ are both mapped to \mathbb{C}/\mathbb{Z} using the canonical projection from \mathbb{C} to the quotient, then each has at least the structure of the other (take $\phi_1(z) = z + 1$ and $\phi_2(z) = z$), while they are not equivalent. However:

Proposition 3 *On the subclass of structures with connected X and at least one marked point, this preorder is an order.*

Proof Assume each of (a, f_1) and (b, f_2) has at least the structure of the other and assume that both X_i are connected and $I \neq \emptyset$. Call $\phi_1 : X_1 \rightarrow X_2$ and $\phi_2 : X_2 \rightarrow X_1$ the two analytic injections. We have to prove that (a, f_1) is structurally equivalent to (b, f_2) . It is sufficient to prove that ϕ_2 is surjective (the inverse of an analytic bijection is analytic). Call $\zeta = \phi_2 \circ \phi_1$. It is injective, satisfies $f_1 \circ \zeta = f_1$ and fixes the marked

points of f_1 . The map f_1 being not locally constant at the marked points, each marked point has a neighborhood on which some iterate ζ^m of the map ζ is the identity, where m is the local degree of f_1 at the marked point. Since there is at least one marked point and since X_1 is connected, $\zeta^m = \text{id}$ holds everywhere by analytic continuation. Hence, ϕ_2 is surjective. The proof is analogous for ϕ_1 . \square

1.3 Universality and Maps with all “the” Structure

In Appendix A, we fix notations and conventions concerning parabolic points, their Fatou coordinates, the extensions thereof, their horn maps and renormalization. In Appendix C, we give a reminder on singular values.

For $d \geq 2$ an integer, let

$$B_d(z) = \left(\frac{z+a}{1+az} \right)^d \text{ with } a = a_d = \frac{d-1}{d+1}.$$

Let

$$B_\infty(z) = \exp \left(2 \frac{z-1}{z+1} \right).$$

They induce unisingular self maps of \mathbb{D} with a unique singular value $z = 0$ in \mathbb{D} and they have a non-linearizable parabolic fixed point on the boundary at $z = 1$ with two attracting petals. Interestingly:

$$B_d \xrightarrow{d \rightarrow +\infty} B_\infty$$

uniformly on compact subsets of \mathbb{D} .

The unit disk is the (immediate) basin of one of the two petals. The inverse of the unit disk is the basin of the other. We let $\Phi_{\text{attr}}[B_d] : \mathbb{D} \rightarrow \mathbb{C}$ be the extended attracting Fatou coordinate for the first petal. The map has also two repelling petals, with vertical axes. We choose the one on the top and let $\Psi_{\text{rep}}[B_d]$ denote the corresponding extended repelling Fatou parametrization. We let $h[B_d] = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$. It is defined on an upper half plane.

The following theorem is an interpretation of a classical theorem of Fatou.

Theorem 4 (Fatou+folk) *Let $f : U \subset \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ a holomorphic map with a non-linearizable parabolic fixed point. Let A be a cycle of immediate parabolic basins associated to this fixed point. Then,*

- *either $U = \widehat{\mathbb{C}}$ and f is a homography*
- *or there is a singular value in A of the restriction of f to A .*

Proof (Sketch) Assume that the second point does not hold. Since the set of singular values is the closure of the union of the set of critical values of and asymptotic values, it follows that A contains no asymptotic nor critical values of $f|_A$. In particular, there is no critical point of f in A so Φ_{attr} has no critical point either.

From the absence of singular value, one gets the following path-lifting property: Given a close-ended path $\gamma : [0, 1] \rightarrow A$ and a point $a \in A$ such that $f(a) = \gamma(0)$, there exists a lift $\tilde{\gamma}$ by f that starts from a and such that $\tilde{\gamma}([0, 1]) \subset A$.

Let $\Phi_{\text{attr}} : A \rightarrow \mathbb{C}$ be the attracting Fatou coordinate extended to A . One can then prove that Φ_{attr} also has a path lifting property: Given a path $\gamma : [0, 1] \rightarrow \mathbb{C}$ and a point $a \in A$ such that $\Phi_{\text{attr}}(a) = \gamma(0)$, there exists a lift $\tilde{\gamma}$ by Φ_{attr} that starts from a and such that $\tilde{\gamma}([0, 1]) \subset A$. The proof consists in finding $n \geq 0$ such that $f^n(a)$ is in an attracting petal \mathcal{P} and $n + \gamma$ is contained in $\Phi_{\text{attr}}(\mathcal{P})$, then applying the first path lifting property n times.

One deduces that Φ_{attr} has no asymptotic values. Since Φ_{attr} cannot have critical values either, it has no singular values and hence, it is a covering from A to \mathbb{C} . Since \mathbb{C} is simply connected, each connected component of A is conformally isomorphic to \mathbb{C} . So each component is $\widehat{\mathbb{C}}$ minus one point, so there is only one such component and the parabolic point p is the missing point, so $U = \widehat{\mathbb{C}}$. The isomorphism Φ_{attr} is a homography, and f restricted to $\mathbb{C} \setminus \{p\}$ is the conjugate of a translation by this homography; hence, f is a homography of $\widehat{\mathbb{C}}$. \square

Proposition 5 (Complement of Theorem 4) *In the second case of Theorem 4, at least one of the two statements below is true*

- *there is a critical point of f in A ,*
- *or there is an open-ended path $\gamma : [0, 1[\rightarrow A$ which leaves every compact of U and whose image $f \circ \gamma$ tends to a point of A .*

Proof Indeed the set of singular values of $f|_U$ is the closure of the set of its critical values and of its asymptotic values. In the case of an asymptotic value v , we will repeat here the analysis done in the proof of Lemma 86: there is a path $\gamma : [0, 1[\rightarrow A$ that leaves every compact of A and with $f \circ \gamma(t) \rightarrow v$ as $t \rightarrow 1$. Such a path must also leave every compact of U for otherwise:

- either $\gamma(t)$ converges in U to a point a that must then be in ∂A but also must satisfy $f(a) \in A$; hence, a whole neighborhood of a in U is contained in the basin, but it also contains points of A so $a \in A$, leading to a contradiction.
- or $\gamma(t)$ has an accumulation set that is bigger than one point. But since f is holomorphic and nowhere constant, this would contradict that $f \circ \gamma(t)$ converges. \square

The following theorem treats the case when there is only one such singular value.

Theorem 6 (folk) *Let f be as in Theorem 4 and A a cycle of immediate basins of its parabolic fixed point. Denote by p the period of A . Assume that one and only one singular value of the restriction of f to A lies in A . Then, the restriction of f^p to any connected component of A is analytically conjugated to the restriction of B_d to \mathbb{D} for some $d \in \{2, 3, \dots\} \cup \{\infty\}$.*

See for instance [15], exposé IX or [21], Theorem 2.9 for similar statements. We will be mainly interested by the case $p = 1$. This has the following consequences, discovered by several authors, including Shishikura (see [25]), and Lanford and Yampolsky (see [21]). See also Part 3 of [6].

Corollary 7 (S., L.-Y.) *With the same notations, call $\zeta : A \rightarrow \mathbb{D}$ the conjugacy from f^P to B_d . Then, there exists a constant $\tau \in \mathbb{C}$ (which depends on the normalizations of the Fatou coordinates) such that $\Phi_{\text{attr}}[B_d] \circ \zeta = \tau + \Phi_{\text{attr}}[f^P]$, where the right hand side is restricted to A .*

Thus, in particular, using the terminology introduced here, $\tau + \Phi_{\text{attr}}[f^P]$ restricted to A is structurally equivalent to $\Phi_{\text{attr}}[B_d]$ over \mathbb{C} . This is illustrated on Figs. 2 and 3, using a widespread visualization technique explained in Sect. 2.

Below is a theorem specifying some structure of renormalizations of maps of Theorem 6. For simplicity, we restrict to the case with only one attracting petal. For this statement and the subsequent ones, we will denote \mathcal{S}_d the set of maps f as in Theorem 6 that satisfy its conclusion with this value of d and that have only one attracting petal. In other words:

Definition 8 Let $d \in \{2, 3, \dots\} \cup \{\infty\}$. We denote \mathcal{S}_d the set of holomorphic maps $f : U \subset \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ with a non-linearizable parabolic fixed point with only one attracting petal, such that if we denote A its immediate parabolic basin, then there is one and only one singular value in A of the restriction of f to A , and such that:

- (if $d = \infty$) this singular value is an asymptotic value of $f|_A$ or
- (if $d < \infty$) there is a critical point of $f|_A$ of degree d .

According to Theorem 6, these cases are mutually exclusive.

There is no specification in the statement below of which normalization is chosen for the parabolic renormalization $\mathcal{R}[f]$ of f .

Theorem 9 (S., L.-Y.) *Consider a map $f \in \mathcal{S}_d$. Then, the upper or lower renormalization $\mathcal{R}[f] : V \rightarrow \widehat{\mathbb{C}}$ is defined on a simply connected subset of \mathbb{C} and has exactly 3 singular values: the asymptotic values $z = 0$, $z = \infty$ and a critical value $z = v \in \mathbb{C}^*$ if d is finite or an asymptotic value $z = v \in \mathbb{C}^*$ otherwise. We have $\mathcal{R}[f]^{-1}(\{0\}) = \{0\}$, $\mathcal{R}[f]^{-1}(\{\infty\})$ is empty.*

- If $d < \infty$, the set $\mathcal{R}[f]^{-1}(\{v\})$ consists in regular points and critical points of degree d , and v is not an asymptotic value.
- If $d = \infty$, the set $\mathcal{R}[f]^{-1}(\{v\})$ consists only in regular points, and v is an asymptotic value.

The following lemma is not optimal but it will be convenient for future reference.

Lemma 10 *If $v \in \mathbb{C}^*$ and $f : U \rightarrow \widehat{\mathbb{C}}$ is such that:*

- *The set of singular values of f is equal to $\{0, v, \infty\}$,*
- *$f(0) = 0$ and $f'(0) = 1$,*
- *$U \subset \mathbb{C}$.*

Then, $f \in \mathcal{S}_d$ for some $d \in \{2, 3, \dots\} \cup \{\infty\}$.

Proof We first prove by contradiction that f cannot be the identity near 0. Otherwise, it would be the identity on the component of U containing 0 by analytic continuation. Then, ∂U would be contained in the set of singular values of f . Hence, $\partial U \subset \{v, \infty\}$.

But then 0 cannot be a singular value, leading to a contradiction. Hence, f has a non-linearizable parabolic point at 0: it has petals.

Since $f'(0) = 1$, the petals have period one. We then prove that f can have only one attracting petal. Let A be the immediate basin of an attracting petal. By Lemma 87, the restriction of $\tilde{f} : A \rightarrow A$ has a set of singular values contained in that of f . But an immediate basin of f cannot contain 0, which is fixed, nor ∞ , which is not in the domain of f , so only v is available as a singular value. Moreover, different immediate basins being disjoint, the singular values that each contains must be distinct. Hence, there can only be one immediate basin, and it contains v but not 0 nor ∞ . \square

This applies in particular to the map $\mathcal{R}[f]$ of Theorem 9. Hence, we get for each value of d a class that is stable by parabolic renormalization:

Corollary 11 (S., L.-Y.) *For every $f \in \mathcal{S}_d$ we have $\mathcal{R}[f] \in \mathcal{S}_d$.*

We make the following conjecture to complement this.

Conjecture 12 *Choose and fix any $v \in \mathbb{C}^*$. All the maps $f \in \mathcal{S}_d$ have structurally equivalent upper renormalizations $\mathcal{R}[f]$, when the latter are normalized so that v is a singular value (0 and ∞ necessarily are). More precisely, they are (I, Y) -structurally equivalent with $Y = \mathbb{C}$, I being a singleton and the marked point being the origin. The same holds for the lower renormalization, and the upper ones are structurally equivalent to the conjugate of the lower ones by the reflection $z \mapsto 1/\bar{z}$.*

There is a special case for which the conjecture is known to hold. To state it we need to introduce the rational map C_d , semi-conjugate of B_d by $z \mapsto -((1-z)/(1+z))^2$. The map C_d has a parabolic point at 0 with one attracting petal, its basin is $A[C_d] = \mathbb{C} \setminus [0, +\infty[$ and the semi-conjugacy is a bijection from \mathbb{D} to $A[C_d]$ (see Sect. 2.2 for details). The horn map of C_d is defined on the complement of the real line and the domain of the renormalization $\mathcal{R}[C_d]$ is \mathbb{D} .

The statement below is a corollary of Proposition 3.16 in [21] together with Theorem 6 above.

Theorem 13 (S., L.-Y.) *Let $d < \infty$. Then for every map $f \in \mathcal{S}_d$ whose immediate basin is a Jordan domain, $\mathcal{R}[f]$ is equivalent to $\mathcal{R}[C_d]$ (for the same notion of equivalence as in Conjecture 12).*

To state the next results, we introduce a temporary notation for the structural equivalence class of Theorem 13.

Definition 14 Let us denote \mathfrak{R}_d the set of holomorphic maps f that are defined on an open subset U of \mathbb{C} , such that $f(0) = 0$, $f'(0) = 1$ and f is (I, Y) -equivalent to $\mathcal{R}[C_d]$ where, as above, $Y = \mathbb{C}$, I is a singleton and 0 is the marked point.

Another way to interpret Theorem 13 is to say that maps $f \in \mathcal{S}_d$ whose immediate basin is a Jordan domain satisfy $\mathcal{R}[f] \in \mathfrak{R}_d$. And Conjecture 12 states that all maps $f \in \mathcal{S}_d$ satisfy $\mathcal{R}[f] \in \mathfrak{R}_d$.

Theorem 13 applies to many maps, for instance:

Theorem 15 (Roesch, Yin) *For any polynomial, for any Fatou component U that is not eventually mapped to a Siegel disk, U is a Jordan domain.*

The case of the cauliflower polynomial $z \mapsto z^2 + 1/2$ is covered by this theorem. For this map, Theorem 13 is illustrated in Fig. 9.

By Lemma 10, we have (see Proposition 25 for a detailed argument)

$$\mathfrak{R}_d \subset \mathcal{S}_d.$$

Let us note the following result, that holds on the more general class \mathcal{S}_d . (Under the restriction that the immediate basin is a Jordan domain, it holds in for more general maps, see Theorem 2.8, in [21].)

Lemma 16 (S., L.-Y.) *For every $f \in \mathcal{S}_d$ the domain of the map $\mathcal{R}[f]$ is equal to its maximal domain of analyticity.*

Proof We include a sketch of the proof with an approach different from [21], to take into account that we do not require the boundary of the immediate basin to be a Jordan curve. Denote $D = \text{Dom } \mathcal{R}[f]$. Recall that by Theorem 9, the map $\mathcal{R}[f]$ is a covering over $V = \hat{\mathbb{C}} \setminus \{0, v, \infty\}$. Let $D' = \mathbb{R}[f]^{-1}(V)$. Since $D' \subset D$, the hyperbolic metric over D' is bigger than over D . It follows that for all $\varepsilon > 0$, there is $M < +\infty$ such that the preimage by \mathcal{R}_d of the ε -neighbourhood in $\hat{\mathbb{C}}$ of $\{0, v, \infty\}$ is a subset of D whose M_ε -thickening for the hyperbolic metric of D covers all D . It implies by commensurability of the hyperbolic metric coefficient to the inverse distance to the boundary, that preimages by $\mathcal{R}[f]$ of points arbitrarily close to either 0, v or ∞ accumulate on every point of ∂D . \square

The next statement is Theorem 2.8 in [21] where the maximal domain of analyticity condition is removed, the proof being the same.

Theorem 17 (L.-Y.) *Consider a holomorphic map f with a one-petal parabolic fixed point whose immediate basin is a Jordan domain. Then, the domain of $\mathcal{R}[f]$ is a Jordan domain.*

The following is Theorem 3.8 in [21] (their class $\hat{\mathbf{P}}$ corresponds to our class \mathfrak{R}_d).

Theorem 18 (L.-Y.) *Let $d < \infty$. Consider a holomorphic map $f \in \mathfrak{R}_d$. Assume that the domain of f is a Jordan domain. Then, the immediate basin of f is a Jordan domain,*

This allows to use Theorem 13, so $\mathcal{R}[f] \in \mathfrak{R}_d$. Together with Theorem 17, this allows in particular to iterate renormalization under the condition that the domain of f is a Jordan domain. This is Theorem 3.7 in [21]:

Theorem 19 (L.-Y.) *Let $d < \infty$. Assume that $f \in \mathfrak{R}_d$ and that its domain is a Jordan domain. Then, $\mathcal{R}[f] \in \mathfrak{R}_d$ and its domain is a Jordan domain.*

In fact, we can get rid of the Jordan domain condition in the theorem above, thanks to a limit argument:

Theorem 20 *Let $d < \infty$. Assume that $f \in \mathfrak{R}_d$. Then, $\mathcal{R}[f] \in \mathfrak{R}_d$.*

Proof In this proof, we normalize the maps $\mathcal{R}[f]$, $f \in \mathfrak{R}_d$, so that their critical value v is a fixed complex number as follows. First, we choose $\mathcal{R}[C_d]$ so that its domain is \mathbb{D} and its critical value v_d is real and positive. Then, we ask that $\mathcal{R}[f]$ has the same critical value v_d (c.f. Theorem 9).

Let $f \in \mathfrak{R}_d$. Up to conjugacy by a rescaling, it takes the form $f = f_0 \circ \phi^{-1}$ where $f_0 = \mathcal{R}[C_d] : \mathbb{D} \rightarrow \mathbb{C}$ and $\phi : \mathbb{D} \rightarrow U[f]$ is a schlicht map. Pick any sequence of schlicht maps $\phi_n : \mathbb{D} \rightarrow U_n$ such that U_n is a Jordan domain and ϕ_n tends to ϕ locally uniformly. For instance $\phi_n(z) = \phi(r_n z)/r_n$ restricted to $z \in \mathbb{D}$, where $r_n < 1$ and $r_n \rightarrow 1$ satisfies these hypotheses. The map $f_n = f_0 \circ \phi_n^{-1}$ is defined on the Jordan domain U_n and tends to f locally uniformly. It follows that $h[f_n]$ tends to $h[f]$ locally uniformly on compact subsets of the domain of $h[f]$. Hence, $\mathcal{R}[f_n] \rightarrow \mathcal{R}[f]$ uniformly on compact subsets of the domain of $\mathcal{R}[f]$. We have seen that $\mathfrak{R}_d \subset \mathcal{S}_d$. By Theorem 18, we can apply Theorem 13 to f_n , i.e., $\mathcal{R}[f_n] = \mathcal{R}[C_d] \circ \psi_n^{-1}$ for some schlicht map ψ_n . The set of schlicht maps is compact and for any cluster value ψ of the sequence ψ_n , we have that $\text{Dom } \mathcal{R}[f] \subset \psi(\mathbb{D})$ and $\mathcal{R}[f](z) = \mathcal{R}[C_d] \circ \psi^{-1}(z)$ holds for all $z \in \text{Dom}(f)$. (It follows that the germ of ψ at 0 is uniquely determined, and by analytic continuation of equalities, that ψ is unique, hence ψ_n converges. However, we do not need that fact). Last, the inclusion $\text{Dom } \mathcal{R}[f] \subset \psi(\mathbb{D})$ must be an equality, for otherwise $\mathcal{R}[f]$ would have a maximal domain of holomorphy that is bigger than its domain, contradicting Lemma 16. \square

For what we are concerned with in this article, this is the base of everything.

1.4 Inou and Shishikura: Giving Up Part of the Structure to Gain Flexibility

Here is the central gear in the work of Inou and Shishikura:

Theorem 21 (Inou Shishikura) *There exists an explicit pair of open subsets A, B of \mathbb{C} and an explicit holomorphic map $f_0 : B \rightarrow \mathbb{C}$ with the following properties:*

- (1) $0 \in A$, A is compactly contained in B ,
- (2) A and B are simply connected,
- (3) f_0 fixes the origin and has derivative 1 there,
- (4) f_0 has exactly one critical point in B ; this critical point has local degree two, belongs to A , and is mapped to $-4/27$ by f_0 ,
- (5) for any upper renormalization $\mathcal{R}[f]$ of a map $f \in \mathcal{S}_d$, there exists a subset U of $\text{Dom } \mathcal{R}[f]$ and a holomorphic bijection $\phi : B \rightarrow U$ with $\phi(0) = 0$ and $\mathcal{R}[f]|_U = f_0 \circ \phi^{-1}$,
- (6) for any univalent map $\phi : A \rightarrow \mathbb{C}$ with $\phi(0) = 0$ and $\phi'(0) = 1$, there exists a univalent map $\psi : B \rightarrow \mathbb{C}$ with $\psi(0) = 0$ and $\psi'(0) = 1$, such that the map $f_0 \circ \phi^{-1}$, which fixes the origin with multiplier one, has an upper renormalization which has a restriction of the form $f_0 \circ \psi^{-1}$.

The map f_0 has a particularly simple expression: $f_0(z) = z(1+z)^2$. It turns out that f_0 commutes with $z \mapsto \bar{z}$; thus, the theorem holds with the same f_0 for lower renormalization.

The statement below is a reformulation of their theorem using the language introduced in the present article. Given a structure \mathcal{B} and a sub-structure \mathcal{A} , we will say that the second is a *relatively compact sub-structure* of the first whenever maps in \mathcal{A} are structurally equivalent to restrictions of maps in \mathcal{B} to relatively compact open subsets of their domains (not just subsets).¹

Theorem 22 (Inou Shishikura, reformulated) *Let I be a singleton and $Y = \mathbb{C}$. There exists an explicit pair of (I, Y) -structures \mathcal{A} and \mathcal{B} with the following properties:*

- (1) *\mathcal{A} is a relatively compact sub-structure of \mathcal{B} and \mathcal{B} is a sub-structure of the universal structure \mathfrak{R}_d of Theorem 20,*
- (2) *$\forall (a, f) \in \mathcal{A}$, the map f is defined on a connected and simply connected Riemann surface and has exactly one critical point, of local degree two; the same holds for \mathcal{B} .*
- (3) *For any map in \mathcal{A} whose domain of definition is a subset of \mathbb{C} and that fixes the marked point with multiplier one, its (suitably normalized) upper parabolic renormalization has at least structure \mathcal{B} .*

It is also the case for the *lower* parabolic renormalization, with the same structures \mathcal{A}, \mathcal{B} .

Definition 23 (*High type numbers*) For $N \in \mathbb{N}^*$, let HT_N be the set of irrationals whose modified continued fraction satisfies $|a_n| \geq N, \forall n \in \mathbb{N}$. In settings where N has been fixed, the set HT_N is often called the set of *high type numbers*. We will call it here the set of numbers of *type* $\geq N$.

To keep it short, the following corollary, also by Inou and Shishikura, is stated here with some imprecision concerning the renormalization:

Corollary 24 (Perturbation) *There exists $N > 0$ such that the class of maps defined in an open subset of \mathbb{C} , with structure \mathcal{A} and fixing the marked point with a rotation number θ of type $\geq N$, is invariant under a cylinder renormalization operator (called the near-parabolic renormalization).*

They prove more: thanks to the compact inclusion of structure \mathcal{A} in \mathcal{B} , there is a form of contraction. Cylinder renormalization was introduced by Yampolsky in the study of analytic circle homeomorphisms with a critical point.

Consequences of this corollary are numerous and are still being harvested. Its main quality is that it allows a fine control on the post-critical set of quadratic polynomials with high type rotation numbers. For instance, Shishikura and Yang Fei proved that in this case, the boundary of the Siegel disk is a Jordan curve [27]. It allows to study the hedgehogs and the size of Siegel disks. In [5], we proved the Marmi Moussa Yoccoz conjecture restricted to high type numbers. Cheraghi has given many other applications [1, 8, 11, 12] and [14] which is a progress on the MLC conjecture. It was also used in [2] to prove the existence of quadratic polynomials with a Julia set of positive Lebesgue measure. We believe that it can also give a new approach to the results of

¹ If it holds for some representatives then it holds for all representatives in the equivalence class.

McMullen [22] on the self-similarity of Siegel disks whose rotation number has an eventually periodic continued fraction expansion² at the critical point. McMullen used Ghys' quasiconformal surgery procedure as a first step in his proofs, to transfer some properties that are easier to prove for circle maps. It would be nice to have a more direct proof, that would adapt to situation, like the exponential maps $z \mapsto e^z + c$, where a quasiconformal surgery does not exist but where self-similarity still seems to occur.

1.5 Main Theorem

In this article, we prove the following extension of Inou and Shishikura's Theorem.

Main Theorem *Let I be a singleton and $Y = \mathbb{C}$. For all $d \in \mathbb{N}$ with $d \geq 2$, there exists (I, Y) -structures \mathcal{A} and \mathcal{B} with the following properties:*

- (1) *\mathcal{A} is a relatively compact sub-structure of \mathcal{B} and \mathcal{B} is a sub-structure of the universal structure \mathfrak{R}_d of Theorem 20,*
- (2) *every map in \mathcal{A} or in \mathcal{B} is defined on a connected and simply connected Riemann surface,*
- (3) *every map in \mathcal{A} or in \mathcal{B} has exactly one critical value, and all critical points have local degree d ,*
- (4) *for any map in \mathcal{A} whose domain of definition is a subset of \mathbb{C} and that fixes the marked point with multiplier one, the upper parabolic renormalization has at least structure \mathcal{B} , and the lower parabolic renormalization has at least structure the conjugate of \mathcal{B} by $z \mapsto \bar{z}$, for appropriate normalizations of the renormalizations.*

The structures \mathcal{A} and \mathcal{B} are obtained by retaining most of the universal structure (call it \mathcal{U}) of \mathfrak{R}_d . More precisely, we choose for \mathcal{B} the restriction of a map in \mathcal{U} to a subset of its domain U defined as points having U -hyperbolic distance $\leq L$ to the point marked by I , and we prove in Sect. 3 that for L big enough, there is a relatively compact sub-structure \mathcal{A} of \mathcal{B} such that the main theorem holds.

Remark It should be noted that for $d = 2$, our theorem can be considered as weaker than Inou and Shishikura's. For one thing, maps in our class have much more structure, so our class is smaller. Second they have many critical points (though only one critical value); whereas, there is only one in Inou and Shishikura's. This should not prevent our class, though, to be applied to $z^d + c$ as we explain now. Note that a similar situation occurs for the IS class: a polynomial $z^2 + c$ with an indifferent fixed point of multiplier close to 1 never has a restriction that belongs to the IS class, but its first cylinder renormalization has some as soon as the multiplier is close to 0. Here it is the same: a map of the form $z^d + c$ never has structure \mathcal{A} or more, but its first cylinder renormalization does if the rotation number is close enough to 0.

It should be easy to check that the analog of Corollary 24 also holds. We believe that many of its consequences for quadratic maps, therefore, carry over to unicritical polynomials.

² These rotation numbers are the quadratic irrationals.

About degree three maps:

In [30], Yang Fei has constructed a renormalization invariant class for $d = 3$, in the spirit of Inou and Shishikura. In particular, it has the advantage of having only one critical value.

About unisingular maps:

We wonder if one can extend the above work to the case $d = \infty$. There are some subtleties occurring here.

We do not believe that one can take for \mathcal{B} a substructure of $g \in \mathcal{R}[\mathcal{S}_d]$ defined by a restriction on a compact subset of the domain of g , like we did in the case $d < \infty$, that would yield a invariant class. The natural idea is to keep instead a whole connected preimage of a neighborhood of the singular value, which adds a subset of $\text{Dom } g$ that is at least as tangent to its boundary as a horocycle. Unfortunately, we realized that this does not provide an invariant class either.

It shall be noted that some consequences of Inou and Shishikura's invariant class for $d < \infty$ will not hold anymore for $d = \infty$: for instance, there are unisingular maps for which the boundary of the Siegel disk is not a Jordan curve. This includes the exponential $z \mapsto \lambda(\exp(z) - 1)$ (or equivalently $z \mapsto e^z + \kappa$) when it has an indifferent periodic point of rotation number in Herman's class.³ Interestingly, there are some other maps with only one free singular value, with $d = \infty$, and for which the Siegel disk seems to be more often locally connected (always): for instance, the semi-conjugate of $z \mapsto e^{i\theta/2} \tan z$ by $z \mapsto z^2$, i.e., $z \mapsto e^{i\theta} (\tan \sqrt{z})^2$. See Fig. 1 for a plot of the Julia set and Siegel disk of $z \mapsto e^{i\theta/2} \tan z$.

It is to be noted that though the two (essentially) unisingular families $\lambda(e^z - 1)$ and $\lambda(\tan tz)^2$ have very different Siegel disks for $\theta =$ the golden mean, computer experiments weakly hint at a possible identical asymptotic limit when zooming at the singular value: there might exist a cylinder renormalization operator with a fixed point capturing both maps.

2 Visualizing Structures with the Chessboard Graph

An often used and very useful technique of visualization of ramified covers (and partial cover structures that are not too messy) consists in cutting the range in domains, often simply connected, along lines joining singular values, and taking the pre-image of these pieces, which gives a new set of pieces. The way they connect together and the way they map to the range give information about the structure.

2.1 Changes of Variables

The map B_d is the composition of the automorphism $\mu : z \mapsto \frac{z+a}{1+az}$ of the disk, with $a = (d-1)/(d+1)$, followed by $\text{pow} : z \mapsto z^d$: $B_d = \text{pow} \circ \mu$. If we conjugate B_d

³ By [18] the Siegel disk is unbounded and then by [4] it is not locally connected. Thanks to Lasse Rempe for pointing this out to me.

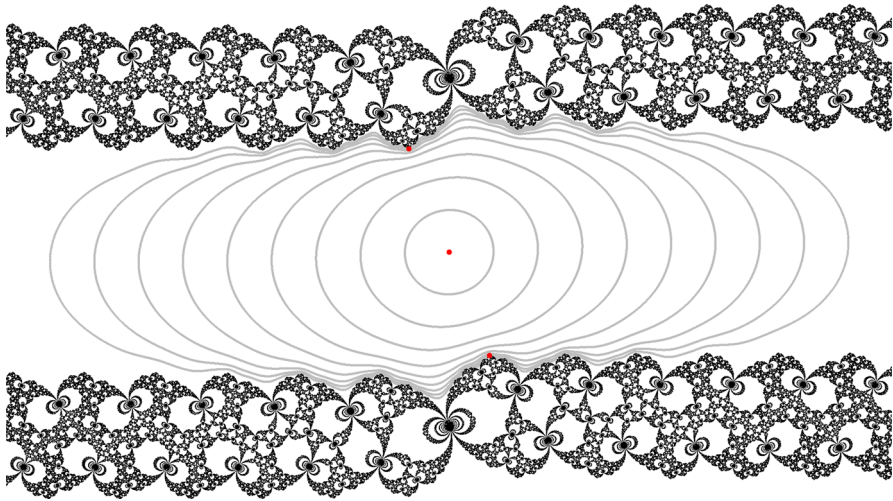


Fig. 1 Rotated by 90° , the Julia set of the map $z \mapsto \lambda \tan z$ with λ so that the origin is indifferent with rotation number $\theta/2$ and $\theta = (\sqrt{5} - 1)/2$ is the golden mean. The Julia set is periodic of period π , we drew only two periods. There also seems to be an asymptotic similarity by some imaginary translation. There are red points at the origin and at the two (symmetric) asymptotic values. A few orbits inside the Siegel disk have been drawn. The Siegel disk seems to be bounded by a Jordan curve (but not a quasicircle: there must be a dense set of cusps). The rotation number is $\theta/2$ but the picture has a symmetry of order 2 and quotienting out, i.e., semi-conjugating by $z \mapsto z^2$, gives a transcendental meromorphic map $z \mapsto \lambda^2 (\tan \sqrt{z})^2$ with rotation number θ at 0, with infinitely many critical points but that all map to 0, and with only one asymptotic value $-\lambda^2$

by μ , we get the map $\mu \circ \text{pow}$:

$$\tilde{B}_d = \mu \circ B_d \circ \mu^{-1} : z \mapsto \frac{z^d + a}{1 + az^d},$$

which is a Blaschke product too and has its critical point at the origin, the parabolic point still being at $z = 1$. As $d \rightarrow \infty$, \tilde{B}_d tends to the constant 1 uniformly on compact subsets of \mathbb{D} .

The map B_∞ is the composition of $\mu : z \mapsto i \frac{1-z}{1+z}$ (mapping the disk to the upper half plane) followed by $z \mapsto \exp(2iz)$. Interestingly, if we conjugate B_∞ with μ , we get the trigonometric map:

$$\mu \circ B_\infty \circ \mu^{-1} : z \mapsto \tan z,$$

whose non-linearizable parabolic fixed point is at the origin and which maps the upper half plane to itself, the asymptotic value in the upper half plane being i .

2.2 Preferred Representative

Theorems 13 and 20 say that all maps satisfying some assumption have structurally equivalent upper parabolic renormalizations (appropriately normalized). Their equiv-

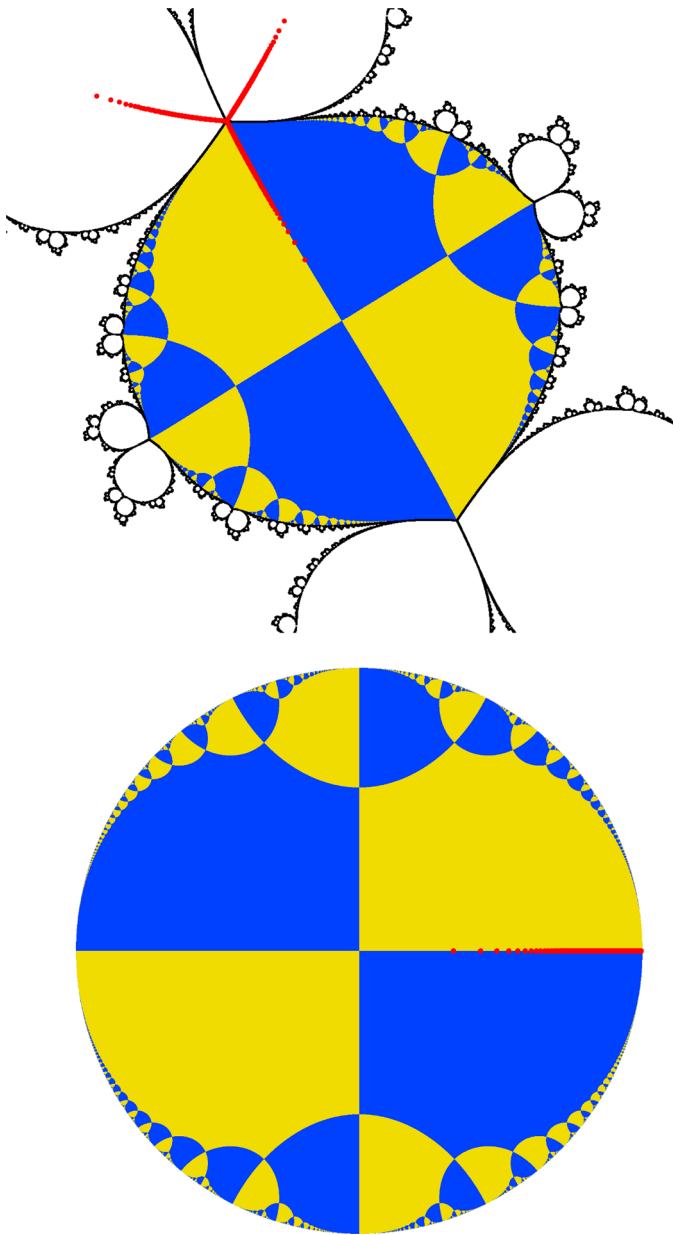


Fig. 2 Illustration of Theorem 6. Above: zoom on the fat Douady rabbit, the Julia set of the quadratic map $P = e^{2i\pi/3}z + z^2$, which has a parabolic fixed point with three attracting petals, and acts transitively on them. The Fatou component U that contains the finite critical point has been colored with the parabolic chessboard, whose definition is recalled later in the present article. In red, the orbit of the critical value. On U , P^3 satisfies the hypotheses of the theorem. According to the conclusion, P^3 is conjugated on U to the Blaschke product $B_2(z) = \left(\frac{z+1/3}{1+z/3}\right)^2$. Below: the chessboard of $\mu_2 \circ B_2 \circ \mu_2^{-1}$, with $\mu(z) = (z + 1/3)/(1 + z/3)$, which is conjugated to B_2 and, hence, to P^3 on U . The conjugacy has to transport the chessboard and the critical orbit

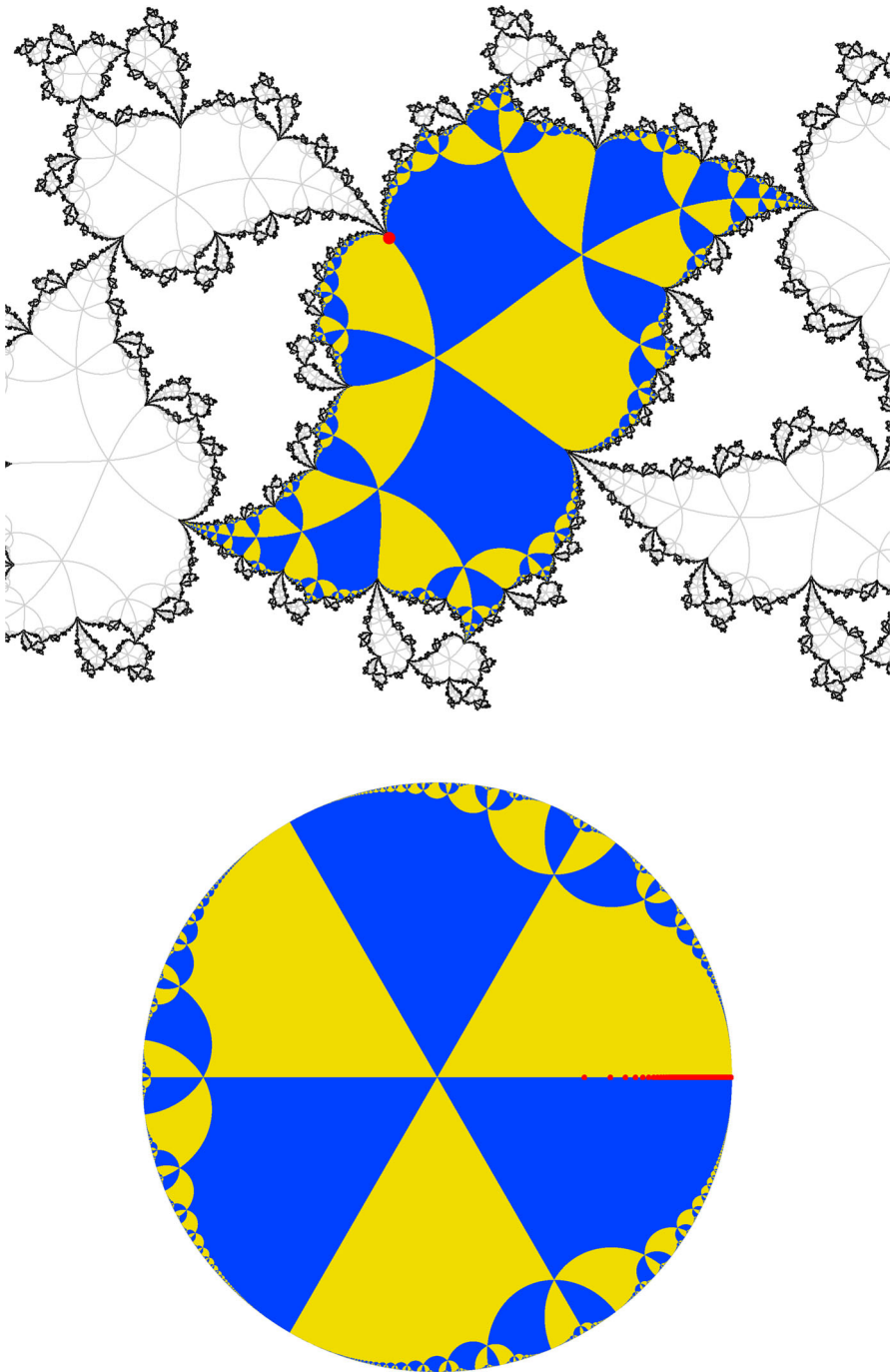


Fig. 3 Another illustration of Theorem 6. This time, $d = 3$. The parabolic point on the first picture is indicated by a red dot. The orbit of the critical value is indicated in red on the second picture

alence class, that depends only on d , is universal in some sense. We will here choose a preferred representative in this equivalence class, and for this use the maps B_d . A defect of the maps B_d and \tilde{B}_d , seen as maps from the Riemann sphere to itself, is that their parabolic point has two attracting petals instead of one. We prefer to work with a semi-conjugate C_d of B_d that we introduce now. The map B_d commutes with $z \mapsto 1/\bar{z}$ and with $z \mapsto \bar{z}$ hence with $z \mapsto 1/z$. It is, therefore, a well-defined map on pairs $\{z, 1/z\}$. A first change of variables $u = (1 - z)/(1 + z)$ maps the unit disk to the right half plane $\{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$ sending the parabolic point to 0 and conjugates B_d to a map which can be formulated as follows:

$$u \mapsto \frac{\text{odd} \left(\left(1 + \frac{u}{d}\right)^d \right)}{\text{even} \left(\left(1 + \frac{u}{d}\right)^d \right)},$$

where odd and even refer to the sum of monomials of odd and even power in u in the polynomial expansion of $(1 + u/d)^d$. For $d = \infty$, we get the ratio of the odd and even parts of the exponential,

$$u \mapsto \tanh(u).$$

Setting $v = -u^2 = -((1 - z)/(1 + z))^2$ identifies pairs $\{z, 1/z\}$ with single values of v . There exists a map C_d , rational of degree d if $d < \infty$, entire transcendental if $d = \infty$, such that the following diagram commutes

$$\begin{array}{ccc} & \xrightarrow{B_d} & \\ S \downarrow & & \downarrow S \\ & \xrightarrow{C_d} & \end{array}$$

where $S(z) = v = -u^2 = (i(1 - z)/(1 + z))^2$. If $d = \infty$ we get $C_\infty(v) = (\tan \sqrt{v})^2$. If $d < \infty$ the formula is more complicated. The map S is a bijection from the unit disk to the complement A of $[0, +\infty]$ in the Riemann sphere, and sends 1 to 0, -1 to ∞ and 0 to -1 . The map C_d has a parabolic fixed point at the origin with one attracting petal, whose immediate basin is A . By construction, C_d is conjugate on A to the restriction of B_d to \mathbb{D} . The extended horn map of C_d is defined on the complement of a horizontal line. Thus, the upper and lower parabolic renormalizations of C_d are defined on round disks centered on the origin. A lengthy computation shows that

$$\gamma[C_d] = \frac{3}{20} \cdot \frac{d^2 + 1}{d^2 - 1}$$

where γ is defined in Appendix A. We have not defined in this article a general theory of Fatou coordinates, horn maps and their normalizations, for parabolic points with more than one attracting petal. However, in the particular case of B_d , which has two attracting petals, we defined in Appendix A the objects $\Phi_{\text{attr}}[B_d]$, $\Psi_{\text{rep}}[B_d]$ and $h[B_d]$. Let us recall this here.

