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RESEARCH CONTRIBUTION

Quantitative Uncertainty Principles Related to Lions Transform

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Abstract
We prove various mathematical aspects of the quantitative uncertainty principles, including Donoho–Stark's uncertainty principle and a variant of Benedicks theorem for Lions transform.

Keywords Lions transform · Benedicks theorem · Donoho–Stark's uncertainty principle

Mathematics Subject Classification 42A38 · 44A35 · 34B30

1 Introduction

The uncertainty principle says that a function and its transform cannot concentrate both on small sets. Depending on the precise way to measure "concentration" and "smallness", this principle can assume different forms. This paper focuses on studying different uncertainty principles for the Jacobi–Dunkl transform, by following the procedures for similar transforms, such as the Fourier transform (the classical setting) we refer to the book [12] and the surveys [5, 9] for further references. The concept of concentration has taken different interpretations in different contexts. For example, Benedicks [2], Slepian and Pollak [17], Landau and Pollak [13], and Donoho and Stark [7] paid attention to the supports of functions and gave quantitative uncertainty principles for the Fourier transforms. Qualitative uncertainty principles are not inequalities, but are theorems that tell us how a function (and its Fourier transform) behave under certain circumstances. For example, Hardy [11], Cowling and Price [6], Beurling [4],

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and Miyachi [14] theorems enter within the framework of the quantitative uncertainty principles. The quantitative and qualitative uncertainty principles have been studied by many authors for various Fourier transforms, cf. for instance [1, 3, 10, 15, 18].
The first principle, studied in Sect. 2, is a Benedicks-type result which shows that two measurable sets (S, Σ) with finite measure form a strong annihilating pair. This means that a function supported in S cannot have an spectrum in Σ giving a quantitative information of the mass of a function whose spectrum is contained in Σ. The approach is based on the corresponding version of this type of principle for the Fourier–Bessel transform, studied in [10].

The second principle that is studied is a Donoho–Stark-type inequality. One can write the classical uncertainty principle in the following way: if a function $f(t)$ is essentially zero outside an interval of length Δt and its Fourier transform $f(w)$ is essentially zero outside an interval of length Δw, then $\Delta t \Delta w \geq 1$. In [7], Donoho and Stark show that it is not necessary to assume that the support and the spectrum are concentrated on intervals and one can replace intervals by measurable sets, and then the length of the interval is naturally replaced by the measure of the set. In Sect. 3, a version of this inequality for the Jacobi–Dunkl transform is given, and, as it appears in [7] it is explained how to reconstruct a signal f from a noisy measurement, knowing that the signal is supported on a set S.

In the last section, after having introduced the notion of ε-concentration we study what is the relation between the measure of the support of the function f and the measure of the support of the Fourier transform of f, that is ε-concentrated in measurable sets giving. More precisely we will prove that $|\text{supp}(f)|.|\text{supp}(\mathcal{F}(f)| \geq 1$.

To describe our results, we first need to introduce some facts about harmonic analysis related to Lions transform. We cite here, as briefly as possible, some properties. For more details, we refer to [20].
The Lions operator Δ defined on $]0, +\infty[$ by

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\mathcal{A}'(x)}{\mathcal{A}(x)} \frac{\partial}{\partial x} + \rho^2,$$

where

$$\mathcal{A}(x) = x^{2\alpha+1} B(x),\ \alpha > \frac{-1}{2}$$

where B is an even C^∞-function on $\mathbb{R}$ such that $B(0) = 1$ and $\rho \geq 0$. Moreover, we assume that $\mathcal{A}$ satisfy the following conditions:

- $\mathcal{A}$ is increasing and $\lim_{x\to\infty} \mathcal{A}(x) = \infty$.
- $\dfrac{\mathcal{A}'}{\mathcal{A}}$ is decreasing and $\lim_{x\to\infty} \frac{\mathcal{A}'(x)}{\mathcal{A}(x)} = 2\rho \geq 0$.
- There exists a constant $\delta > 0$ such that

$$\frac{B'(x)}{B(x)} = 2\rho - \frac{2\alpha+1}{x} + D(x)\exp(-\delta x), \quad \rho \geq 0$$
$$\frac{B'(x)}{B(x)} = D(x)\exp(-\delta x), \quad \rho = 0$$

where D is an infinitely differentiable function bounded together with its derivatives. For all $\lambda \in \mathbb{C}$, the equation

$$\begin{cases} \Delta u = -\lambda^2 u \\ u(0) = 1, \quad u'(0) = 0. \end{cases} \tag{1}$$

admits a unique C^∞ solution on $\mathbb{R}^+$ denoted φ_λ.

Equation (1) possesses also two solutions $\phi_{\pm\lambda}$ linearly independent having the following behavior at infinity: $\phi_{\pm\lambda}(x) \sim e^{(\pm i\lambda-\rho)x}$. Then, there exists a function C such that

$$\varphi_\lambda(x) = C(\lambda)\phi_\lambda(x) + C(-\lambda)\phi_{-\lambda}(x).$$

For $\lambda \in \mathbb{C}$ and $x \geq 0$ such that $|\mathrm{Img}(\lambda)| \leq \rho$, we have

$$|\varphi_\lambda(x)| \leq 1.$$

We denote by $L^p(\mathbb{R}^+, \mu)$, $1 \leq p \leq \infty$ the space of measurable functions f on $\mathbb{R}^+$ such that

$$\|f\|_{L^p(\mathbb{R}^+,\mu)} = \left(\int_{\mathbb{R}^+} |f(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} < +\infty, \quad \text{if } 1 \leq p < +\infty,$$
$$\|f\|_\infty = \operatorname{ess}\sup_{x\in\mathbb{R}^+} |f(x)| < +\infty, \quad \text{if } p = \infty$$

where

$$\mathrm{d}\mu(x) = \mathcal{A}(x)\mathrm{d}x.$$

The Lions transform $\mathcal{F}$ is defined on $L^1(\mathbb{R}^+, \mu)$ by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}^+} f(x)\varphi_\lambda(x)\mathrm{d}\mu(x), \text{ for all } \lambda \in \mathbb{R}^+.$$

Let ν be the measure defined on $[0, \infty[$ by

$$\mathrm{d}\nu(\lambda) = \frac{\mathrm{d}\lambda}{C(\lambda)^2}.$$

Let $L^p(\mathbb{R}^+, \nu)$, $1 \leq p \leq \infty$, the space of measurable functions f on $[0, \infty[$, such that $\|f\|_{L^p(\mathbb{R}^+,\nu)} < \infty$.

Plancherel theorem. The Lions transform $\mathcal{F}$ extends uniquely to an isometric isomorphism of $L^2(\mathbb{R}^+, \mu)$ onto $L^2(\mathbb{R}^+, \nu)$

$$\int_{\mathbb{R}^+} |f(x)|^2 \mathrm{d}\mu(x) = \int_{\mathbb{R}^+} |\mathcal{F}(f)(\lambda)|^2 \mathrm{d}\nu(\lambda). \tag{2}$$

Inversion theorem. Let $f \in L^1(\mathbb{R}^+, \mu)$ such that $\mathcal{F}(f) \in L^1(\mathbb{R}^+, \nu)$. Then,

$$f(x) = \int_{\mathbb{R}^+} \mathcal{F}(f)(\lambda)\varphi_\lambda(x)\mathrm{d}\nu(\lambda), \text{ a.e. } x \in \mathbb{R}^+.$$

Definition 1 Let $S,\ \Sigma$ be two measurable subsets of $\mathbb{R}^+$. Then, (S, Σ) is called a strong annihilating pair for the Lions transform if there exists a constant $C(S, \Sigma)$ such that for all function $f \in L^2(\mathbb{R}^+, \mu)$, with supp $\mathcal{F}(f) \subset \Sigma$,

$$\|f\|_{L^2(\mathbb{R}^+,\mu)} \leq C(S, \Sigma)\|f\|_{L^2(S^c,\mu)} \tag{3}$$

where $S^c = \mathbb{R}^+ \backslash S$ and supp $f = \{x : f(x) \neq 0\}$.

Lemma 1 *Let $\rho > 0$, $1 \leq p < 2$ and D_p be the strip in the complex ξ-plane defined by*

$$D_p = \left\{\xi \in \mathbb{C} : |Img(\xi)| < \rho\left(\frac{2}{p} - 1\right)\right\}.$$

For any function $f \in L^p(\mathbb{R}^+, \mu)$, its Lions transform $\mathcal{F}(f)$ is well defined and holomorphic in D_p and for all $\xi \in D_p$,

$$|\mathcal{F}(f)(\xi)| \leq \|f\|_{L^p(\mathbb{R}^+,\mu)}\|\varphi_\xi\|_{L^q(\mathbb{R}^+,\nu)}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof See Lemma 3.1 in [8]. □

As $|\varphi_\xi(x)| \leq 1$, then if $f \in L^1(\mathbb{R}^+, \mu)$, $\mathcal{F}(f)$ is continuous also in the closure $\overline{D}_1$ of D_1 and for all $\xi \in \overline{D}_1$

$$\|\mathcal{F}(f)\|_\infty \leq \|f\|_{L^1(\mathbb{R}^+,\mu)}, \tag{4}$$

where $\|.\|_\infty$ is the usual essential supremum norm.