The unit disk is the (immediate) basin of one of the two attracting petals of B_d . We let $\Phi_{\text{attr}}[B_d] : \mathbb{D} \rightarrow \mathbb{C}$ be the extended attracting Fatou coordinate for this petal. The map has also two repelling petals, with vertical axes. We choose the one on the top and let $\Psi_{\text{rep}}[B_d]$ denote the corresponding extended repelling Fatou parametrization. We then let $h[B_d] = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$. It is defined on an upper half plane.

The object for B_d are related to those of C_d as follows:

$$\begin{aligned} T \circ \Phi_{\text{attr}}[B_d] &= \Phi_{\text{attr}}[C_d] \circ S|_{\mathbb{D}} \\ S \circ \Psi_{\text{rep}}[B_d] &= \Psi_{\text{rep}}[C_d] \circ T' \\ T \circ h[B_d] &= h[C_d]|_H \circ T' \end{aligned}$$

where H is the upper half plane on which $h[B_d]$ is defined, T and T' are translations that depends on normalizations, and S is the 2:1 rational map defined a few lines above, that semi conjugates B_d to C_d . If we choose a normalization for the objects associated to C_d , this induces a normalization for the objects associated to B_d by declaring that T and T' must be the identity:⁴

$$\begin{aligned} \Phi_{\text{attr}}[B_d] &= \Phi_{\text{attr}}[C_d] \circ S|_{\mathbb{D}} \\ S \circ \Psi_{\text{rep}}[B_d] &= \Psi_{\text{rep}}[C_d] \\ h[B_d] &= h[C_d]|_H \end{aligned}$$

Let $\mathcal{H}[B_d]$ denote the semi-conjugate of h by E , that we complete by $\mathcal{H}[B_d](0) = 0$. Then, $\mathcal{H}[B_d]$ is defined and holomorphic on the unit disk. We now define $\mathcal{R}[B_d]$ and a preferred normalization for it:

$$\mathcal{R}[B_d] = A \circ \mathcal{H}[B_d] \circ B,$$

with A and B linear such that: $B = \text{id}$, and $\mathcal{R}[B_d]'(0) = 1$.

2.3 Dynamical and Structural Chessboards

Let f be as in Theorem 13. Let v_f denote the unique singular value of $f|_A$ in the immediate parabolic basin A . The set of singular values of h over \mathbb{C} is of the form $v' + \mathbb{Z}$ for $v' = \Phi(v_f)$. Let us cut the range along the horizontal line $v' + \mathbb{R}$ passing through them. To understand the shape of the preimages of this line and of the upper and lower half planes it bounds, it is useful to work first with the map B_d . Recall: h is the horn map associated with a dynamical system f with an immediate parabolic basin A , on which there is a conjugacy $\zeta : A \rightarrow \mathbb{D}$ to the map B_d , and $\Phi[B_d] \circ \zeta = \tau + \Phi[f]$. Thus, the preimage $\Phi[f]^{-1}(v' + \mathbb{R})$ is mapped by the isomorphism ζ to a universal shape, that depends only on d . The set $\Phi[f]^{-1}(v' + \mathbb{R})$ is called the *parabolic chessboard graph* of

⁴ There is a way to extend convention 2 on page 80 for the normalizations, using asymptotic expansions of Fatou coordinates and the general definition of the iterative residue, see for example, [7], Chapter 1. A remarkable fact is then that these normalizations of B_d and C_d are compatible: T and T' are automatically the identity.

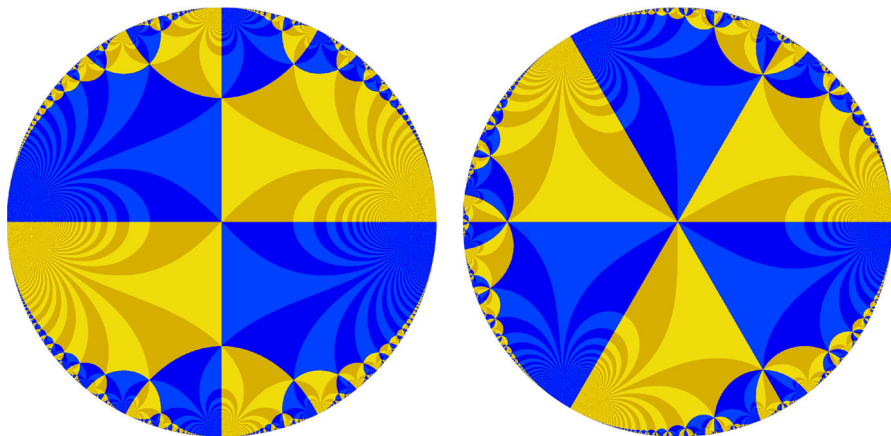


Fig. 4 Light and dark stripes, preimages of vertical strips of width 1 under the extended attracting Fatou coordinate Φ , for \tilde{B}_2 and \tilde{B}_3

f on A . The connected components of its complement in A are called the *chessboard boxes* (in an actual chessboard they are called squares but here they have infinitely many corners and not just four). The chessboard is the name of this decomposition of A into a graph and boxes. Since the chessboard is universal, it can be well understood by looking only at the maps B_d . Note that these maps have a singular orbit contained in $[0, 1[$ and that they send reals to reals; thus, the chessboard graph is also the union of the preimages of $[0, 1[$.

Each chessboard box of f is mapped by $\Phi = \Phi[f]$ to the upper or the lower half plane delimited by $v' + \mathbb{R}$, and we colored them accordingly (yellow and blue in most illustrations of the present article). The set of singular values of Φ is precisely $\{v' - 1, v' - 2, v' - 3, \dots\}$. These singular values, however, have also regular preimages, so these universal structures we are considering are not so simple as ramified covers. Under the dynamics of f , each box is mapped to a box of the same color, and there is exactly one box of each color that is fixed by f : these are the ones that have the singular value in their boundaries. The Fatou coordinate Φ conjugates the dynamics of f on these two fixed boxes to the dynamics of the translation by 1 on the upper and lower half planes. The chessboard also tells us about the structure of Φ as defined in Sect. 1.2. In view of this, the chessboard in the immediate basin A is both a dynamical object w.r.t. f and a structural object w.r.t. Φ .

The figures can be enhanced a little bit: let us use two shades of yellow and two shades of blue in the range of Φ . Use the light shade if the floor integer part $\lfloor \operatorname{Re}(z - v') \rfloor$ is even, and the dark shade otherwise. Color points in A according to $\Phi(z)$. Then, we get Fig. 4 for $f = \tilde{B}_2, \tilde{B}_3$ and B_∞ . This color scheme is useful to visualize the pull-back by $\Phi[f]$ of the vertical direction. Under f , a light stripe is mapped to a dark stripe and vice versa.

The chessboard graph has no endpoint, and it is closed in A but not compact. Since we considered the chessboard graph as a subset of \mathbb{C} endowed with its topology, not as a combinatorial object, there is an ambiguity outside branching points concerning which

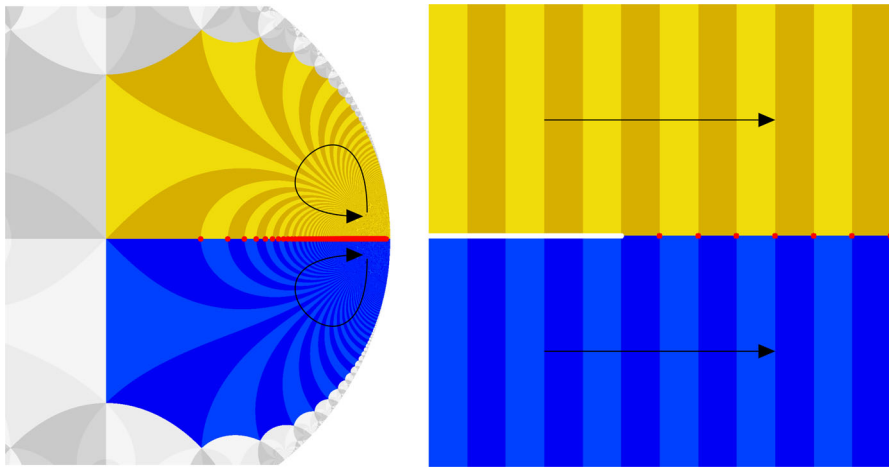


Fig. 5 The extended attracting Fatou coordinates of \tilde{B}_2 conjugate the restriction of \tilde{B}_2 to the two principal chessboard boxes and the segment $]0, 1[$, indicated on the left, to the translation $z \mapsto z + 1$ on a slit plane, as on the right

points are vertices of valence 2 and which points belong to edges: the singular value is one such point w . So let us define an abstract graph with vertices at all preimages of the singular value by $\pi \circ \Phi[f] : A \rightarrow \mathbb{C}/\mathbb{Z}$, where $\pi : \mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$ is the canonical projection, and edges as preimages of the horizontal circle through it.

Remark We will not make use of it, but it would make sense to consider some supplementary topological information on the abstract graph, like the cyclic order induced by the embedding in the plane on edges at every vertex.

The abstract graph and the way it is embedded in A tell us how are glued together pieces obtained by cutting A along the preimages of the horizontal circle through the critical value of $\pi \circ \Phi[f]$. They also tell us how are glued together pieces obtained by cutting A along the preimages of the vertical line through this critical value: each of these piece is a union of two light stripes or two dark stripes, glued along a segment of the graph.

Figure 5 explains how the union of the edges that touch points in the orbit of the singular value forms an infinite line in the graph, and how the union of this line and of the two chessboard boxes whose closures contain the line, make a domain where the dynamics is conjugated to the translation by 1 restricted to $\mathbb{C} \setminus]-\infty, 0[$. The bright and dark stripes help to figure out how things are mapped and what the dynamics is within this domain. This would work for any $d \geq 2$, including ∞ .

Let us now introduce the chessboard associated to the horn map h . It is defined using the pre-image of the horizontal line through the set of singular values of h , and of the upper and lower half plane cut by this line. From the definitions, it follows that it is also equal to the pre-image by Ψ (the extended repelling Fatou parametrization) of the chessboard of f in the full parabolic basin (the union of all iterated preimages of A by f). This time, it is not a dynamically invariant object, but it gives information on the *structure* of h as defined in Sect. 1.2.

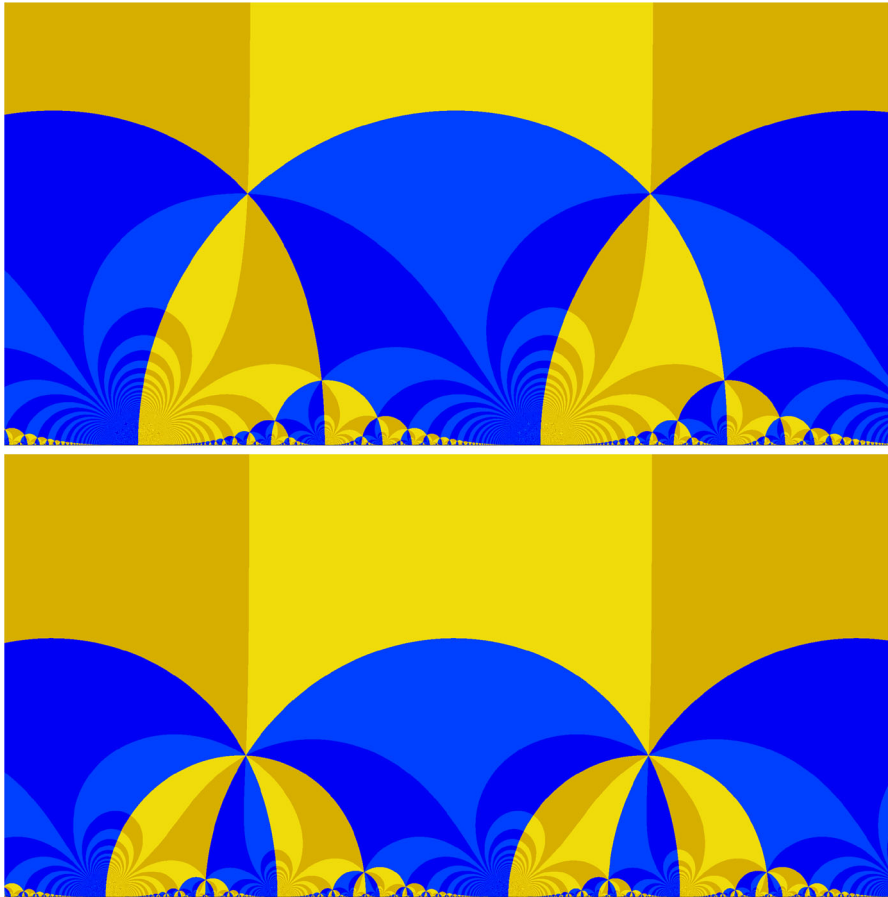


Fig. 6 These two pictures show the structures of the extended horn maps h of respectively B_2 and B_3 . They are all defined on the complement of a horizontal line; in each case, we only drew the picture above this line; the full picture is obtained by reflection through this line, permuting blue \leftrightarrow yellow. The same coloring conventions apply as in the previous figures: yellow boxes map by h to the upper half plane delimited by the horizontal line through the singular value, blue boxes to the lower half plane. The boundaries between dark and light shades of a given color are mapped by $\pi \circ h$ to the vertical line through the critical value, where $\pi : \mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$ denotes the canonical projection

The next set of pictures, in Fig. 6, shows the structure of the horn maps of B_d . The image of these three pictures by the exponential map $z \mapsto \exp(2\pi iz)$ is shown on Figs. 7 and 8 and gives us information about the structure of the upper renormalization $\mathcal{R}[B_d]$ of B_d . One sees that it is also defined on a disk centered on the origin. For the beauty of the thing, we replaced the dark and light strips by a lighting scheme that gives the illusion of a texture made of cylinders.⁵ A more shameful reason for this change is that the light and dark stripe scheme does not pass to the quotient.

⁵ The trick to produce such a computer picture is called *normal mapping*, it is the same trick used to give a realistic look in 3D imaging to texture-mapped polygons subjected to a light source. Some specular reflection reinforces the feeling of relief.

Recall that $\mathcal{H}[B_d]$ denotes the semi-conjugate of $h[B_d]$ by $E : z \mapsto e^{2\pi iz}$, i.e., $\mathcal{H} \circ E = E \circ h$, completed by $\mathcal{H}[B_d](0) = 0$, and that $\mathcal{R}[B_d] = A \circ \mathcal{H}[B_d]$ for some linear map A . Recall also that there are three singular values of $\mathcal{R}[B_d]$: 0 , ∞ and some third point v . The only preimage of 0 is 0 .

The chessboard decomposition for $\mathcal{R}[B_d]$ has two equivalent definitions. First as the preimage by $\mathcal{R}[B_d]$ of the decomposition of \mathbb{C} into the circle of center 0 going through v and the two connected components of its complement, yellow corresponding to the inside and blue to the outside. But it is also the image by E of the chessboard decomposition of $h[B_d]$.

With our coloring convention, the box surrounding 0 is yellow, every blue box is mapped by $\mathcal{R}[B_d]$ to the set $|z| > |v|$ as a universal cover, the yellow box containing the origin is mapped $1 : 1$ to $|z| < |v|$ and every other yellow box is mapped as a universal cover to $|z| < |v|$ minus the origin.

This can be generalized to the renormalization $\mathcal{R}[f]$ for any $f \in \mathcal{S}_d$. It is both: on one hand, the preimage of the decomposition of \mathbb{C} into the circle of center 0 going through the singular value v of $\mathcal{R}[f]$ and the two components of its complement; on the other hand, the image by E of the chessboard of h , subject to the same restriction, rescaling and possibly inversion if we are performing lower parabolic renormalization instead of upper, as were done to pass from h to $\mathcal{R}[f]$, and completed by adding 0 . The partition of the domain of $\mathcal{R}[f]$ into two colors and the graph separating them is called the *structural chessboard* of $\mathcal{R}[f]$. It is different from the *dynamical chessboard* of $\mathcal{R}[f]$, which is defined only in the basin of its parabolic point $z = 0$ and only if the normalization of $\mathcal{R}[f]$ is such that 0 is a parabolic point of $\mathcal{R}[f]$. In particular, unlike the dynamical chessboard, the structural chessboard is *not* dynamically invariant.

In the case $d < \infty$, notice that there is a tiny loop bounding a small yellow box containing the origin and that looks like a droplet. When d increases, the angle at the tip of the loop decreases and the tip gets closer to the boundary of the domain of definition of the map. In the case $d = \infty$, the droplet touches the boundary.

The next picture illustrates Theorems 9 and 13. Figure 9 shows the famous case dubbed the Cauliflower: this is the Julia set of $z \mapsto z^2 + 1/4$. We removed the colors and drew the boundaries between boxes and the boundaries of the definition domains. The six images are ordered in a 2×3 grid whose first column figures the dynamical chessboard of f atop and of B_2 below. The next column represents views of their chessboards in repelling Fatou coordinates or, more precisely, two periods of their preimage by Ψ_{rep} . The last column is the projection to \mathbb{C}^* of the middle column by the map $b[f]^{-1}E : z \mapsto b[f]^{-1} \exp(2\pi iz)$ where $b[f]$ is some constant used in normalizing \mathcal{R} . The vertical arrows are isomorphisms between the three pairs of domains, mapping graph to graph, respecting box colors (not figured here) and even better: they are structure isomorphisms for the following respective maps (properly normalized): the attracting Fatou coordinate for the first column, the horn map for the second column and the parabolic renormalization for the last one. This diagram is also commutative if one adds the following self maps of the six sets: column 1: f , B_2 , column 2: T_1 , T_1 , column 3: Id , Id . From all this, we can build a big commutative diagram, but we do not think that it would be much readable. Note that the tiny loops in the last column are the images of the big unbounded square that lie above in both middle images. The image of this square by Ψ_{rep} is one of the two f -invariant



Fig. 7 Structure of $\mathcal{R}[B_2]$

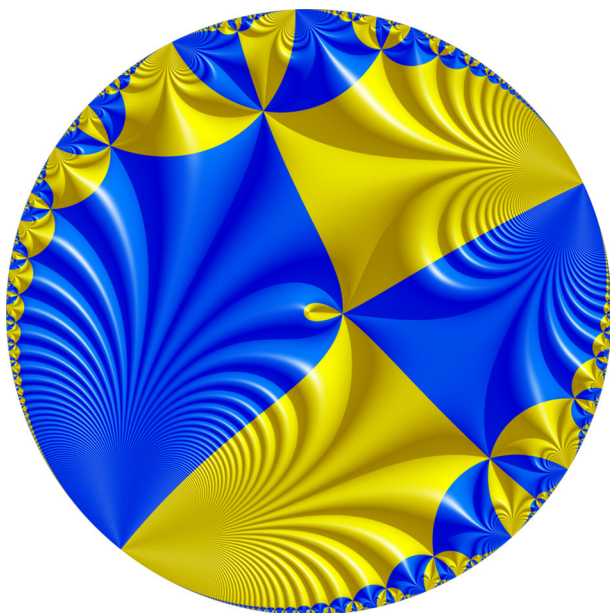


Fig. 8 Structure of $\mathcal{R}[B_3]$

(resp. B_2 -invariant) squares (they touch the fixed point), but the latter has many other preimages by Ψ_{rep} .

2.4 Inou and Shishikura's Sub-structure

To finish this visualization chapter, we present here the structure \mathcal{B} of Theorem 22 (Inou–Shishikura's theorem), and how \mathcal{B} and its sub-structure \mathcal{A} fit as sub-structures of $\mathcal{R}[B_2]$.

The first set of drawings shows one of the ways Inou and Shishikura used to present it. They defined a Riemann surface with a natural projection over \mathbb{C}/\mathbb{Z} as follows: cut the cylinder \mathbb{C}/\mathbb{Z} so as to retain only the part where $\text{Im}(z) > -\eta$ with $\eta = 2$.⁶ Slit this cylinder along the vertical segment from 0 to $-\eta i$. To this, glue the rectangle $\text{Re}(z) \in]-1, 1[$ and $\text{Im}(z) \in]-\eta, \eta[$, cut along the same segment. As usual with Riemann surfaces, we glue each side of the segment in one piece to the opposite side on the other piece. This is represented on the upper left part of Fig. 10. This method is reminiscent of the way Perez-Marco uses to build structures in his work. Below it in the same figure, is a tentative to picture the way it projects to the cylinder \mathbb{C}/\mathbb{Z} , while on its right, there is a planar open set isomorphic to it (conformal moduli are not respected in the figure). In the lower right corner, there is the image of the lower left by $z \mapsto \exp(2\pi iz)$ (rotated by 90 degrees). The right column is a map f with structure \mathcal{B} (the marked point is $z = 0$). The left part of Fig. 11 accurately shows how \mathcal{B} sits as a substructure of the structure of $\mathcal{R}[B_2]$. The right part identifies the pieces.

Structure \mathcal{A} is a substructure of \mathcal{B} , obtained by mapping conformally the domain of f minus the origin to the complement of the closed unit disk and removing the interior of some specific and explicitly defined ellipse (see [19]). The result, mapped to the set of Fig. 11, is shown in Fig. 12.

Recall that $\mathcal{R}[B_d]$ is the unit disk \mathbb{D} . Let $U \subseteq V \subseteq \mathbb{D}$ be the sub-domains corresponding to respectively \mathcal{A} and \mathcal{B} . Inou and Shishikura worked with the particular sets we just described. It is more natural, though not easy, to take for U and V a pair of disks centered on the origin. The objective of the present article is to prove that this works. The downside is that we lose unicriticality of maps in the class we construct. Yet, it still applies to unicritical polynomials, after taking one renormalization (they become multicritical, with only one critical value); recall that even Inou and Shishikura need to take first one iteration of renormalization of to get a map in their class from a quadratic polynomial anyway. The upside is that our approach will work for critical points of any degree.

3 Proof

The element of hyperbolic metric of a connected open subset U of \mathbb{C} will be denoted by $\rho_U(z)|dz|$ and the corresponding hyperbolic distance by $d_U(z, z')$.

⁶ There is some flexibility in the value of this lower bound. In [19], they proved that their theorem holds for any real η between 13 and 2 included. Here, we drew the domain only for their original value $\eta = 2$.

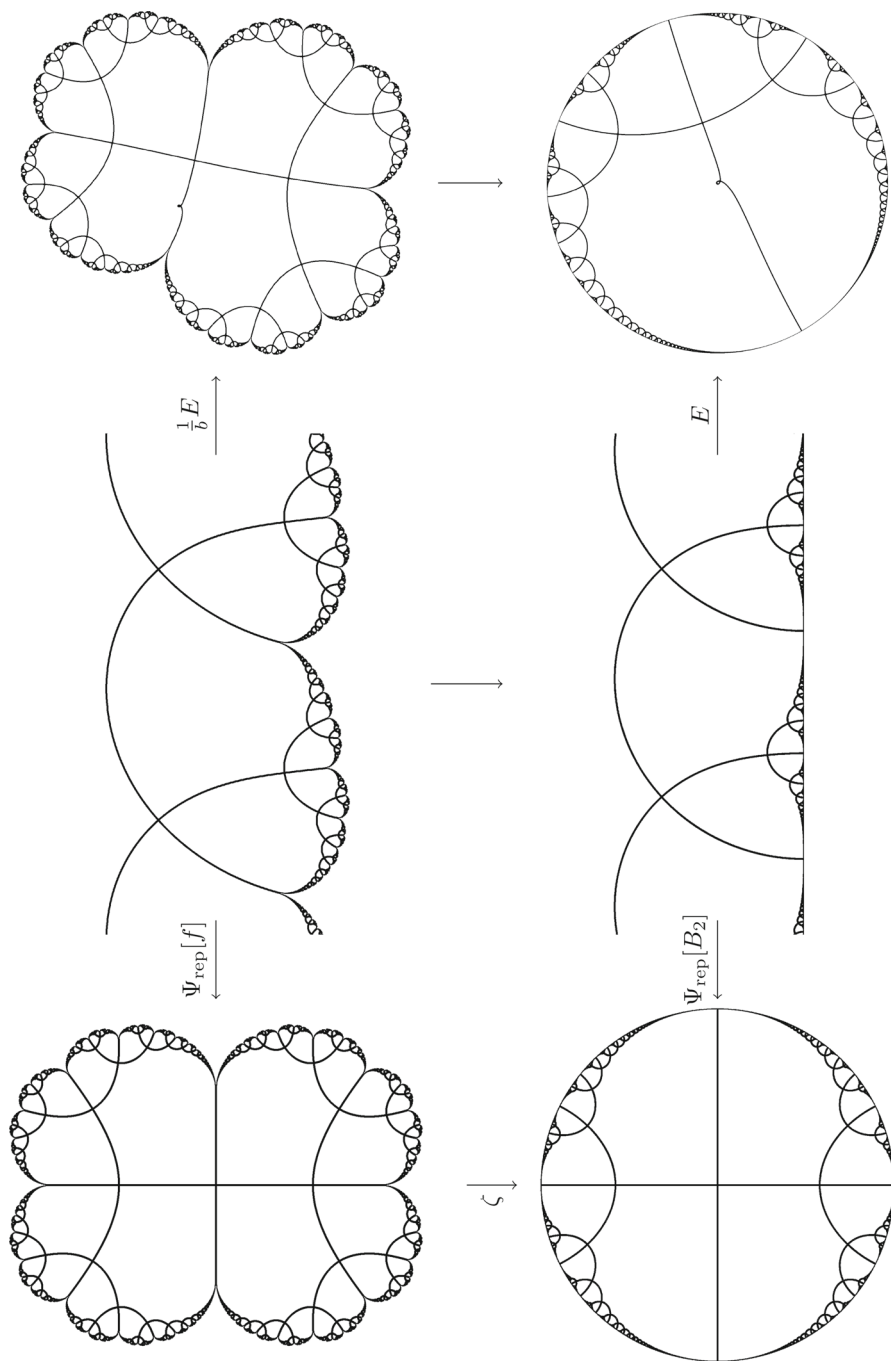
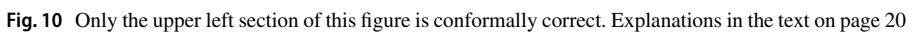


Fig. 9 (Rotated 90°) Illustration of Theorems 9 and 13 for $f(z) = z^2 + 1/4$. See the text page 19 for a description

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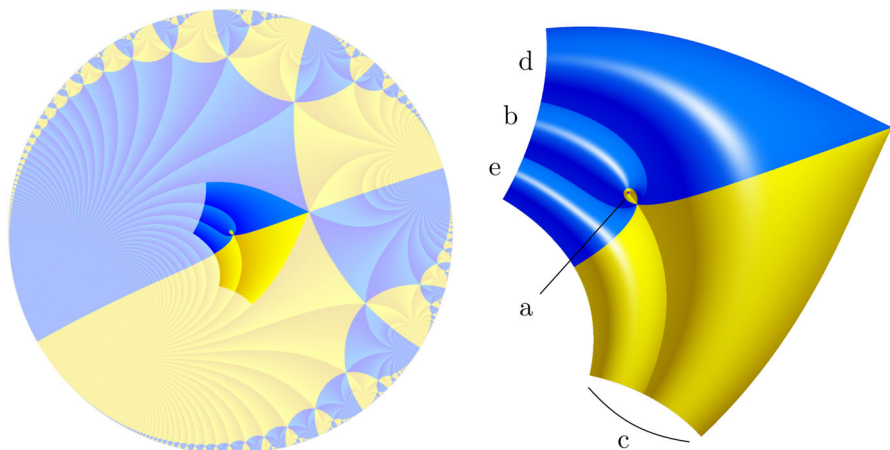


Fig. 11 Caption in the text. Note that compared to the upper right part of Fig. 10, there is a supplementary corner. The picture is accurate

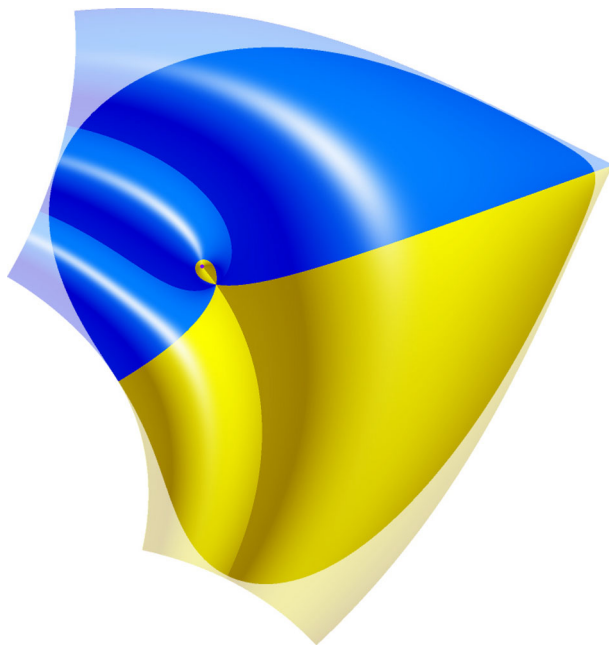


Fig. 12 Comparison of \mathcal{A} and \mathcal{B} . The picture is accurate. Though it is hard to see, the boundary of the light-toned domain and the boundary of the color-saturated domain are disjoint

3.1 A Convenient Notation

Given $r \in]0, 1[$ and a subset U of \mathbb{C} conformally equivalent to \mathbb{D} and containing 0, we will denote

$$U \odot r = \{z \in U \mid d_U(0, z) < d_{\mathbb{D}}(0, r)\}.$$

Note that $U \odot r = \phi(B(0, r))$, where $\phi : \mathbb{D} \rightarrow U$ is a conformal isomorphism mapping 0 to 0.

Recall that we denoted $E(z) = e^{2\pi iz}$, which is a universal cover from \mathbb{C} to \mathbb{C}^* . Given a set of the form $V = E^{-1}(U)$ where U is as above, we will denote

$$V \Vdash r = E^{-1}(U \odot r).$$

3.2 Outline

Our main theorem will be proved in two steps. Let us fix in this section some $d \in \mathbb{N}$ with $d \geq 2$. From now on, parabolic renormalization refers to upper parabolic renormalization. In Sect. 2.2 we defined the objects $\Phi_{\text{attr}}[B_d]$, $\Psi_{\text{rep}}[B_d]$, $h[B_d]$, $\mathcal{H}[B_d]$ and $\mathcal{R}[B_d]$ and adopted specific normalizations for them (except for $\mathcal{R}[B_d]$). In particular, we chose to define $h[B_d]$ on the upper half plane \mathbb{H} only. The map $\mathcal{H}[B_d]$ is defined on \mathbb{D} , maps 0 to 0 and satisfies $\mathcal{H}[B_d] \circ E = E \circ h[B_d]$. The map $\mathcal{R}[B_d]$ is defined as

$$\mathcal{R}[B_d] = b\mathcal{H}[B_d],$$

where $b \in \mathbb{C}^*$ is chosen so that $\mathcal{R}[B_d]'(0) = 1$. In Sect. 2.2 we introduced a semi-conjugate C_d of B_d by a 2:1 rational map, such that C_d has only one attracting petal, and we gave relations between the objects for B_d and the objects for C_d . Note that $\mathcal{R}[B_d]$ coincides with $\mathcal{R}[C_d]$ and

$$\text{Dom } \mathcal{R}[B_d] = \mathbb{D}.$$

In the introduction, we introduced, Definition 14, a class \mathfrak{R}_d that we will also denote by \mathcal{F} :

$$\mathcal{F} = \mathfrak{R}_d = \{\mathcal{R}[B_d] \circ \phi^{-1} \mid \phi : \mathbb{D} \rightarrow \mathbb{C} \text{ is univalent and } \phi(z) = z + \mathcal{O}(z^2)\}.$$

Let

$$\mathcal{F}_\varepsilon = \{\mathcal{R}[B_d] \circ \phi^{-1} \mid \phi : B(0, 1 - \varepsilon) \rightarrow \mathbb{C} \text{ is univalent and } \phi(z) = z + \mathcal{O}(z^2)\}.$$

In other words, \mathcal{F} is the invariant class of Shishikura, Lanford, Yampolsky consisting of maps f with a fixed point at the origin tangent to the identity and such that f is structurally equivalent to the renormalization of the Blaschke product B_d , and \mathcal{F}_ε is a class of maps having only a subset of this structure. The smaller the ε , the more

the structure. Note that $\mathcal{F} = \mathcal{F}_0$. To be more precise and to stick to the language introduced in Sect. 1.2, let I be a singleton. If we mark the origin by the unique map $I \rightarrow \{0\}$, maps in \mathcal{F}_ε are all $(I, \widehat{\mathbb{C}})$ -structurally equivalent.

The maps $f \in \mathcal{F}$ have the same set of singular values as $\mathcal{R}[B_d]$ and they have the same nature: $\{0, v, \infty\}$ for some $v \in \mathbb{C}^*$ that depends only on d , with 0 and ∞ two asymptotic values, $\mathcal{R}[f]^{-1}(\{0\}) = \{0\}$, $\mathcal{R}[f]^{-1}(\{\infty\}) = \emptyset$ and v is a critical value that is not an asymptotic value, and $\mathcal{R}[f]^{-1}(\{v\})$ consists in regular points and critical points of degree d .

All maps $f \in \mathcal{F}$ are tangent to the identity at the origin. For the following statement, recall that \mathcal{S}_d is the class of Definition 8.

Proposition 25

$$\mathcal{F} \subset \mathcal{S}_d.$$

Proof This has already been mentioned in the introduction and follows from Lemma 10, let us give details: $f \in \mathcal{F}$ has exactly the same singular values as $\mathcal{R}[B_d]$, hence $f \in \mathcal{S}_{d'}$ for some d' . We also have to check that $d = d'$. The singular value of f in the immediate basin A is v which is not an asymptotic value hence $d' \neq \infty$ (see Definition 8). Then by Definition 8 again, f has a critical point of degree d' in A , hence $d' = d$. \square

We will prove the following more precise version of the main theorem:

Theorem 26 *The main theorem page 9 holds with \mathcal{B} = the structure of maps $f \in \mathcal{F}_{\varepsilon_1}$ with marked point 0 and \mathcal{A} = the substructure $\mathcal{F}_{\varepsilon_0}$, for some pair $\varepsilon_0 > \varepsilon_1$.*

The class of Schlicht maps is denoted by \mathcal{SL} , thus $\mathcal{F} = \{\mathcal{R}[B_d] \circ \phi^{-1} \mid \phi \in \mathcal{SL}\}$. The two steps are the following:

- (1) Contraction: for $f \in \mathcal{F}$ denote $f = \mathcal{R}[B_d] \circ \phi_1^{-1}$, $\phi_1 \in \mathcal{SL}$. Then by Theorem 20, with an appropriate normalization, $\mathcal{R}[f]$ is of the form $\mathcal{R}[B_d] \circ \phi_2^{-1}$, $\phi_2 \in \mathcal{SL}$. We will prove that “the definition of $\mathcal{R}[f]$ on $\text{Dom}(\mathcal{R}[f]) \odot (1 - \varepsilon)$ uses only iteration of f on $\text{Dom}(f) \odot (1 - \varepsilon')$ where $\varepsilon' \gg \varepsilon$ ”.
- (2) Perturbation: for a map $f \in \mathcal{F}$, we will define a continuous deformation $f_t \in \mathcal{F}_t$. Every map in \mathcal{F}_t will be a deformation of a map in \mathcal{F} . We will prove that $\mathcal{R}[f_t]$ has structure at least \mathcal{F}_ε , provided $t \leq \varepsilon'/K$ for some $K > 1$, where ε' is given by the first step.

Let us give a slightly more detailed formulation of these two steps; we leave here some imprecisions; they will be fully stated and proven in details in Sect. 3.7 to 3.10.

Step 1: Let $E(z) = e^{2\pi iz}$, Φ_{attr} the extended attracting Fatou coordinate of f , Ψ_{rep} the extended repelling inverse Fatou coordinate of f , and recall that $\mathcal{R}[f](z)$ can be defined (up to pre and post composition by two linear maps) as

$$E(\Phi_{\text{attr}}(f^m(\Psi_{\text{rep}}(u)))),$$

where $u \in E^{-1}(z)$ is chosen so that it belongs to the image of the repelling petal by the repelling Fatou coordinates and $m \in \mathbb{N}$ is chosen so that $f^m(\Psi_{\text{rep}}(u))$ belongs to

the attracting petal. So we are following the orbit of $w = \Psi_{\text{rep}}(u)$ under iteration of f from the repelling petal to the attracting petal. The claim is that this orbit stays in $\text{Dom}(f) \odot 1 - \varepsilon'$. Now recall that by the properties of the extended repelling Fatou parametrization, we have $f^k(w) = \Psi_{\text{rep}}(u + k)$ and that the domain of definition of h is invariant by the translation T_1 . Therefore, using that $E^{-1}(\text{Dom} \mathcal{R}[f])$ is equal to the translate by an appropriate complex constant of the domain of the horn map h , point (1) above can be stated as follows:

$$\Psi_{\text{rep}}(\text{Dom}(h) \parallel 1 - \varepsilon) \subset \text{Dom}(f) \odot 1 - \varepsilon'.$$

The relation $\varepsilon' \gg \varepsilon$ will take the form:

$$\log \frac{1}{\varepsilon'} \leq c' + c \log \left(1 + \log \frac{1}{\varepsilon} \right)$$

for some positive constants c, c' (Proposition 45).

Step 2: In the perturbation part, given $r = 1 - t_0$ and $f \in \mathcal{F}_r$, we define an element $f_0 \in \mathcal{F}$ together with a smooth interpolation $f_t, t \in [0, t_0]$, between f_0 and $f = f_{t_0}$. It has the following form:

$$f_t = \mathcal{R}[B_d] \circ \phi_t^{-1}.$$

The map ϕ_t is a univalent map, defined on $B(0, 1 - t)$ with $\phi_t(0) = 0$ and $\phi_t'(0) = 1$ and is defined as follows: let $r_t = 1 - t$, decompose $f(z) = \mathcal{R}[B_d] \circ \tilde{\phi}^{-1}$, let $\phi(z) = r_{t_0}^{-1} \tilde{\phi}(r_{t_0} z)$, whence $\phi \in \mathcal{SL}$, and define

$$\phi_t(z) = r_t \phi(r_t^{-1} z).$$

The map ϕ_t is an isomorphism from $B(0, 1 - t)$ to $r_t \cdot \text{Dom}(f)$. In particular ϕ_t is *not* the restriction⁷ of ϕ to $B(0, 1 - t)$, and

$$\text{Dom } f_t = r_t \cdot \text{Dom } f,$$

and is, thus, usually *not* equal to $\text{Dom}(f) \odot r_t$.

Now since $f_0 = \mathcal{R}[B_d] \circ \phi^{-1}$ belongs to \mathcal{F} , its renormalization $\mathcal{R}[f_0]$ decomposes as $\mathcal{R}[B_d] \circ \phi_2^{-1}$ for some Schlicht map ϕ_2 . By the first step, given $\varepsilon > 0$ and a point $z \in \text{Dom } \mathcal{R}[f_0] \odot (1 - \varepsilon) = \phi_2(B(0, 1 - \varepsilon))$, we know that the value of $\mathcal{R}[f_0]$ is obtained through iteration under f_0 of a point w in the repelling petal of f_0 , point whose orbit remains in $\text{Dom } f_0 \odot 1 - \varepsilon' = \phi(B(0, 1 - \varepsilon'))$ with $\varepsilon' \gg \varepsilon$. We will then vary t from 0 to t_0 and follow by continuity the points in the orbit of w , not by fixing the initial value, but instead by imposing that their attracting Fatou coordinate stays the same, where we normalize the attracting Fatou coordinates (it varies with t

⁷ It would be too much to ask for an interpolation $f_t = \mathcal{R}[B_d] \circ \phi_t^{-1}$ for which ϕ_t is the restriction of some ϕ to $B(0, 1 - t)$. Look for instance at $t = r$: this would mean that the initial univalent map $\tilde{\phi}$ is the restriction to $B(0, r)$ of the univalent map ϕ . But there are plenty of univalent maps $\tilde{\phi}$ on $B(0, r)$ that are not the restriction of a univalent map defined on $B(0, 1)$.

since f_t does) by putting its critical values at the nonnegative integers. In particular, w moves with t . A local study shows that the tail of the orbit will not move much. The motion of the other points will be bounded from above inductively by iterating backwards along the orbit, until we reach w . We will measure the motion in terms of the hyperbolic metric on the complement in \mathbb{C} of the post-critical orbit of f_0 . The study will show (Proposition 66) that there is some $K > 0$ independent of f (necessarily $K > 1$) such that, provided ε' is small enough, an orbit that is initially completely contained in $\text{Dom}(f_0) \odot 1 - \varepsilon'$ survives all the way as t varies from 0 to ε'/K . Thus, $\mathcal{R}[f_t]$ has at least structure \mathcal{F}_ε provided $t \leq \varepsilon'/K$. The main theorem thus holds for $\mathcal{A} = (0, f \in \mathcal{F}_{\varepsilon_0})$ and $\mathcal{B} = (0, f \in \mathcal{F}_{\varepsilon_1})$ with $\varepsilon_0 = \varepsilon'/K$ and $\varepsilon_1 = \varepsilon$ with ε small enough, as $\varepsilon' \gg \varepsilon$ will imply $\varepsilon_0 > \varepsilon_1$.

3.3 Normalizations

In the rest of Sect. 3, i.e., in the proof of the main theorem, more precisely of Theorem 26,

- normalized Fatou coordinates refer to the normalization by the asymptotic expansion at infinity, convention numbered 2 on page 80,
- Φ_{attr} will refer to extended attracting Fatou coordinates, normalized according to the same convention,
- Ψ_{rep} will refer to extended inverse of the repelling Fatou coordinates that are normalized according to the same convention,
- $h[f] = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$,
- $\mathcal{R}[f]$ is the parabolic renormalization, normalized by the critical value (convention numbered 3 on page 81); see details below,
- in the second step, we will use the notation Ψ_t and Φ_t to denote the extended repelling inverse Fatou coordinate and the extended attracting Fatou coordinate of f_t , normalized *not* by their asymptotic expansion but according to a convention analog to number 3.

Let f satisfy the hypotheses of Theorem 20. Let us call (only in this paragraph) U the connected component of $\text{Dom}(h[f])$ that contains an upper half plane and Ξ the map such that

$$E \circ h[f]|_U = \Xi \circ E.$$

Then, $\mathcal{R}[f] = M_a \circ \Xi \circ M_b^{-1}$ for a pair of linear maps $M_a : z \mapsto az$ and $M_b : z \mapsto bz$ that depend on f , hence

$$M_a \circ E \circ h[f]|_U = \mathcal{R}[f] \circ M_b \circ E.$$

The constants a and b depend on f .

By Theorem 20, there exists a choice of a and b in the equation above, such that

$$\mathcal{R}[f] = \mathcal{R}[B_d] \circ \phi^{-1},$$

i.e., such that $\mathcal{R}[f] \in \mathcal{F}$. This is the normalization that we choose for $\mathcal{R}[f]$. We have $\mathcal{R}[f]'(0) = 1$ and $\mathcal{R}[f]$ and $\mathcal{R}[B_d]$ have the same (unique) critical value, and these two conditions characterize this choice of normalization. It coincides with convention numbered 3 on page 81. The class \mathcal{F} is stable by renormalization with this convention:

$$\mathcal{R} : \mathcal{F} \rightarrow \mathcal{F}.$$

3.4 Universality

For later reference, let us mention and prove the following two universality statements for the repelling extended Fatou coordinates and the horn map, analog to Theorems 19 and 20.

Proposition 27 (S., L.-Y.) *Let $d < \infty$. Recall $h = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$. Let f be either a holomorphic map as in Theorem 13, i.e., $f \in \mathcal{S}_d$ and its immediate basin is a Jordan domain, or as in Theorem 20, i.e. $f \in \mathcal{F} = \mathfrak{R}_d$. Let $U[f]$ denote the component of the domain of $h[f]$ that contains an upper (resp. a lower) half plane. (Up to a complex rescaling, resp. an inversion and a complex rescaling, the image of $U[f]$ by $E : z \mapsto e^{2\pi iz}$ is the domain of the renormalization of f .) Then, there is a conformal isomorphism $\phi_0 : U[f] \rightarrow U[B_d]$ that commutes with T_1 and such that $\Psi_{\text{rep}}[B_d] \circ \phi_0 = \zeta \circ \Psi_{\text{rep}}[f]$, where $\zeta : A[f] \rightarrow A[B_d] = \mathbb{D}$ is the conjugacy on the immediate parabolic basins of the respective fixed attracting petals, mentioned in Theorem 6.*

Proof First case: for maps as in Theorem 13, this is proved in [21]. Second case: for maps $f \in \mathfrak{R}_d$, the proof proceeds similarly as the proof of Theorem 20: we introduce the same sequence f_n to which the first case applies. We get a sequence of conformal isomorphisms $\phi_n : U[f_n] \rightarrow U[B_d]$ that commute with T_1 and such that $\Psi_{\text{rep}}[B_d] \circ \phi_n = \zeta_n \circ \Psi_{\text{rep}}[f_n]$, $\zeta_n : A[f_n] \rightarrow A[B_d]$. We normalize $\Psi_{\text{rep}}[f_n]$ by the expansion (convention numbered 2 on page 80), in particular, it tends to $\Psi_{\text{rep}}[f]$ on every compact subset of the domain of $\Psi_{\text{rep}}[f]$. By extracting a limit as in the proof of Theorem 20 we have a conformal map ϕ commuting with T_1 from a simply connected subset U of \mathbb{C} with $T_1(U) = U$ with $\text{Dom } \Psi_{\text{rep}}[f] \subset U$ and $\Psi_{\text{rep}}[B_d] \circ \phi = \zeta \circ \Psi_{\text{rep}}[f]$. Again, the inclusion must be an equality for otherwise, by post-composing by $\Phi_{\text{attr}}[B_d]$ we would get that $\mathcal{R}[f]$ is not defined on its maximal domain of holomorphy, in contradiction with Lemma 16. \square

Lemma 28 *For $f \in \mathcal{F}$, let v_f denote the critical value of f and $v'_f = \Phi_{\text{attr}}(v_f)$. There is a conformal map ϕ from the upper component $U[B_d]$ of $\text{Dom}(h[B_d])$ to the upper component $U[f]$ of $\text{Dom}(h[f])$ that commutes with T_1 and such that*

$$T_\tau \circ h[f]|_{U[f]} = h[B_d] \circ \phi^{-1},$$

with $\tau = v'_{B_d} - v'_f$ and $T_\tau(z) = z + \tau$.

Proof By Corollary 7, $\Phi_{\text{attr}}[B_d] \circ \zeta = \tau + \Phi_{\text{attr}}[f]|_A$ for some $\tau \in \mathbb{C}$ and $\zeta : A \rightarrow \mathbb{D}$ the conjugacy from f on its immediate parabolic basin to B_d . By applying to the unique

critical value of $f|_A$, we get $\tau = v'_{B_d} - v'_f$. By Proposition 27, $\Psi_{\text{rep}}[B_d] \circ \phi^{-1} = \zeta \circ \Psi_{\text{rep}}[f]$ for some conformal isomorphism $\phi : U[B_d] \rightarrow U[f]$ that commutes with T_1 . We conclude using $h = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$. \square

3.5 Chessboards

Just before we begin the proofs, let us recall that maps $f \in \mathcal{F}$ have a *structural chessboard* and a *dynamical chessboard*. The first is a partition of $\text{Dom } f$ that is a pre-image by f of the partition of \mathbb{C}^* cut by the circle of center 0 and passing through the critical value of f . The second is a partition of the basin (or of the immediate basin) of the parabolic point $z = 0$ of f , and is f -invariant. The second is also a structural object w.r.t. $\Phi_{\text{attr}}[f]$. See Sect. 2.3 for more details.

We defined a chessboard for the horn maps h associated with parabolic points of maps $f \in \mathcal{F}$ (more generally to maps f in the class \mathcal{S}_d of Definition 8). It is the preimage in repelling Fatou coordinates of the dynamical chessboard of f and it is also the preimage by h of the partition of its range cut by a horizontal line. There is a box that contains an upper half plane, we call it the *main upper box* of h . Similarly, the box that contains a lower half plane is called the *main lower box* of h .

The map ϕ introduced in Lemma 28 maps the chessboard decomposition of $h[B_d]$ to the chessboard decomposition of $h[f]$.

3.6 Toolkit

In this section, we redo classical computations on Fatou coordinates and first terms of their expansion. We add dependence on a map staying in a compact class and put the emphasis on uniformity of the bounds obtained. The section mainly serves as a reference for the rest of the text. The trusting reader may skip it.

3.6.1 Compact Classes of Parabolic Maps with One Attracting Petal

Proposition 29 Assume \mathcal{G} is a set of holomorphic maps $g : \mathbb{D} \rightarrow \mathbb{C}$ with $g(z) = z + c_g z^2 + \dots$, that \mathcal{G} is compact for the topology of local uniform convergence and that c_g is never 0, i.e., that g has one attracting petal. Denote γ_g the iterative residue of g . Let \log_p be the principal branch of the complex logarithm. Then, there exists r_0 such that $\forall g \in \mathcal{G}$

- the disk D_{attr} of diameter $[0, r_0 e^{i\alpha}]$, where α is the direction of the attracting axis of g , is contained in the parabolic basin of g ; $g(D_{\text{attr}}) \subset D_{\text{attr}}$ and every orbit in the parabolic basin eventually enters D_{attr} ;
- the extended attracting Fatou coordinate of g is injective on D_{attr} and maps D_{attr} to a set of the form $\{z \in \mathbb{C} \mid \text{Re}(z) > \zeta(\text{Im}(z))\}$ with $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ an analytic function (that depends on g) satisfying $\zeta(x)/x \rightarrow 0$ when $x \rightarrow \pm\infty$;
- on D_{attr} , the normalized attracting Fatou coordinates Φ of g and the map $\tilde{\Phi} : z \mapsto \frac{-1}{c_g z} - \gamma_g \log_p \frac{-1}{c_g z}$ have a difference uniformly bounded by a quantity that is independent of g .

The above points also hold if r_0 is replaced by any smaller positive real.

Proof The techniques in this proof are standard (see [7,15,20,26]). We will insist here on providing uniformity of the bounds as g varies in \mathcal{G} .

By compactness, uniformly on \mathcal{G} :

- c_g is bounded away from 0: $\exists \varepsilon > 0$ such that $\forall g \in \mathcal{G}$, $|c_g| \geq \varepsilon$;
- g is bounded on $B(0, 1/2)$: $\exists K > 0$ such that $\forall g \in \mathcal{G}$, $|g| \leq K$ on $B(0, 1/2)$.

Also, by Cauchy's inequality,

$$|c_g| \leq 4K.$$

Since $|g(z) - z| \leq K + 1/2$ on $B(0, 1/2)$, we get $|g(z) - z| \leq K'|z|^2$ with $K' = 4K + 2$, and in particular g does not vanish on $B(0, 1/K')$ except at the origin.

We will make a series of change of variables $z \mapsto u \mapsto w \mapsto \xi$ with

$$u = \frac{-1}{c_g z}, \quad w = u - \gamma_g \log_p u, \quad Z = \Phi(z),$$

where \log_p denotes the principal branch of the logarithm. We will denote $z' = f(z)$ and use the notation $u \mapsto u'$, ..., $Z \mapsto \mathbb{Z}'$ for the dynamical systems $z \mapsto z'$ will be conjugated to.

The first change of variable is injective on \mathbb{C}^* . It maps D_{attr} to the half plane

$$H_{\text{attr}} : \operatorname{Re}(u) > U_0(g) = 1/r_0|c_g|.$$

We have the following asymptotic expansion

$$u'_{\infty} = u + 1 + \frac{\gamma_g}{u} + \mathcal{O}(u^{-2}).$$

The condition $z \in B(0, 1/K')$ is equivalent to $|u| > K'/|c_g|$. Under this condition the map $u \mapsto u'$ is holomorphic, and depends continuously on g . From compactness of \mathcal{G} , it follows that these restrictions form a compact family too. In particular, if we further restrict to $|u| > 1 + K'/|c_g|$, we get by a simple application of the maximum principle that

$$\begin{aligned} |u' - (u + 1)| &\leq M_1/u \\ |u' - (u + 1 + \frac{\gamma_g}{u})| &\leq M_2/u^2, \end{aligned}$$

for some constants M_1, M_2 independent of $g \in \mathcal{G}$. Thus for $r_0 \leq 1/(|c_g| \max(1 + K'/|c_g|, 4/M_1))$, we have

$$\frac{M_1}{|u|} \leq \frac{1}{4},$$

thus,

$$|u' - (u + 1)| \leq 1/4,$$

thus, the set H_{attr} is invariant under the dynamics of $u \mapsto u'$, so D_{attr} is invariant under $z \mapsto z'$. It is also easy to see that in the u -coordinate, an orbit tending to ∞ must eventually get into H_{attr} . The right-hand side of the condition $r_0 \leq 1/(|c_g| \max(1 + K'/|c_g|, 4/M_1))$ depends continuously on g and reaches, thus, a positive minimum: it is satisfied as soon as $r_0 \leq r_1$, where r_1 is independent of g .

The constant γ_g is finite and depends continuously⁸ on g . Thus, it is bounded over \mathcal{G} , say by Γ :

$$|\gamma_g| \leq \Gamma.$$

The change of variable $w = u - \gamma_g \log_p u$ has derivative $1 - \gamma_g/u$. It is, thus, injective on the convex set $\text{Re}(u) > 2|\gamma_g|$. Thus, when $r_0 \leq r_2$, where $r_2 = \min_{g \in \mathcal{G}} (1/2|\gamma_g c_g|) > 0$, then $\forall g \in \mathcal{G}$, the map $u \mapsto w$ is injective on H_{attr} . We will require in fact a bit more: $r_0 \leq r'_0 = r_2/2$, so that $|\frac{\partial w}{\partial u} - 1| \leq \frac{1}{4}$. This implies that the image of H_{attr} by $u \mapsto w$ is a set that is of the form $\text{Re}(w) > \zeta(\text{Im}(w))$ for some analytic function $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ that depends on g and satisfies $|\zeta'(y)| < 1/\sqrt{15}$. Moreover, $\zeta(y)/y \rightarrow 0$ when $y \rightarrow \pm\infty$ because $w \sim u$ when $|u| \rightarrow \infty$. In this new coordinates, we get

$$w' - w = \int_{[u, u']} \left(1 - \frac{\gamma_g}{a}\right) da,$$

whence

$$w' - w = 1 + \frac{\gamma_g}{u} + \frac{\leq M_2}{u^2} - \gamma_g \log \left(1 + \frac{1}{u} + \frac{\leq M_1}{u^2}\right),$$

where $\leq M_2$ means a complex number that depends on u but whose module is at most M_2 ; we require $r_0 \leq r_3$, where r_3 is chosen independent of g and so that the quantity $\frac{1}{u} + \frac{\leq M_1}{u^2}$ has necessarily modulus $< 1/2$: recall that $1/u = -c_g z$ and that $|c_g| \leq 4K$. We can then apply the following estimate: $|a| < 1/2 \implies |\log_p(1+a) - a| \leq L_0|a|^2$ for some $L_0 > 0$. Hence, (thanks to a cancellation of the term γ_g/u)

$$w' - w = 1 + \frac{\leq M_2}{u^2} + \gamma_g \frac{\leq M_1}{u^2} + \gamma_g \frac{\leq (1 + 1/4)^2 L_0}{u^2},$$

(recall that $M_1/|u| < 1/4$). Thus for some constant M_3 independent of g :

$$|w' - (w + 1)| \leq \frac{M_3}{u^2}.$$

⁸ Because it is equal to $1 - a_3/c_g^2$ if we denote $g(z) = z + c_g z^2 + a_3 z^3 + \dots$.

The Fatou coordinates can be defined by

$$\Phi(z) = \mu + \lim(w_n - n),$$

where μ is a constant (that depends on the normalization) and w_n is the n -th iterate of w under the dynamics. Since $\operatorname{Re}(u_n) > \operatorname{Re}(u_0) + \frac{3}{4}n$ and $\operatorname{Re}(u_0) \geq \frac{1}{r_0|c_g|}$, using $r_0 \leq r_4 = \min(r_1, r'_2, r_3)$, we, thus, get

$$\lim |w_n - (w_0 + n)| \leq \sum \frac{M_3}{|u_n|^2} \leq \sum \frac{M_3}{\left(\frac{1}{4Kr_4} + \frac{3/4}{n}\right)^2} = M_4.$$

Thus, $|\Phi(z) - (\mu + w)| \leq M_4$ holds on D_{attr} for all g . The normalizing constant μ is so that $\Phi(z) = w + o(1)$ as $z \rightarrow 0$ (iff. $w \rightarrow \infty$) and therefore, $|\mu| \leq M_4$ whence: $\forall g \in \mathcal{G}, \forall z \in D_{\text{attr}}$,

$$|\Phi(z) - w| \leq 2M_4.$$

Recall that H_{attr} is the image of D_{attr} in the u -coordinate and that it is equal to the half plane $\operatorname{Re}(u) > U_0(g) = 1/r_0|c_g|$. Let $U_4(g) = 1/r_4|c_g|$ and H_4 be defined by $\operatorname{Re}(u) > U_4(g)$. Let $\Theta : H_4 \rightarrow \mathbb{C}, u \mapsto \Phi(z)$. Then $|\Theta(u) - (u - \gamma_g \log_p(u))| \leq 2M_4$ and by Cauchy's inequality, $|\Theta'(u) - (1 - \gamma_g/u)| \leq 2M_4/(\operatorname{Re}(u) - u_4)$. In particular, the image of H_{attr} by $u \mapsto Z = \Phi(z)$ is of the form $\operatorname{Re}(Z) > \zeta(\operatorname{Im}(Z))$ for some function $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ provided $r_0 \leq r_5 = r_4/(1 + 8M_4)$ so that $2M_4/(\operatorname{Re}(u) - u_4) \leq 1/4$ and provided $r_0 \leq r_6 = 1/16K\Gamma$ so that $|\gamma_g/u| \leq 1/4$. The fact that $\zeta(y)/y \rightarrow 0$ as $y \rightarrow \pm\infty$ follows again from $|\Theta(u) - (u - \gamma_g \log_p(u))| \leq 2M_4$.

We can now fix the value of r_0 to $\min(r_5, r_6)$ (or any smaller value) and this gives us a set D_{attr} that satisfies all points stated in the proposition. \square

For reference, let us extract the following point (far from being optimal) from the proof:

Lemma 30 *Under the assumptions of Proposition 29, the change of variable $u = -1/c_g z$ conjugates $z \mapsto z' = g(z)$ to $u \mapsto u'$ satisfying:*

$$\forall z \in B(0, r_0), \forall g \in \mathcal{G}, |u' - (u + 1)| \leq \frac{1}{4}.$$

Similar arguments provide:

Proposition 31 *Under the same assumptions as in the Proposition 29, let*

$$D_{\text{rep}} = -D_{\text{attr}}.$$

Then, for r_0 small enough, the following holds: $\forall g \in \mathcal{G}$,

- *there is a branch ℓ of g^{-1} defined on a neighborhood of 0 containing D_{rep} such that $\ell(D_{\text{rep}}) \subset D_{\text{rep}}$, D_{rep} is contained in the parabolic basin of ℓ , every orbit in the parabolic basin of ℓ eventually enters D_{rep} ;*

- a normalized repelling Fatou coordinate Φ_{rep} for g is defined on D_{rep} ; it is injective on this set and maps it to a domain of the form $\text{Re}(z) < \zeta(\text{Im}(z))$ for some analytic function ζ ;
- $\Phi_{\text{rep}} - \tilde{\Phi}_{\text{rep}}$ is uniformly bounded on D_{rep} by a constant M_{rep} independent of g , where $\tilde{\Phi}_{\text{rep}} = \frac{-1}{c_g z} - \gamma_g \log_p \frac{1}{c_g z}$ (notice the change of sign inside the log compared to attracting Fatou coordinates);

We will also need a control on the inverse Fatou coordinates, that we easily deduce from the control on the Fatou coordinates:

Proposition 32 *Using the notations of Proposition 29, provided r_0 was chosen small enough, then for all $g \in \mathcal{G}$:*

- Let $\Psi = \Phi^{-1}$. Then, the difference between $-1/c_g \Psi(Z)$ and $Z + \gamma_g \log Z$ is bounded by a quantity independent of g and of $Z \in \Phi(D_{\text{attr}})$.
- The domain of definition of Φ^{-1} , i.e., $\Phi(D_{\text{attr}})$, contains the set

$$\{Z \in \mathbb{C} \mid \text{Re } Z > \xi(\text{Im } Z)\},$$

where ξ is a function independent of g and satisfying $\xi(y) = \mathcal{O}(\log |y|)$ as $y \rightarrow \pm\infty$.

Proof We will use the notations of the proof of Proposition 29. There was a change of variables $u = s(z) = -1/c_g z$ and a bound

$$|Z - (u - \gamma_g \log_p u)| \leq M,$$

for some constant M independent of g , where Z .

$$|Z - (u - \gamma_g \log_p u)| \leq M.$$

There exists $C > 0$ such that for $|z| > C$ then $\Gamma |\log_p z| + M < |z|/4$ (recall $\Gamma = \sup_{g \in \mathcal{G}} |\gamma_g|$), whence if $r_0 < 1/C \sup_{g \in \mathcal{G}} |c_g|$ then H_{attr} is contained in $|u| > C$ and thus: $|\Theta(u) - u| < |u|/4$ i.e. $|\Theta(u)/u - 1| < 1/4$, i.e.,

$$\forall Z \in \Phi(D_{\text{attr}}), |Z/u - 1| < 1/4.$$

Now

$$\begin{aligned} |u - (Z + \gamma_g \log_p Z)| &\leq |Z - (u - \gamma_g \log_p u)| + |\gamma_g| |\log_p u - \log_p Z| \\ &\leq M + \sup |\gamma_g| \left| \log_p \frac{u}{Z} \right| \\ &\leq M + \sup |\gamma_g| \log \frac{3}{2}. \end{aligned}$$

The proof of the second point is similar. Recall that H_{attr} depends on g , and is defined by $\text{Re } z > a_g$, where

$$a_g = 1/r_0 |c_g|.$$

The image $\Phi(D_{\text{attr}})$ is of the form $\{z \in \mathbb{C} \mid \text{Re } z > \zeta(|\text{Im } z|)\}$ where $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ is an analytic function that depends on g . Let $\Theta(u) = Z = \Phi(s^{-1}(u))$. Then, $\Phi(D_{\text{attr}}) = \Theta(H_{\text{attr}})$. The map Θ extends to a neighborhood of the closure of H_{attr} and still satisfies $|\Theta(u) - (u - \gamma_g \log_p u)| \leq M$ on this closure. The curve $\zeta(\mathbb{R})$ is the image of ∂H_{attr} under this extension of Θ . Let $b \in \mathbb{R}$ parameterize a point $u = a_g + ib$ varying on ∂H_{attr} and denote $x + iy = \Theta(a_g + ib)$. Then, $\log_p u = \log |u| + i \arg_p(u)$ and $\arg_p(u) < \pi/2$ thus the bound $|\Theta(u) - (u - \gamma_g \log_p u)| < M$ yields for the real and imaginary parts:

$$\begin{aligned} |x - (a_g - \text{Re}(\gamma_g) \log |a_g + ib|)| &\leq M' := M + \Gamma\pi/2, \\ |y - (b - \text{Im}(\gamma_g) \log |a_g + ib|)| &\leq M'. \end{aligned}$$

There exists $C' > 0$, independent of g , such that for all $b \in \mathbb{R}$, $|\text{Im}(\gamma) \log_p |a_g + ib|| \leq |b|/2 + C'$. The second line thus yields $|b| \leq |b|/2 + C' + |y| + M'$ i.e. $|b| \leq 2|y| + M''$ for some M'' . Whence $x \leq \xi(y) := \sup(a_g) + M' + \Gamma \log |\sup(a_g) + i(2|y| + M'')|$, which is independent of g and has the right order of growth w.r.t. y . \square

Proposition 33 *Under the same assumptions, there exists $h > 0$ such that for all $g \in \mathcal{G}$, the normalized extended repelling inverse Fatou coordinate Ψ_{rep} and the normalized extended horn map $h[g]$ are defined on a set containing the half planes $\text{Im}(z) > h$ and $\text{Im}(z) < -h$, and injective on the union of those half planes. Moreover, for all $r > 0$, there exists $h > 0$ such that for all $g \in \mathcal{G}$, Ψ_{rep} maps these half planes inside the disk $B(0, r)$.*

Proof Let us continue with the notations of the proof of Proposition 29. Note that, decreasing the value of r_0 , we can assume that the maps $g \in \mathcal{G}$ are all injective on $B(0, r_0)$. Without loss of generality, we assume $r < r_0$. Let us again work in the coordinates

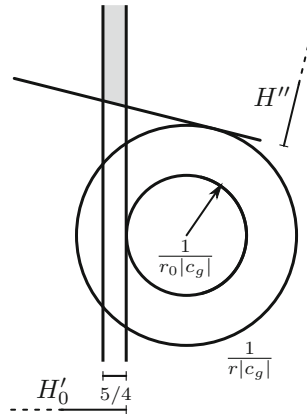
$$u = s(z) = -1/c_g z.$$

Let $D'(r)$ be the disk of diameter $[0, re^{i\alpha}]$ where α is the repelling direction of f . (In particular $D_{\text{rep}} = D'(r_0)$.) The set $D'(r)$ is transformed by s into the half plane $H' : \text{Re}(z) < -1/r |c_g|$. Let us also denote

$$H'_0 : \text{Re}(z) < -1/r_0 |c_g|.$$

Recall that if $|u| > 1/r_0 |c_g|$ then $|u' - (u + 1)| < 1/4$. To shorten formulas, we will work with $\Phi_{\text{rep}}^u(u) = \Phi_{\text{rep}} \circ s^{-1}(u)$, $\tilde{\Phi}_{\text{rep}}^u(u) = \tilde{\Phi}_{\text{rep}} \circ s^{-1}(u) = u - \gamma_g \log_p(-u)$ and $\Psi_{\text{rep}}^u(Z) = s \circ \Psi_{\text{rep}}(Z)$. Consider the line of slope $-1/\sqrt{15}$ that is tangent to the disk $B(0, 1/r |c_g|)$. Consider open half plane H'' above this line: it does not contain

this disk. In particular, $|u' - (u + 1)| < 1/4$ holds on U and thus H'' is stable: $u \in H'' \implies u' \in H''$. Consider now the vertical bi-infinite strip S of width $5/4$ whose rightmost bounding line is the boundary of H'_0 . Its image in repelling Fatou coordinates contains a fundamental domain for the translation $z \mapsto z + 1$. The intersection of S with H'' contains all points $u \in S$ with $\text{Im}(u) > h_1$ for some h_1 that depends on r and r_0 and the lower bound ε on $|c_g|$ mentioned at the beginning of the proof of Proposition 29. Using $|\Phi_{\text{rep}}^u - \tilde{\Phi}_{\text{rep}}^u| < M_{\text{rep}}$ and the upper bound $|\gamma_g| \leq \Gamma$, we deduce that $\Phi_{\text{rep}}^u(S \cap H'')$ contains every point of $\Phi_{\text{rep}}^u(S)$ with imaginary part $\geq h$, where h depends on r and on the other constants but not on g .



Recall that Φ_{rep}^u maps the vertical line bounding H'_0 to a y-graph, i.e., a curve which crosses each horizontal line exactly once. The translate by -1 of this curve is the image by Φ_{rep}^u of a curve C , preimage in H'_0 of $\partial H'_0$ by $u \mapsto u'$. Because of the inequality $|u' - (u + 1)| < 1/4$, we get $C \subset S$. Thus, $\Phi_{\text{rep}}^u(S)$ contains a domain bounded by a y-graph and its translate by -1 , i.e., a fundamental domain for the translation by -1 .

Let us prove that the domain of the extended normalized inverse repelling Fatou coordinate Ψ_{rep} contains all points at height $> h$. Recall Ψ_{rep} is defined by extending Φ_{rep}^{-1} , which is defined only on $\Phi_{\text{rep}}^u(H')$, by setting $\Psi_{\text{rep}}(Z) = g^n(\Phi_{\text{rep}}^{-1}(Z - n))$ for all $n \geq 0$ and all $Z \in \mathbb{C}$ such that the right-hand side is defined. Consider now $Z \in \mathbb{C}$. By the fundamental domain property proved above, there exists $m \in \mathbb{Z}$ such that $Z - m \in \Phi_{\text{rep}}^u(S)$. If $m \leq 0$, then $Z \in \Phi_{\text{rep}}^u(H') = \text{Dom}(\Phi_{\text{rep}}^{-1})$, hence $Z \in \text{Dom } \Psi_{\text{rep}}$. If $m \geq 0$ and $\text{Im}(Z) > h$ then $\text{Im}(Z - m) = \text{Im } Z > h$ and thus we have seen that $u_{-m} := (\Phi_{\text{rep}}^u)^{-1}(Z - m)$ belongs to $H'' \cap S$. Since H'' is stable, the whole forward orbit of u_{-m} belongs to H'' . In particular $g^m(\Phi_{\text{rep}}^{-1}(Z - m))$ is defined, hence $Z \in \text{Dom } \Psi_{\text{rep}}$. We have proven that the half plane $\{Z \in \mathbb{C} \mid \text{Im}(Z) > h\}$ is contained in $\text{Dom } \Psi_{\text{rep}}$.

Let now $Z \in \mathbb{C}$ with $\text{Im}(z) > h$ and let us prove that $\Psi_{\text{rep}}(Z) \in B(0, r)$ and to the parabolic basin. Again consider $m \in \mathbb{Z}$ such that $Z - m \in \Phi_{\text{rep}}^u(S)$. Then, in the case $m \geq 0$ we just saw that the whole orbit of u_{-m} is in H'' , in particular the m -th iterate, which is equal to $\Psi_{\text{rep}}^u(Z)$. Thus the point $\Psi_{\text{rep}}(Z) = s^{-1}(\Psi_{\text{rep}}^u(Z))$

belongs to $B(0, r)$. Also, the orbit of u tends to ∞ hence $\Psi_{\text{rep}}(Z)$ belongs to the basin of the parabolic point of g . In the case $m \leq 0$, then $Z \in \Phi_{\text{rep}}(D'(r))$ and thus, $\Psi_{\text{rep}}(Z) = \Phi_{\text{rep}}^{-1}(Z) \in D'(r) \subset B(0, r)$. Since moreover $Z - m$ satisfies the first case and, thus, the point $\Psi_{\text{rep}}(Z - m)$ belongs to the parabolic basin, we get that $\Psi_{\text{rep}}(Z)$ also belongs to the basin, as it is mapped to the former point by the $|m|$ -th iterate of g .

The proofs for the lower half plane $\{z \in \mathbb{C} \mid \text{Im } z < -h\}$ are similar. Let us prove injectivity of Ψ_{rep} on the union V of $\{z \in \mathbb{C} \mid \text{Im } z < -h\}$ and $\{z \in \mathbb{C} \mid \text{Im } z > h\}$. First, it is injective on $U = \Phi_{\text{rep}}(D'(r))$, because it is equal to Φ_{rep}^{-1} there. The map g is injective on $\Psi_{\text{rep}}(V)$ because the latter is contained in $B(0, r_0)$. The set $\Psi_{\text{rep}}(V)$ is also stable by g , thus g^n is also injective on it. Then, for each n , the map $g^n \circ \Phi_{\text{rep}}^{-1} \circ T_{-n}$ is a composition of injective maps on $T_n(U) \cap V$, and coincides there with Ψ_{rep} . Since the union over n of $T^n(U)$ is the whole complex plane, the claim follows.

Injectivity of $h[g]$ on V is similar, since $h[g]$ is the union over $n \geq 0$ of the maps $T^{-n} \circ \Phi_{\text{attr}}|_{D_{\text{attr}}} \circ g^n \circ \Psi_{\text{rep}}$, which are injective when restricted to V . \square

Let us introduce a weak notion of convergence of analytic maps: let X, Y be connected Riemann surfaces and let $f_n : U_n \rightarrow Y$ and $f : U \rightarrow Y$ be analytic with U and U_n open subsets of X . Endow Y with any metric compatible with its topology. Let us say that f_n tends to f if for all compact subset K of U , K is eventually contained in U_n and f_n tends to f uniformly on K . This does not depend on the choice of the metric.⁹ Note that this does not prevent U_n to have a bigger limit than U . In particular, limits are not unique. We will use the following notation:

$$f_n \rightrightarrows f,$$

which is chosen so to express the fact that f can be contained in limits with a bigger domain. We do not define an associated topology but we will use the notion of sequential continuity with respect to that notion of convergence, as illustrated by the following two properties, whose proofs are left to the reader:

- (1) The composition $f \circ g$ depends continuously on the pair f, g : if $f_k \rightrightarrows f$ and $g_k \rightrightarrows g$ then $f_k \circ g_k \rightrightarrows f \circ g$.
- (2) For a fixed n , f^n depends continuously on f : if $f_k \rightrightarrows f$ then $f_k^n \rightrightarrows f^n$.

For the next statement, recall that Φ_{attr} and ψ_{rep} denote the *extended* Fatou functions.

Proposition 34 (Continuous dependence) *Assume $g_n : U_n \rightarrow \mathbb{C}$ is a sequence of holomorphic maps defined on an open subset U_n of \mathbb{C} containing the origin, with expansion $g_n(z) = z + c_n z^2 + \dots$ at 0, and with $c_n \neq 0$. Assume g is also of this form with $c_g \neq 0$ and that $g_n \rightrightarrows g$. Then $\Phi_{\text{attr}}[g_n] \rightrightarrows \Phi_{\text{attr}}[g]$, $\Psi_{\text{rep}}[g_n] \rightrightarrows \Psi_{\text{rep}}[g]$ and $h[g_n] \rightrightarrows h[g]$.*

Proof The claim on $h = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$ follows from the claims on Φ_{attr} and Ψ_{rep} .

⁹ This definition has the following equivalent topological formulation. Let $X' = \{0, 1, 1/2, 1/3, 1/4, \dots\} \times X \subset \mathbb{R} \times X$ and embed X' with the topology induced by $\mathbb{R} \times X$. Let $W \subset X'$ be defined by $(0, z) \in W \iff z \in U$ and $(1/n, z) \in W \iff z \in U_n$. Let $F : W \rightarrow Y$ defined by $F(0, z) = f(z)$ and $F(1/n, z) = f_n(z)$. Then $f_n \rightrightarrows f \iff [W \text{ is open relative to } X \text{ and } F \text{ is continuous}]$.

Recall that $D_{\text{attr}}[g]$ is the disk of diameter $[0, r_0 e^{i\alpha[g]}]$ where $\alpha[g]$ is the direction of the attracting axis of g , and that r_0 is independent of g . Hence, $D_{\text{attr}}[g]$ depends continuously on g . A compact set K contained in the parabolic basin of g is mapped in $D_{\text{attr}}[g]$ by an iterate g^k . The latter depends continuously on g when k is fixed. Since the center and radius of $D_{\text{attr}}[g]$ depend continuously on g , $g_n^k(K) \subset D_{\text{attr}}[g_n]$ for all n big enough. Continuity, as a function of g , of the restriction of Φ_{attr} to D_{attr} , follows for instance from the third point of Proposition 29 combined with uniqueness of Fatou coordinates: the sequence $\Phi_{\text{attr}}[g_n]$ forms a normal family, and any extracted limit is a Fatou coordinate for g because the functional equation $\Phi_{\text{attr}}[g_n] \circ g_n = T_1 \circ \Phi_{\text{attr}}[g_n]$ passes to the limit, and uniqueness of the normalized Fatou coordinates implies uniqueness of the extracted limit. From the convergence of $\Phi_{\text{attr}}[g_n]$ to $\Phi_{\text{attr}}[g]$ on $D_{\text{attr}}[g]$, we deduce the convergence of $\Phi_{\text{attr}} = \Phi_{\text{attr}}[g_n] \circ g_n^k - k$ to $\Phi_{\text{attr}}[g] \circ g^k - k = \Phi_{\text{attr}}[g]$ on $g^{-k}(D_{\text{attr}}[g])$, and hence on the whole parabolic basin of g .

The proof for Ψ_{rep} is similar. \square

3.6.2 Transferring to \mathcal{F}

Fix some $d \in \{2, 3, \dots, \infty\}$ and recall the definition $\mathcal{F} = \{\mathcal{R}[B_d] \circ \phi^{-1} \mid \phi \in \mathcal{SL}\}$. The conclusions of the previous propositions hold for \mathcal{F} . Indeed, the set of restrictions to \mathbb{D} of maps $A \circ f \circ A^{-1}$ with $A(z) = 4z$ satisfies the assumptions of the propositions. First, the set of Schlicht maps \mathcal{SL} is compact, and by Koebe's one quarter theorem, the domain of their reciprocal contains $B(0, 1/4)$. The restriction of these reciprocals on $B(0, 1/4)$ forms a compact family. We saw in Proposition 25 that $\mathcal{F} \subset \mathcal{S}_d$. In particular, maps in \mathcal{F} have only one attracting petal. This is, therefore, also the case for the conjugate map $A \circ f \circ A^{-1}$. This proves the claim.

Call \tilde{f} the restriction of $A \circ f \circ A^{-1}$ to \mathbb{D} . The conclusions of the previous propositions are easily transposed from \tilde{f} back to f because they were all local (except for Proposition 34, which directly applies): for instance, normalized Fatou coordinates satisfy $\Phi_{\text{attr}}[\tilde{f}](z) = \Phi_{\text{attr}}(A(z))$ for all z in the domain of the left hand side (it is contained in the domain of the right-hand side but not necessarily equal to it, because \tilde{f} is a restriction). Proposition 34 did not assume that the maps are defined on the unit disk, it applies directly, so continuous dependence of $\Phi_{\text{attr}}[f]$ and $\Psi_{\text{rep}}[f]$ holds without restricting the domain.

Recall $h[f]$ has the following expansion:

$$h[f](z) = z + a_{\text{up/down}} + o(1),$$

as $\text{Im}(z) \rightarrow \pm\infty$, where a_{up} and a_{down} are two complex constants. For any map in the class \mathcal{F} , denote $\{0, v_f, \infty\}$ its singular values.¹⁰ The corresponding map $h[f]$ has a set of singular values of the form $v_h + \mathbb{Z}$ where

$$v_h = v_h[f] = \Phi_{\text{attr}}[f](v_f).$$

By Proposition 33 there is a uniform $h > 0$ such that for all $f \in \mathcal{F}$, the domain of $h[f]$ contains the half planes $\text{Im } z > h$ and $\text{Im } z < -h$.

¹⁰ It turns out that v_f is independent of f for a fixed d , but we will not use that fact.

For any $d \in \{2, 3, \dots, \infty\}$, the set \mathcal{F} is sequentially compact, for the notion of convergence defined above. By this, we mean that every sequence $f_n \in \mathcal{F}$ has a subsequence f_k such that $f_k \rightrightarrows f$.¹¹ A sequentially continuous real-valued function over a sequentially compact set is bounded. This implies the following proposition.

Proposition 35 *For any $d \in \{2, 3, \dots, \infty\}$ over the class \mathcal{F} , the following holds:*

- (1) *(bound in the normalized attracting Fatou coordinates)*
 $\exists M$ such that $\forall f \in \mathcal{F}$, $|\operatorname{Im}(v_h)| \leq M$.
- (2) *(bound on the horn map at the ends of the cylinder)*
 $\exists M$ such that $\forall f \in \mathcal{F}$, $|a_{\text{up}}[f]| \leq M$ and $|a_{\text{down}}[f]| \leq M$.
- (3) *(bound in the normalized repelling Fatou coordinates)*
 $\exists M$ such that $\forall f \in \mathcal{F}$, the main¹² upper and lower chessboard boxes of $h[f]$ respectively contain the half planes $\operatorname{Im}(z) > M$ and $\operatorname{Im}(z) < -M$.

Proof The map $f \in \mathcal{F} \mapsto v_h \in \mathbb{C}$ is sequentially continuous by Proposition 34. The set \mathcal{F} being sequentially compact, its image by $f \mapsto v_h$ is sequentially compact in \mathbb{C} (i.e., compact), thus bounded. The first point follows.

For the second, by periodicity and the maximum principle and according to the expansion, the distance $|h[f](z) - z|$ is bounded over $\operatorname{Im}(z) > h+1$ by its supremum over a segment of length 1 inside the line $\operatorname{Im}(z) = h+1$, for instance the segment $[i(h+1), 1+i(h+1)]$. Continuous dependence implies the distance is uniformly bounded as f varies in \mathcal{F} . Since a_{up} is the limit of this difference as $\operatorname{Im}(z) \rightarrow +\infty$, this implies the bound on a_{up} . (Alternatively, one can use the fact that $a_{\text{up}} = \int_{ih+i}^{ih+i+1} (h(z) - z) dz$.) The proof is similar for a_{down} .

For the third, we will use the following trick: first $h[f]$ is an analytic isomorphism commuting with $T_1(z) = z + 1$ from the upper and the lower structural boxes to one of the half plane delimited by $v_h + \mathbb{R}$. By Koebe's one quarter theorem, the upper box must contain $\operatorname{Im}(z) > \frac{\log 4}{2\pi} + \operatorname{Im}(v_h) - \operatorname{Im}(a_{\text{up}})$. The previous bounds allows to conclude. The proof is similar for the other half plane. \square

Let us now prove an independent proposition. Let f be a map in \mathcal{S}_d . Then we can apply Lemma 28 about universality and we know that $\Phi_{\text{attr}} : A \rightarrow \mathbb{C}$ is structurally equivalent to $\Phi_{\text{attr}}[B_d]$ for some $d \in \{2, 3, \dots, \infty\}$. The singular values of Φ_{attr} are ∞ and the points of the form $\Phi_{\text{attr}}(v) - n$ with $n > 0$ (see for instance Proposition 2 in [3], where a notion of *ramified cover* is used: their proposition implies that Φ_{attr} is a cover outside ∞ and the critical values).

Proposition 36 *Under these conditions, the preimage Γ by Φ_{attr} of the horizontal half line $\Phi_{\text{attr}}(v) + [0, +\infty[$ has a connected component \mathcal{C} that is a curve starting from the singular value of f in A and ending at the parabolic point. It is stable: $f(\mathcal{C}) \subset \mathcal{C}$, and contained in the common boundary of the two principal dynamical chessboard boxes of f .*

This curve will be called the *principal curve*. It contains in particular the orbit of the singular value of f . Note that all connected components of Γ are curves since the horizontal half line considered contains no singular value of Φ_{attr} .

¹¹ Note that if we restrict our notion of convergence to \mathcal{F} , we recover uniqueness of the limit.

¹² terminology introduced in Sect. 3.5.

Proof It is enough to prove the proposition for B_d , which is easy because the latter map is real preserving and its singular value is on the real line. Then it transfers to f by universality: all claims are immediate except the statement that \mathcal{C} tends to the parabolic point. The latter follows for instance from \mathcal{C} being the concatenation of the successive images by f of its part from v_f to $f(v_f)$. \square

Let us go back to maps $f \in \mathcal{F}$. As we remarked before, convergence of maps $f_n \rightrightarrows f$ where f_n and f belong to \mathcal{F} is well behaved: limits are unique and in fact it is equivalent to the classical notion of convergence of a sequence with respect to a (metrizable) topology making \mathcal{F} compact: Indeed, let $f_n, f \in \mathcal{F}$. Write $f_n = \mathcal{R}[B_d] \circ \phi_n^{-1}$ and $f = \mathcal{R}[B_d] \circ \phi^{-1}$ with ϕ_n and $\phi \in \mathcal{SL}$ (uniquely determined). Then, the following are equivalent:

- (1) $f_n \rightrightarrows f$,
- (2) for some $\varepsilon > 0$, the map f_n tends to f uniformly on $B(0, \varepsilon)$,
- (3) for some $\varepsilon > 0$, the map ϕ_n tends to ϕ uniformly on $B(0, \varepsilon)$,
- (4) ϕ_n tends to ϕ uniformly on every compact subsets of \mathbb{D} .

A proof of (3) \implies (4) is, for instance, given by compactness of \mathcal{SL} together with analytic continuation of equalities. The last three notions of convergence are easily metrized and all endow \mathcal{F} with the *same* topology. It is Hausdorff and compact for this topology. The map $\mathcal{SL} \rightarrow \mathcal{F}, \phi \mapsto \mathcal{R}[B_d] \circ \phi^{-1}$ is hence a homeomorphism.

Recall that we denote $0, v_f, \infty$ the singular values of f over $\widehat{\mathbb{C}}$. It turns out that the class \mathcal{F} has been defined so that v_f does not depend on f , but let us temporarily ignore that.

Lemma 37 (Uniform bound on the trapping time) *For any $r > 0$, denote $D_r[f]$ the disk of diameter $[0, re^{i\alpha}]$, where α is the direction of the attracting axis of f . There exists $n_0 \in \mathbb{N}$ such that $\forall f \in \mathcal{F}, f^{n_0}(v_f) \in D_r[f]$.*

Proof Consider $r' = \min(r, r_0)$, where r_0 is provided by Proposition 29. The set $D_{r'}[f]$ is an attracting petal for f and is contained in $D_r[f]$. For each $f \in \mathcal{F}$ it takes a finite number of iterates for v_f to be trapped by $D_{r'}[f]$. The same number of iterates is enough for nearby¹³ maps in \mathcal{F} . By compactness¹⁴ of \mathcal{F} , it follows that there is $n_0 \in \mathbb{N}$ such that $\forall f \in \mathcal{F}, \exists n \leq n_0$ such that $f_n(v_f) \in D_{r'}[f]$. Since $D_{r'}[f]$ is a trap this implies $f^{n_0}(z) \in D_{r'}[f]$ and thus $\in D_r[f]$. \square

We will also use a slightly stronger statement:

Lemma 38 *There exists $n_0 \in \mathbb{N}$ and $\eta_0 > 0$ such that $\forall f \in \mathcal{F}, f^{n_0}(B(v_f, \eta_0)) \in D_r[f]$.*

Proof Done by compactness as above, using the following modification of the local statement, which is immediate by continuity for \rightrightarrows of $f \mapsto f^n$ for a fixed n : for each $f \in \mathcal{F}$ and each n such that $f^n(v_f) \in D_r[f]$, there is $\eta > 0$ such that for all maps $g \in \mathcal{F}$ close enough to f the n -th iterate of g sends $B(v_g, \eta)$ in $D_r[g]$. \square

¹³ We may use the topology on \mathcal{F} , in which case it means that the same iterate is enough for all maps in a neighborhood. Or we may use the notion \rightrightarrows , in which case it means that for all sequence $f_n \rightrightarrows f$, this iterate is eventually enough.

¹⁴ cover argument or sequence argument.

We have not checked if all compactness arguments in the rest of the article can be reformulated using \Rightarrow only. This is not the main point, however. Moreover, since there is on \mathcal{F} a topology for which convergence of sequences is equivalent to \Rightarrow , in the sequel we will use compactness of \mathcal{F} for this topology and convergence of sequences in \mathcal{F} w.r.t. this topology. Recall it is a metrizable topology for which \mathcal{F} is compact.

Below, $d_{\mathbb{C}}$ refers to the Euclidean distance on \mathbb{C} and if U is a open subset of \mathbb{C} whose complement has at least two points, d_U denotes the hyperbolic distance on U . Let $\mathcal{C} = \mathcal{C}[f]$ be the curve introduced in Proposition 36.

Lemma 39 *For $f \in \mathcal{F}$, let $PS(f)$ the orbit of the singular value v_f of f . The following holds:*

- (1) *The sets $\mathcal{C}[f]$ and $\overline{PS}(f)$ depend continuously on f for the Hausdorff topology on compact subsets of \mathbb{C} .*
- (2) $\sup \{|z| \mid z \in PS(f), f \in \mathcal{F}\} < +\infty$
- (3) $\sup \{d_{\text{Dom}(f)}(0, z) \mid z \in PS(f), f \in \mathcal{F}\} < +\infty$
- (4) $\inf \{d_{\mathbb{C}}(z, \partial \text{Dom}(f)) \mid z \in PS(f), f \in \mathcal{F}\} > 0$

Proof Let us use the same notations as in Lemma 37. For any $r \leq r_0$, denote $D_r = D_r[f]$: it is an attracting petal for f . Let $n_0(r) = n_0$ be provided by Lemma 37. For a fixed $m < n_0(r)$, $f^m(v_f)$ depends continuously on f . The rest of the orbit of v_f is contained in D_r . Continuity of $\overline{PS}(f) = PS(f) \cup \{0\}$ follows, as well as the point 2. For point 4, note that $B(0, 1/4) \subset \text{Dom}(f)$ (this follows from Koebe's 1/4 theorem). Choose now $r = \min(r_0, 1/8)$. For each fixed $m < n_0 = n_0(r)$, the distance from $f^m(v_f)$ to $\partial \text{Dom}(f)$ reaches a positive minimum as f varies in \mathcal{F} , again by continuity and compactness. For $m \geq n_0$, this distance is $\geq 1/8$. For point 3 first note that, on one hand, for $m \geq n_0$, $f^m(v) \in B(0, 1/8)$ and thus $d_{\text{Dom}(f)}(0, f^m(v)) \leq 1$ (better constants can be computed but that is not the point here). Let us now use the sets O and O_{attr} introduced in Proposition 36. The map Φ_{attr} is a holomorphic bijection from O_{attr} to $O = \mathbb{C} \setminus]-\infty, v' - 1]$ and the set $X = \{f^m(v) \mid 0 \leq m < n_0\}$ is the preimage by this map of $v' + \{0, 1, \dots, n_0 - 1\}$. Therefore, the $\text{Dom}(f)$ hyperbolic distance from X to $B(0, 1/8)$ is \leq the hyperbolic distance in O from v' to $v' + n_0$, which is itself $< n_0$. \square

3.6.3 Lemmas for the Second Step

Consider again a class of maps \mathcal{G} as in Proposition 29, i.e., \mathcal{G} is a set of holomorphic maps $g : \mathbb{D} \rightarrow \mathbb{C}$ with $g(z) = z + c_g z^2 + \dots$, that is compact for the topology of local uniform convergence and such that c_g is never 0, i.e., g has one attracting petal. In the second step of the proof of the main theorem, we will need some control on the variation of Fatou coordinates in terms of the variation of the map. For this, we first need to extend Fatou coordinates to bigger petals, as in [26].

Let $\theta \in [\pi/2, \pi]$ and $\Omega_{\theta}(r)$ denote the following domain: it contains a right half plane and is bounded by the arc of circle of center 0, radius r and argument ranging from $-(\theta - \pi/2)$ to $\theta - \pi/2$, and by the two half lines continuing this arc tangentially

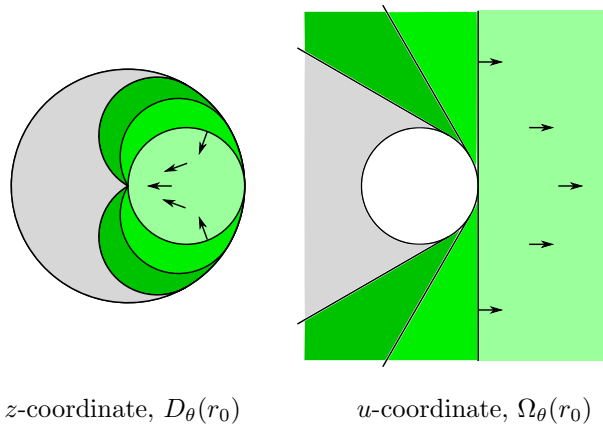


Fig. 13 Bigger domains for Fatou coordinates. On the left : z -coordinate and different domains $D_\theta(r_0)$ (in this example, the attracting axis is the positive reals), $\theta - 90^\circ = 0, 30^\circ, 60^\circ$, difference between these regions are highlighted in different colors. On the right, the u -coordinate, with $u = -1/c_f z$, and the corresponding domains $\Omega_\theta(R_0)$. In light green are D_{attr} and H_{attr}

to the circle (see Fig. 13). For $\theta = \pi/2$ and $r = R_0[g] = 1/|c_g|r_0$, this domain is exactly the half plane, image of D_{attr} in the u -coordinates of g .

Lemma 40 *There is r_0 such that for all $g \in \mathcal{G}$, the change of variable $w = u - \gamma_g \log_p u$ is injective on the non-convex set $\Omega_\pi(R_0[g])$.*

Proof Let us give a computational but elementary proof of this fact. Write $u = re^{i\alpha}$ and $u' = r'e^{i\alpha'}$ with $\alpha, \alpha' \in]-\pi, \pi[$ and note that $r, r' \geq R_0[g] \geq 1/r_0 \inf_{g \in \mathcal{G}} |c_g|$. Then, $|r - r'| \leq |u - u'|$ and $|e^{i\alpha} - e^{i\alpha'}| \leq |u - u'| / \min(r, r')$. If $|\alpha - \alpha'| \leq \pi$ (case 1), then $|\alpha - \alpha'| \leq (\pi/2)|e^{i\alpha} - e^{i\alpha'}|$. Otherwise (case 2), let us just use that $|\alpha - \alpha'| \leq 2\pi$. Now $w = w'$ means $u - u' = \gamma_g(\log r' - \log r) + \gamma_g i(\alpha' - \alpha)$ whence (case 1) $|u - u'| \leq \frac{|\gamma_g|(1+\pi/2)}{\min(r, r')} |u - u'|$ therefore $u - u' = 0$ provided r_0 was chosen big enough (independently of g). Or (case 2) $|u - u'| \leq \frac{|\gamma_g|}{\min(r, r')} |u - u'| + 2\pi |\gamma_g|$. In the second case, choose r_0 small enough (independently of g) so that $\frac{|\gamma_g|}{\min(r, r')} \leq 1/2$. Then, $|u - u'| \leq 4\pi |\gamma_g|$. Since $\alpha - \alpha' > \pi$ the points u and u' must have opposite imaginary part and one of them at least has negative real part. Since they belong to $\Omega_\pi(R_0[g])$, which does not contain the half strip of equation $\{z \in \mathbb{C} \mid \operatorname{Re} z \leq 0 \text{ and } -R_0[g] \leq \operatorname{Im} z \leq R_0[g]\}$, we get in particular that $|u - u'| > R_0[g]$. So if we choose r_0 small enough so that, $\forall g \in \mathcal{G}$, $R_0[g] > 4\pi |\gamma_g|$, this is impossible. \square

Proposition 41 *Let $\theta \in [\pi/2, \pi[$. Proposition 29 still holds if we replace D_{attr} with the domain $D_\theta(r_0)[g]$ whose image in the u -coordinate is $\Omega_\theta(R_0[g])$, where $R_0[g] = 1/|c_g|r_0$, and if we replace the condition on ζ by $\zeta(x) = -|x \tan(\theta - \pi/2)| + o(x)$. Similar statements hold for repelling Fatou coordinates.*

Proof The proof carries over with little modification. The constant $1/4$ has to be replaced by a smaller constant (by $\sin \theta$) when θ is too close to π . Injectivity of the

change of variable $w = u - \gamma_g \log_p u$ on the non-convex set $\Omega_\theta(R_0)$ follows from the previous lemma. For the uniform bound on $\sum M_3/|u_n|^2$: divide the orbit of u_n into three parts, according to $\operatorname{Re}(u_n)$ being in $]-\infty, -R_0[$, in $[-R_0, R_0]$, or in $]R_0, +\infty[$. In the central part, there is a uniformly bounded number of u_n . The two other parts are bounded exactly like before. \square

In particular, $\Phi_{\text{attr}}[g]$ is defined on a set containing the image of $\Omega_\theta(R_0[g])$ by $u \mapsto -1/c_g z$. Recall that $R_0[g]$ is bounded, hence the sets $\Omega_\theta(R_0[g])$ all contain $\Omega_\theta(R_0)$ for some R_0 independent of g . Choose any θ with $\pi/2 < \theta < \pi$. Let

$$\Xi[g](w) = \Phi(z),$$

where $u = -1/c_g z$, $w = u - \gamma_g \log_p(u)$, and Φ is the attracting Fatou coordinate: we take Ξ defined on the image of $\Omega_\theta(R_0[g])$ by $u \mapsto w$. Note that this change of coordinates depends on g , but if one chooses any other $\theta' < \theta$, there exists $R'_2 > 0$ such that for all $g \in \mathcal{G}$, it contains $\Omega_{\theta'}(R'_2)$. By Proposition 41, $\Xi[g](w) - w$ is bounded by a constant independent of g . Hence there exists $R_2 > 0$ such that for all $g \in \mathcal{G}$, the domain and the range of $\Xi[g]$ contains $\Omega_{\theta'}(R_2)$. Recall that maps in \mathcal{G} are assumed to be defined on \mathbb{D} .

The next three lemmas express a form of Lipschitz dependence with respect to g for $\Xi[g]$, $\Xi[g]^{-1}$ and $\Phi[g]$.

Proposition 42 *Let R_2 as above. Let $r' \in]0, 1[$. There exists $M > 0$, $R_1 > R_2$ and $\varepsilon_0 > 0$ such that for all $f, g \in \mathcal{G}$ with $\sup_{B(0, r')} |f - g| \leq \varepsilon_0$ then $\forall w \in \mathbb{C}$ with $w \in \Omega_{\theta'}(R_1)$, $|\Xi[f](w) - \Xi[g](w)| \leq M \sup_{B(0, r')} |f - g|$.¹⁵*

Proof A trick to shorten the proof is to use holomorphic dependence of Fatou coordinates w.r.t. the map. Let $\|f - g\| = \sup_{B(0, r')} |f - g|$. Let $c_0 = \inf |c_g|$ over all $g \in \mathcal{G}$. Let first ε_0 be such that the sum h of a map in \mathcal{G} with a holomorphic map defined on $B(0, r')$ and with a double root at the origin and sup norm $\leq 2\varepsilon_0$, satisfies $|c_h| > c_0/2$. Let \mathcal{G}' be the union of \mathcal{G} and of all the maps of the form $h_t = f + t \frac{2\varepsilon_0}{\|f - g\|} (g - f)$, where $t \in \mathbb{D}$, $f, g \in \mathcal{G}$ and $\|f - g\| \leq \varepsilon_0$. Then \mathcal{G}' is compact (for the topology associated to uniform convergence on compact subsets of $B(0, r')$) and, conjugating its members by $z \mapsto z/r'$ and restricting to \mathbb{D} , gives a family satisfying the hypotheses of Propositions 29 and 41. Using the latter and the same analysis as in the paragraph that follows it, we see that maps $h \in \mathcal{G}'$ all have a function $\Xi[h]$ that is defined on a set containing $\Omega_{\theta'}(R_1)$ for some R_1 independent of h . Moreover, this function depends holomorphically on $t \in \mathbb{D}$ (recall the definition of Φ as a limit of $w_n - n$ and realize that w_n depends holomorphically on w_n) and its difference with $w \mapsto w$ is uniformly bounded, hence $\Xi[h_t](w) - \Xi[f](w)$ is also bounded. The proposition follows by Schwarz's inequality¹⁶ applied to $t \mapsto \Xi[h_t](w) - \Xi[f](w)$, specialized to $t = \|f - g\|/2\varepsilon_0$. \square

Similarly, Proposition 42 holds word for word with Ξ replaced by Ξ^{-1} , i.e.:

¹⁵ A better bound holds, that decays when w tends to infinity, but it will not be used here.

¹⁶ We mean: if $f : \mathbb{D} \rightarrow \mathbb{C}$ is holomorphic and satisfies $f(0) = 0$ and $\sup |f| < +\infty$ then $|f(z)| \leq |z| \sup |f|$.

Proposition 43 *There exists $M > 0$, $R_1 > R_2$ and $\varepsilon_0 > 0$ such that for all $f, g \in \mathcal{G}$ with $\sup_{B(0, r')} |f - g| \leq \varepsilon_0$, then $\forall Z \in \mathbb{C}$ with $Z \in \Omega_{\theta'}(R_1)$, $|\Xi^{-1}[f](Z) - \Xi^{-1}[g](Z)| \leq M \sup_{B(0, r')} |f - g|$.*

Proof This follows from the above proposition applied to some θ'' between θ and θ' , and from the fact that, the derivative of Ξ_g is uniformly bounded¹⁷ over $g \in \mathcal{G}$. Computations are left to the reader. \square

Remark Note that since the maps Ξ_g and Ξ_g^{-1} all differ from identity by a bounded amount that is independent of $g \in \mathcal{G}$, it follows that in both propositions, by increasing the value of M , we can remove the assumption $\sup_{B(0, r')} |f - g| \leq \varepsilon_0$.

From Proposition 42, we deduce the following control, which is somewhat weaker:

Proposition 44 (variation of Fatou coordinates) *Let $r' \in]0, 1[$. Let R_1 be given by Proposition 42. Let $\theta'' < \theta'$. Then, there exists $M > 0$, $R_3 > R_1$ and $\varepsilon_0 > 0$ such that for all $f, g \in \mathcal{G}$ with $\sup_{B(0, r')} |f - g| \leq \varepsilon_0$ and $\forall z \in \mathbb{C}$ with $-1/c_f z \in \Omega_{\theta''}(R_3)$, then $-1/c_g z \in \Omega_{\theta'}(R_1)$ and*

$$|\Phi_{\text{attr}}[f](z) - \Phi_{\text{attr}}[g](z)| \leq \frac{M}{|z|} \sup_{B(0, r')} |f - g|.$$

The same holds for repelling Fatou coordinates.

Proof Let $d = \sup_{B(0, r')} |f - g|$. The claim $-1/c_g z \in \Omega_{\theta'}(R_1)$ follows from continuity of $g \mapsto c_g$ and its non-vanishing: given any $R_3 > 1$ and $\theta'' < \theta'$, a small enough d will ensure that the quotient c_g/c_f is close enough to 1 so that an element of $\Omega_{\theta''}(R_3)$ multiplied by c_f/c_g is still in $\Omega_{\theta'}(R_1)$. Now $\Phi_{\text{attr}}[f](z) = \Xi[f](w_1)$ and $\Phi_{\text{attr}}[g](z) = \Xi[g](w_2)$ with $w_1 = u_1 - \gamma[f] \log_p(u_1)$ and $w_2 = u_2 - \gamma[g] \log_p(u_2)$ with $u_1 = -1/c_f z$ and $u_2 = -1/c_g z$. The constants c , $1/c$ and γ are Lipschitz functions of $f \in \mathcal{G}$ w.r.t. the distance d . Now, under the assumption $d \leq \varepsilon_0$, we successively get $|u_1 - u_2| \leq M_1 d/|z|$, $|w_1 - w_2| \leq M_2 d/|z|$ (use that $|\log_p u| \leq M'_2/|z|$ for some constant and that $u_2/u_1 = c_f/c_g$ is close enough to 1), then we decompose $|\Xi[f](w_1) - \Xi[g](w_2)| \leq |\Xi[f](w_1) - \Xi[g](w_1)| + |\Xi[g](w_1) - \Xi[g](w_2)|$. The first term is dealt with using Proposition 42 and the second term using the fact that there is a uniform bound on Ξ' . \square

Remark • Here, the condition $\sup_{B(0, r')} |f - g| \leq \varepsilon_0$ cannot be removed.

- Also, in the conclusion $|\Phi_{\text{attr}}[f](z) - \Phi_{\text{attr}}[g](z)| \leq \frac{M}{|z|} \sup_{B(0, r')} |f - g|$, the factor $1/|z|$ cannot be removed because $\Phi_{\text{attr}}[f](z) \sim -1/c_f z$ and c_f varies with f .

Let us stress again that, though maps in \mathcal{F} are not defined on the unit disk, they are all defined in $B(0, 1/4)$ and the results above easily transfer to \mathcal{F} by a homothety. (See Sect. 3.6.2.)

¹⁷ Increase R_2 by 1 and use Cauchy's formula and the uniform bound on $\Xi - \text{id}$.

3.7 Step 1: Contraction Argument (i.e., There is a Lot of Room)

Fix $d \in \mathbb{N}$ with $2 \leq d < \infty$: we now exclude $d = +\infty$. In this section we will define constants c_1, c_2, \dots . They all depend on d but not on $f \in \mathcal{F}$.

Recall that $\mathcal{R}[B_d]$ is defined on the unit disk and has derivative one at the origin. Recall the definition of the set \mathcal{SL} of Schlicht maps: univalent holomorphic maps $\phi : \mathbb{D} \rightarrow \mathbb{C}$ such that $\phi(z) = z + \mathcal{O}(z^2)$. Recall that $\mathcal{F} = \{\mathcal{R}[B_d] \circ \phi^{-1} \mid \phi \in \mathcal{SL}\}$, and that for all $f \in \mathcal{F}$, the map $\mathcal{R}[f]$ is again in \mathcal{F} , for an appropriate normalization. Since all maps in \mathcal{F} have the same unique critical value, this normalization coincides with the one numbered 3 on page 81, which we called “by the critical value”.

Let $f \in \mathcal{F}$:

$$f = \mathcal{R}[B_d] \circ \phi_1^{-1},$$

where $\phi_1 \in \mathcal{SL}$. Denote

$$U_1 = \phi_1(\mathbb{D}) = \text{Dom}(f).$$

Let $L(\varepsilon)$ be the hyperbolic radius in \mathbb{D} of the Euclidean ball $B(0, 1 - \varepsilon)$:

$$L(\varepsilon) = \tanh^{-1}(1 - \varepsilon) = \frac{1}{2} \log \frac{2 - \varepsilon}{\varepsilon}. \quad (1)$$

In particular,

$$\frac{1}{2} \log \frac{1}{\varepsilon} \leq L(\varepsilon) \leq \frac{\log 2}{2} + \frac{1}{2} \log \frac{1}{\varepsilon}.$$

Since $\mathcal{R}[f]$ belongs to \mathcal{F} (Theorem 6), there exists $\phi_2 \in \mathcal{SL}$ such that:

$$\mathcal{R}[f] = \mathcal{R}[B_d] \circ \phi_2^{-1}.$$

The map ϕ_2 is an isomorphism from \mathbb{D} to the domain of definition of $\mathcal{R}[f]$.

Denote by $A \subset U_1$ the immediate basin of the parabolic fixed point 0 of f . Let U_u denote the connected component of $\text{Dom}(h[f])$ that contains an upper half plane. It is also equal to the connected component of $\Psi_{\text{rep}}^{-1}(A)$ that contains an upper half plane. Denote by $C \subset A$ the following set, which is the object under study in the present section:

$$C = C[f] = \Psi_{\text{rep}}(U_u \Vdash (1 - \varepsilon)).$$

(The notation \Vdash has been introduced in Sect. 3.1). We claim it can be rewritten as

$$C = \phi_3(C[B_d]),$$

where $\phi_3 : \mathbb{D} \rightarrow A$ is the conformal isomorphism conjugating B_d to $f|_A$. Indeed, according to Proposition 27, $\phi_3 \circ \Psi_{\text{rep}}[B_d] = \Psi_{\text{rep}}[f] \circ \phi_4$ for some conformal map

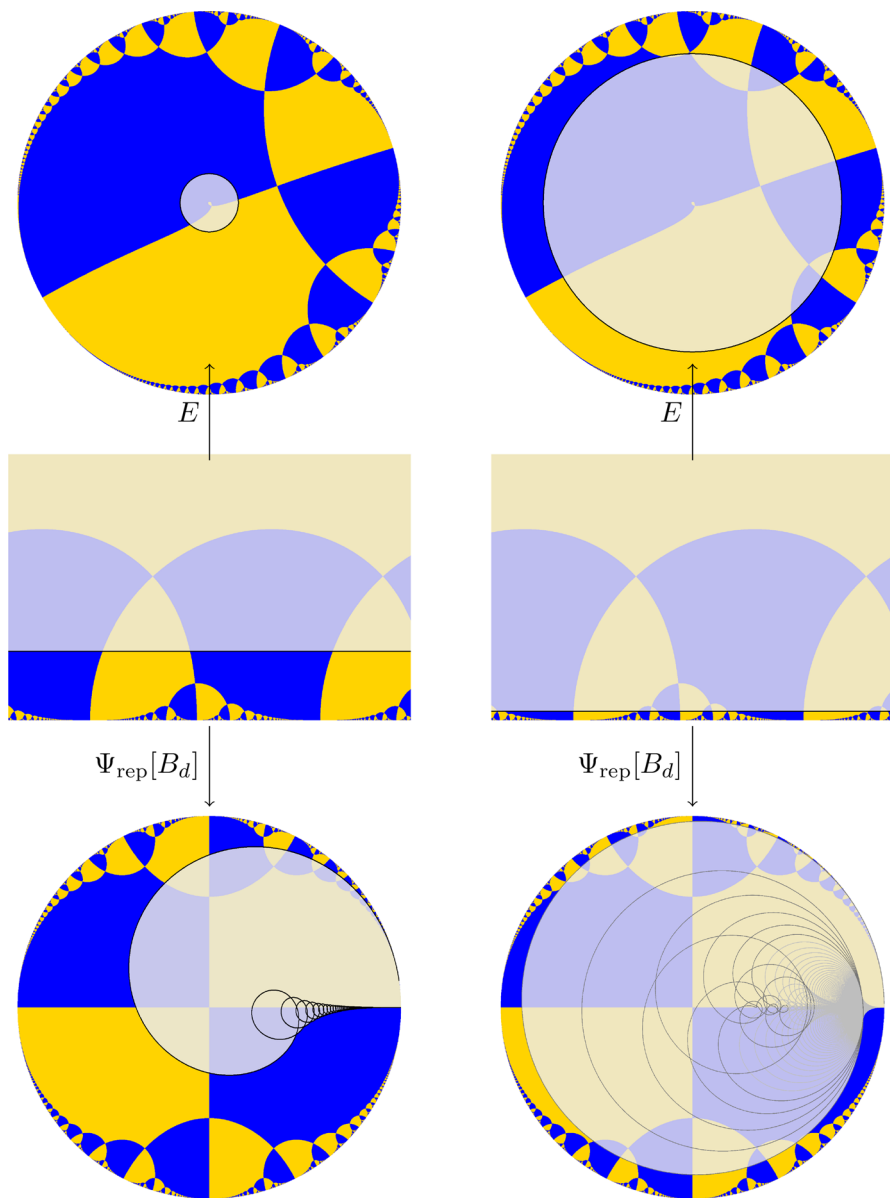


Fig. 14 The set $C[B_d]$ (bottom row, light tones) for $d = 2$ and (left) $\varepsilon \approx 0.85$ (this is quite high a value for an ε) and (right) $\varepsilon \approx 0.22$

ϕ_4 from $\mathbb{H} = U_u[B_d]$ to $U_u[f]$, commuting with T_1 , thus $\phi_4(U_u[B_d]) \models (1 - \varepsilon) = U_u[f] \models (1 - \varepsilon)$.

The set $C[B_d]$, which is equal to $\Psi_{\text{rep}}[B_d](H(\varepsilon))$ where $H(\varepsilon) = E^{-1}(B(0, 1 - \varepsilon))$ is the half plane defined by $\text{Im } z > \frac{1}{2\pi} \log \left(\frac{1}{1 - \varepsilon} \right)$, depends only on d and ε , not on f , and is forward invariant under B_d . Figure 14 shows examples of sets $C[B_d]$.

In this section, we will prove:

Proposition 45 *There exists c, c' and $\xi > 0$ (these constants depend on d) such that for all $\varepsilon < \xi$, there exists $\varepsilon' > 0$ satisfying*

$$\log \frac{1}{\varepsilon'} \leq c' + c \log \left(1 + \log \frac{1}{\varepsilon} \right)$$

such that for all $f \in \mathcal{F}$,

$$C \subset \text{Dom}(f) \odot (1 - \varepsilon').$$

Above, the set $C = C[f]$ depends also on ε but we did not figure it in the notation, to avoid clutter. The notation $U \odot r$ has been introduced in Sect. 3.1.

We begin with an easy lemma (recall L was defined near (1)):

Lemma 46 *The set $C[B_d]$ is contained within hyperbolic \mathbb{D} -distance $\leq c_2 + L(\varepsilon)$ of the upper main chessboard box of B_d .*

Proof The upper chessboard box of B_d is the image by $\Psi_{\text{rep}}[B_d]$ of an open set that contains a half plane $\{z \in \mathbb{C} \mid \text{Im}(z) > M_d\}$ and is contained in another half plane strictly smaller than \mathbb{H} . Recall that $C[B_d] = \Psi_{\text{rep}}[B_d](H(\varepsilon))$ with $H(\varepsilon) = \{z \in \mathbb{C} \mid \text{Im}(z) > \frac{1}{2\pi} \log \left(\frac{1}{1 - \varepsilon} \right)\}$. For ε big, $H(\varepsilon) \subset \{z \in \mathbb{C} \mid \text{Im}(z) > M_d\}$. For other values of ε , every point in $H(\varepsilon)$ can be joined to $\{z \in \mathbb{C} \mid \text{Im}(z) > M_d\}$ by a vertical segment of hyperbolic length in \mathbb{H} at most $\frac{1}{2} \left(\log M_d - \log \frac{\log \frac{1}{1 - \varepsilon}}{2\pi} \right)$. Since $\Psi_{\text{rep}}[B_d] : \mathbb{H} \rightarrow \mathbb{D}$ contracts hyperbolic metrics and $\frac{1}{2} \log \frac{1}{\log \frac{1}{1 - \varepsilon}} \leq \frac{1}{2} \log \frac{1}{\varepsilon} \leq L(\varepsilon)$, the lemma follows. \square

Note that $\phi_3 : \mathbb{D} \rightarrow A$ is an isometry for the respective hyperbolic metrics, and that the upper main chessboard box of B_d is mapped by ϕ_3 to the main upper dynamical chessboard box of A , call it \mathfrak{B} :

$$\mathfrak{B} = \phi_3(\mathfrak{B}[B_d]).$$

See Fig. 15. From the lemma above, it follows that the set $C = C[f]$ under study is contained within A -hyperbolic distance $c_2 + L(\varepsilon)$ of \mathfrak{B} . To prove an estimate concerning the latter set, we first need the following easy consequence of the compactness of \mathcal{F} :

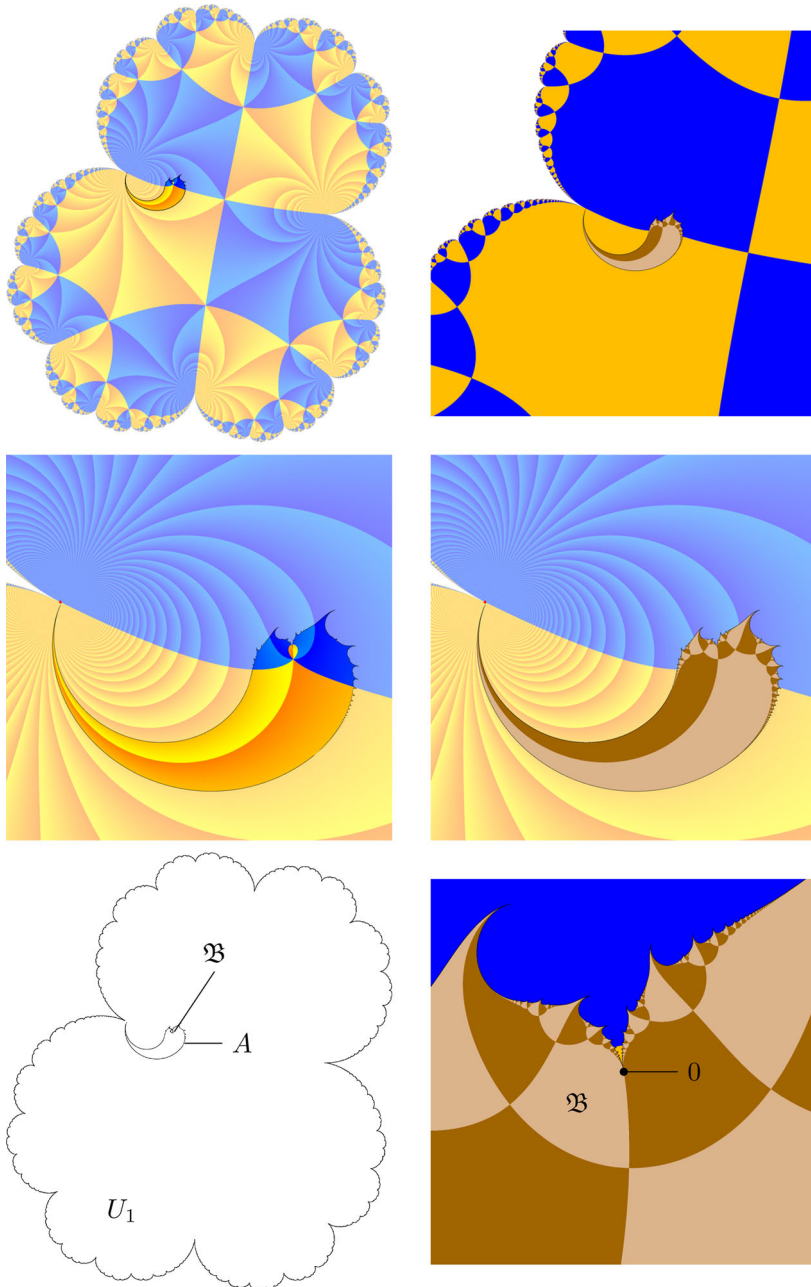


Fig. 15 Some open sets associated with $\mathcal{R}[P]$ with $P : z \mapsto z + z^2$: its domain U_1 , its parabolic immediate basin A , and the latter's main upper dynamical box \mathfrak{B} . The rightmost column features the dynamical chessboard of A in shades of brown. The blue and yellow shades depict the structural chessboard of U_1

Lemma 47 *For all $M > 0$ there exists $c' > 0$ such that for all $f \in \mathcal{F}$, the upper main and the lower main chessboard boxes of $h[f]$ are both at hyperbolic $\text{Dom}(h[f])$ -distance $\leq c'$ from respectively the half planes $\text{Im}(z) > M$ and $\text{Im}(z) < -M$ (intersected with $\text{Dom}(h[f])$) if necessary).*

Proof The extended normalized horn map of B_d is defined on $\mathbb{C} \setminus \mathbb{R}$. The upper/lower main chessboard boxes of $h[B_d]$ are at positive Euclidean distance from \mathbb{R} . Recall (see Sect. 3.3, in particular Lemma 28) that we have the following: $h[f] \circ \phi = T_{w[f]} \circ h[B_d]$ where $w[f] = v_{h[f]} - v_{h[B_d]}$ and ϕ is an isomorphism commuting with T_1 from \mathbb{H} , which is the upper connected component of $\text{Dom } h[B_d]$, to the upper connected component of $\text{Dom } h[f]$, and that ϕ maps the chessboard graph of $h[B_d]$ to that of $h[f]$. Therefore, it is enough to prove that $\phi^{-1}(\{z \in \mathbb{C} \mid \text{Im}(z) > M\})$ contains an upper half plane independent of f , and a similar statement for the lower part. Let us write, as $\text{Im}(z) \rightarrow +\infty$:

$$\phi(z) = z + \tau_f + o(1).$$

From the first point of Proposition 35 it follows that $|\text{Im}(w[f])|$ is bounded over \mathcal{F} . From this and the second point, it follows that $|\text{Im}(\tau_f)|$ is bounded over \mathcal{F} . Now one of Koebe's inequalities states that $\forall f \in \mathcal{SL}$, $|f(z)| \leq \frac{|z|}{(1-|z|)^2}$. Equivalently, $\forall r \in]0, 1[$, $f^{-1}(B(0, r/(1-r)^2)) \supset B(0, r)$. The map $T_{-\tau_f} \circ \phi$ is semi-conjugate by E to a Schlicht map thus: $\phi^{-1}(\{z \in \mathbb{C} \mid \text{Im}(z) > M\})$ contains the half plane $\{z \in \mathbb{C} \mid \text{Im}(z) > M'\}$ where $M' = M'[\tau_f] > 0$ is related to $M \in \mathbb{R}$ by $e^{2\pi(M+\text{Im } \tau_f)} = e^{2\pi M'} + e^{-2\pi M'} - 2 = 2(\cosh(2\pi M') - 1)$. Since τ_f is bounded, the constant $M'[\tau_f]$ is bounded too. The proof for the lower box is similar. \square

Recall U_1 denotes the domain of f .

Lemma 48 *The box \mathfrak{B} is contained in a hyperbolic U_1 -ball of uniform diameter c_7 .*

Proof Choose r small enough so that $B(0, 2r) \subset U_1$ for all $f \in \mathcal{F}$. By Proposition 33, there is some $h > 0$ such that for all $f \in \mathcal{F}$, the half planes $\text{Im}(z) > h$ and $\text{Im}(z) < -h$ are mapped by $\Psi_{\text{rep}}[f]$ inside $B(0, r)$. From Lemma 47 the upper box is at distance $\leq c_7$ from $\{z \in \mathbb{C} \mid \text{Im}(z) > h\}$ for the hyperbolic metric of $\text{Dom}(h[f])$. The map $\Psi_{\text{rep}} : \text{Dom}(h[f]) \rightarrow A$ is holomorphic thus a contraction for hyperbolic metrics, thus the image by Ψ_{rep} of the upper chessboard box is at bounded A -hyperbolic distance of $B(0, r)$ (the latter is not contained in A but it does not matter) and, thus, at U_1 -hyperbolic distance even smaller, since the inclusion of A in U_1 is a contraction too. \square

By Lemmas 46 and 48, to fulfill the objectives of Step 1, it is enough to prove that a path starting from \mathfrak{B} , contained in A and of A -hyperbolic length $\leq c_2 + L(\varepsilon)$ has a U_1 -hyperbolic length much smaller than $c_2 + L(\varepsilon)$. The precise bound obtained will yield Proposition 45. Note that we will in fact bound the U_1^* -hyperbolic length, which is bigger than the U_1 -hyperbolic length, where

$$U_1^* = U_1 \setminus \{0\}.$$

Let us make the following change of coordinates: $w = \log(z)/2i\pi$. Let \tilde{A} be a lift of A : it is a connected and simply connected subset of \mathbb{C} that does not intersect its translates $\tilde{A} + k$ when $k \in \mathbb{Z}$ is non-zero. As a consequence, each horizontal intersects this open set along a union of open segments of length at most 1 (in fact the sum of lengths is at most 1). Thus, the Euclidean distance from any $z \in \tilde{A}$ to the boundary of \tilde{A} is $\leq 1/2$. This implies by Koebe's $1/4$ theorem:

$$\rho_{\tilde{A}}(z) \geq 1/2$$

(a better bound holds but we do not need it; recall $\rho_U(z)|dz|$ designates the infinitesimal element of hyperbolic metrics on U).

Remark The set \tilde{A} is unbounded upwards, since the image in \tilde{A} of an attracting petal in A is an infinite finger-shaped domain extending upwards. See Fig. 16 for examples. One should not expect \tilde{A} to be bounded in the other directions either. Recall that $f \in \mathcal{F}$ is characterized by the choice of its domain U_1 , which can be any simply connected domain containing the origin with conformal radius 1 w.r.t. the origin. For well-chosen unbounded U_1 , the set \tilde{A} is unbounded downwards. One could object that since in the applications, the renormalization operator is iterated, we could restrict to maps in $\mathcal{R}[\mathcal{F}]$ instead of \mathcal{F} , and that maps in $\mathcal{R}[\mathcal{F}]$ all have a uniformly bounded domain of definition, as follows for instance from Proposition 35. But this will not prevent unboundedness in the horizontal direction: even for bounded U_1 , provided its boundary winds infinitely many times around 0, carefully chosen U_1 will yield a set \tilde{A} whose projection on the real line is unbounded. The latter case is not just a curiosity but does happen for $f = \mathcal{R}[z \mapsto ze^z]$, i.e., the first renormalization of the map $g(z) = ze^z$ which has a non-linearizable parabolic point at the origin, and whose set of singular values are the two asymptotic values $\infty, 0$ and the image $g(-1)$ of the unique critical point -1 . Its immediate basin must contains a singular value, and the only possible one is $g(-1)$. Hence, the map g satisfies the hypotheses of Theorem 6, thus $f = \mathcal{R}[g] \in \mathcal{F}$. A careful study shows that the domain of definition of $\mathcal{R}[g]$ swirls like above, more precisely that its lifted immediate basin \tilde{A} has infinitely many accesses to infinity by curves asymptotic to some common horizontal line. The map g does not belong to \mathcal{F} but we believe that for all $n > 0$, $\mathcal{R}^n[g]$, that belongs to \mathcal{F} , will have a set \tilde{A} with the same properties. To prove this, one may try and see if there is invariance by \mathcal{R} of the following property for $f \in \mathcal{F}$: let c be the main critical point of f (the one on the boundary of the main upper structural box); let \tilde{f} be a lift of f and let γ be the lift by \tilde{f} starting from c , of the horizontal half line $\tilde{f}(c) + [0, +\infty[$, such that γ intersects the boundary of the upper box only at c ; then $\text{Re}(\gamma)$ tends to infinity.

Now consider a point $z_0 \in C = \phi_3(C[B_d])$ and consider a path γ of A -length at most $c_2 + L(\varepsilon)$ from \mathfrak{B} to z_0 . Let us apply f once. Then A is mapped to itself and so are C and \mathfrak{B} . The path γ is mapped to a path $f(\gamma)$ contained in A , from \mathfrak{B} to $z_1 = f(z_0)$, and by the Schwarz-Pick inequality, the A hyperbolic length of $f(\gamma)$ is \leq that of γ . Consider a lift γ_2 of $f \circ \gamma$ by E (the path $f(\gamma)$ is contained in A , thus

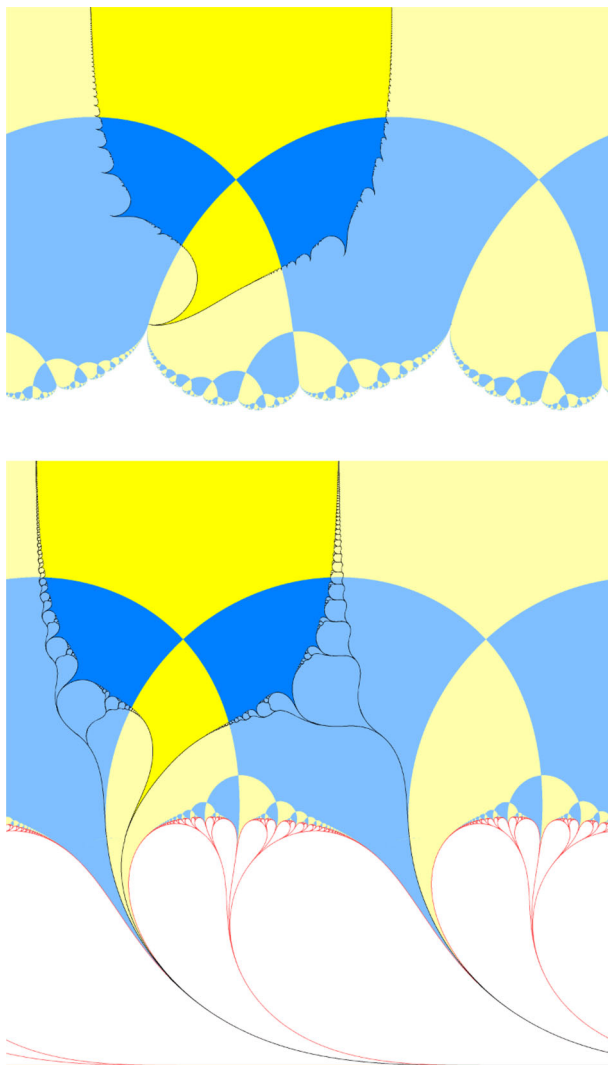


Fig. 16 Two examples of lifted immediate parabolic basins \tilde{A} for maps $f \in \mathcal{F}$. Top: $f = \mathcal{R}(z \mapsto z + z^2)$, bottom: $f = \mathcal{R}(z \mapsto ze^z)$

does not meet the origin). The Euclidean length of γ_2 is equal to

$$\int_{\gamma_2} |dz| = \int_{\gamma_2} \frac{\rho_{\tilde{A}}(z)|dz|}{\rho_{\tilde{A}}(z)} \leq 2 \int_{\gamma_2} \rho_{\tilde{A}}(z)|dz| \leq 2(c_2 + L(\varepsilon)). \quad (2)$$

Let us now relate the element of length $\rho_{U_1^*}(z)|dz|$ to $|\mathrm{d} \log f(z)/2\pi|$. Let \tilde{f} be the continuous lift of f by E that fixes \tilde{A} : $\tilde{f}: \tilde{U}_1 \stackrel{\mathrm{def}}{=} E^{-1}(U_1) \rightarrow \mathbb{C}$ and $E \circ \tilde{f} = f \circ E$. The inverse of E is the multivalued function $E^{-1}(z) = \frac{1}{2\pi i} \log z$. Let $\tilde{v} + \mathbb{Z}$ be the

set of critical values of \tilde{f} . The map \tilde{f} has no asymptotic value over \mathbb{C} . Denote by \mathbb{C}^\pm the upper half plane and the lower half plane delimited by the horizontal line through these critical values. For every point z mapped to \mathbb{C}^\pm by any branch of $\frac{1}{2\pi i} \log f$, the latter map has inverse branches defined in \mathbb{C}^\pm , with image the f -structural chessboard box containing z . This inverse branch is univalent, except for z in the little loop around 0 where it is infinite-to one. In all cases, these inverse branches map in U_1^* and are non-expanding for the respective hyperbolic metrics as follows:

$$\rho_{U_1^*}(z)|dz| \leq \rho_{\mathbb{C}^\pm}(\zeta)|d \log f(z)/2\pi|, \quad (3)$$

where ζ is the image of z by the considered branch of $\frac{1}{2\pi i} \log f$.

Near the boundary of \mathbb{C}^\pm , better estimates hold. For instance:

Lemma 49 *There exists $c_3 > 0$ such that for all $f \in \mathcal{F}$, the following holds. Let \tilde{v} be a critical value of \tilde{f} and V be any connected component of the pre-image by \tilde{f} of the square $\tilde{v} + I + iI$ where $I = [-1/2, 1/2]$. Then, the hyperbolic diameter in U_1^* of $E(V)$ is $\leq c_3$.*

Proof Recall the critical values of \tilde{f} , are the elements of $\tilde{v} + \mathbb{Z}$ and that its only asymptotic value over $\hat{\mathbb{C}}$ is ∞ . Consider the disk $\tilde{v} + \mathbb{D}$ and the component U of \tilde{f}^{-1} that contains V . Then, \tilde{f} factors on U as $a \circ \text{pow} \circ b$ where $\text{pow} : \mathbb{D} \rightarrow \mathbb{D}$ is either the identity or the map $z \mapsto z^d$, where $a(z) = \tilde{v} + z$ and where b is an isomorphism from U to \mathbb{D} . Then, $a^{-1}(V) = I + iI \subset B(0, 1/\sqrt{2})$ thus $(a \circ \text{pow})^{-1}(V)$ is contained in the Euclidean ball $B(0, \left(\frac{1}{\sqrt{2}}\right)^{1/d})$. The map $b^{-1} : \mathbb{D} \rightarrow E^{-1}(U_1)$ is non-expanding for the respective hyperbolic metrics, and $E : E^{-1}(U_1) \rightarrow U_1^*$ also is, thus, the lemma holds with $c_3 =$ the hyperbolic distance in \mathbb{D} from 0 to the d -th root of $1/\sqrt{2}$. \square

Another easy lemma:

Lemma 50 *Let a, b be two points in the hyperbolic plane \mathbb{H} :*

$$\text{Im}(a) \geq \frac{1}{2} \text{ and } \text{Im}(b) \geq \frac{1}{2} \implies d_{\mathbb{H}}(a, b) \leq \log(1 + 2|a - b|).$$

Proof Use the following formula for the hyperbolic distance in \mathbb{H} :

$$d_{\mathbb{H}}(a, b) = \text{argsh} \frac{|b - a|}{2\sqrt{\text{Im } a \text{ Im } b}},$$

and the inequality $\text{argsh } t \leq \log(1 + 2t)$. \square

So for instance, the hyperbolic distance from i to $i + x$ is a $\mathcal{O}(\log x)$ when $x \rightarrow +\infty$, thus much smaller than x . Recall that the geodesic between a and b in \mathbb{H} is an arc of Euclidean circle. For the hyperbolic metric, this arc turns out to be much shorter than the straight euclidean line.

Let β_0 be the structural U_1 chessboard box that is a punctured neighborhood of the origin. Recall that we denote $U_1^* = U_1 \setminus \{0\}$. Consider any structural U_1 chessboard box

β . Let us call U_1 -box the set $\bar{\beta} \cap U_1^*$. Let us endow $U_1^* \setminus f^{-1}(v)$ with the infinitesimal metric induced by pulling back the Euclidean metric by $\frac{1}{2\pi i} \log f$. We call this the *flat metric*. It has a regular and locally flat extension to a neighborhood of the non-critical preimages of v and is singular precisely at the critical preimages of v , where it has a conical point of angle $2\pi d$. Let us call *box-Euclidean distance* the distance induced on U_1^* by this flat metric. Recall that if $\beta \neq \beta_0$, then $\frac{1}{2i\pi} \log f$ is well defined on β and maps it to a half plane \mathbb{C}^\pm . It also maps the U_1 -box $\bar{\beta} \cap U_1^*$ to the closure of this half plane.

In the sequel, we call b_* the U_1 -box that contains a punctured neighborhood of the origin: $b_* = \bar{\beta}_0 \cap U_1^*$.

Corollary 51 *Consider two points in a U_1 -box b . Denote d_e the distance between these two points for the metric induced by the flat metric restricted to b and d_h the distance between these two points for hyperbolic metric on U_1^* . Then,*

$$d_h \leq c'_5 + \log(1 + c_5 d_e).$$

Proof Let us apply $\frac{1}{2\pi i} \log f$ so as to work in a half plane, and to fix ideas, let us assume it is the half plane \mathbb{C}^+ . If the U_1 -box b is b_* , then when we lift the two points we choose these lifts so that their euclidean distance is minimal, so as to coincide with d_e . If any of the two points is at distance $\leq 1/2$ from the boundary of \mathbb{C}^+ then move it up so that it is at distance $1/2$: we get a new pair of points in \mathbb{C}^+ that corresponds to a new pair of points in b . By Lemma 49, each new point is at U_1^* -hyperbolic distance $\leq c_3$ from the former so the U_1^* -hyperbolic distance between the the points in the pair has changed by at most c_3 , and by at most $2c_3$ if we needed to move both points. Similarly, the Euclidean distance between the points in \mathbb{C}^+ has changed by at most 1. By Equation (3), the U_1^* -hyperbolic distance between the two (possibly) new points will be at most their \mathbb{C}^+ -hyperbolic distance. Using Lemma 50, on the latter, we get $d_h \leq 2c_3 + \log(1 + 2(d_e + 1)) = (2c_3 + \log 3) + \log(1 + \frac{2}{3}d_e)$. \square

Let \tilde{U}_1 -boxes be defined similarly: these are sets of the form $\bar{b} \cap \tilde{U}_1$, where b is a structural chessboard box of \tilde{f} . The map \tilde{f} is a bijection from such a set to the closed upper or lower half plane. We can endow \tilde{U}_1 with an infinitesimal box-Euclidean metric, by pulling-back by \tilde{f} the canonical Euclidean metric element $|dz|$ on the complex plane. Recall that $f \circ E = \tilde{E} \circ \tilde{f}$, thus we get the following compatibility statements. The projection by E of an \tilde{U}_1 -box is a U_1 -box.¹⁸ The box-Euclidean metric element on \tilde{U}_1 is the pull-back by E of the box-Euclidean metric element on U_1^* .

The following result is not used here, but we find it interesting:

Lemma 52 *A connected union of U_1 -boxes that includes b_* is simply connected if we add $\{0\}$ to the union.*

Proof Remove the loop from the parabolic structural chessboard graph of U_1 . Then we get a tree (an infinite tree), on which the union retracts to a connected subset, which is thus simply connected and homotopically equivalent to the union. \square

¹⁸ The connected components of the preimage of a U_1 -box by \tilde{f} are \tilde{U}_1 -boxes with one notable exception where we get a chain of U_1 -boxes that meet at corners.

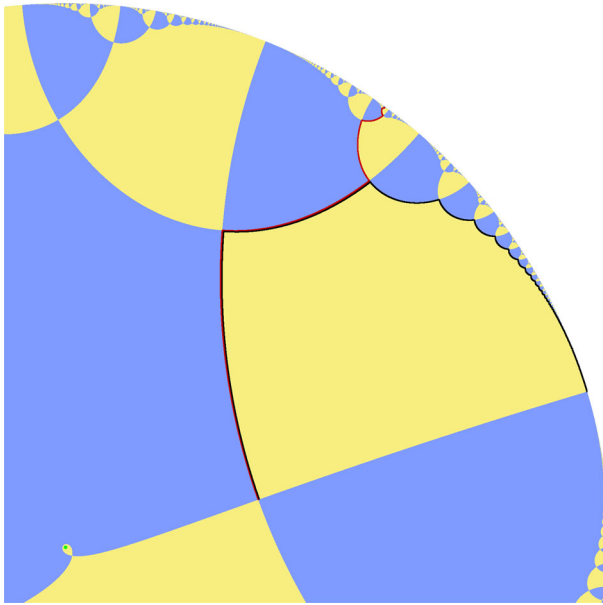


Fig. 17 A slow path in black, a quick path in red. The first one stays on the boundary of a single box. The other one turns alternately left and right at every corner. Here, speed is to be understood as the order of magnitude of the hyperbolic distance from the origin, when the curve is followed at constant box-Euclidean speed (on this picture, it takes the same time to get from a corner to the next one): in the first case it is logarithmic, in the second case linear

Note that there are paths in U_1 reaching the boundary, and whose compact subsets are of hyperbolic diameter comparable to their box euclidean length: see Fig. 17. An important task is thus to formulate and prove a combinatorial statement (Lemma 54) about the U_1 -boxes that the immediate basin A may cross, and that prevents this kind of behaviour for paths contained in A .

Define a chain of boxes to be a finite sequence b_0, b_1, \dots, b_n of U_1 -boxes such that two consecutive elements have non empty intersection, i.e. consecutive boxes are equal or share a side or a corner within U_1 . The integer n is called the length of the chain. With our convention there are $n + 1$ U_1 -boxes in a chain of length n . Define the *combinatorial distance* between U_1 -boxes as the minimal length of chains from one to the other. With our convention, this is a distance.

Lemma 53 *Let b, b' be U_1 -boxes and consider points $x \in b$ and $x' \in b'$. Then the box-Euclidean distance L between x and x' and the combinatorial distance n between b and b' satisfy:*

$$n \leq \lfloor L \rfloor + 1.$$

Proof First case: $L < 1$. Recall that the set of critical values of \tilde{f} is of the form $\tilde{v} + \mathbb{Z}$ for some \tilde{v} and that \tilde{f} has no asymptotic value over \mathbb{C} . Consider a path γ from x to x' and of box-Euclidean length < 1 . Let $\tilde{\gamma}$ be a lift of γ by E . The image of $\tilde{\gamma}$ by

\tilde{f} has Euclidean length < 1 in the plane. There will, therefore, exist $k \in \mathbb{Z}$ such that $\tilde{\gamma}$ is completely contained in the plane minus the translate of $] - \infty, -1] \cup [1, +\infty[$ by $\tilde{v} + k$. The connected components of the pre-image by \tilde{f} of such a slit plane are contained in unions of 2 or $2d$ of \tilde{U}_1 -boxes that touch at a common point: this is because there is at most one critical value (and no asymptotic value) of \tilde{f} in the slit plane. Now, $\tilde{\gamma}$ is contained in such a component, hence $n \leq 1$.

In the general case, we could use that there is a shortest path from x to x' (see Lemma 55) but we can do here without that information: consider a path γ from x to x' of length close enough to L so as to have the same integer part as L . Let $\varepsilon > 0$ and cut the path into pieces of length $1 - \varepsilon$, except maybe for the last piece for which we require length $\leq 1 - \varepsilon$. Let k be the number of pieces obtained: if ε small enough, $k = \lfloor L \rfloor + 1$. Let x_0, \dots, x_k denote the sequence of starting and end points of these pieces. Let $b_0 = b, b_k = b'$ and for $0 < n < k$ let b_n be a U_1 -box containing x_n . From the first case, we get that the combinatorial distance between b_n and b_{n+1} is ≤ 1 for $0 \leq n < k$. The combinatorial distance between b and b' is, thus, $\leq k$. \square

For $n \geq 0$, consider the set \mathcal{B}_n of U_1 -boxes at combinatorial distance $\leq n$ of the U_1 -box b_* . Note that for $n \geq 2$, the set \mathcal{B}_n is a infinite union of U_1 -boxes. The next lemma is illustrated by Fig. 18.

Lemma 54 *There exists $c_4 \in \mathbb{N}$ such that $\forall f \in \mathcal{F}, A \subset \mathcal{B}_{c_4}$.*

Proof Let us consider the principal curve $\mathcal{C} = \mathcal{C}[f]$ defined in Proposition 36, starting from $v = v_f$ and ending at 0. We recall it is a connected component of the preimage by $\Phi_{\text{attr}} = \Phi_{\text{attr}}[f]$ of $\Phi_{\text{attr}}(v) + [0, +\infty[$, that starts from v and ends on the parabolic point (and is contained in the common boundary in A of the two principal dynamical chessboard boxes of f), that $f(\mathcal{C}) \subset \mathcal{C}$ and hence \mathcal{C} contains the orbit of the critical value. Let us use Proposition 29, that provides a disk $D_{\text{attr}} = D_{\text{attr}}[f]$ of uniform diameter r_0 contained in the basin of f and which eventually traps any orbit in the parabolic basin. By Lemma 37, the number of iterates needed for the critical value v to enter D_{attr} is bounded over \mathcal{F} : $\exists n_0 \geq 0$ such that $\forall f \in \mathcal{F}, f^{n_0}(v) \in D_{\text{attr}}[f]$. The second point of Proposition 29 implies that the subset \mathcal{C}' of \mathcal{C} corresponding to $\Phi_{\text{attr}}(v) + [n_0, +\infty[$ satisfies $\mathcal{C}' \subset D_{\text{attr}}$.¹⁹

We will also require $r_0 < |v|$. Then, the set $\mathcal{C}' \subset D_{\text{attr}}$ does not cross the circle of equation $|z| = |v|$. Now \mathcal{C} is the union of \mathcal{C}' and of a connected component of the preimage by Φ_{attr} of the segment $\Phi_{\text{attr}}(v) + [0, n_0]$. As f varies in \mathcal{F} , the maps $\Phi_{\text{attr}} - \Phi_{\text{attr}}(v)$ all have an inverse branch defined on a common open connected neighborhood V of the segment $S = [0, n_0]$, mapping $0 = \Phi_{\text{attr}}(v) - \Phi_{\text{attr}}(v)$ back to v . This family is normal.²⁰ It also avoids 0. Take a lift $\tilde{\mathcal{C}}$ of \mathcal{C} by $E : z \mapsto e^{2\pi i z}$. This is

¹⁹ The restriction of Φ_{attr} to the union W of the principal chessboard boxes of A and their common boundary (minus endpoints) maps W univalently to the complement V of $\Phi_{\text{attr}}(v) +] - \infty, -1]$. The normalized attracting Fatou coordinate Φ of Proposition 29 necessarily coincides with Φ_{attr} on some smaller petal. Because of the shape of $U = \Phi(D_{\text{attr}})$, we necessarily have $U \subset V$, for otherwise, one can prove there would be a critical point of f in U , leading to a contradiction with injectivity of Φ claimed in the second point of Proposition 29. It follows that $D_{\text{attr}} \subset W$, and then that $\Phi = \Phi_{\text{attr}}$ on D_{attr} .

²⁰ There are many reasons for this; for instance, one can use continuity of $f \mapsto \Phi_{\text{attr}}[f]$ together with compactness of \mathcal{F} ; or remark that it is 1-Lipschitz, hence equicontinuous, from the hyperbolic norm on the chosen neighborhood V of the segment S to the metric $|dz|/4|z|$, because it maps in the simply connected set A that avoids 0 so one can use the Schwarz–Pick inequality and Koebe’s one quarter theorem.

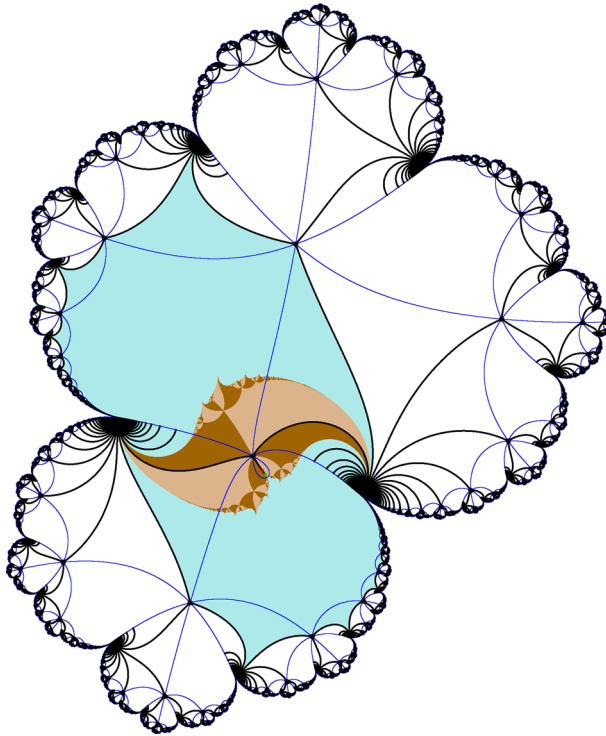


Fig. 18 Example for $d = 3$. We chose some $f \in \mathcal{F}$ (more precisely we took the first renormalization of $z \mapsto z^3 + c$ with c so that there is a fixed point tangent to the identity). The blue graph is the structural chessboard of f . The origin is marked by a tiny green dot and the critical value of f by a red one. It is a non-linearizable parabolic point of f . We drew in brown shades the dynamical chessboard of f in the immediate basin A of this point. The dark lines are the set $f^{-1}(\mathcal{C})$, where \mathcal{C} is the principal curve (see the text). The light blue set is the component containing A of $\text{Dom}(f)$ minus all the dark lines that are not contained in A . The picture has been accurately drawn, the curve \mathcal{C} is a small edge part in the black graph, from the green dot to the red one. It is very close to be a segment. As a consequence, $f^{-1}(\mathcal{C})$ is formed of curves that are very close to intrinsic verticals of U_1 -boxes. It seems therefore that the light blue is completely contained in \mathcal{B}_2 . This is probably the case for all maps in \mathcal{F} for $d = 3$ because the loop is very small. It may still hold when d gets close to ∞ , but that would require a more detailed specific analysis as in [19], starting from the fact that v_f is close to 0 ($|v_f| = u_d$ only depends on $d, u_d \in [u_2, u_\infty]$ and $u_2 \sim 1/140, u_\infty \sim 1/20$). We decided instead to resort to general arguments instead: in the proof of Lemma 54 we consider cases where \mathcal{C} may be very far from a segment

a curve starting from a preimage \tilde{v} of v and ending at ∞ tangentially to a vertical line. The part corresponding to \mathcal{C}' lives in the upper half plane $\{\text{Im}(z) \mid \text{Im}(\tilde{v})\}$ because we took $r_0 < |v|$. The rest is the image of S by a normal family defined in V . In particular it has bounded Euclidean length. Let L_1 be a bound, independent of $f \in \mathcal{F}$.

There are infinitely many connected component of $f^{-1}(\mathcal{C})$. Consider any. It consists either in a single curve or in a union of d curves starting from a common critical point of f . Each of these curves has a part mapped in $\mathcal{C} \setminus \mathcal{C}'$ by f that has box-Euclidean length $\leq L_1$, and a part mapped to \mathcal{C}' by f that is completely contained in one box.

By Lemma 53, the union of U_1 -boxes visited by the full curve has

$$\text{combinatorial diameter} \leq \lfloor L_1 \rfloor + 1. \quad (4)$$

The lifted immediate basin \tilde{A} contains exactly one component of $E^{-1}f^{-1}(\mathcal{C})$ and is disjoint from all other components. We claim that A is contained in $\mathcal{B}_{\lfloor L_1 \rfloor + 2}$: indeed consider the union G_1 of the $2d - 1$ U_1 -boxes which contain the critical point in A . It is contained in \mathcal{B}_1 . The immediate basin A contains exactly one component of $f^{-1}(\mathcal{C})$. Let G_2 be the component containing A of the complement in U_1^* of the union of all other components of $f^{-1}(\mathcal{C})$ (Fig. 18 may help). It is enough to prove that G_2 is contained in $\mathcal{B}_{\lfloor L_1 \rfloor + 2}$.

The boundary of G_2 in U_1^* consists in curves all of whose starting points s are preimages of v . We claim that they all belong to \mathcal{B}_1 . Indeed, the curve \mathcal{C} is isotopic in \mathbb{C} to the straight segment from v to 0 by an isotopy that does not move its endpoints. This isotopy extends to the whole Riemann sphere into an isotopy fixing ∞ . The singular values of f are $\{0, v, \infty\}$ and, thus, the isotopy does not move the singular values of f . Hence, the extended isotopy lifts by f to an isotopy of U_1 . This lifted isotopy does not move the points in $f^{-1}(v)$. Now, a starting point s as above can be linked to the unique critical point $c_0 \in b_*$ by a path within G_2 (except at its starting point $s \in \partial G_2$). The lifted isotopy deforms this path into a path with the same endpoints and that is completely contained in the complement of $f^{-1}([0, v])$. The image by f of the new path is contained in $\mathbb{C} \setminus [0, v]$ and goes from v to v . It is homotopic to a path completely contained $|z| > 1$. The homotopy lifts by f . Hence, s and c_0 are linked by a path contained in a U_1 -box. Whence the claim.

Consider any point $z \in G_2$. If z belongs to $f^{-1}(\mathcal{C})$ then it belongs to the unique component of $f^{-1}(\mathcal{C})$ in G_2 , which is the one attached to the critical point in A , which belongs to b_* . Hence $z \in \mathcal{B}_{\lfloor L_1 \rfloor + 1}$ by the bound (4) above. Otherwise, $f(z) \notin \mathcal{C}$. Then $f(z) \in H$ for $H = \mathbb{D} \setminus \{0\}$ or $H = \mathbb{C} \setminus \mathbb{D}$ (if $|f(z)| = 1$ then either can be chosen). There is a path $\gamma \subset H$ from $f(z)$ to a point of $\mathcal{C} \setminus \{0\}$ (which may be its endpoint v). Let b be the (unique) U_1 -box containing z and such that $f(b) = H$. The path γ lifts by f to a path within b from z to a point w in $f^{-1}(\mathcal{C})$, and w is either in G_2 or in ∂G_2 . We saw that the component of $f^{-1}(\mathcal{C})$ that w belong to is attached to a point in $f^{-1}(v)$ that belongs to \mathcal{B}_1 . By the bound (4), we get that $b \in \mathcal{B}_{\lfloor L_1 \rfloor + 2}$. This ends the proof that $G_2 \subset \mathcal{B}_{\lfloor L_1 \rfloor + 2}$. \square

Let the combinatorial distance between two points of $\text{Dom } \tilde{f}$ be the smallest combinatorial distance of boxes containing them. Two important facts used in the proof of the following lemma are that the chessboard graph of $\text{Dom } \tilde{f}$ is a tree and that the boundary in $\text{Dom } \tilde{f}$ of a \tilde{U}_1 -box is a connected subset of this graph.

Lemma 55 *For any two points $w, z \in \text{Dom } \tilde{f}$, then there is a unique shortest path γ' from w to z for the box-Euclidean distance. If m denotes the combinatorial distance from w to z then γ' can be cut into $\leq m + 1$ connected pieces, each of which stays in some U_1 -box.*

Proof ²¹ Let us define a projection from $\text{Dom } \tilde{f}$ to the chessboard graph of \tilde{f} as follows. Recall that each \tilde{U}_1 -box b is homeomorphically mapped by \tilde{f} to a closed half plane and that the box-Euclidean metric element is sent to the canonical Euclidean element $|dz|$ of \mathbb{C} . The vertical projection on this half plane to its boundary is 1-Lipschitz and can be conjugated back to a projection from b to its boundary in $\text{Dom } \tilde{f}$. The union of all these projections for all U_1 -boxes b is easily seen to match at the boundary points and corners, and yields a projection function from $\text{Dom } \tilde{f}$ to the chessboard graph, which is locally 1-Lipschitz for the box-Euclidean metric (the only place where checking this claim is not trivial is at corners). In particular, it is 1-Lipschitz for path-length.

Given any path γ from w to z , if the path meets the chessboard graph then the part from its first intersection with the graph to its last can be projected as above. The new path is strictly shorter unless the part was already contained in the graph. This part can be further simplified into an injective path within the graph, strictly shorter unless it was already injective.

If moreover both w and z belong to a given U_1 -box c , the first and last point in the graph are also in c , and since the graph is a tree and the boundary of a U_1 -box is a connected subset of this tree, the simplified part is necessarily contained in this boundary, hence the simplified path is contained in c . We have, thus, in particular proved that for any path that is not completely contained in c there is a strictly shorter path contained in c . Hence the straight segment γ'' from w to z for the Euclidean structure on c is the unique shortest box-Euclidean path from w to z within $\text{Dom } \tilde{f}$. The other conclusions of the lemma are trivial in this case: $m = 0$ and γ'' does not need to be cut.

In the rest of the proof of the lemma, we assume that there is no U_1 -box containing both w and z .

Then, given the simplification of path constructed above, it follows that the infimum of box-Euclidean lengths of paths between w and z is the same as the infimum over the set \mathcal{A} of paths defined below, and that a path that is not in \mathcal{A} cannot be minimal. The set \mathcal{A} consist in paths that are a straight box-Euclidean line from w to the boundary of its box if w is in the interior of a box, then an injective path within the graph, then similarly a straight box-Euclidean line to z if z is in the interior of a box. From the form of \mathcal{A} and the fact that the distance along the graph between two points a and b of the graph is a continuous function of the pair (a, b) , the fact that a minimal distance is reached on \mathcal{A} easily follows. Let us sum up what we have proved so far: there is at least one shortest path, all shortest paths are in \mathcal{A} .

Let I_w be defined as follows: if w is in the graph we let $I_w = \{w\}$; otherwise we let I_w be the boundary in $\text{Dom } \tilde{f}$ of the unique U_1 -box containing w . In the latter case, I_w is an infinite curve in the graph. The set $I_w \cap I_z$ is either empty or a point or a connected curve, of finite or infinite box-Euclidean length.

First case: $I_w \cap I_z$ is empty or a point. Then, there is a unique shortest path γ'' within the graph from the set I_w to the set I_z ; we allow γ'' to be reduced to a single point to include the case when $I_w \cap I_z$ is a single point. We call w' the initial point of γ'' and z' the endpoint; as we explained, z' may be equal to w' . It is also possible that $w = w'$, similarly $z = z'$ is possible. If $w \neq w'$ then there is a unique U_1 -box

²¹ Special thanks to Arnaud Mortier for a great help in this proof.

containing both. Similarly for z and z' . In all cases, the box-straight path from w to w' , followed by γ'' , followed by the box-straight path from z' to z is the unique shortest path in \mathcal{A} , and thus the unique shortest path within $\text{Dom } \tilde{f}$. Call it γ' .

Consider now any U_1 -box chain b_0, \dots, b_m with $w \in b_0$ and $z \in b_m$. We claim that this chain necessarily covers γ' . Let us prove this claim. Note that the part of γ' from w to w' is contained in b_0 and the part from z' to z in b_m . Each b_m is path connected, so by definition of a chain, the union of the b_m is path connected. Consider path from w' to z' within this union. Project it on the graph and simplify it as above. This leads to an injective path from w' to z' contained in the graph. By uniqueness of injective paths in a tree, this path is equal to γ'' . This proves the claim.

The intersection of a U_1 -box with the graph is a connected subset of this tree (it is a curve, infinite in both directions). It follows that the intersection of γ' with a U_1 -box is necessarily a connected portion of γ' . Let us now split γ' as follows: choose any U_1 -box b_i containing w , define $i_1 = i$ and cut γ' at the last point where it is contained in b_{i_1} . Note that the part before the cut is entirely contained in b_{i_1} . If this cutpoint is not the endpoint of γ' , then a non-trivial sub-part of the path starting from b_{i_1} belongs to another U_1 -box $b_{i'}$. Define $i_2 = i'$ and cut the remaining part of the path at the last point where it is contained in b_{i_2} . And so on. This process necessarily ends (because, for instance, the cut points are contained in a discrete set, because they are either branch points of the graph or the first or the last intersection of γ' with the graph). So we get a finite sequence of U_1 -boxes $b_{i_1}, b_{i_2}, \dots, b_{i_{m'}}$ for some $m' \in \mathbb{N}^*$ and a splitting $\gamma'_1, \dots, \gamma'_{m'}$ of γ' into connected pieces with $\gamma'_j \subset b_{i_j}$ for all $j \leq m'$. By construction $b_{i_{j+1}} \neq b_{i_j}$. Now by the property that the intersection of a U_1 -box with γ' is necessarily connected, and the it follows that no two U_1 -boxes b_{i_j} and b_{i_k} can be equal for $j \geq k + 2$, for otherwise the whole part of the path between γ'_j and γ'_k (included) would be contained in b_{i_k} , contradicting the way we built the splitting. Hence, $m' \leq m + 1$.

Second case: $I_w \cap I_z$ is a connected curve in the graph. Let b be the unique U_1 -box containing w and b' be the same for z . Note that b and b' are adjacent, and $m = 1$. The union $b \cup b'$ is connected. It consists in the interior of b , the interior of b' , the common curve, and at most four disjoint pieces of curves in the boundaries of b or b' , attached to an end point of the common curve. Because the graph is a tree, all paths in \mathcal{A} are contained in $b \cup b'$ and all paths in \mathcal{A} must meet the common curve, possibly at an end thereof. It follows that the shortest path in \mathcal{A} from w to z is a straight segment to a point in the common curve, followed by a straight segment. We have, thus, cut the shortest path in two pieces satisfying the conclusion of the lemma, since $m + 1 = 2$. \square

Let us now go back to the situation we were studying: recall $L(\varepsilon)$ was defined at the beginning of Sect. 3.7; for convenience we denote $L = L(\varepsilon)$; we had a path γ of A -length at most $c_2 + L$, starting from \mathfrak{B} and going to some point z_0 . We are ready to prove that:

$$d_{U_1^*}(\gamma(0), \gamma(1)) \leq c'_6 + c_6 \log(1 + L). \quad (5)$$

Recall that there is a special U_1 -box b_* that is a punctured neighborhood of the origin. Note that b_* is the only U_1 -box that has a unique lift by E , which we denote \tilde{b}_* . Let $\tilde{\gamma}$ be a lift of γ by E and $\gamma_2 = \tilde{f} \circ \tilde{\gamma}$. Then γ_2 is also a lift of $f \circ \gamma$ by E . The U_1^* -hyperbolic length of γ is equal to the $\text{Dom}(\tilde{f})$ -hyperbolic length of $\tilde{\gamma}$. The box-Euclidean length of $\tilde{\gamma}$ is equal to Euclidean length of γ_2 and is thus $\leq 2(c_2 + L)$ by Eq. (2). By Lemma 54, the path γ is contained in \mathcal{B}_{c_4} . There is, thus, for any $t \in [0, 1]$, a chain of U_1 -boxes of length at most c_4 from some U_1 -box containing $\gamma(t)$ to b_* . This chain lifts by E into a chain of U_1 -boxes from $\tilde{\gamma}(t)$ to \tilde{b}_* . Applying this to $t = 0$ and $t = 1$, we get that the combinatorial distance²² from $\tilde{\gamma}(0)$ to $\tilde{\gamma}(1)$ is $\leq 2c_4$. Consider the path γ' provided by Lemma 55, from $\tilde{\gamma}(0)$ to $\tilde{\gamma}(1)$, of box-Euclidean length at most that of $\tilde{\gamma}$, and consisting in $p \leq 2c_4 + 1$ parts γ'_i each contained in some U_1 -box. Denote L_i the box-Euclidean length of γ'_i . Then, $\sum_{i=1}^p L_i \leq 2(c_2 + L)$ and in particular $L_i \leq 2(c_2 + L)$. By Corollary 51, the endpoints of γ'_i sit at U_1^* -hyperbolic distance $\leq c'_5 + \log(1 + c_5 L_i)$ from each other. Thus, putting it all together:

$$\begin{aligned} d_{U_1^*}(\gamma(0), \gamma(1)) &\leq d_{\text{Dom } \tilde{f}}(\tilde{\gamma}(0), \tilde{\gamma}(1)) \\ &\leq \sum_{i=1}^p c'_5 + \log(1 + c_5 L_i) \\ &\leq p c'_5 + p \log(1 + 2c_5(L + c_2)) \\ &\leq (2c_4 + 1)c'_5 + (2c_4 + 1) \log(1 + 2c_5 L + 2c_5 c_2) \\ &\leq c'_6 + c_6 \log(1 + L) \end{aligned}$$

for some constants c_6, c'_6 that depend only on c_2, c_4, c_5 and c'_5 , which proves (5).

Because of the inclusion $U_1^* \subset U_1$, the U_1 -hyperbolic distance between $\gamma(0)$ and $\gamma(1)$ will be even shorter. Using Lemma 48, we get that z_0 belongs to the U_1 -hyperbolic ball of center 0 and radius

$$L' = c_7 + c'_6 + c_6 \log(1 + L).$$

Hence, the set C object of Proposition 45, which we are proving, is contained in this hyperbolic ball (see the discussion after Lemma 48). Recall that $L = L(\varepsilon) = \tanh^{-1}(1 - \varepsilon)$. Introduce $\varepsilon' \in]0, 1[$ such that $\tanh^{-1}(1 - \varepsilon') = L'$. Then

$$C \subset \phi_1(B(0, 1 - \varepsilon')).$$

Now, $\varepsilon' = 2/(e^{2L'} + 1) \geq e^{-2L'}$ and $L' = c_7 + c'_6 + c_6 \log(1 + L)$ and $L \leq c_1 + \frac{1}{2} \log \frac{1}{\varepsilon}$ so $L' \leq c'_8 + c_8 \log(1 + \log(1/\varepsilon))$, thus

$$\boxed{\log \frac{1}{\varepsilon'} \leq c'_9 + c_9 \log \left(1 + \log \frac{1}{\varepsilon} \right)}. \quad (6)$$

²² Notion defined just before Lemma 55.

In particular, as $\varepsilon \rightarrow 0$, ε' also tends to 0 but remains much bigger than ε . This proves Proposition 45.

3.8 Step 2, I: Perturbation Argument

Let us recall the notations introduced in Sect. 3.2:

$$\mathcal{F} = \{\mathcal{R}[B_d] \circ \phi^{-1} \mid \phi : \mathbb{D} \rightarrow \mathbb{C} \text{ is univalent and } \phi(z) = z + \mathcal{O}(z^2)\},$$

and

$$\mathcal{F}_\varepsilon = \{\mathcal{R}[B_d] \circ \phi^{-1} \mid \phi : B(0, 1 - \varepsilon) \rightarrow \mathbb{C} \text{ is univalent and } \phi(z) = z + \mathcal{O}(z^2)\},$$

where $\mathcal{R}[B_d]$ is the (upper) parabolic renormalization of the Blaschke product, normalized to be defined on the unit disk. In particular,

$$\mathcal{F}_0 = \mathcal{F}.$$

Last, for $X \subset [0, 1]$, we will denote

$$\mathcal{F}_X = \bigcup_{x \in X} \mathcal{F}_x.$$

3.8.1 An Interpolation

Let $\varepsilon_1 > 0$ and $f \in \mathcal{F}_{\varepsilon_1}$:

$$f = \mathcal{R}[B_d] \circ \tilde{\phi}^{-1}.$$

For convenience, we will denote

$$r' = 1 - \varepsilon_1$$

and $\phi(z) = \frac{1}{r'} \tilde{\phi}(r'z)$. Then $\phi \in \mathcal{SL}$ (the class of Schlicht maps) and

$$f(z) = \mathcal{R}[B_d](r' \phi^{-1}(z/r')).$$

Let $U = \phi(\mathbb{D})$. We will interpolate smoothly between f , which belongs to $\mathcal{F}_{\varepsilon_1}$, and an element of \mathcal{F} as follows: for $t \in [0, 1[$, let

$$\phi_t(z) = r_t \phi(z/r_t) \text{ with } r_t = 1 - t.$$

Then the map ϕ_t is an isomorphism from $B(0, r_t)$ to $r_t U$. Let

$$f_t(z) = \mathcal{R}[B_d] \circ \phi_t^{-1} : r_t U \rightarrow \mathbb{C}.$$

Then,

$$f_{\varepsilon_1} = f,$$

and

$$f_t \in \mathcal{F}_t \text{ thus } f_0 \in \mathcal{F}.$$

In the sequel, we will start from knowledge about f_0 and transfer it to f_{ε_1} , by continuously increasing t from 0 to ε_1 .

Using the language of structures that we introduced in Sect. 1.2, let us stress that maps in \mathcal{F}_t are all $(I, \widehat{\mathbb{C}})$ -structurally equivalent (I being a singleton and the origin being the marked point). For $t' > t$, the structure of maps in $\mathcal{F}_{t'}$ is a sub-structure of that of maps in \mathcal{F}_t .

Remark Though, for $t' > t$, $f_{t'}$ is a sub-structure of f_t , it is very unlikely that the map $f_{t'}$ would be conjugate to a restriction of f_t .

Let us show a non-commuting diagram that the reader may find useful in order to follow the arguments.

$$\begin{array}{ccc} & \xleftarrow{\cdot/r_t} & \\ \phi_0^{-1} \downarrow & & \uparrow \mathcal{R}[B_d] \\ & \xrightarrow{r_t \times \cdot} & \end{array}.$$

The map f_t consists in turning once around this diagram, starting from the upper right corner.²³

3.8.2 About the Critical Value

Let T_0 be one minus the absolute value of the critical point of $\mathcal{R}[B_d]$ that is closest to 0. Then for all $t \in [0, T_0]$, maps in \mathcal{F}_t have a unique critical value.

Lemma 56 *There exists $T'_1 \in]0, T_0[$ such for all maps $f \in \mathcal{F}_{[0, T'_1]}$, the critical value is attracted to 0.*

Proof By Fatou's theorem (Theorem 4), this is the case for all maps in \mathcal{F}_0 . The existence of T'_1 then follows from compactness of \mathcal{F}_0 and the fact that for a parabolic map with one petal attracting a given point, nearby parabolic maps will attract nearby points. \square

²³ It may at first seem to be better to start from the upper left corner, since the corresponding composition has a domain U that does not depend on t . However, when we iterate these maps, we basically go in round circles along a non-commuting diagram again and again, and the author thinks that it would not simplify the proof that much.

A consequence of the uniqueness of the critical value is that the extended attracting Fatou coordinate $\Phi_{\text{attr}}[f_t]$ has a set of critical values contained in $\{v' - n \mid n > 0\}$, where $v' = \Phi_{\text{attr}}[f_t](v)$ and v is the critical value of f_t . Unlike the case $t = 0$, when $t > 0$ the map $\Phi_{\text{attr}}[f_t]$ probably has a big set of asymptotic values (it is likely that it contains curves).

3.9 Step 2, II: Following Fibers

3.9.1 A Motion of the Fibers of the Fatou Coordinates and of the Renormalized Map

The point of view outlined in Sect. 3.8.1 can be reversed and we may start from any map $f_0 = \mathcal{R}[B_d] \circ \phi_0^{-1} \in \mathcal{F}$, which has the full structure of $\mathcal{R}[B_d]$ and perturb it into the map $f_t \in \mathcal{F}_t$ as before, which has less and less structure as $t \in [0, 1[$ increases. Let us recall how f_t is defined:

$$f_t(z) = \mathcal{R}[B_d] \circ \phi_t^{-1} \quad \text{with} \quad \phi_t(z) = r_t \phi_0(z/r_t) \quad \text{and} \quad r_t = 1 - t.$$

Studying the survival of (part of) the structure of the parabolic renormalization $\mathcal{R}[f_t]$ as t increases means following fibers of $\mathcal{R}[f_t]$.

Recall that $\mathcal{R}[f_t]$ is defined by

$$(a^{-1} \circ \mathcal{R}[f_t] \circ b) \circ E = E \circ (\Phi_{\text{attr}}[f_t] \circ \Psi_{\text{rep}}[f_t])|_{W_t},$$

with $E(z) = e^{2\pi iz}$, W_t is some domain, and a and b are linear maps that depend on f_t and on normalization conventions. Recall that we chose to normalize Fatou coordinates by their expansion at infinity, and to normalize $\mathcal{R}[f_t]$ by fixing its critical value. See Sect. 3.3 for more details.

To lighten the expressions, let us abbreviate $R_t = \mathcal{R}[f_t]$ and introduce extended Fatou coordinates Φ_t and Ψ_t of f_t , normalized differently from $\Phi_{\text{attr}}[f_t]$ and $\Psi_{\text{rep}}[f_t]$, and so that

$$R_t \circ E = E \circ \Phi_t \circ \Psi_t|_{W_t}.$$

We defined in Sect. 3.8.2 two constants T_0 and $T'_1 < T_0$ such that:

- For $t \leq T_0$, for all $f \in \mathcal{F}$, f_t has a unique critical value. Let us denote it by v_t .
- For $t \leq T'_1$, this point v_t is in the domain of definition of $\Phi_{\text{attr}}[f_t]$.

Let

$$\Phi_t(z) = \Phi_{\text{attr}}[f_t](z) + \beta_t,$$

where $\beta_t = \sigma_d - \Phi_{\text{attr}}[f_t](v_t)$, so that $\Phi_t(v_t)$ does not depend on t and where σ_d is a constant that depends only on d and is chosen so that $E(\sigma_d)$ is the critical value of $\mathcal{R}[B_d]$.²⁴ For the repelling inverse Fatou coordinate (whose normalization is less

²⁴ One can for instance take $\sigma_d = i\pi\gamma[B_d]$ but we will not use this fact.

important), we let

$$\Psi_t(z) = \Psi_{\text{rep}}[f_t](z - \beta'_t),$$

for $\beta'_t = \beta_t - i\pi\gamma[f_t]$ (recall γ is the iterative residue, see Appendix A). Let $\Phi : (t, z) \mapsto \Phi_t(z)$, that we define on

$$\text{Dom } \Phi = \{(t, z) \in [0, T'_1[\times \mathbb{C} \mid z \in \text{Dom}(\Phi_t)\}.$$

It is an open subset of $[0, T'_1[\times \mathbb{C}$, and Φ is a continuous function of (t, z) by Proposition 34 (in fact, it is analytic, see [28]). Similarly, let

$$R : \begin{cases} \text{Dom } R \rightarrow \mathbb{C} \\ (t, z) \mapsto R_t(z) \end{cases}$$

The domain of R is an open subset of $[0, T'_1[\times \mathbb{C}$ and R is continuous, analytic w.r.t. z for fixed values of t . (It is also analytic w.r.t. (t, z) but we will not use this fact.)

The critical values of Φ_t and R_t do not move when t varies (even when some critical points vanish). It has the following consequence:

Proposition 57 (following part of the structure) *Let $F = \Phi$ or $F = R$. Then,*

- (Lemma 59) *fibers of Φ form a foliation that is locally parallelizable over the first coordinate.*

It follows that there exists a function $\tau : \text{Dom } F_0 \rightarrow]0, T'_1[$ (survival time) and function $\zeta(t, z)$ (fiber follower) such that:

- $\text{Dom } \zeta = \{(t, z) \in [0, T'_1[\times \text{Dom}(F_0) \mid t \in [0, \tau(z)[\}$
- *the map τ is lower semi continuous, i.e., for all $t \in [0, T'_1[$, the set $U_t = \tau^{-1}(]t, T'_1[) \subset \mathbb{C}$ is open*
- *the above two points imply that $\text{Dom } \zeta$ is an open subset of $[0, T'_1[\times \mathbb{C}$ and $\text{Dom } \zeta = \{(t, z) \in [0, T'_1[\times \mathbb{C} \mid z \in U_t\}$*
- *the map ζ is continuous*
- *for each fixed $t \in [0, T'_1[$, the map $z \in U_t \mapsto \zeta(t, z)$ is holomorphic and injective*
- $\forall (t, z) \in \text{Dom } \zeta$, $F_0(z) = F_t(\zeta(t, z))$, i.e., *the map $t \in [0, \tau(z)[\mapsto (t, \zeta(t, z))$ follows a fiber of F*
- (maximality and uniqueness) *consider any continuous map following a fiber of F as t varies from 0 to some t_0 , starting from $(0, z) \in \text{Dom } F$; then $\tau(z) > t_0$ and the continuous map must coincide with $t \in [0, t_0] \mapsto \zeta(t, z)$.*

The rest of the present section (Sect. 3.9.1) is devoted to the proof of the above proposition. The proof is written for Φ but is the same, word for word, for R .

Lemma 58 *Let $(t_0, z_0) \in \text{Dom } \Phi$ and assume that z_0 is a critical point of Φ_{t_0} . Then, there exists a connected neighborhood I of t_0 in $[0, T'_1[$, and $r_0 > 0$ such that for all $t \in I$, Φ_t has a unique critical point in $B(z_0, r_0)$, it moves continuously with t and its multiplicity does not change.*

Proof We apply Hurwitz's theorem²⁵ to Φ'_t and to Φ_t (note that Φ'_t also depends continuously on t , by Cauchy's estimates): let $u = \Phi_{t_0}(z_0)$. Take $r_0 > 0$ small enough so that z_0 is the only critical point of Φ_{t_0} in $\overline{B} := \overline{B}(z_0, r_0)$, the only solution of $\Phi_{t_0}(z) = u$ in \overline{B} , and such that Φ_{t_0} maps this disk in $B(u, 1/2)$; there exists ε_0 such that for all $t \in [0, T'_1[$ with $|t - t_0| < \varepsilon_0$, Φ_t is defined on \overline{B} and maps it in $B(u, 1/2)$; then by Hurwitz's theorem, there exists $0 < \varepsilon < \varepsilon_0$ such that for $|t - t_0| < \varepsilon$, $\Phi_t - u$ has $d - 1$ critical points counted with multiplicity in B and d roots in B . Now recall, we normalized the maps Φ_t so that all critical values belong to $\mathbb{Z} + u$ and u does not depend on t . Since $\Phi_t(\overline{B}) \subset B(u, 1/2)$, this implies that all critical points of Φ_t in \overline{B} map to u . Thus, the sum of local degrees of Φ_t at preimages of u in \overline{B} equals d , and the sum of local degrees minus one equals $d - 1$: there is exactly one preimage of u , thus exactly one critical point. Moreover, its local degree is d , thus its multiplicity is constant. Continuous dependence is a classical application of Hurwitz's theorem and is left to the reader.²⁶ \square

Now consider the fibers of Φ : $X_c = \{(t, z) \in \text{Dom } \Phi \mid \Phi(t, z) = c\}$. They form a collection of disjoint closed subsets of $\text{Dom } \Phi$. We will prove that this collection is a locally trivial foliation, in the following precise sense:

Lemma 59 (local trivialization) *All $(t_0, z_0) \in \text{Dom } \Phi$ has an open neighborhood V in $\text{Dom } \Phi$ on which a change of variable $U : V \rightarrow V' \subset_{\text{open}} [0, T'_1[\times \mathbb{C}$ of the form*

$$U : (t, z) \mapsto (t, u(t, z))$$

is defined,

- (1) U is a homeomorphism to V' ,
- (2) for all t , $z \mapsto u(t, z)$ is holomorphic,
- (3) $\forall c \in \mathbb{C}$, $U(X_c)$ is the intersection of a horizontal with V' : it is of the form $V' \cap ([0, T'_1[\times \{w\})$ for some $w \in \mathbb{C}$.

Proof Case 1: z_0 is not a critical point of Φ_{t_0} . It is an application of Hurwitz's theorem. Since the family Φ_t depends continuously on t and Φ_{t_0} is not locally constant near z_0 , one can deduce from Hurwitz's theorem that the map $U = \Phi$ itself, restricted to an appropriate neighborhood V , will be a local trivialization. Details are left to the reader.

Case 2: z_0 is a critical point of Φ_{t_0} . A consequence of Lemma 58, is that we can factor $\Phi_t(z) = (z - c_t)^d h_t(z)$ where $h_t(z)$ is a holomorphic function in z , continuous in (t, z) , defined locally and non-vanishing. The map $g(t, z) = \sqrt[d]{h_t(z)}$ is defined locally,

²⁵ There seems to be several statements called Hurwitz's theorem. We are referring to the following: for a sequence of holomorphic functions f_n converging uniformly on compact subsets of an open subset U of \mathbb{C} , call its limit f . If D is a disk compactly contained in U and f does not vanish on the boundary of D then for all n big enough, f and f_n have the same number of zeroes in D , counted with multiplicity.

²⁶ There is a more direct proof, with Hurwitz's theorem used only at the end to deduce continuity. From the fact that z_0 is in a parabolic basin and that all critical points of f_t map to the same point, it follows that the orbit of z_0 hits the set of critical points only once. Then, one uses that $\Phi_{\text{attr}} = -n + \Phi_{\text{attr}} \circ f^n$, and that Φ_{attr} is injective in the petal $D_{\text{attr}}[f_t]$ and that the latter moves continuously with t . Similar arguments can be carried out for R in place of Φ .

and we leave to the reader to check that the map $(t, z) \mapsto (t, (z - c_t)g(t, z))$ is a local trivialization. \square

Hence, connected components of fibers are graphs of continuous functions $t \mapsto z(t)$ defined on connected open subsets of $[0, T'_1[$. Now, given any $z \in \text{Dom}(\Phi_0)$, we follow its fiber as t increases from 0 as long as possible: this gives a maximal continuous function $t \in [0, \tau(z)[\mapsto \zeta_z(t)$ such that $\zeta_z(0) = z$ and $\Phi_t(\zeta_z(t))$ is constant. The real number $\tau(z)$ belongs to $]0, T'_1]$. Uniqueness and maximality (last point of Proposition 57) follow easily. In the lemma below, the point $\zeta_z(t)$ is denoted

$$z\langle t \rangle.$$

Lemma 60 *The following holds:*

- (1) *The function τ is lower semi-continuous, i.e. for all $t \in [0, T'_1[$, the set $U_t = \tau^{-1}(]t, T'_1]) \subset \mathbb{C}$ is open.*
- (2) *On U_t , the function $z \mapsto z\langle t \rangle$ is holomorphic.*

Proof For a given $z \in U_t$, since $t < \tau(z)$, cover the compact set $[0, t]$ by open subsets on which there is a local trivialization of the fiber z belongs to. Extract a finite cover. From it, one can build a trivialization like in the previous lemma, but in a whole neighborhood of $z\langle [0, t] \rangle$ relative to $[0, t] \times \mathbb{C}$. The lemma follows. \square

This ends the proof of the Proposition 57.

3.9.2 Objectives

Let $f \in \mathcal{F}$ and denote by $\tau_R[f]$ the τ function corresponding to R in Proposition 57: i.e. $\tau_R[f](z)$ is the time up to which the fiber of $(t, z) \mapsto R_t(z)$ that contains $(0, z)$ can be followed. Recall that $\mathcal{R}[f_t]$ denotes the parabolic renormalization of f_t , normalized so that the critical value does not move as t varies, and recall that f_t is a specific perturbation of $f_0 = f$. Consider the parabolic renormalization $\mathcal{R}[f_0]$ of f_0 .

Lemma 61 *If $\forall z \in \text{Dom}(\mathcal{R}[f_0]) \odot (1 - \varepsilon_1)$, $\tau_R[f_0](z) > \varepsilon_0$ then $\mathcal{R}[f_{\varepsilon_0}]$ has a restriction that belongs to $\mathcal{F}_{\varepsilon_1}$.*

Proof The map $\mathcal{R}[f_0]$ belongs to \mathcal{F} , thus it can be written as $\mathcal{R}[f_0] = \mathcal{R}[B_d] \circ \phi_2^{-1}$ where $\phi_2 : \mathbb{D} \rightarrow \mathbb{C}$ is univalent and $\phi_2(z) = z + \mathcal{O}(z^2)$. By hypothesis, the set U_{ε_0} contains $\text{Dom}(\mathcal{R}[f_0]) \odot (1 - \varepsilon_1) = \phi_2(B(0, 1 - \varepsilon_1))$ (the sets U_t were defined in Proposition 57 and Lemma 60). According to Proposition 57, the map $\zeta^t : z \in U_t \mapsto \zeta(t, z)$ is a holomorphic bijection to its image, and $\mathcal{R}[f_t](\zeta^t(z)) = \mathcal{R}[f_0](z)$ holds on U_t . Apply this to $t = \varepsilon_0$: let $V = \zeta^{\varepsilon_0}(\phi_2(B(0, 1 - \varepsilon_1)))$, then $\zeta^{\varepsilon_0} \circ \phi_2$ is a structural equivalence, with 0 as a marked point, between the restriction of $\mathcal{R}[f_{\varepsilon_0}]$ to V and the restriction of $\mathcal{R}[B_d]$ to $B(0, 1 - \varepsilon_1)$. \square

So the Main theorem (more precisely Theorem 26) will be proved if we can prove the following claim:

Proposition 62 (survival of fibers of R) *There exists a pair $\varepsilon_1 < \varepsilon_0$ with $\varepsilon_0 < T'_1$ such that for all $f_0 \in \mathcal{F}$, for all $z \in \text{Dom}(\mathcal{R}[f_0]) \odot (1 - \varepsilon_1)$,*

$$\tau_R[f_0](z) > \varepsilon_0.$$

The constant T'_1 was defined in Sect. 3.8.2. We will in fact prove more: for all ε_0 small enough, there exists $\varepsilon_1 < \varepsilon_0$ such that the conclusion of the proposition holds. Better: we can take $\varepsilon_1 \ll \varepsilon_0$ (see details in Sect. 3.11).

3.9.3 Restatement of the Objectives

Let $\varepsilon > 0$ and consider some

$$z \in \text{Dom}(\mathcal{R}[f_0]) \odot (1 - \varepsilon).$$

The map $R_t = \mathcal{R}[f_t]$ is the semi-conjugate by E of the composition $\Phi_t \circ \Psi_t$, but it can also be viewed differently: recall that the extended Fatou coordinates Φ_t and extended inverse Ψ_t are defined via iteration of f_t , using bijective Fatou coordinates in petals as a starting point. Let \mathcal{P}_{rep} be a repelling petal and Φ_{rep} be a repelling Fatou coordinate such that $\Psi_t = \Phi_{\text{rep}}^{-1}$ holds on $\Phi_{\text{rep}}(\mathcal{P}_{\text{rep}})$. The value $R_t(z)$, thus, decomposes as follows (see Fig. 22 in Appendix A):

$$R_t(z) = E(\Phi_t(f_t^{m_0}(\Psi_t(u)))),$$

where $E(z) = e^{2\pi iz}$, $u \in E^{-1}(z) \cap \Phi_{\text{rep}}(\mathcal{P}_{\text{rep}})$ and $m_0 = m_0(z) \in \mathbb{N}$ is chosen so that $f_t^{m_0}(\Psi_t(u))$ belongs to the attracting petal. Let us now focus on the initial situation, at $t = 0$: consider the f_0 bilateral orbit

$$(n \in \mathbb{Z}) \quad \omega_n := \Psi_0(u + n).$$

It depends on z and on the choice of $u \in E^{-1}(z) \cap \Phi_{\text{rep}}(\mathcal{P}_{\text{rep}})$. Interestingly, if one chooses another $u \in E^{-1}(z) \cap \Phi_{\text{rep}}(\mathcal{P}_{\text{rep}})$, we get the same orbit, but with the index n shifted. According to the first step, if $z \in \text{Dom}(\mathcal{R}[f_0]) \odot (1 - \varepsilon)$ then the orbit ω_n is contained in $\text{Dom}(f_0) \odot (1 - \varepsilon') = \phi_0(B(0, 1 - \varepsilon'))$ with $\varepsilon' \gg \varepsilon$:

$$\forall n \in \mathbb{Z}, \quad \omega_n \in \text{Dom}(f_0) \odot (1 - \varepsilon').$$

Let us again insist on our interpretation of this fact, that is the central idea of the whole machinery: given $f_0 \in \mathcal{F}$, the restriction of its renormalized map R_0 to a map with substructure \mathcal{F}_ε , can be defined using a restriction of the map f_0 that has structure $\mathcal{F}_{\varepsilon'}$, i.e. much less structure. If all maps with structure $\mathcal{F}_{\varepsilon'}$ were restrictions of maps in \mathcal{F} we would be done (the main theorem would follow at once), but this is of course not the case, and this is the reason why we introduced the interpolation f_t . The idea is then the following: since $\varepsilon \ll \varepsilon'$, for t at most ε or just slightly bigger, the map f_t will be extremely close to f_0 on a set slightly bigger than $\text{Dom}(f_0) \odot (1 - \varepsilon')$. The

task is then to check that this is close enough so that the fibers attached to the orbits ω_n survive and thus the \mathcal{F}_ε -structure of the parabolic renormalization survives.

Let us now denote $\tau_\Phi[f]$ the τ function corresponding to Φ in Proposition 57. This proposition also provides a map $(t, z) \mapsto \zeta(t, z)$, to be interpreted as a motion of z as t varies. For convenience, in the sequel, we will use the notation

$$z\langle t \rangle = \zeta(t, z).$$

Lemma 63 (the motion is compatible with the dynamics) $\forall z \in U_0$, $\tau_\Phi(f_0(z)) \geq \tau_\Phi(z)$ and $\forall t < \tau_\Phi(z)$, $f_t(z\langle t \rangle) = f_0(z)\langle t \rangle$.

Proof By construction of the extended Fatou coordinates, if $(t, z) \in \text{Dom } \Phi$ then $(t, f_t(z)) \in \text{Dom } \Phi$ and $\Phi(t, f_t(z)) = 1 + \Phi(t, z)$. By hypothesis, the graph of $t \in [0, \tau_\Phi(z)) \mapsto z\langle t \rangle$ is contained in $\text{Dom } \Phi$ hence so is the graph of $t \in [0, \tau_\Phi(z)) \mapsto f_t(z\langle t \rangle)$ and $\Phi(t, f_t(z\langle t \rangle)) = 1 + \Phi(t, z\langle t \rangle)$, and, thus, remains constant as t varies, by construction of the motion $z\langle t \rangle$. This means that $t \in [0, \tau_\Phi(z)) \mapsto f_t(z\langle t \rangle)$ is in the unique fiber of Φ containing $f_0(z)$: hence, $f_t(z\langle t \rangle) = f_0(z)\langle t \rangle$. \square

Now for a given t , consider the sequence

$$\omega_n\langle t \rangle.$$

It is an orbit of f_t , though, depending on t , it may not be defined for all n :

Lemma 64 For all $t \in [0, T'_1]$:

- if $\omega_n\langle t \rangle$ is defined (i.e. $\tau_\Phi(\omega_n) > t$) then $\omega_{n+1}\langle t \rangle$ is defined and $\omega_{n+1}\langle t \rangle = f_t(\omega_n\langle t \rangle)$,
- $\omega_n\langle t \rangle$ is defined when n is big enough.

Proof Since $\omega_n\langle 0 \rangle = \omega_n$ is an orbit for f_0 : $\omega_{n+1}\langle 0 \rangle = f_0(\omega_n\langle 0 \rangle)$. The first point follows from the previous lemma. Informally, the second point states that points deep enough in the attracting petal can be followed for a long time. Let us apply Proposition 32 and its companion Proposition 29 to the family of maps $\mathcal{G} = \{f_s \mid s \in [0, t]\}$. The Fatou coordinates in this proposition are normalized by the expansion. They, thus, differ from Φ_s by the constant $\beta_s = \sigma_d - \Phi_{\text{attr}}[f_t](v_t)$ of Sect. 3.9.1, which is bounded for $s \in [0, t]$. Hence, there is a map ξ , independent of $s \in [0, t]$, such that the domain of equation $\text{Re}(z) > \xi(\text{Im}(z))$ is contained in the image by Φ_s of the attracting petal $D_{\text{attr}}[f_s]$ (defined in Proposition 29). Choose N_1 so that $\omega_{N_1}\langle 0 \rangle \in D_{\text{attr}}[f_0]$. For $N = N_1 + k \geq N_1$, we have $\omega_N\langle 0 \rangle \in D_{\text{attr}}[f_0]$ and $\Phi_0(\omega_N\langle 0 \rangle) = \Phi_0(\omega_{N_1}) + k$, hence there is some $N_2 \geq N_1$ such that for all $n \geq N_2$, $\omega_n\langle 0 \rangle$ is in the domain of equation $\text{Re}(z) > \xi(\text{Im}(z))$. Let us call $\Psi_{\text{attr},s}$ the inverse of the restriction of Φ_s to the petal. The function $s \mapsto \Psi_{\text{attr},s}(\Phi_0(\omega_n\langle 0 \rangle))$ then defines a motion of $\omega_n\langle 0 \rangle$ within a fiber of Φ , whence the conclusion by the uniqueness point of Proposition 57. \square

The sequence $\omega_n\langle t \rangle$ is, thus, defined either for all $n \in \mathbb{Z}$ or for all $n \geq N \in \mathbb{Z}$, where N depends both on t and on the orbit $\omega_n = \omega_n\langle 0 \rangle$. Proposition 31 provides a

repelling petal $D_{\text{rep}}[f_t]$ of diameter r_0 that varies continuously with f_t . Here, r_0 can be any small enough constant independent of f_t . Proposition 62, and thus, the main theorem (more precisely Theorem 26) will follow from:

Proposition 65 (survival of orbits as fibers of Φ , and control) *There exists $r'_0 < r_0$ and a pair $\varepsilon_1 < \varepsilon_0$ with $\varepsilon_0 < T'_1$ such that for all $f_0 \in \mathcal{F}$, for all $z \in \text{Dom}(\mathcal{R}[f_0]) \odot (1 - \varepsilon_1)$, if we consider the orbit ω_n associated with z , then*

- for all $n \in \mathbb{Z}$

$$\tau_{\Phi}[f_0](\omega_n) > \varepsilon_0,$$

- there exists $M \in \mathbb{Z}$ such that $(t \leq \varepsilon_0 \text{ and } n \leq M) \implies \omega_n \langle t \rangle \in D_{\text{rep}}[f_t](r'_0)$.

Indeed, let $\Phi_{+,t}$ be the repelling Fatou coordinates on $D_{\text{rep}}[f_t]$ such that $\Psi_t \circ \Phi_{+,t}(z) = z$ holds on $D_{\text{rep}}[f_t]$. It depends continuously on t . Consider a point $z \in \text{Dom}(R_0) \odot 1 - \varepsilon_1$ and the f_0 -orbit ω_n associated to z . Let then $z(t) = E(\Phi_{+,t}(\omega_M \langle t \rangle))$. Then, $z(t) \in \text{Dom } R_t$ and $\forall n \geq M$, $R_t(z(t)) = E(\Phi_{+,t}(\omega_M \langle t \rangle)) = E(\Phi_t(\omega_n \langle t \rangle) + M - n) = E(\Phi_t(\omega_n \langle t \rangle)) = E(\Phi_0(\omega_n \langle 0 \rangle))$ (the last equality because we follow a fiber of Φ), i.e., $R_t(z(t))$ is constant as t varies. Since $z(0) = z \langle 0 \rangle$, we have followed the R -fiber associated to z : $z(t) = z \langle t \rangle$. In particular $\tau_R(z) > \varepsilon_0$.

Again, we will get slightly stronger information on the valid pairs $(\varepsilon_0, \varepsilon_1)$ for Proposition 65, see Sect. 3.11.

3.10 Step 2, III: Survival of Fibers

In this section, we will prove the following proposition (see the paragraph just before Proposition 65 for information about the constant r_0):

Proposition 66 *There exists $K > 0$, $r'_0 < r_0$ and ε'_0 such that for all $\varepsilon' < \varepsilon'_0$, for all $f_0 \in \mathcal{F}_0$, for all f_0 -orbit ω_n indexed by $I = \mathbb{Z}$ that tends to 0 in the future (in an attracting petal) and in the past (in a repelling petal), if the orbit (ω_n) is completely contained in $\text{Dom}(f) \odot (1 - \varepsilon')$, then its survival time is at least ε'/K :*

$$\forall n \in \mathbb{Z}, \tau_{\Phi}(\omega_n) > \varepsilon'/K.$$

Moreover²⁷ there is some $M \in \mathbb{Z}$ such that $\forall n \in \mathbb{Z}$ with $n \leq M$ and $\forall t \leq \varepsilon'/K$, $\omega_n \langle t \rangle \in D_{\text{rep}}[f_t](r'_0)$.

Here we do *not* need to assume that ε' is related to some $\varepsilon > 0$ like in Proposition 45.

3.10.1 Local Orbits

We first consider those orbits that stay near the parabolic point, and prove their survival for some uniform time.

²⁷ This constant M will of course *not* be independent of the orbit (ω_n) .

Lemma 67 (Survival of local orbits) *For all $T_3 < T'_1$ there exists $r_1 > 0$ such that for all $f_0 \in \mathcal{F}_0$ and for all f_0 -orbit ω_n indexed by $I = \mathbb{Z}$ or $I = \mathbb{N}$, if the sequence (ω_n) is contained in $B(0, r_1)$, then*

- for all $n \in I$, $\tau_\Phi[f_0](\omega_n) > T_3$,
- if $I = \mathbb{Z}$, then there exists $N \in \mathbb{Z}$ such that $\forall n \in \mathbb{Z}$ with $n \leq N$ and $\forall t \in [0, T_3]$, $\omega_n \langle t \rangle \in D_{\text{rep}}[f_t](r_0)$.

Proof Recall the statements and notations of Propositions 29 and 32 and apply them to the compact set of maps $\mathcal{F}_{[0, T_3]}$, which yields a value r_0 . In their proofs, we introduced the right half plane $H_{\text{attr}}[f]$, image of the disk $D_{\text{attr}}[f]$ by $z \mapsto s(z) = -1/c_f z$. The boundary of H_{attr} is a vertical line of abscissa $1/r_0|c_f|$. Call R_0 the supremum of $1/r_0|c_f|$ when f varies over \mathcal{F}_0 . The function Ψ_{attr} was the inverse of $\Phi_{\text{attr}} : D_{\text{attr}} \rightarrow \Psi_{\text{attr}}(D_{\text{attr}})$. It is important to note a difference: the Fatou coordinates were normalized by they asymptotic expansion in these propositions; whereas, here, they are normalized using the critical value $v[f_t]$: $\Phi_t(z) = \Phi_{\text{attr}}[f_t](z) + \beta_t$ where $\beta_t = \beta[f_t] = \sigma_d - \Phi_{\text{attr}}[f_t](v[f_t])$. Let

$$z \mapsto s_t(z) = -1/c[f_t]z.$$

Let $\tilde{\Psi}_t = \Phi_t^{-1}$ defined on $\Phi_t(D_{\text{attr}}[f_t])$. Choose any $T'_3 \in]T_3, T'_1[$. The following three bounds are finite:

$$B = \sup_{f \in \mathcal{F}_{[0, T'_3]}} |c_f|, \quad B' = \sup_{f \in \mathcal{F}_{[0, T'_3]}} |\beta_t| \quad \text{and} \quad \Gamma = \sup_{f \in \mathcal{F}_{[0, T'_3]}} |\gamma[f]|.$$

Since $B' < +\infty$, one can translate the estimates given in Propositions 29 and 32 into estimates on $\tilde{\Psi}_t$ and Φ_t as follows:

$$\begin{aligned} |s_t(f_t(z)) - (s_t(z) + 1)| &\leq 1/4 & (\forall z \in B(0, r_0)) \\ |\Phi_t(s_t^{-1}(u)) - (u - \gamma \log_p u)| &\leq M_1 \\ |s_t \circ \tilde{\Psi}_t(Z) - (Z + \gamma \log_p Z)| &\leq M_2 \\ \text{Dom}(\tilde{\Psi}_t) &\supset \{Z \in \mathbb{C} \mid \text{Re } Z > \xi(\text{Im } Z)\} \\ \xi(y) &\underset{y \rightarrow \pm\infty}{=} \mathcal{O}(\log |y|), \end{aligned}$$

where $s_t, \gamma = \gamma[f_t], \Phi_t$ and $\tilde{\Psi}_t$ all depend on f_t , but the function ξ and the constants M_1, M_2 are independent of f_0 and of t . Consider now a real number $a > R_0$ and the sector $S \subset H_{\text{attr}}$ defined by $\arg(z - a) < \pi/3$. By the first estimate above, $s_t^{-1}(S)$ is stable by f_t . By the other estimates, if a is big enough, for all $f_t \in \mathcal{F}_{[0, T'_3]}$, for all $z \neq 0$, if $s_0(z) \in S$, then $\tilde{\Psi}_t(\Phi_0(z))$ is defined. It follows a fiber of Φ , hence by uniqueness in Proposition 57, $\tau_\Phi(z) \geq T'_3$ and

$$z \langle t \rangle = \tilde{\Psi}_t(\Phi_0(z)).$$

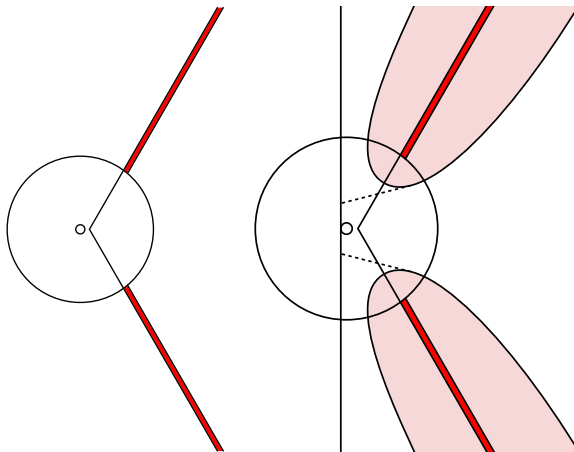


Fig. 19 Illustration of the proof of Lemma 67. Both pictures live in the u -plane. The small circle has radius R_0 , the big circle radius R_1 , both are centered on the origin. The sector S has apex having some real affix $a > R_0$, which we depicted closer to R_0 than to R_1 . See the text for further description

Using the estimate above on $\tilde{\Psi}_t$, we get $\forall t \in [0, T'_3[, s_t(z\langle t \rangle) \in H_{\text{attr}}[f_t]$ provided $a \geq A'$ for some A' independent of f_0, t and z . Let

$$u(t) := s_t(z\langle t \rangle) = s_t \circ \tilde{\Psi}_t(\Phi_0(z)).$$

In particular $u(0) = s_0(z)$. We then get the following bound on the motion:

$$|u(t) - u(0)| \leq M_4 \log(M'_4 + |u(0)|),$$

where M_4 and M'_4 are independent of t, f_0 and z . Indeed, we start from $|\log_p(x)| \leq \pi + \log|x|$ when $\log|x| > 0$. We then use the estimates above to first get $|\Phi_0(z)| \leq M_1 + |u(0)| + \Gamma\pi + \Gamma \log|u(0)|$ (we can ensure $\log|u(0)| > 0$ by taking $a > 1$) and $|\Phi_0(z)| > 1$ (take a big enough). Then, $|u(t)| \leq M_2 + |\Phi_0(z)| + \Gamma\pi + \Gamma \log|\Phi_0(z)| \leq M + M'|u(0)|$ for a pair (M, M') independent of t, f_0, z . Then, $|u(t) - u(0)| \leq |u(t) - \Phi_0(z)| + |\Phi_0(z) - u(0)|$. Last, we use for $t' = t$ and $t' = 0$ that $|u(t') - \Phi_0(z)| \leq M_1 + \Gamma \log|u(t')|$.

So far, we have proved survival of points z with $s_0(z) \in S$, i.e. $\tau_\Phi(z) \geq T'_3 > T_3$. Figure 19 illustrates the next step of the proof. Let r_1 to be chosen later, with $r_1 < r_0$. Let $R_1 = \inf(1/|c_f r_1|) = 1/(r_1 \sup|c_f|)$ where the extrema are taken over $f \in \mathcal{F}_{[0, T'_3]}$. Assume z_n is an orbit of f_0 indexed by \mathbb{N} that is contained in $B(0, r_1)$. Then, the sequence $u_n = s_0(z_n)$ is contained in $\{u \in \mathbb{C} \mid |u| > R_1\}$. If $u_0 \in S$ then $\forall n \geq 0, \tau_\Phi(z) \geq T'_3$. If $u_0 \notin S$, let n_0 be the smallest positive integer such that $u_{n_0} \in S$ (there is one, by the first estimate in the list). Since $u_{n_0-1} \in \{u \in \mathbb{C} \mid |u| > R_1\} \setminus S$ and $u_{n_0} \in \{u \in \mathbb{C} \mid |u| > R_1\} \cap S$, the first estimate in the list gives, again, that u_{n_0} must belong to the set Λ , depicted in red in Fig. 19, intersection of $\{u \in \mathbb{C} \mid |u| > R_1\}$ with the set of points in S at distance $\leq 5/4$ from ∂S . By the bound on the motion, $\forall t \in [0, T'_3[, u_{n_0}(t)$ belongs to the set Λ' , depicted in light red, union of balls of center

$u \in \Lambda$ and of radius $M_4 \log(M'_4 + |u|)$. The map f_t still satisfies the first inequality in the list, hence, provided R_1 is big enough then for all f_0 and for all $t \in [0, T'_3]$, and for all sequence u_n as above, there is an inverse orbit of the conjugate of f_t by s_t , starting from $u_{n_0}(t)$ and remaining in $\{u \in \mathbb{C} \mid |u| > R_0\}$, in fact remaining above or below a domain delimited by the dotted line on the figure (on which we interrupted the dotted line when it reaches the repelling petal, delimited by the vertical plain line). By continuity, this orbit is equal to $s_t(z_n(t))$ and $\tau_\Phi(z_n) \geq T'_3 > T_3$, for all $n \in I = \mathbb{Z}$ or \mathbb{N} . If $I = \mathbb{Z}$, let as above n_0 be the smallest relative integer such that $u_{n_0} \in S$. By the first inequality it exists, and moreover the inverse orbit $u_n(t)$, n negative, must enter the repelling petal (and stay there) as soon as $|u_{n_0}| + M_4 \log(M'_4 + |u_{n_0}|) + \frac{3}{4}(n - n_0) < -R_0$. \square

We can in fact bound their motion.

Lemma 68 (Bound on the motion of local orbits) *The following can be added to the conclusions of Lemma 67:*

- $\forall t \in [0, T_3], \forall n \in I$, let $z = \omega_n$: $|z(t) - z| \leq K_1 |z| t$.

The constant K_1 is independent of f_0 , t and z but may depend on T_3 .

Proof To shorten the proof, we will use holomorphic motions:²⁸ let us extend the deformations f_t to complex values of t in an open neighborhood V of $[0, T_3]$ that does not depend on $f_0 \in \mathcal{F}$. The hyperbolic length of $[0, T_3]$ in V is thus independent of f_0 . For those values of t such that $|r_t| > 1$, where $r_t = 1 - t$, the map f_t is only defined on $r_t \phi_0(r_t^{-1} \mathbb{D})$ instead of $r_t \phi_0(\mathbb{D})$ when $|r_t| \leq 1$. Those sets contain a common ball $B(0, r)$ for some r independent of f . By compactness, an analog of Lemma 67 still holds. The function $t \mapsto z(t)$ is defined on V and holomorphic.²⁹ Consider the cone of vertex 0, axis \mathbb{R}_+ and angle 3π : this is a Riemann surface over \mathbb{C}^* that is bijectively parameterized in polar coordinates (r, θ) by $]0, +\infty[\times] - 3\pi/2, 3\pi/2[$. The study made in the previous lemma shows that, for r_1 small enough, the points ω_n satisfying the assumptions of the theorem have a motion $\omega_n(t)$ such that $u_n(t) := -1/c[f_t]\omega_n(t)$ stays in this cone when t varies. The element of hyperbolic metric on the cone has expression $c(\theta)|du|/r$ where $c(\theta) \geq c(0) > 0$. The movement of u is holomorphic, hence bounded in this metric by the hyperbolic length of $[0, T_3]$ in V . In Euclidean terms, $u_n(t)$ has moved by at most $Kt|u_n(0)|$ for some K independent of f_0 . Moreover, $|u_n(t)|$ and $|u_n(0)|$ are of comparable size. Going back to $\omega_n(t) = -1/c[f_t]u_n(t)$, we get $|\omega_n(t) - \omega_n(0)| \leq |1/c[f_t]u_n(t) - 1/c[f_t]u_n(0)| + |1/c[f_t]u_n(0) - 1/c[f_0]u_n(0)| \leq |u_n(0) - u_n(t)|/|c[f_t]u_n(0)u_n(t)| + |1/c[f_t] - 1/c[f_0]|/|u_n(0)|$. One concludes recalling $c[f_t]$ is not too close to 0 and depends holomorphically on t . \square

3.10.2 Contraction

Arguments in this section are standard in holomorphic dynamics in complex dimension one.

²⁸ It is possible to avoid holomorphic motions completely, using Propositions 42 and 43 and the remark that follows, which can themselves be proved without holomorphic motions. However, that is much longer.

²⁹ Hence we have a holomorphic motion, because it is injective w.r.t. z , but we will not use that fact.

Let $PC(f_0)$ denote the post critical set of f_0 , i.e. the orbit of the (unique) critical value. Since this orbit tends to 0, the closure $\overline{PC}(f_0)$ equals $PC(f_0) \cup \{0\}$. Let

$$W_0 = \mathbb{C} \setminus \overline{PC}(f_0).$$

It is well known that inverse branches of f_0 are locally contracting for the hyperbolic metric of W_0 . Let us recall the argument: f_0 is a cover from $W'_0 := f_0^{-1}(W_0)$ to W_0 . As such, it is an isometry, at the infinitesimal level, from the hyperbolic metric of W'_0 to that of W_0 . Now $W'_0 \subset W_0$, and strict inclusion maps are locally contracting. Recall that for a hyperbolic domain U of \mathbb{C} we denote $\rho_U(z)|dz|$ the element of hyperbolic metric of U . For $z \in W'_0$, let us denote $\lambda(z)$ the contraction factor of f_0^{-1} from $f_0(z)$ to z , measured with the hyperbolic metric element of W_0 :

$$\lambda(z) = \frac{\rho_{W_0}(z)}{\rho_{W_0}(f_0(z))} \left| \frac{dz}{df_0(z)} \right|;$$

it is also equal to the contraction factor at z of the inclusion map from W'_0 to W_0 :

$$\lambda(z) = \frac{\rho_{W_0}(z)}{\rho_{W'_0}(z)}.$$

The function λ is continuous and takes values in $]0, 1[$.

Let us recall that a hyperbolic open subset of the Riemann sphere with an isolated point a in its complement has a hyperbolic metric coefficient $\rho(z) \sim \frac{1}{2|z-a|\log \frac{1}{|z-a|}}$ as $z \rightarrow a \neq \infty$, or $\rho(z) \sim \frac{1}{2|z|\log |z|}$ as $z \rightarrow a = \infty$.

Lemma 69 *Let $z_n \in W'_0$ be a sequence.*

- (1) *If z_n leaves every compact subset of the open set $W'_0 \cup \overline{PC}(f_0)$, then $\lambda(z_n) \rightarrow 0$.*
- (2) *If $\lambda(z_n) \rightarrow 1$ then $z_n \rightarrow \overline{PC}(f_0)$.*

Proof We may extract a subsequence and assume z_n convergent in the Riemann sphere.

Point (1): If $z_n \rightarrow \infty$ then $\rho_{W_0}(z) \sim \frac{1}{2|z|\log |z|}$, whereas $\rho_{W'_0}(z) \geq \rho_{\text{Dom}(f_0)}(z)$ and the latter is $\geq \frac{1}{4d_{\mathbb{C}}(z, \partial \text{Dom}(f_0))}$ by Koebe's one quarter theorem. Now since the domain of f_0 is the image of \mathbb{D} by a Schlicht map, there is at least one point in its complement that is at distance at most 1 from 0. Hence, $d_{\mathbb{C}}(z, \partial \text{Dom}(f_0)) \leq 1 + |z|$. Putting it all together, we get that $\rho_{W_0}(z)/\rho_{W'_0}(z) \rightarrow 0$ as $|z| \rightarrow +\infty$. In the remaining case: $\lim z_n \neq \infty$ so $\rho_{W_0}(z)$ converges to a constant, whereas $\rho_{W'_0}(z) \rightarrow +\infty$.

Point (2): The function λ is continuous and $\lambda(z) < 1$ thus if $\lambda(z_n)$ tends to 1 then z_n leaves every compact subset of W'_0 , and we conclude by the previous point. \square

Lemma 70 (Definite contraction factor at definite distance of PC) *For all $\delta > 0$, there exists $\Lambda(\delta) < 1$ such that $\forall f \in \mathcal{F}, \forall z \in W'_0$, if $d_{\mathbb{C}}(z, \overline{PC}(f)) \geq \delta$, then $\lambda(z) \leq \Lambda(\delta)$.*

Proof If not, there would be sequences $f_n = \mathcal{R}[B_d] \circ \phi_n^{-1} \in \mathcal{F}$ and $z_n \in W'_0[f_n]$ such that $d_{\mathbb{C}}(z_n, \overline{PC}(f_n)) \geq \delta$ but $\lambda[f_n](z_n) \rightarrow 1$. Let us extract convergent subsequences and assume that $z_n \rightarrow z' \in \widehat{\mathbb{C}}$ and $\phi_n \rightarrow \phi$, thus $f_n \rightarrow f = \mathcal{R}[B_d] \circ \phi^{-1}$.

Since $\overline{PC}(f)$ is contained in a ball $B(0, R)$ with R independent of $f \in \mathcal{F}$ (Point 2 of Lemma 39), $W_0(f)$ contains $V := \mathbb{C} \setminus \overline{B}(0, R)$, hence $\rho_{W_0}(z) \leq \rho_V(z) \sim 1/2|z| \log |z|$ as $z \rightarrow \infty$. This gives an upper bound like in Point (1) of Lemma 69, but moreover independent of $f \in \mathcal{F}$. It follows that $z' \neq \infty$. By Lemma 39, $\overline{PC}(f)$ depends continuously on ϕ thus $d_{\mathbb{C}}(z', \overline{PC}(f)) \geq \delta$. Hence $z' \in W_0[f]$. Now, the marked domains $(W_0[f_n], z_n)$ converge for the Caratheodory topology on marked domains. Hence their universal cover from $(\mathbb{D}, 0)$ with real positive derivative at the origin converge, and the coefficient of the hyperbolic metric converges locally uniformly: $\rho_{W_0[f_n]}(z_n) \rightarrow \rho_{W_0[f]}(z')$. Concerning the marked domains $(W'_0[f_n], z_n)$, there are two cases: either $z' \in W'_0[f]$ in which case there is Caratheodory convergence to $(W'_0[f], z')$ and thus $\rho_{W'_0[f_n]}(z_n) \rightarrow \rho_{W'_0[f]}(z')$; or $z' \notin W'_0[f]$ in which case we will prove in the next paragraph the following claim: $\rho_{W'_0[f_n]}(z_n) \rightarrow +\infty$. In the first case, $\lambda[f_n](z_n) \rightarrow \lambda[f](z') < 1$. In the second case, $\lambda[f_n](z_n) \rightarrow 0$. Both cases lead to a contradiction.

Let us prove the claim. There exists then a point $x_n \in \mathbb{C} \setminus W'_0[f_n]$ such that $x_n \rightarrow z'$. Let $r' = |z'|$ and let $r'' \geq 1$ be any real such that $r'' \neq r'$, for instance $r'' = r' + 1$. Since the conformal radius w.r.t. 0 of the simply connected set $\text{Dom } f_n$ is 1, there exists a point in $\mathbb{C} \setminus \text{Dom } f_n$ of any modulus ≥ 1 , in particular a point y_n of modulus r'' . Let $V_n = \mathbb{C} \setminus \{x_n, y_n\}$. Then, $\rho_{W'_0[f_n]}(z_n) \geq \rho_{V_n}(z_n)$. Let ϕ_n be the unique \mathbb{C} -affine map sending 0 to x_n and 1 to y_n and let $u_n = \phi_n^{-1}(z_n)$. Then $\phi'_n = y_n - x_n$ and $\rho_{\mathbb{C} \setminus \{0,1\}} = \phi_n^*(\rho_{V_n}) = |\phi'_n| \times \rho_{V_n} \circ \phi_n$. For n big enough, the sequence $x_n - y_n$ is bounded away from 0 (and ∞) thus $u_n \rightarrow 0$ thus $\rho_{\mathbb{C} \setminus \{0,1\}}(u_n) \rightarrow +\infty$ and also $\rho_{V_n}(z_n) = \rho_{\mathbb{C} \setminus \{0,1\}}(u_n)/|y_n - x_n| \rightarrow +\infty$. \square

3.10.3 Putting Back the Post Critical Set

The following easy lemma will be useful in several places.

Lemma 71 *There exists a function $\delta > 0 \mapsto M(\delta) > 0$ such that the following holds. For all $f \in \mathcal{F}$, for all $z \in \text{Dom}(f)$, if $d_{\mathbb{C}}(z, PC(f)) \geq \delta$, then*

$$\frac{\rho_{W_0(f)}(z)}{\rho_{\text{Dom}(f)}(z)} \leq M(\delta).$$

Proof In this proof, the notation $B(z, r)$ denotes the euclidean ball and $PC = PC(f)$. By Lemma 39, there is $R > 0$ such that for all $f \in \mathcal{F}$, $PC \subset \overline{B}(0, R)$. Let $U = \mathbb{C} \setminus \overline{B}(0, R)$. Then for $|z| > R$:

$$\rho_{W_0}(z) \leq \rho_U(z) = \frac{1}{2|z| \log \frac{|z|}{R}}.$$

For any $z \in W_0$, since the disk D of center z and radius $d_{\mathbb{C}}(z, PC)$ is contained in W_0 , we get

$$\rho_{W_0}(z) \leq \rho_D(z) = \frac{1}{d_{\mathbb{C}}(z, PC)}.$$

By the theory of univalent functions,

$$\rho_{\text{Dom}(f)}(z) \geq \frac{1}{4(1 + |z|)}.$$

The lemma follows. \square

3.10.4 Homotopic Length and Decomposition

I was introduced to the notion of *homotopic length* by reading [13].

For γ a path defined on an interval I containing $[a, b]$, let us denote its restriction to $[a, b]$ by

$$\gamma|_{[a, b]}.$$

Let us similarly denote

$$\omega_n \langle [0, t] \rangle : s \in [0, t] \mapsto \omega_n \langle s \rangle,$$

where ω_n is an orbit of f_0 as in Sect. 3.9.3.

To bound the motion of $\omega_n \langle t \rangle$, we will look at the *homotopic length* of the path $\omega_n \langle [0, t] \rangle$ for the hyperbolic metric on $W_0 = \mathbb{C} \setminus \overline{PC}(f_0)$. Homotopic length of a path γ refers to the infimum of W_0 -hyperbolic lengths of paths homotopic to γ in W_0 , where the ends of the path are fixed. It will be denoted

$$\text{hlen}_{W_0}(\gamma).$$

By contrast, we denote as follows the usual length of a rectifiable path for the hyperbolic metric of W_0 :

$$\text{len}_{W_0}(\gamma).$$

Last, we will call *extent* of a path γ defined on $[0, t]$ the quantity

$$\text{extent}_{W_0}(\gamma) = \sup_{t' \in [0, t]} \text{hlen}_{W_0}(\gamma|_{[0, t]}).$$

Remark Homotopic length is also the hyperbolic distance between the starting point and the end point of a lift of the curve to the universal cover. There are in particular shortest homotopic paths. The extent of a curve is the smallest radius of a ball in the universal cover containing a lift of the curve and centered on the initial point of this lift. If U is connected and $\gamma \subset U \subsetneq V$ then the V -homotopic length of γ is strictly smaller than its U -homotopic length: consider, for instance, the shortest homotopic path for V ; its U -length is strictly shorter. If U and V are hyperbolic Riemann surfaces and $f : U \rightarrow V$ is a cover then $\text{hlen}_U(\gamma) = \text{hlen}_V(f \circ \gamma)$.

Remark The sequence $(\omega_n \langle t \rangle)_{n \in \mathbb{N}}$ is an orbit of f_t , not f_0 . It may therefore seem unnatural to measure the motion of $t \mapsto \omega_n \langle t \rangle$ using the hyperbolic metric on the complement of $\overline{PC}(f_0)$. However, we found the proof simpler to write that way. Note that the motion will be evaluated only at some distance from the post critical points, and in the end it will be small.

Recall that $f_0 \in \mathcal{F}$ decomposes as

$$f_0 = \mathcal{R}[B_d] \circ \phi_0^{-1}$$

with $\phi_0 : \mathbb{D} \rightarrow U_0$ a Schlicht map. Let us decompose the map f_t as follows:

$$f_t = f_0 \circ \sigma_t,$$

where

$$\sigma_t(z) = \phi_0 \circ r_t \circ \phi_0^{-1} \circ r_t^{-1}$$

with the notations of Sect. 3.8.1 and letting r_t denote the multiplication by

$$r_t = 1 - t.$$

The map σ_0 is the identity restricted to $\text{Dom } f_0$. If we interpret $\sigma_t(z)$ as a motion of z as t varies, then it can be viewed as the composition of two motions: $(t, z) \mapsto (t, r_t^{-1}z)$ followed by the conjugate by ϕ_0 of the radial motion $(t, z) \mapsto (t, r_t z)$ on the unit disk:

$$\sigma_t = \mu_t \circ r_t^{-1}$$

with

$$\mu_t = \phi_0 \circ r_t \circ \phi_0^{-1}.$$

The domain of definition of the reciprocal σ_t^{-1} equals $\phi_0(B(0, r_t)) = \text{Dom}(f_0) \odot r_t$ and thus, as t varies away from 0, it shrinks.

One way to get a control $\omega_{n-1} \langle s \rangle$ is to do it inductively from a control on $\omega_n \langle s \rangle$, using the relation $f_s(\omega_{n-1} \langle s \rangle) = \omega_n \langle s \rangle$ of Lemma 64. Consider the case where $\omega_n \langle 0 \rangle$ is not equal to 0 nor to the singular value v of f_0 . Then, $\omega_n \langle s \rangle \notin \{0, v\}$, because 0 and v do not move under the fiberwise motion, and Φ -fibers are disjoint. Recall that the singular values of f_0 are precisely 0, ∞ and v . We claim that, under some condition stated below, the path $s \in [0, t] \mapsto \omega_{n-1} \langle s \rangle$ is homotopic (with endpoints fixed) in W_0 to the concatenation of the following two paths (see Fig. 20):

- The first path, denoted $\gamma_1 = f_0^* \omega_n$ by a slight abuse of notation, is parameterised by $s \in [0, t]$ and is defined by continuity by $\gamma_1(0) = \omega_{n-1} \langle 0 \rangle$ and $f_0(\gamma_1(s)) = \omega_n \langle s \rangle$, i.e. we replaced f_s by f_0 in $f_s(\omega_{n-1} \langle s \rangle) = \omega_n \langle s \rangle$. Existence of this path follows from $\omega_n \langle s \rangle$ never hitting the singular values of f_0 . It ends at some point w' (which depends on t);

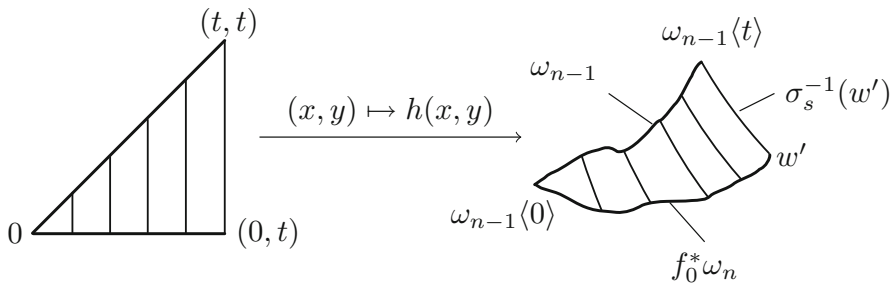


Fig. 20 The map $h(x, y) = \sigma_y^{-1}(f_0^* \omega_n(x))$ defined on the triangle of inequations $x \in [0, t]$, $y \in [0, t]$, $y \leq x$ induces a homotopy between ω_{n-1} on $[0, t]$ and the concatenation of $f_0^* \omega_n$ and $s \in [0, t] \mapsto \sigma_s^{-1}(w')$

- The second path is $\gamma_2 : s \in [0, t] \mapsto \sigma_s^{-1}(w')$. For it to be defined up to $s = t$, we need to assume that $w' \in \text{Dom}(f_0) \odot (1 - t) = \phi_0(B(0, 1 - t))$.

The homotopy will be defined by means of a map h defined on the set of $(x, y) \in [0, t]^2$ such that $y \leq x$ by

$$h(x, y) = \sigma_y^{-1}(\gamma_1(x)).$$

For it to be well defined, we will make assumptions on t , on the length of ω_n and on the ε such that $\omega_{n-1} \langle 0 \rangle \in \text{Dom}(f_0) \odot (1 - \varepsilon)$. For it to be a homotopy in W_0 , we need to prove that its support does not intersect $\overline{PC}(f_0)$ and for this, we will make further assumptions on t , on the length of ω_n and on the Euclidean distance from $\omega_{n-1} \langle 0 \rangle$ to $\overline{PC}(f_0)$.

To state these sufficient conditions, we will introduce the following objects and quantities. For $\delta > 0$ let $V_\delta[f]$ denote the δ -neighborhood of $PC(f)$, i.e. the set of points whose Euclidean distance to $PC(f)$ is $< \delta$ (see Fig. 21). According to Lemma 39, the following quantity is positive:

$$\delta_1 := \inf_{f_0 \in \mathcal{F}_0} d_{\mathbb{C}}(PC(f_0), \mathbb{C} \setminus \text{Dom } f_0),$$

where $d_{\mathbb{C}}$ refers to the Euclidean distance, and the following are finite:

$$\begin{aligned} R_1 &:= \sup \{|z| \mid z \in PC(f_0), f_0 \in \mathcal{F}_0\}, \\ R_2 &:= \sup \{d_{\text{Dom } f_0}(0, z) \mid z \in PC(f_0), f_0 \in \mathcal{F}_0\}. \end{aligned}$$

Lemma 72 *For all (δ, δ') with $\delta' < \delta < \delta_1$, there exists $T = T(\delta, \delta') > 0$ such that $\forall f_0 \in \mathcal{F}_0, \forall t < T$:*

- $\mu_t^{-1}(\mathbb{C} \setminus V_\delta[f_0]) \cap V_{\delta'}[f_0] = \emptyset$,
- $r_t(\mathbb{C} \setminus V_\delta[f_0]) \cap V_{\delta'}[f_0] = \emptyset$,
- $\sigma_t^{-1}(\mathbb{C} \setminus V_\delta[f_0]) \cap V_{\delta'}[f_0] = \emptyset$.

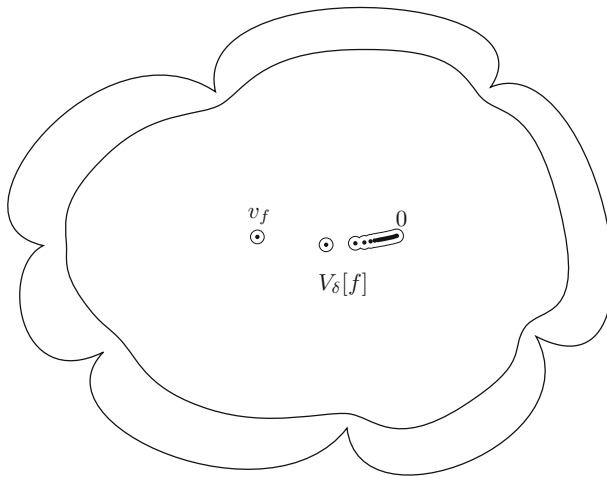


Fig. 21 A schematic illustration of $\text{Dom}(f)$, $\text{Dom}(f) \odot r$ and $V_\delta[f]$. Scales are not respected. The outer curve represents the boundary of the domain of some $f \in \mathcal{F}$, the nearby smooth curve the boundary of the sub-domain $\text{Dom}(f) \odot 1 - \varepsilon$. The post-critical set is indicated by dots, its δ -neighborhood for the Euclidean metric is $V_\delta[f]$ and its boundary is indicated by thin curves

Proof We can deduce the third point from the first two, using an intermediary value δ'' . This may not be optimal³⁰ but it is not the point here. For the second point, an explicit valid value of T can easily be computed using Lemma 39: assume $z \in r_t(\mathbb{C} \setminus V_\delta[f_0]) \cap V_{\delta'}[f_0]$. Then, there exists $z' \in PC(f_0)$ such that $|z - z'| < \delta'$, thus $|z| < R_1 + \delta'$. Then $|z - r_t^{-1}z| \leq (R_1 + \delta')(r_t^{-1} - 1)$. If T is chosen so that $(R_1 + \delta')(r_T^{-1} - 1) < \delta - \delta'$, then $r_t^{-1}z$ cannot belong to $\mathbb{C} \setminus V_\delta[f_0]$. For the first point, let us work by contradiction and assume there is $f_n \in \mathcal{F}_0$, $a_n \in \mathbb{C} \setminus V_\delta[f_n]$, $b_n \in V_{\delta'}[f_n]$ and $t_n > 0$ such that $t_n \rightarrow 0$ and $a_n = \mu_{t_n}(b_n)$. We may assume that $f_n \rightarrow f \in \mathcal{F}_0$, $a_n \rightarrow a \in \widehat{\mathbb{C}}$ and $b_n \rightarrow b \in \mathbb{C}$. From $|a_n - b_n| > \delta - \delta'$ we get $|a - b| \geq \delta - \delta'$. Write $f_n = \mathcal{R}[B_d] \circ \phi_n^{-1}$ and $f = \mathcal{R}[B_d] \circ \phi^{-1}$. From $\delta' < \delta_1$ and $R_2 < +\infty$, we deduce that $\phi_n^{-1}(b_n)$ remains in a compact subset of \mathbb{D} , thus b belongs to $\text{Dom}(f)$, but then $a = \mu_0(b) = b$, a contradiction. \square

We will later choose some

$$\delta < \delta_1.$$

Let then

$$d_1 = d_1(\delta) = \inf_{f_0 \in \mathcal{F}_0} d_{W'_0}(V_{\delta/3}[f_0], \mathbb{C} \setminus V_{\delta/2}[f_0])$$

³⁰ Near $z = 0$, the Euclidean motion of σ_t is of order $|z|^2$, thus smaller than the sum of the motions of μ_t and of r_t , which are both of order $|z|$.

with $W'_0 := f_0^{-1}(W_0)$. Let also

$$d''_1 = d''_1(\delta) = \inf_{f_0 \in \mathcal{F}_0} d_{f^{-1}(\mathbb{C} \setminus \{0, v\})}(V_{\delta/3}[f_0], \mathbb{C} \setminus V_{\delta/2}[f_0]),$$

v being the critical value of f_0 , and note that $d''_1 < d_1$. Using the notation of Lemma 72 let

$$T_4(\delta) = T(\delta/3, \delta/4)$$

so that $\forall f_0 \in \mathcal{F}_0, \forall t < T_4(\delta), \sigma_t^{-1}(\mathbb{C} \setminus V_{\delta/3}[f_0]) \cap V_{\delta/4}[f_0] = \emptyset$. Let $\ell(x)$ denote the hyperbolic distance from 0 to x in \mathbb{D} :

$$\ell(x) = d_{\mathbb{D}}(0, x) = \operatorname{argh}(x).$$

It is a bijection from $[0, 1[$ to $[0, +\infty[$. For a given $\varepsilon' > 0$, let $T_5 = T_5(\delta, \varepsilon') \in]0, 1[$ be the unique solution to

$$\ell(1 - T_5) = d_1(\delta) + \ell(1 - \varepsilon').$$

Note that the solution T''_5 of $\ell(1 - T''_5) = d''_1(\delta) + \ell(1 - \varepsilon')$ satisfies $T''_5 > T_5$. We will later look at how $T_5(\delta, \varepsilon')$ varies as $\varepsilon' \rightarrow 0$ for a fixed δ . Recall the definition of extent given at the beginning of the present section on page 66.

Proposition 73 *Let $t > 0$. If we assume that*

- (1) $\tau_{\Phi}(\omega_n\langle 0 \rangle) > t$,
- (2) *the path $s \in [0, t] \mapsto \omega_n\langle s \rangle$ is contained in W_0 ,*
- (3) $\operatorname{extent}_{W_0}(\omega_n\langle [0, t] \rangle) \leq d_1(\delta)$,
- (4) $\omega_{n-1}\langle 0 \rangle \in \operatorname{Dom}(f_0) \odot (1 - \varepsilon')$,
- (5) $\omega_{n-1}\langle 0 \rangle \notin V_{\delta/2}[f_0]$,
- (6) $t \leq T_4(\delta)$,
- (7) $t \leq T_5(\delta, \varepsilon')$,

then $\tau_{\Phi}(\omega_{n-1}\langle 0 \rangle) > t$ and the function h mentioned above is well defined and has support in W_0 (even better: it avoids $V_{\delta/4}[f]$). In particular, $s \in [0, t] \mapsto \omega_{n-1}(s)$ is homotopic in W_0 to the concatenation $\gamma_1 \cdot \gamma_2$, of the two paths defined earlier, page 66. We also have $\gamma_1 \subset \operatorname{Dom}(f_0) \odot (1 - T_5)$.

Proof By (2) the path ω_n is contained in $\mathbb{C} \setminus \{0, v\}$ thus the path γ_1 , defined as the pull-back by f_0 of $\omega_n\langle \cdot \rangle$ starting from $\omega_{n-1}\langle 0 \rangle$, is well defined. Let $t' \in [0, t]$:

$$\operatorname{hlen}_{\operatorname{Dom} f_0}(\gamma_1|_{[0, t']}) < \operatorname{hlen}_{W'_0}(\gamma_1|_{[0, t']}) = \operatorname{hlen}_{W_0}(\omega_n\langle \cdot \rangle|_{[0, t']}) \leq d_1$$

(the first inequality comes from the strict inclusion $W'_0 \subset \operatorname{Dom} f_0$, the equality follows from f_0 being a cover from W'_0 to W_0 , the second inequality comes from point (3)). In particular, the $\operatorname{Dom} f_0$ -hyperbolic distance from $\gamma_1(0)$ to $\gamma_1(t')$ is $\leq d_1$. Since moreover by (4), $d_{\operatorname{Dom} f_0}(0, \omega_{n-1}\langle 0 \rangle) \leq \ell(1 - \varepsilon')$ we get that γ_1 is contained in the

$\text{Dom } f_0$ hyperbolic ball of center 0 and radius $d_1 + \ell(1 - \varepsilon') = \ell(1 - T_5)$. Hence, $\gamma_1 \subset \text{Dom}(f_0) \odot (1 - T_5)$. Hence, by (7), γ_2 and the map h defined at the same place are well defined. Let us check that h takes values in W_0 , i.e., that it avoids $\overline{PC}(f_0)$. Note that we have already proved that $\text{hlen}_{W'_0}(\gamma_1|_{[0, t']}) \leq d_1$. In particular, the W'_0 -hyperbolic distance from $\gamma_1(0)$ to $\gamma_1(t')$ is $\leq d_1$. Together with point (5) and the definition of d_1 , it implies that γ_1 is contained in $\mathbb{C} \setminus V_{\delta/3}[f_0]$. Point (6) then implies that γ_2 and h take value in $\mathbb{C} \setminus V_{\delta/4}[f_0]$, which is contained in W_0 . The points $h(s, s)$ and $\omega_{n-1}\langle s \rangle$ are both mapped by f_s to the same point: $\omega_n\langle s \rangle$, for which we recall that $\Phi_s(\omega_n\langle s \rangle)$ stays constant when s varies. The uniqueness statement in Proposition 57 applied to Φ then implies that $\tau_\Phi(\omega_{n-1}) > t$ and that the functions defined on $[0, t]$, $s \mapsto h(s, s)$ and $s \mapsto \omega_{n-1}\langle s \rangle$, are in fact equal. \square

We have the following variation with W_0 replaced by $\mathbb{C} \setminus \{0, v\}$ in the hypotheses, but not in the conclusion:

Proposition 74 *Let $t > 0$. If we assume that*

- (1) $\tau_\Phi(\omega_n\langle 0 \rangle) > t$,
- (2) *the path $s \in [0, t] \mapsto \omega_n\langle s \rangle$ is contained in $\mathbb{C} \setminus \{0, v\}$,*
- (3) $\text{extent}_{\mathbb{C} \setminus \{0, v\}}(\omega_n\langle [0, t] \rangle) \leq d_1''(\delta)$,
- (4) $\omega_{n-1}\langle 0 \rangle \in \text{Dom}(f_0) \odot (1 - \varepsilon')$,
- (5) $\omega_{n-1}\langle 0 \rangle \notin V_{\delta/2}[f_0]$,
- (6) $t \leq T_4(\delta)$,
- (7) $t \leq T_5''(\delta, \varepsilon')$,

then $\tau_\Phi(\omega_{n-1}\langle 0 \rangle) > t$ and the function h is well defined and avoids $V_{\delta/4}[f]$. In particular, it has support in W_0 and the path $s \in [0, t] \mapsto \omega_{n-1}\langle s \rangle$ is homotopic in W_0 to $\gamma_1 \cdot \gamma_2$. We also have $\gamma_1 \subset \text{Dom}(f_0) \odot (1 - T_5')$.

Proof As in the previous proof. \square

Lemma 75 *Under the conditions of Proposition 73, the W_0 -homotopic length of γ_1 is at most $\Lambda(\delta/3)$ times the W_0 -homotopic length of ω_n , where $\Lambda(\delta/3) < 1$ is given by Lemma 70.*

Proof We have seen that $\text{hlen}_{W'_0}(\gamma_1) \leq d_1$. Consider a shortest path γ homotopic to γ_1 in W'_0 : $\text{len}_{W'_0}(\gamma) = \text{hlen}_{W'_0}(\gamma_1)$. It is a geodesic for the hyperbolic metric, in particular all its points are at W'_0 -hyperbolic distance $\leq d_1$ from its starting point. By the definition of d_1 this implies that γ is disjoint from $V_{\delta/3}[f_0]$. By Lemma 70, we have $\lambda(z) \leq \Lambda(\delta/3)$ for z in the support of γ , with $\lambda(z) = \rho_{W_0}(z)/\rho_{W'_0}(z)$. Thus, $\text{hlen}_{W_0}(\gamma_1) \leq \text{len}_{W_0}(\gamma) \leq \Lambda(\delta/3) \text{len}_{W'_0}(\gamma) = \Lambda(\delta/3) \text{hlen}_{W'_0}(\omega_n)$. \square

Lemma 76 *Under the conditions of Proposition 74, the W_0 -homotopic length of γ_1 is at most $M(\delta/3)$ times the $\mathbb{C} \setminus \{0, v\}$ -homotopic length of ω_n , where $M(\cdot \cdot \cdot)$ is given in Lemma 71.*

Proof This is done as in the previous lemma, with W'_0 replaced by $f^{-1}(\mathbb{C} \setminus \{0, v\})$, d_1 by d_1'' and $\lambda(z)$ by $\rho_{W_0}(z)/\rho_{f^{-1}(\mathbb{C} \setminus \{0, v\})}(z)$. By inclusion, the latter quantity is $\leq \rho_{W_0}(z)/\rho_{\text{Dom } f}(z)$, thus $\leq M(\delta/3)$. \square

The W_0 -homotopic length of γ_2 will be controlled using Lemma 77 below. To state it, we need to introduce another quantity. By Lemma 72, there exists $T_6 = T_6(\delta)$ such that $\forall f_0 \in \mathcal{F}_0, \forall t < T_6, \mu_t^{-1}(\mathbb{C} \setminus V_{\delta/4}[f_0]) \cap V_{\delta/5}[f_0] = \emptyset$ and $r_t(\mathbb{C} \setminus V_{\delta/5}[f_0]) \cap V_{\delta/6}[f_0] = \emptyset$.

Lemma 77 *For all $\delta < \delta_1$, there exists $K_0 = K_0(\delta)$ such that under the conditions of Proposition 73 or 74, and assuming moreover*

- $t \leq T_6(\delta)$ and $t \leq T_5(\delta, \varepsilon')/2$,

then the W_0 -homotopic length of γ_2 is $\leq K_0 t / T_5(\delta, \varepsilon')$.

Proof Similarly to the proof of the propositions, the condition $t < T_6$ ensures that there is a homotopy in W_0 between γ_2 and $\gamma_3 \cdot \gamma_4$ where $\gamma_3(s) = \mu_s^{-1}(\gamma_2(0))$ and $\gamma_4(s) = r_s w''$ where w'' is the endpoint of γ_3 . The motion μ_s is the conjugate by ϕ_0 of the radial motion and we have seen that $x := |\phi_0^{-1}(\gamma_2(0))| \leq 1 - T_5$ (in the case of Proposition 74 we have $x \leq 1 - T_5' < 1 - T_5$), thus the length of γ_3 for the hyperbolic metric of $\phi_0(\mathbb{D}) = \text{Dom } f_0$ is $\leq d_{\mathbb{D}}(x, \frac{x}{1-t}) \leq d_{\mathbb{D}}(1 - T_5, \frac{1-T_5}{1-t}) = \frac{1}{2} \log \left(\frac{1 - \frac{t}{T_5}}{1 - \frac{t}{T_5}} \right) \leq \frac{1}{2} \times -\log(1 - \frac{t}{T_5}) \leq t \log(2)/T_5$, the latter because $t/T_5 \leq 1/2$. Lemma 71 implies that its W_0 -length is at most $M(\delta/5)$ times this quantity. To bound the W_0 -length of γ_4 , note that it is contained in the complement of $V_{\delta/6}$, thus $\forall z \in \gamma_4, \rho_{W_0}(z) \leq 6/\delta$. Also, $\rho_{W_0}(z) \leq 1/(|z| - R_1)$ where $R_1 = \sup \{|z| \mid f \in \mathcal{F}, z \in PC(f)\}$. If $|w''| > 4R_1$ then since $t \leq 1/2$, the whole path γ_4 is contained in the complement of $B(0, 2R_1)$ and thus $\rho_{W_0} \leq 2/|z|$ whence a W_0 -length of γ_4 that is $\leq \int_{|z|}^{\frac{|z|}{1-t}} \frac{2}{x} dx = 2 \log(1/(1-t)) \leq 4t \log 2$ because $t \leq 1/2$. If $|w''| \leq 4R_1$, then the whole euclidean length of γ_4 is $\leq 4R_1 t$; hence, the W_0 -length is $\leq (6/\delta)4R_1 t$. Recall that $T_5 < 1$. \square

Remark The linearity of the bound w.r.t. t is not crucial for this article: weaker orders of convergence to 0 would work for our purpose, thanks to the fact that in Proposition 45, ε' is much bigger than ε . What will be important is that values of t for which the bound is a given small constant are much bigger than ε . So how T_5 depends on ε will be important too (recall δ will be fixed).

Remark Lemmas 75 to 77 give an upper bound on the W_0 -homotopic length of the curve $s \in [0, t] \mapsto \omega_{n-1}\langle s \rangle$ on each of its subsegments $s \in [0, t']$ for $t' < t$, by applying Proposition 73 to t' instead of t . So we get in fact bounds on the extent. This allows for induction.

3.10.5 Visits in the Repelling Petal

Lemma 78 *There exists $K_4 > 0$ such that $\forall f \in \mathcal{F}, \forall z \in W_0$, if $|z| \leq 1$, then $\rho_{W_0}(z) \geq 1/K_4|z|$.*

Proof Let us work by contradiction and assume that there exist $f_n \in \mathcal{F}$ and $z_n \in \mathbb{D}$ such that $z_n \in W_0[f_n]$ and $\rho_{W_0[f_n]}(z_n)|z_n| \rightarrow 0$. Consider the dilatation by $1/z_n$: the set $z_n^{-1}W_0[f_n] \subset \mathbb{C}$ does not contain 0, but it contains the point 1 and has a

hyperbolic metric coefficient at this point tending to 0 as $n \rightarrow +\infty$. There would, thus, exist $R_n, r_n > 0$ such that $R_n > |z_n| > r_n$, such that $W_0[f_n]$ contains the annulus $\{z \in \mathbb{C} \mid r_n < |z| < R_n\}$ and such that $R_n/|z_n| \rightarrow +\infty$ and $r_n/|z_n| \rightarrow 0$ (this can be proved by contradiction, using that \mathbb{C} minus two points is hyperbolic, and that inclusion is non-expanding for the hyperbolic metric).

Let us apply Lemma 37 to $r = r_0$ where r_0 is provided by Proposition 29. The point $f^{n_0}(v_f)$ belongs to $D_{r_0}[f]$. It depends continuously on f and thus it remains in a compact subset of $\mathbb{C} \setminus \{0\}$. Let $a_n[f] = -1/c_f f^n(v_f)$ where c_f is defined in Proposition 29 and is bounded away from 0 and ∞ as f varies in \mathcal{F} . Then, $a_{n_0}[f]$ also belongs to a compact set. By Lemma 30, $\forall n \geq 0$, $3n/4 - A \leq |a_{n_0+n}[f]| \leq A + 5n/4$ for a constant $A > 0$ that is independent of f . It follows that there is $A', A'' > 0$ and $n_1 \geq n_0$ such that for all $f \in \mathcal{F}$ and for all $n \geq n_1$, $\frac{A'}{n} \leq |f^n(v_f)| \leq \frac{A''}{n}$.

Hence, the aforementioned sequence of annuli cannot exist, which yields a contradiction. \square

In coordinates $u = -1/c_f z$, this reads $\rho_{-1/(c_f W_0)}(u) \geq 1/K_4 |u|$.

Proposition 79 *There exists r_2, T_8 and d'_1 , positive reals, such that for all $f_0 \in \mathcal{F}$, for all $n_0, n_1 \in \mathbb{Z}$ with $n_0 < n_1$, for all f_0 -orbit ω_n indexed by $\mathbb{Z} \cap [n_0, +\infty[$, and for all $t > 0$, if*

- (1) $\omega_{n_0}\langle 0 \rangle, \dots, \omega_{n_1}\langle 0 \rangle \in D_{\text{rep}}[f_0](r_2)$,
- (2) $\tau(\omega_{n_1}) > t$,
- (3) $t \leq T_8$,
- (4) $\text{extent}_{W_0}(\omega_{n_1}\langle [0, t] \rangle) < d'_1$,

then $\tau(\omega_{n_0}) > t$, the paths γ_1 and γ_2 defined below are well defined, and $s \in [0, t] \mapsto \omega_{n_0}\langle s \rangle$ is homotopic in $W_0[f_0]$ to their concatenation $\gamma_1 \cdot \gamma_2$. The path γ_1 is the pull-back of $s \in [0, t] \mapsto \omega_{n_1}\langle s \rangle$ by $f_0^{n_1-n_0}$ that starts from $\omega_{n_0}\langle 0 \rangle$; the path $\gamma_2 : [0, t] \rightarrow \mathbb{C}$ is the continuous solution, starting from $\gamma_1(t)$, of $f_s^{n_1-n_0}(\gamma_2(s)) = \text{const} = f_0^{n_1-n_0}(\gamma_1(t)) = \omega_{n_1}\langle t \rangle$.

Proof Consider the domains $\Omega_\theta(R)$ and $D_\theta(r_0)[g]$ introduced in Sect. 3.6.3, with $-1/c_g D_\theta(r_0) = \Omega_\theta(1/|c_g|r_0)$. We will take some $T_8 \leq 1/2$. The class of maps $\mathcal{F}_{[0,1/2]}$ is compact and the domain of its members all contain $B(0, 1/8)$, so we can apply Propositions 41 and 44 to the restriction to \mathbb{D} of the conjugates of maps in this class by $z \mapsto 8z$. Choose $\theta = 3\pi/4$, $\theta' = (\theta + \pi/2)/2$, $\theta'' = (\theta' + \pi/2)/2$, so that $\pi/2 < \theta'' < \theta' < \theta$. It was proved in Proposition 41 that for $r_0 > 0$ small enough, the (invertible) repelling Fatou coordinates of $g \in \mathcal{F}_{[0,1/2]}$ extend to $-D_\theta(r_0)[g]$ for some $r_0 > 0$, and that $-D_\theta(r_0)[g]$, $-D_{\theta'}(r_0)[g]$ and $-D_{\theta''}(r_0)[g]$ are all invariant by a branch of g^{-1} . By compactness, for r_0 small enough, there is only one such branch. Also, provided r_0 has been chosen small enough, it can be checked using Proposition 29 and Lemma 37 that $PC[g]$ does not intersect $-D_\theta(r_0)[g]$.

Now choose any $r_1 < r_0$, for instance $r_1 = r_0/2$ and impose $r_2 \leq r_1$. Let

$$\gamma_0 : [0, t] \rightarrow \mathbb{C}, \quad s \mapsto \omega_{n_1}\langle s \rangle.$$

By assumption, its initial point is contained in $D_{\pi/2}[f_0](r_2)$. By Lemma 78, for d'_1 small enough, we are ensured that γ_0 is contained in $-D_{\theta''}(r_0)[f_0]$ (this is more

easily seen in coordinates $u = -1/c_{f_0}z$: the path stays in a ball of center its initial point u_0 and radius $\mathcal{O}(d'_1|u_0|)$. Since $-D_{\theta''}(r_0)[f_0]$ is stable by a branch of f_0^{-1} , the path γ_1 is well defined and contained in $-D_{\theta''}(r_0)[f_0]$. Now, as in the proof of Proposition 73, we set up a triangular homotopy $h(x, y)$ for $y \leq x \leq t$ with $f_y^{n_1-n_0}(h(x, y)) = f_0^{n_1-n_0}(\gamma_1(x)) = \gamma_0(x)$ and $h(x, 0) = \gamma_1(x)$. Taking T_8 small enough, we get $-D_{\theta''}(r_0)[f_0] \subset -D_{\theta''}(r_0)[f_y]$ for all $y \leq T_8$ and all $f_0 \in \mathcal{F}$. In particular $\gamma_0 \subset -D_{\theta''}(r_0)[f_y]$. Since the latter is invariant by a branch of f_y^{-1} , unique and continuously depending on y , it follows that h is well defined, continuous, and has support in $-D_{\theta''}(r_0)[f_y]$. For T_8 small enough, $-D_{\theta''}(r_0)[f_y] \subset -D_{\theta}(r_0)[f_0]$, hence h takes values in $W_0[f_0]$. \square

This proof yields more:

Proposition 80 (Complement of Proposition 79) *There is some $r_4 > 0$ and $\theta' > \pi/2$ such that under the conditions of the proposition above, and $\forall s \in [0, t]$, $-D_{\theta'}[f_s](r_4)$ is a repelling petal for f_s and for all k with $n_0 \leq k \leq n_1$, $\omega_k\langle s \rangle \in -D_{\theta'}[f_s](r_4)$.*

Proof Change the value of n_0 to that of k in the previous proposition. Its proof provided some quantities called r_0 and θ' , and proved the claim of the complement for $r_4 = r_0$ and the same value of θ' . \square

Note that by infinitesimal contraction of f_0^{-1} for the hyperbolic metric of W_0 ,

$$\text{hlen}_{W_0}(\gamma_1) < \text{hlen}_{W_0}(s \in [0, t] \mapsto \omega_{n_0}\langle s \rangle).$$

Since γ_2 stays far from the boundary of W_0 , the control we get on its homotopic length is better than in Lemma 77:

Lemma 81 *We can add the following conclusion to the previous lemma*

$$\text{hlen}_{W_0}(\gamma_2) \leq K_5 t.$$

Proof In this proof, we will say that a constant is independent if it is independent of f , of t , of the chosen orbit ω_n and of the length $n_1 - n_0$. We will use $\mathcal{O}(\text{expression})$ to express a quantity that is at most the expression times a constant that is independent. We will write that two quantities are comparable when their quotient is bounded away from 0 and ∞ with bounds that are independent of f , of t , of the chosen orbit ω_n and of the length $n_1 - n_0$. Let us continue with the notations of the previous proof. In particular, $\theta = 3\pi/4$. Note that $\gamma_2(y) = h(t, y)$ and $\gamma_2(t) \in -D_{\theta}(r_0)[f_0]$. Since there are sectors $-D_{\theta_3}(r_3)[f_0]$ contained in $W_0[f_0]$ for $\theta_3 = (\theta + \pi)/2 > \theta$ with r_3 independent of f_0 , by imposing $r_0 < r_3$, we have $\forall z \in -D_{\theta}(r_0)[f_0]$, $B(z, |z|/K) \subset W_0[f_0]$ for some $K > 1$. Hence, it is enough to prove that for $y \leq t$,

$$|\gamma_2(y) - \gamma_2(0)| = \mathcal{O}(K't|\gamma_2(0)|),$$

in which case, for $t < T_8$, T_8 is small enough, the euclidean ball $B(\gamma_2(0), K't|\gamma_2(0)|)$ is contained in $W_0[f_0]$ and contains γ_2 thus γ_2 is homotopic in $W_0[f_0]$ to the straight

segment from $\gamma_2(0)$ to $\gamma_2(t)$ and the latter has a $W_0[f_0]$ -hyperbolic length at most its $B(\gamma_2(0), |\gamma_2(0)|/K)$ -hyperbolic length, thus at most $K_5 t$ for T_8 small enough. As in the proof of Proposition 79, let

$$\gamma_0(s) = \omega_{n_1}\langle s \rangle,$$

and let $\Phi_{\text{rep}}[f_y]$ be a repelling Fatou coordinate on $-D_\theta(r_0)[f_y]$, normalized by the expansion. We have seen in this former proof that γ_1 and γ_2 and γ_0 are contained in $-D_{\theta'}(r_0)[f_y]$ for all $y \leq T_8$, for some constant $\theta' = 5\pi/8 < \theta = 3\pi/4$. Then,

$$\Phi_{\text{rep}}[f_y](\gamma_2(y)) = \Phi_{\text{rep}}[f_y](\gamma_0(t)) - (n_1 - n_0).$$

Let us denote $\text{Rep}_y z = \Phi_{\text{rep}}[f_y](z)$. By taking r_0 small enough we can ensure that for all $z \in -D_\theta(r_0)[f_y]$, the quantity $\text{Rep}_y z$ is comparable to $1/z$ and the quantity $\text{Rep}'_y(z)$ is comparable to $1/z^2$ (use the bound on $\tilde{\Phi}$ given in Proposition 29 that extends to Ω_θ according to Proposition 41). For $y \leq 1/2$, we have $\sup_{|z| < 1/16} |f_0(z) - f_y(z)| \leq Ky$ for some K independent of f . We have $|\text{Rep}_y \gamma_2(y) - \text{Rep}_0 \gamma_2(0)| = |\text{Rep}_y \gamma_0(t) - \text{Rep}_0 \gamma_0(t)|$. Provided r_2 has been chosen small enough, Proposition 44 gives $|\text{Rep}_y \gamma_0(t) - \text{Rep}_0 \gamma_0(t)| = \mathcal{O}(y/|\gamma_0(t)|)$, as Φ_{rep} is normalized by the expansion. Let $u_x = -1/(c[f_0]\gamma_0(x))$ and $Z_x = \text{Rep}_0 \gamma_0(x)$. The size of the quantities Z_x , Z_0 , u_x , u_0 , $1/\gamma_0(x)$ and $1/\gamma_0(0)$ are all comparable. Similarly, $|1/\gamma_2(0)|$ is comparable to $|\text{Rep}_0 \gamma_2(0)| = |Z_t - (n_1 - n_0)|$. Note that the positive integer $n_1 - n_0$ can be arbitrarily large. However since Z_t is contained in $-\Omega_{3\pi/4}(10)$ (provided r_2 is small enough), there is an independent lower bound on $|Z_t - (n_1 - n_0)|/|Z_t|$ thus $y/|\gamma_0(t)| = \mathcal{O}(y|Z_0|) = \mathcal{O}(y|Z_t|) \leq \mathcal{O}(y|Z_t - (n_1 - n_0)|) = \mathcal{O}(y|\text{Rep}_0 \gamma_2(0)|) = \mathcal{O}(y/|\gamma_2(0)|)$: for some $M > 0$

$$|\text{Rep}_y \gamma_2(y) - \text{Rep}_0 \gamma_2(0)| \leq My/|\gamma_2(0)|.$$

Then, by Proposition 44, again, we get $|\text{Rep}_y \gamma_2(0) - \text{Rep}_0 \gamma_2(0)| \leq M'y/|\gamma_2(0)|$ thus $|\text{Rep}_y \gamma_2(y) - \text{Rep}_y \gamma_2(0)| \leq |\text{Rep}_y \gamma_2(y) - \text{Rep}_0 \gamma_2(0)| + |\text{Rep}_0 \gamma_2(0) - \text{Rep}_y \gamma_2(0)| \leq (M + M')y/|\gamma_2(0)|$. The straight segment from $\text{Rep}_y \gamma_2(y)$ to $\text{Rep}_y \gamma_2(0)$ is contained in the subset $-\Omega_{\theta'}(R_2)$ of the domain of Rep_y^{-1} and $|(\text{Rep}_y^{-1})'(Z)|$ is comparable to $1/|Z|^2$ for $Z \in -\Omega_{\theta'}(R_2)$. Using moreover that $\text{Rep}_y(Z)$ is comparable to $1/Z$, we get: provided T_8 was chosen small enough, for all $y \leq t$, $|\gamma_2(y) - \gamma_2(0)| \leq K'y/|\gamma_2(0)|$. \square

Lemma 82 *If in Proposition 79 we take $n_0 = -\infty$, i.e., start from an orbit indexed by \mathbb{Z} such that $\omega_n\langle 0 \rangle \in D_{\text{rep}}[f_0](r_2)$ for all $n \leq n_1$, and leave the other three assumptions unchanged, then for all $\alpha > 0$ and all $r > 0$, $\exists n' \in \mathbb{Z}$ such that $\forall n \leq n'$, $\forall s \in [0, t]$, $\omega_{n_0}\langle s \rangle$ belongs to the sector of apex 0, radius r , and angle α around the repelling axis of f_s .*

Proof In the course of the proof of Proposition 79, we proved that $\gamma_0 : s \in [0, t] \mapsto \omega_n\langle s \rangle$ has a support contained in $-D_{\theta'}[f_y](r_0)$ for all $y \leq t$. In particular the function

$\chi : s \mapsto -1/c_{f_s} \gamma_0(s)$ takes values in $-\Omega_{\theta'}(1/|c_{f_s} r_0|)$. Recall that on this set, the dynamics differs from the translation by 1 by at most $1/4$. The path χ has compact image. The lemma follows. \square

3.10.6 Bounding the Motion of Orbits (Putting It All Together)

We now have the tools to prove Proposition 66.

Recall that we are considering an orbit ω_n indexed by \mathbb{Z} of a map $f_0 \in \mathcal{F}$, eventually captured by an attracting petal in the future, by a repelling petal in the past, and defined a movement $\omega_n \langle t \rangle$ of this sequence, for which it remains an orbit of f_t and so that its attracting Fatou coordinate, normalized by immobilizing the image of the critical value, remains constant. The starting hypothesis is that ω_n is entirely contained in $\text{Dom}(f_0) \odot (1 - \varepsilon') = \phi_0(B(0, 1 - \varepsilon'))$. In particular condition (4) of Proposition 73 and its analog in Proposition 74 are satisfied for all $n \in \mathbb{Z}$ by the assumption.

We will now compute a lower bound for the survival time $\tau(\omega_n)$, that depends only on ε' .

This will be done by decreasing induction on n , using Propositions 73, 74 and 79 and their complements Lemmas 75–77 and 81. The induction hypothesis will be that the motion of $t \mapsto \omega_n \langle t \rangle$, measured with the hyperbolic metric of the set $W_0[f_0]$, more precisely what we called the extent at the beginning of Sect. 3.10.4, is smaller than the constants d_1, d'_1 and d''_1 appearing in the propositions. The complements then give an upper bound on the motion of $t \mapsto \omega_{n-1} \langle t \rangle$. We will show that for t small enough, this bound is also less than d_1, d'_1 and d''_1 , so that the induction can go on, and we will give a lower bound on how small t needs to be.

Recall r_0 is a small enough constant provided by Proposition 29–33, and 41.

By Lemma 67, we know the survival of local orbits. More precisely let us choose $T_3 = T'_1/2$. Lemma 67 yields a value r_1 . If the whole orbit $(\omega_n \langle 0 \rangle)_{n \in \mathbb{Z}}$ is contained in $B(0, r_1)$ then we get the lower bound $\tau(\omega_n) \geq T_3$. In this simple case, the lower bound is independent of ε' , so it is even better. In the sequel, we assume that we are not in this case, i.e., that the orbit $(\omega_n \langle 0 \rangle)_{n \in \mathbb{Z}}$ leaves $B(0, r_1)$ at least once.

Recall that maps $f \in \mathcal{F}$ all have the same critical value v . We have already remarked that by compactness of \mathcal{F} and Proposition 34 (see also Lemma 38), there exists η_0 such that $\forall f \in \mathcal{F}$, $B(v, \eta_0)$ is contained in the basin of the parabolic point. Recall $D_{\text{rep}}(r) = D_{\text{rep}}[f](r)$ denotes the disk of diameter $[0, r e^{i\theta}]$ where $e^{i\theta}$ points in the direction of the repelling axis of f . Let $f^{\mathbb{N}}(B(v, r))$ denote the union of $B(v, r)$ and of all its images by iteration of f .

Lemma 83 *There exists $r_3 > 0$ and $\eta'_0 < \eta_0$ such that $\forall r \leq r_3, \forall f \in \mathcal{F}$, the set $f^{\mathbb{N}}(B(v, \eta'_0))$ is disjoint from $f(B(0, r)) \setminus B(0, r)$ and from $f(D_{\text{rep}}(r))$.*

Proof Let r_0 be provided by Proposition 29: for all $f \in \mathcal{F}$, and all $r \leq r_0$, $D_{\text{attr}}(r)$ is stable by f and contained in the parabolic basin. Note that for some r' small enough, then for all r small enough, then for all $f \in \mathcal{F}$, $f(B(0, r)) \setminus B(0, r)$ and $f(D_{\text{rep}}(r))$ are disjoint from $D_{\text{attr}}(r')$, as easily follows from Lemma 30 and the fact that in the change of variable $u = -1/c_f z$, the factor c_f is bounded away from 0 and from ∞ . By Lemma 38 there is some n_0 and some $\eta'_0 > 0$ such that

$\forall f \in \mathcal{F}$, $f^{n_0}(B(v, \eta'_0)) \subset D_{\text{attr}}(r')$, and hence $\forall n \geq n_0$, $f^n(B(v, \eta'_0)) \subset D_{\text{attr}}(r')$. By compactness of \mathcal{F} again, there is a uniform lower bound on the distance from 0 to $f^n(B(v, \eta'_0))$ as n varies between 0 and $n_0 - 1$ and f varies in \mathcal{F} . So the lemma will hold for r small enough. \square

Let T_8 , d'_1 and r_2 be provided by Proposition 79. Let

$$r'_0 = \min(r_0, r_1, r_2, r_3),$$

and denote

$$D_{\text{rep}} = D_{\text{rep}}[f] = D_{\text{rep}}[f](r'_0).$$

We introduced earlier the δ -neighborhood $V_\delta[f]$ of $PC(f)$. Let $\tilde{B}(r) = \tilde{B}(r)[f]$ be the set of points in $B(0, r) \setminus \{0\}$ whose forward orbit by f is contained in $B(0, r)$. Let

$$\tilde{V}_\eta = \tilde{V}_\eta[f] = \tilde{B}(\eta) \cup f^{\mathbb{N}}(B(v, \eta)).$$

By construction, $f(\tilde{V}_\eta) \subset \tilde{V}_\eta$ (do not forget that there is no other preimage of the origin than itself³¹).

Lemma 84 *The following holds, where $D = D_{\text{rep}}[f](\eta)$:*

- (1) $\forall \eta > 0$, $\exists \delta > 0$ s.t. $\forall f \in \mathcal{F}$, $V_\delta[f] \subset \tilde{V}_\eta[f] \cup (D \cap f^{-1}(D))$,
- (2) $\exists \eta_2 > 0$, $\forall \eta \leq \eta_2$, $\exists \delta > 0$ s.t. $\forall f \in \mathcal{F}$, $V_\delta[f] \cap f^{-1}(\tilde{V}_\eta[f]) \subset \tilde{V}_\eta[f]$.

Proof These are again proved by compactness arguments. Let r_0 be provided by Proposition 29 applied to \mathcal{F} . Then, $D_{\text{attr}}[f](r)$ is an attracting petal for all $r \leq r_0$.

- (1) The set $\tilde{B}(\eta) \cup (D_{\text{rep}}[f](\eta) \cap f^{-1}(D_{\text{rep}}[f](\eta))) \subset \tilde{V}_\eta[f] \cup (D_{\text{rep}}[f](\eta) \cap f^{-1}(D_{\text{rep}}[f](\eta)))$ is a neighborhood of 0 thus contains a ball $B(0, r)$. We can take a uniform value of r for maps $f \in \mathcal{F}$ (this can be seen in coordinates $u = -1/c_f z$ as in the proof of Proposition 29: the constant c_f is bounded away from 0 and ∞ and the map f is conjugated to a map $u \mapsto u'$ defined on a uniform neighborhood of ∞ and with $|u' - (u + 1)| < 1/4$). We impose $\delta \leq r/2$. By Lemma 37 for some n_0 we have $\forall f \in \mathcal{F}$, $f^{n_0}(v_f) \in D_{\text{attr}}[f](r)$ and thus $\forall n \geq n_0$, $B(f^n(v_f), \delta) \subset \tilde{V}_\eta[f] \cup (D_{\text{rep}}[f](\eta) \cap f^{-1}(D_{\text{rep}}[f](\eta)))$. Finally by compactness, there is a lower bound on $\inf \{\delta > 0 \mid \forall f \in \mathcal{F}, \forall k < n_0, B(f^k(v_f), \delta) \subset f^k(B(v_f, \eta))\}$.
- (2) Let $\eta_0 > 0$ to be determined below and set $\eta_2 = \eta_0/2$. Let us assume by contradiction that for some $\eta \leq \eta_0/2$, there exists sequences $\delta_n \rightarrow 0$, $f_n \in \mathcal{F}$, z_n such that $z_n \in V_{\delta_n}[f_n]$, $f_n(z_n) \in \tilde{V}_\eta[f_n]$, $z_n \notin \tilde{V}_\eta[f_n]$. We may extract a subsequence so that $f_n \rightarrow f_0$, and $z_n \rightarrow z_0$. If $z_0 \neq 0$ then $z_0 \in PC(f_0)$ (see point (1) of Lemma 39), a fortiori $z_0 \in f_0^{\mathbb{N}}(B(v_{f_0}, \eta))$ and thus for n big enough $z_n \in f_0^{\mathbb{N}}(B(v_{f_n}, \eta))$ by Hurwitz's theorem, thus $z_n \in \tilde{V}_\eta[f_n]$, leading to a contradiction. If $z_n \rightarrow 0$ then for n big enough, let us prove the statement

³¹ And even if there were, it would be sufficient to assume η small enough.

$f_n(z_n) \in \tilde{V}_\eta[f_n] \implies z_n \in \tilde{V}_\eta[f_n]$, which leads to a contradiction. Indeed either $f_n(z_n) \in B(\eta)[f_n]$ but then as soon as $|z_n| \leq \eta$, the whole orbit of z_n by f_n is in $B(0, \eta)$ and thus $z_n \in \tilde{B}(\eta)[f_n]$ thus $z_n \in \tilde{V}_\eta[f_n]$. Or $f_n(z_n) \in f_n^{\mathbb{N}}(B(v_{f_n}, \eta))$, say $f_n(z_n) \in f_n^{k_n}(B(v_{f_n}, \eta))$. For a fixed k , by compactness, there is a lower bound on the distance from 0 to $f^{k'}(B(v_f, \eta_0/2))$ for $k' < k$ and $f \in \mathcal{F}$. So $k_n \rightarrow +\infty$. Now f_n is injective on $B(0, r)$ for some uniform $r \leq r_0$. By Lemma 38 there is some n_0 and $\eta_0 > 0$ such that $\forall f \in \mathcal{F}$, we have $f^{n_0}(B(v_f, \eta_0)) \subset D_{\text{attr}}[f](r)$. As soon as $k_n \geq n_0 + 1$, both $f_n^{k_n-1}(B(v_{f_n}, \eta))$ and $f_n^{k_n}(B(v_{f_n}, \eta))$ are contained in $D_{\text{attr}}[f](r) \subset B(0, r)$, and z_n also belongs to $B(0, r)$ for n big enough. Hence, $f(z_n) \in f_n^{k_n}(B(v_{f_n}, \eta)) \implies z_n \in f_n^{k_n-1}(B(v_{f_n}, \eta))$.

□

Let

$$\eta_1 = \min(\eta_0/2, r_0, r_1, r_2, r_3, \delta_1/2, \eta'_0, \eta_2),$$

where δ_1 was defined just before Lemma 72, r_2 in Proposition 79, η_0, r_0 and $r'_0 = \min(r_0, r_1, r_2, r_3)$ at the beginning of the current section (Sect. 3.10.6), η'_0 and r_3 in Lemma 83, η_2 in Lemma 84.

Let δ be the smallest of the two values associated to $\eta = \eta_1$ by points (1) and (2) of Lemma 84. Since $\eta_1 \leq r'_0$ we get $D_{\text{rep}}[f](\eta_1) \subset D_{\text{rep}}[f](r'_0) = D_{\text{rep}}[f]$ and thus: $\forall f \in \mathcal{F}$,

$$V_\delta[f] \subset \tilde{V}_{\eta_1}[f] \cup (D_{\text{rep}}[f] \cap f^{-1}(D_{\text{rep}}[f])), \quad (7)$$

$$f^{-1}(\tilde{V}_{\eta_1}[f]) \cap V_\delta[f] \subset \tilde{V}_{\eta_1}[f]. \quad (8)$$

Let $d_1 = d_1(\delta)$, $d''_1 = d''_1(\delta)$ and $T_4 = T_4(\delta)$ be the values associated to δ just before Proposition 73, and $T_6 = T_6(\delta)$ defined just before Lemma 77.

Just before Proposition 73 we also defined $T_5(\delta, \varepsilon')$, by $\ell(1 - T_5(\delta, \varepsilon')) = d_1(\delta) + \ell(1 - \varepsilon')$ where $\ell(x) = d_{\mathbb{D}}(0, x)$. Since we just have fixed δ , let us denote $T_5(\varepsilon') = T_5(\delta, \varepsilon')$. Then

$$T_5(\varepsilon') \underset{\varepsilon' \rightarrow 0}{\sim} K_3 \varepsilon'$$

with $K_3 = e^{-2d_1(\delta)}$ (the value of this constant is not important, nor is its dependence on δ).

Lemma 85 *There exists $K_2 > 0$ and $T_7 > 0$ such that for all $f_0 \in \mathcal{F}$, for all $z \in \tilde{V}_{\eta_1}[f_0]$, $\tau(z) > T_7$ and for all $t \leq T_7$, the length of the curve $x \in [0, t] \mapsto z\langle x \rangle$ is $\leq K_2 t$ when measured with the hyperbolic metric of $\mathbb{C} \setminus \{v, 0\}$.*

Proof If the starting point $z\langle 0 \rangle$ belongs to the part $\tilde{B}(\eta_1)$ of \tilde{V}_{η_1} of points whose orbit never leaves $B(0, \eta_1)$, this follows from Lemma 68 since $\eta_1 \leq r_1$ and since $\rho(z) := \rho_{\mathbb{C} \setminus \{0, v\}}(z) = o(1/|z|)$ near 0 thus $\rho(z)|z|$ is bounded on $B(0, \eta_1)$ (note that $\eta_1 < \eta_0 < |v|$). Otherwise the starting point $z\langle 0 \rangle$ belongs to $f_0^{\mathbb{N}}(B(v, \eta_1))$. Note first

that only a finite number of iterates of $B(v, \eta_1)$, bounded independently of f_0 , are not already contained in the first part. Moreover, let $m - 1$ be such a bound. Then for all $k \leq m$, for all $z \in f_0^k(B(v, \eta_1))$, $z\langle t \rangle = f_t^{-(m-k)} \circ \Phi_t^{-1} \circ \Phi_0 \circ f_0^{m-k}(z)$ for some inverse branch of $f_t^{(m-k)}$. Since we do not hit a critical point, everything moves differentiably w.r.t. the pair (t, z) . We thus get³² the claimed bound on the hyperbolic length of the curve $z\langle t \rangle$ away from v , i.e., if $z\langle 0 \rangle \notin B(v, \eta_1)$. Last, for starting points $z\langle 0 \rangle$ near v , i.e., in $B(v, \eta_1)$, note first that v does not move at all: $v\langle t \rangle = v$. Then, $|z\langle t \rangle - z| \leq K|z - v|t$ since the function $(z, t) \mapsto z\langle t \rangle - z$ is at least C^2 and vanishes whenever $t = 0$ or $z = v$. Since $\rho(z) = o(1/|z - v|)$ near v , the lemma follows. \square

Recall that we are dealing with the case where the sequence $n \in \mathbb{Z} \mapsto \omega_n\langle 0 \rangle$ is not completely contained in $B(0, r_1)$. Together with Lemma 83 and $\eta_1 \leq r_1$, this implies that the first point in this orbit that does not belong to $B(0, r_1)$ also does not belong to $\tilde{V}_{\eta_1}[f_0]$. On the other hand the orbit tends to 0 thus eventually stays in $B(0, \eta_1)$ hence in $\tilde{B}(\eta_1)[f_0] \subset \tilde{V}_{\eta_1}[f_0]$. The set $\tilde{V}_{\eta_1}[f_0]$ is mapped in itself by f_0 . Therefore, there is a unique $n_+ \in \mathbb{Z}$ such that

$$\omega_n\langle 0 \rangle \in \tilde{V}_{\eta_1}[f_0] \iff n \geq n_+.$$

If we follow the orbit in the past, it eventually stays in $D_{\text{rep}} = D_{\text{rep}}[f_0](r'_0)$ in the past. There is, thus, a maximal $n_- \in \mathbb{Z}$ such that $\forall n \leq n_-$, $\omega_n\langle 0 \rangle \in D_{\text{rep}}$. Moreover, $n_- + 1 < n_+$ because by Lemma 83, $\omega_{n_-+1}\langle 0 \rangle$ cannot belong to $f^{\mathbb{N}}(B(v, \eta_1))$ and if $\omega_{n_-+1}\langle 0 \rangle$ were in $B(0, \eta_1)$, then the whole orbit would be contained in $B(0, r_1)$.

Between n_- and n_+ , the orbit may visit and leave the repelling petal several times. Let J denote the set of $n \in \mathbb{Z}$ with $n_- < n < n_+$ and $\omega_n\langle 0 \rangle \notin D_{\text{rep}}$. This set is non-empty and its extreme values are $n_- + 1$ and $n_+ - 1$ (these two values may be equal).

Denote as follows the constant provided by Lemma 70 and used in Lemma 75:

$$\Lambda := \Lambda(\delta/3) < 1.$$

Let now $t_{\max} \leq \min(T_3, T_4, T_5/2, T_6, T_7, T_8)$ to be determined later. Let us work with $t \in [0, t_{\max}]$ and let us do a finite decreasing induction on J . In the process, more conditions will be imposed on t_{\max} .

Initialization: By Lemma 85, $\tau(\omega_{n_+}) > t_{\max}$ and for all $t \leq t_{\max}$, the length of $\gamma : s \in [0, t] \mapsto \omega_{n_+}\langle s \rangle$ is $\leq K_2 t$ when measured with the hyperbolic metric on $\mathbb{C} \setminus \{0, v\}$. Provided $K_2 t_{\max} \leq d_1''$, we can apply Proposition 74 (in particular condition (5) of this proposition follows from Equation (8)), thus $\tau(\omega_{n_+-1}\langle 0 \rangle) > t_{\max}$ and $\forall t \in [0, t_{\max}]$, the path $s \in [0, t] \mapsto \omega_{n_+-1}\langle s \rangle$ is homotopic in W_0 to $\gamma_1 \cdot \gamma_2$, where γ_1 and γ_2 are defined in Proposition 74. By Lemma 76, $\text{hlen}_{W_0}(\gamma_1) \leq M(\delta/3) \text{hlen}_{\mathbb{C} \setminus \{0, v\}}(\gamma)$ thus $\leq M_0 K_2 t$ with $M_0 = M(\delta/3)$. And by Lemma 77, $\text{hlen}_{W_0}(\gamma_2) \leq K_0 t/T_5$. Let us sum up: we assumed $K_2 t_{\max} \leq d_1''$ and got $\forall t \in [0, t_{\max}]$, $\text{hlen}_{W_0}(\omega_{n_+-1}|_{[0, t]}) \leq M_0 K_2 t + K_0 t/T_5$. In particular

$$\text{extent}_{W_0}(\omega_{n_+-1}([0, t_{\max}])) \leq M_0 K_2 t_{\max} + K_0 t_{\max}/T_5.$$

³² Here we are *not* using complex values of t .

Let us assume moreover that

$$M_0 K_2 t_{\max} + K_0 t_{\max} / T_5 \leq \min(d_1, d'_1),$$

so that $\text{extent}_{W_0}(\omega_{n-1}([0, t_{\max}])) \leq \min(d_1, d'_1)$.

Induction: Let $n \in \mathbb{Z}$ satisfying $n_- + 1 < n \leq n_+ - 1$ and either $n \in J$ or $n - 1 \in J$ and assume that we have proved $\tau(\omega_n([0, t_{\max}])) > t_{\max}$ and $\text{extent}_{W_0}(\omega_n([0, t_{\max}])) \leq \min(d_1, d'_1)$.

By Eq. (7), $\omega_{n-1}([0, t_{\max}]) \notin V_\delta[f]$ thus condition (5) of Proposition 73 is satisfied. Hence we can apply it, and its complements Lemmas 75 and 77 and we get $\text{hlen}_{W_0}(\omega_{n-1}|_{[0, t]}) \leq \Lambda \min(d_1, d'_1) + K_0 t / T_5$. Let us impose on t_{\max} that $\Lambda \min(d_1, d'_1) + K_0 t_{\max} / T_5 \leq \min(d_1, d'_1)$, so that we get $\text{extent}_{W_0}(\omega_{n-1}([0, t_{\max}])) \leq \min(d_1, d'_1)$. If $n - 1 \in J$, we can carry on the induction with $n - 1$. If $n - 1 \notin J$, let n' be the first element of J below n and let $n_1 = n - 1$ and $n_0 = n' + 1$: $n_0 \leq n_1$. If $n_0 = n_1$ we can also carry on the induction with $n - 1$, because $(n - 1) - 1 \in J$. If $n_0 < n_1$ we can apply Proposition 79 and its complement Lemma 81: $\text{hlen}_{W_0}(\omega_{n_0}|_{[0, t]}) \leq \text{hlen}_{W_0}(\omega_{n_1}|_{[0, t]}) + K_5 t$. Then, we can carry on the induction with n' , provided we require on t_{\max} that $\Lambda \min(d_1, d'_1) + K_0 t_{\max} / T_5 + K_5 t_{\max} \leq \min(d_1, d'_1)$.

In all cases, for the induction to carry on it is enough to assume that

$$\Lambda \min(d_1, d'_1) + K_0 t_{\max} / T_5 + K_5 t_{\max} \leq \min(d_1, d'_1).$$

Post induction: we now know that $\text{extent}_{W_0}(\omega_n([0, t_{\max}])) \leq \min(d_1, d'_1)$ holds for $n = n_- + 1$. We can apply once more Proposition 79 and we get that the rest of the orbit (for all $n \in \mathbb{Z}$ with $n \leq n_-$) is defined at least up to time t_{\max} . Moreover, by Lemma 82, we get that for all n below some relative integer, possibly much smaller³³ than n_- , the full motion takes place in the petal: one of the conclusions of Proposition 66.

Taking everything into account, we get that the full orbit ω_n survives for any time t satisfying $t \leq t_{\max}$ for any t_{\max} satisfying $t_{\max} \leq \min(T_3, T_4, T_5/2, T_6, T_7, T_8)$, $t_{\max} \leq d'_1/K_2$, $t_{\max} \leq \min(d_1, d'_1)/(M_0 K_2 + K_0/T_5)$ and $t_{\max} \leq \min(d_1, d'_1)(1 - \Lambda)/(K_0/T_5 + K_5)$.

Recall that δ is fixed but not ε' . All constants depend only on δ thus are fixed, except, as we saw earlier, $T_5 \sim K_3 \varepsilon'$ (K_3 also depends on δ , thus is fixed). As $\varepsilon' \rightarrow 0$, the biggest t_{\max} we can take is equivalent to $K_6 T_5$ where $K_6 = \min(1/2, \min(d_1, d'_1)(1 - \Lambda)/K_0)$.

Hence, for ε' small enough, the survival time of the full orbit is $> K_6 \varepsilon'$:

$$\boxed{\forall n \in \mathbb{Z}, \tau_\Phi(\omega_n) > K_6 \varepsilon'}.$$

This completes the proof of Proposition 66 with $K = 1/K_6$.

³³ Proposition 66 claims uniformity w.r.t. t , but not w.r.t. f .

3.11 Step 2, Conclusion

Here, we will prove Proposition 65 (which is what is left to prove the main theorem, more precisely Theorem 26), whose statement we recall:

Proposition *There exists $r'_0 < r_0$ and a pair $\varepsilon_1 < \varepsilon_0$ with $\varepsilon_0 < T'_1$ such that for all $f_0 \in \mathcal{F}$, for all $z \in \text{Dom}(\mathcal{R}[f_0]) \odot (1 - \varepsilon_1)$, if we consider the orbit ω_n associated with z , then*

- for all $n \in \mathbb{Z}$

$$\tau_\Phi[f_0](\omega_n) > \varepsilon_0,$$

- there exists $M \in \mathbb{Z}$ such that $(t \leq \varepsilon_0 \text{ and } n \leq M) \implies \omega_n \langle t \rangle \in D_{\text{rep}}[f_t](r'_0)$.

Consider $\varepsilon_1 \in]0, 1[$ to be determined later. Let $f_0 \in \mathcal{F}$, and $z \in \text{Dom}(\mathcal{R}[f]) \odot (1 - \varepsilon_1)$ and apply Proposition 45 to $\varepsilon = \varepsilon_1$. For this, we have to assume $\varepsilon_1 < \xi$ for some $\xi > 0$ given by the proposition. We obtain some $\varepsilon' = \varepsilon'(\varepsilon_1) > 0$ such that the associated orbit $\omega_n \langle 0 \rangle$ of f_0 is contained in $\text{Dom}(f_0) \odot (1 - \varepsilon')$. By the previous section (Proposition 66), $\forall n \in \mathbb{Z}$, $\tau_\Phi(\omega_n) > \varepsilon'(\varepsilon_1)/K$. We can take $\varepsilon_0 = \min(T''_1, \varepsilon'(\varepsilon_1)/K)$ where $T''_1 < T'_1$ is any chosen constant. Since $\varepsilon'(\varepsilon) \gg \varepsilon$, for small enough values of ε_1 we have $\varepsilon_0 > \varepsilon_1$. Proposition 66 also provides the second claim in Proposition 65. \square

Now comes a final set of remarks. Let us call $(\varepsilon_0, \varepsilon_1)$ a valid pair whenever $\varepsilon_1 < \varepsilon_0 < T'_1$ and the proposition holds with these values. Given ε_1 small enough, the set of valid values for ε_0 includes the interval $]\varepsilon_1, \varepsilon'(\varepsilon_1)/K[$. As the right bound is $\gg \varepsilon_1$, it is easy to deduce that: $\forall \varepsilon_0$ there exists ε_1 such that $(\varepsilon_0, \varepsilon_1)$ is a valid pair. Moreover, we can take $\varepsilon_1 = o(\varepsilon_0)$.

This implies that if one iterates renormalization starting from a map in \mathcal{F}_ε with ε small enough, the map $\mathcal{R}^n[f]$ will have at least structure $\mathcal{F}_{\varepsilon_n}$ with $1/\varepsilon_n$ increasing faster than any exponential: the structure tends rapidly to the full structure \mathcal{F} .

Now, given the specific formula in Proposition 45:

$$\log \frac{1}{\varepsilon'(\varepsilon_1)} \leq c' + c \log \left(1 + \log \frac{1}{\varepsilon_1} \right),$$

and the computations above, we get that we can take $\varepsilon_1 \leq \exp(\beta - \alpha/\varepsilon_0)$ for some constants $\alpha, \beta > 0$, i.e. $1/\varepsilon_n$ increases at least like an *iterated* exponential.

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Appendix A: Parabolic points

The present section is given only to fix notations and normalizations and is not an introduction to the subject of parabolic points. We recommend learning it in any of the classic books introducing holomorphic dynamics, or in [16,31]. The article [3] is also instructive and very well illustrated. There is no claim that any of the statements given in this section is due to the author.

What is understood under the terminology *parabolic point* has variations, according to whether or not linearizable maps are allowed, and according to whether the allowed values of the multiplier should be 1 or any root of unity. So, here, we will try and avoid solely mentioning parabolic points and use instead the following terminology:

- *Tangent to identity*: fixed point whose multiplier is equal to 1.
- *Rationally indifferent*: fixed (or periodic) point whose multiplier is a root of unity.
- *Non-linearizable parabolic point*: rationally indifferent fixed (or periodic) point which is not linearizable.

Non-linearizability is the condition to have petals. A parabolic point *with petals* will thus be a synonym for a non-linearizable parabolic point.

We will often denote $\mathcal{P}_{\text{attr}}$ and \mathcal{P}_{rep} attracting and repelling petals of non-linearizable parabolic points. We denote Φ_{attr} and Φ_{rep} the attracting and repelling coordinates, defined on such petals. The extended attracting Fatou coordinate will be denoted by Ψ_{attr} too (context should make it clear which one is referred to). The extended repelling Fatou parametrization will be denoted Ψ_{rep} : it satisfies

$$\Psi_{\text{rep}} \circ T_1 = f \circ \Psi_{\text{rep}},$$

in the sense that the domains of the two hand sides of the equality are equal and that the equality holds on this set.

Recall that Fatou coordinates are unique up to addition of constants. The choice of a Fatou coordinate among $\Phi_{\text{attr}} + c$ or $\Phi_{\text{rep}} + c$, $c \in \mathbb{C}$, is called a *normalization* thereof. Normalizing Φ_{rep} normalizes Ψ_{rep} .

One petal: The rest of the present section focuses on non-linearizable parabolic fixed points with only one attracting petal, i.e. in some chart f has expression

$$f(z) = z + a_2 z^2 + \mathcal{O}(z^3)$$

with $a_2 \neq 0$.

Extended horn maps and parabolic renormalization: In this case, the *extended horn map* is the composition

$$h = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$$

of these extensions. Changing the normalizations of the Fatou coordinates replaces h with its pre composition and post composition with two unrelated translations. In Appendix B we discuss examples of what horn maps can be used for.

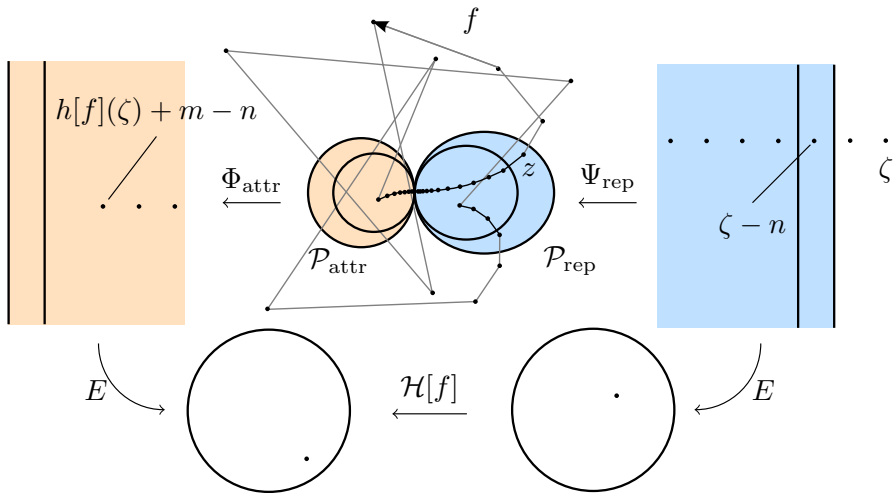


Fig. 22 Decomposing $h[f]$ and $\mathcal{H}[f]$. For convenience, we have chosen petals $\mathcal{P}_{\text{attr}}$ and \mathcal{P}_{rep} whose images in Fatou coordinates are right and left half planes. Note that the orbit may visit the repelling petal more than one time, and does not necessarily enter the attracting petal by its leftmost part (the crescent shaped fundamental domain)

To define a renormalization, we proceed as follows. This definition does not pretend to be the best one, it is well suited to our purposes. The map h commutes with T_1 and its domain of definition is T_1 -invariant and contains an upper and a lower half plane. There is, thus, a quotient map $\text{Dom}(h)/\mathbb{Z} \rightarrow \mathbb{C}/\mathbb{Z}$. Conjugate it by $E : z \mapsto e^{2i\pi z}$ to a map defined on an open subset of \mathbb{C}^* containing a neighborhood of 0 and ∞ . It can be holomorphically extended at these points, and the extension fixes 0 and ∞ . We will denote \mathcal{H} this extension, or $\mathcal{H}[f]$ to emphasize its dependence on f . For the *upper parabolic renormalization* of f , denoted $\mathcal{R}[f]$, consider the restriction of this extension to the connected component of its domain of definition that contains 0, and possibly pre and post compose it with two linear maps ($z \mapsto az$ and $z \mapsto bz$) to be chosen according to conventions. For the *lower parabolic renormalization* of f , conjugate first the extension by $z \mapsto 1/z$, then restrict it to the connected component of the domain of definition containing 0 and finally compose with linear maps. The reason why we allow for these linear maps is that we will find it convenient later to use a different normalization for parabolic renormalization than for Fatou coordinates and the associated horn map.

Another point of view on extended horn maps, and parabolic renormalization: Since Φ_{attr} and Ψ_{rep} are defined beyond the petal $\mathcal{P}_{\text{attr}}$ and beyond $\Phi_{\text{rep}}(\mathcal{P}_{\text{rep}})$ by using iteration of f , the definition of $h[f]$ can be reformulated as follows:

- for $\zeta \in \text{Dom}(h[f])$, there exists $n \in \mathbb{N}$ such that $\zeta - n \in \Phi_{\text{rep}}(\mathcal{P}_{\text{rep}})$,
- $\zeta - n = \Phi_{\text{rep}}(z)$ for a unique $z \in \mathcal{P}_{\text{rep}}$,
- there exists $m \in \mathbb{N}$ such that $f^m(z) \in \mathcal{P}_{\text{attr}}$,
- $h(\zeta) = \Phi_{\text{attr}}(f^m(z)) - m + n$.

We have illustrated a possible orbit on Fig. 22.

The iterative residue: Let

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

be the power series expansion of f . The iterative residue of f is the quantity $\gamma = 1 - \frac{a_3}{a_2^2}$. It is related to the residue at 0 of the meromorphic form $\frac{dz}{f(z)-z}$ by the following formula: $\frac{1}{2\pi i} \oint \frac{dz}{f(z)-z} = \gamma - 1$. In fact the (multivalued near the origin) primitive $\int \frac{dz}{f(z)-z} + \frac{dz}{z}$ turns out to be an interesting approximation of the Fatou coordinates, as their expansions share the same first two terms: as z tends to 0 within a closed sector avoiding the repelling axis for $\Phi = \Phi_{\text{attr}}$ or the attracting axis for $\Phi = \Phi_{\text{rep}}$:

$$\Phi(z) = \frac{-1}{a_2 z} - \gamma \log z + \text{constant} + o(1).$$

Another characterization is in terms of the horn map: there are expansions

$$\begin{aligned} h(z) &= z + a_{\text{up}} + o(1) \quad \text{as } \text{Im}(z) \rightarrow +\infty, \\ h(z) &= z + a_{\text{down}} + o(1) \quad \text{as } \text{Im}(z) \rightarrow -\infty. \end{aligned}$$

The constants a_{up} and a_{down} depend on the normalization of Fatou coordinates, but not the quantity $a_{\text{up}} - a_{\text{down}}$. It turns out that

$$a_{\text{up}} - a_{\text{down}} = -2\pi i \gamma.$$

Interestingly, if we consider the horn map with the normalization number 2 presented below, then $a_{\text{up}} = -\pi i \gamma$ and $a_{\text{down}} = \pi i \gamma$.

Some normalizations: We will give here three examples of normalizations for the upper parabolic renormalization $\mathcal{R}[f]$ of f . The first two work well for germs,³⁴ the third makes strong structural assumptions on f . Recall that $\mathcal{H}[f]$ denotes the semi-conjugate of the horn map by the map $E : z \mapsto e^{2\pi i z}$, extended at 0 and ∞ by fixing them, and that

³⁴ We use the word *germ* in the following meaning: an equivalence class of holomorphic maps defined near the origin, with $f \sim g$ if they coincide in some neighborhood of 0. This is equivalent to f and g having the same power series expansion at the origin.

$$\mathcal{R}[f] = A \circ \mathcal{H}^r[f] \circ B(z),$$

for some linear maps

$$A : z \mapsto az \quad \text{and}$$

$$B : z \mapsto bz,$$

where $\mathcal{H}^r[f]$ denotes the restriction of³⁵ $\mathcal{H}[f]$ to the component containing 0 of its domain. Let

$$\begin{aligned} f(z) &= z + a_2 z^2 + a_3 z^3 + \cdots \\ \mathcal{H}[f](z) &= b_1 z + b_2 z^2 + \cdots \\ \mathcal{R}[f](z) &= b'_1 z + b'_2 z^2 + \cdots \end{aligned}$$

be their power series expansions. We have $b_1 \in \mathbb{C}^*$, and $b_2 \in \mathbb{C}$. The constants b'_1 and b'_2 are expressed from a, b, b_1 and b_2 as follows $b'_1 = abb_1$ and $b'_2 = ab^2 b_2$. Here are our examples of normalizations:

- (1) By imposing $b'_1 = 1$ and $b'_2 = 1$: this first approach is easier but assumes that $b_2 \neq 0$. Then there is a unique pair of linear maps A, B such that $\mathcal{R}[f](z) = z + z^2 + \mathcal{O}(z^3)$.
- (2) By normalizing the expansion of the Fatou coordinates and taking $\mathcal{R} = \mathcal{H}$: Fatou coordinates are unique up to addition of a constant. Moreover, the following limited expansion is valid (even though there is not a *convergent* power series expansion in general): on all closed sectors avoiding respectively the repelling and the attracting axis, we have, as $z \rightarrow 0$:

$$\begin{aligned} \Phi_{\text{attr}}(z) &= \frac{-1}{a_2 z} - \gamma \log_p \frac{-1}{a_2 z} + \text{constant} + o(1), \\ \Phi_{\text{rep}}(z) &= \frac{-1}{a_2 z} - \gamma \log_p \frac{1}{a_2 z} + \text{constant} + o(1), \end{aligned}$$

where \log_p denotes the principal branch of the logarithm. The normalization just consists in adding constants to both Fatou coordinates so as to cancel the two constants in the above expansions. This normalizes $h = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$ and we then choose $\mathcal{R}[f] = \mathcal{H}[f]$. Note that with this normalization,

$$\begin{aligned} h(z) &= z - i\pi\gamma + o(1) \text{ as } \text{Im } z \rightarrow +\infty \text{ and} \\ h(z) &= z + i\pi\gamma + o(1) \text{ as } \text{Im } z \rightarrow -\infty. \end{aligned}$$

where γ is the iterative residue.

³⁵ For lower renormalization instead of upper, replace $\mathcal{H}[f]$ with $s \circ \mathcal{H}[f] \circ s$, where $s(z) = 1/z$.

- (3) By the singular value: we meet in this article a class of maps whose renormalizations have a unique critical value.³⁶ Fix a preferred complex number $v \in \mathbb{C}^*$. We then choose the linear map A so as to place the critical value of $\mathcal{R}[f] = A \circ \mathcal{H}^r[f] \circ B$ at v and then B so that $A \circ \mathcal{R}[f] \circ B$ has derivative 1 at the origin.

Each of these conventions has its own advantages. Conventions number 1 and 3 have the property that $\mathcal{R}[g \circ f \circ g^{-1}] = \mathcal{R}[f]$ in a neighborhood of 0 for all holomorphic maps g fixing the origin with $g'(0) \neq 0$. They also give back a germ $\mathcal{R}[f]$ tangent to the identity. Number 2 does not necessarily, but it is defined for all f . We will work with a class of maps satisfying number 3. Our choice in most of the article will be to normalize Fatou coordinates, the horn map and $\mathcal{H}[f]$ according to number 2, and the parabolic renormalization $\mathcal{R}[f]$ according to number 3.

Appendix B: What are horn maps good for?

Horn maps occur in at least two ways:

- First as local conjugacy invariants. A complete local conjugacy invariant of a non-linearizable parabolic germ with one attracting petal is more or less given by the data of the pair of germs of its horn maps at both ends of the cylinder (see [29] for precise statements; [23] gives an interesting equivalent point of view).
- Second as limits of cylinder renormalization. If a sequence of maps f_n tends to f and fix the origin with multiplier λ_n and if $2\pi i/(\lambda_n - 1) = N_n + a + o(1)$ with $N_n \in \mathbb{Z}$, $N_n \rightarrow \pm\infty$ and $a \in \mathbb{C}$, under some mild supplementary assumptions, the fixed point of f at the origin is the limit of a pair of fixed points of f_n , the origin and another one, and is possible to draw crescent shaped domains with tips at these two fixed points delimited by a curve C_n and its image $f_n(C_n)$. The quotient of this domain by identifying $z \in C_n$ with $f_n(z)$ is isomorphic as a Riemann surface to the cylinder \mathbb{C}/\mathbb{Z} . The first return map from the cylinder to itself then tends, as $n \rightarrow +\infty$, to the horn map (up to pre and post composition with translations). See [16, 19, 20, 25, 26].

The second point justifies why it makes sense to iterate horn maps.

A very important application comes from Lavaurs' theorem: let $\sigma \in \mathbb{C}$ and let the Lavaurs map g_σ be defined as

$$g_\sigma = \Psi_{\text{rep}} \circ T_\sigma \circ \Phi_{\text{attr}},$$

where $T_\sigma(z) = z + \sigma$. Then under the same assumptions as above, $f_n^{N_n} \rightarrow g_\sigma$ for some value of σ that depends on a (and on the chosen normalizations of the Fatou coordinates). This is why the Lavaurs maps are also called *geometric limits* by analogy with the field of Kleinian groups. Application of Lavaurs's theorem include parabolic enrichments (understanding the Hausdorff limits of Julia sets of a sequence

³⁶ Or a unique non-zero singular value.

of polynomials tending to one with a non-linearizable parabolic point), non local connectedness of some bifurcation loci, and several discontinuity theorems.

Now, horn maps are closely related to Lavaurs maps because each are semi-conjugate of the other. More precisely, consider the following non-commuting diagram:

$$\begin{array}{ccc} & \mathbb{C} & \\ \Psi_{\text{rep}} \nearrow & & \searrow \Phi_{\text{attr}} \\ \mathbb{C} & \xleftarrow{T_\sigma} & \mathbb{C} \end{array}$$

The map g_σ is the composition obtained by starting from the top node, and following the arrows in a loop back to the starting node. The map $h_\sigma := T_\sigma \circ h$ is the same but starting from the lower left corner.

Following one resp. two arrows from one corner to another gives a semi-conjugacy from h_σ to g_σ resp. from g_σ to h_σ . The first advantage of horn maps over Lavaurs maps is that they are easier to understand and have better covering properties in many applications (the best is to project the extended horn maps, they commute with T_1 , down to a dynamical system on \mathbb{C}/\mathbb{Z}). From this stems a second advantage: the invariance under parabolic renormalization of some classes of maps, as explained in Sects. 1.3 to 1.5.

Appendix C: A reminder about singular values of maps

Let $f : X \rightarrow Y$ be a holomorphic map where X and Y are Riemann surfaces. Let us recall that a *singular value* of f , as a map from X to Y , is an element $z \in Y$ which has no open neighborhood over which f is a cover³⁷. Every critical value is singular, as is every asymptotic value³⁸, and it is a simple yet very useful theorem that the set of singular values is the closure of the set of all critical and asymptotic values (see for instance³⁹ [17] or Corollary 2.7 in [24]).

³⁷ I.e. there is no open subset V of Y containing z s.t. f is a cover from $f^{-1}(V)$ to V . The definition is equivalent if we consider only neighborhoods V of z homeomorphic to disks.

³⁸ A point $z \in Y$ is an asymptotic value whenever there exists a continuous path $\gamma : [0, t[\rightarrow X$ that leaves every compact of X and whose image by f tends to z .

³⁹ The language is slightly different in [17] since he calls singular values the critical or asymptotical ones. But his Proposition 1 amounts to our claim.

It shall be noted that restricting the domain of a map will likely introduce a lot of singular values: if $U \subset X$, every point in $f(\partial U)$ will be a singular value of f as a map from U to Y . In fact:

Lemma 86 (folk.) *Let $F : X \rightarrow Y$ be holomorphic and denote S its set of singular values. Assume $A \subset X$ and $B \subset Y$ are open X and that $f(A) \subset B$. Then the set of singular values of the restriction $\tilde{f} : A \rightarrow B$ of f contains $B \cap \overline{f(\partial A)}$ and is contained in $\overline{f(\partial A)} \cup S$.*

Proof First inclusion: The set of points in ∂A that are accessible from A is dense in ∂A . If $b = f(a) \in B$ with $a \in \partial A$ then a is the limit of $a_n \in \partial A$ which is accessible and $f(a_n)$ is an asymptotic value of \tilde{f} and tends to b , hence b is a singular value.

Second inclusion: it is enough to prove it for critical values and critical points, since the set of singular values of \tilde{f} is the closure of their union. All critical values of \tilde{f} are of course critical values of f , hence in S . Consider an asymptotic value b of \tilde{f} and let $\gamma : [0, 1[\rightarrow A$ with $f \circ \gamma(t) \xrightarrow{t \rightarrow 1} b$ and $\gamma(t)$ leaves every compact of A . If the set of accumulation points of γ in X contains more than one point, then f must be constant on the connected component of $\text{Dom } f$ containing γ , and then b is a singular value. Otherwise either γ leaves every compact of X , and then b is an asymptotic value of f hence in S , or γ converges to a point $a \in \partial A$, whence $b = f(a) \in \overline{f(\partial A)}$. \square

Similarly, enlarging the range Y of $f : X \rightarrow Y$ will introduce singular values at boundary points.

As a corollary of Lemma 86, if we restrict f to a parabolic immediate basin, we do not introduce new singular values:

Lemma 87 (folk.) *If $f : U \subset X \rightarrow X$ is a holomorphic map with a non-linearizable parabolic fixed point p , and if A denotes the union of a cycle of immediate basins of p , then the set of singular values of $f|_A : A \rightarrow A$ is contained in the intersection of A with the set of singular values of f .*

Proof Indeed $\overline{f(\partial A)} \cap A = \emptyset$ (here ∂ is relative to U). \square

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