2 Uncertainty Principles

In this section, we will give some remarks about Annihilating sets.

Proposition 1 *Let $f \in L^2(\mathbb{R}^+, \mu)$ has non empty support, then*

$$\nu(supp\ \mathcal{F}(f))\mu(supp\ f) \geq 1.$$

In particular, if :

$$\mu(supp\ f)\nu(supp\ \mathcal{F}(f)) < 1 \text{ then } f = 0.$$

Proof If the function $f \in L^2(\mathbb{R}^+, \mu)$ has non empty support, by the Cauchy–Schwartz inequality and (4), we have

$$\begin{aligned}\|\mathcal{F}(f)\|^2_{L^2(\mathbb{R}^+,\nu)} &\leq \nu(\text{supp } \mathcal{F}(f))\|\mathcal{F}(f)\|^2_\infty \\ &\leq \nu(\text{supp } \mathcal{F}(f))\|f\|^2_{L^1(\mathbb{R}^+,\mu)} \\ &\leq \nu(\text{supp } \mathcal{F}(f))\mu(\text{supp } f)\|f\|^2_{L^2(\mathbb{R}^+,\mu)}.\end{aligned}$$

Using Plancherel's theorem (2), we have the following quantitative uncertainty inequality connecting the support of f and the support of its Lions transform $\mathcal{F}$:

$$\nu(\text{supp}\mathcal{F}(f))\mu(\text{supp} f) \geq 1. \tag{5}$$

It follows that if :

$$\mu(\text{supp} f)\nu(\text{supp}\mathcal{F}(f)) < 1 \text{ then } f = 0.$$

□

We consider a pair of orthogonal projections on $L^2(\mathbb{R}^+, \mu)$. The first is the operator defined by

$$E_S f = \chi_S f \tag{6}$$

and the second is the operator defined by

$$F_\Sigma f = \mathcal{F}^{-1}\left[\chi_\Sigma \mathcal{F}(f)\right], \tag{7}$$

where S and Σ are measurable subsets of $\mathbb{R}^+$, and χ_S denote the characteristic function of S.

Let $0 < \varepsilon_S, \varepsilon_\Sigma < 1$ and let $f \in L^p(\mathbb{R}^+, \mu)$, be a nonzero function where $1 < p \leq 2$. We say that f is ε_S-concentrated on S if:

$$\|E_{S^c} f\|_{L^2(\mathbb{R}^+,\mu)} \leq \varepsilon_S \|f\|_{L^2(\mathbb{R}^+,\mu)}. \tag{8}$$

Similarly, we say that f is ε_Σ-concentrated on Σ for the Lion transform if

$$\|F_{\Sigma^c} f\|_{L^2(\mathbb{R}^+,\mu)} \leq \varepsilon_\Sigma \|f\|_{L^2(\mathbb{R}^+,\mu)}.$$

We define the norm of E_S as following:

$$\|E_S\| = \sup_{f \in L^2(\mathbb{R}^+,\mu)} \frac{\|E_S(f)\|_{L^2(\mathbb{R}^+,\mu)}}{\|f\|_{L^2(\mathbb{R}^+,\mu)}}.$$

In the same way, the norm of F_Σ is defined by

$$\|F_\Sigma\| = \sup_{f \in L^2(\mathbb{R}^+,\mu)} \frac{\|F_\Sigma(f)\|_{L^2(\mathbb{R}^+,\mu)}}{\|f\|_{L^2(\mathbb{R}^+,\mu)}}.$$

Since E_S and F_Σ are projections, it is clear that $\|E_S\| = \|F_\Sigma\| = 1$.

Lemma 2 *Let (S, Σ) be two measurable subsets of $\mathbb{R}^+$. Then, the following assertions are equivalent:*

(i) $\|E_S F_\Sigma\| < 1$.
(ii) (S, Σ) *is strongly annihilating pair for the Lions transform. Moreover, we have*

$$\|f\|^2_{L^2(\mathbb{R}^+,\mu)} \le (1 - \|E_S F_\Sigma\|)^{-2} \left(\|E_{S^c} f\|^2_{L^2(\mathbb{R}^+,\mu)} + \|F_{\Sigma^c} f\|^2_{L^2(\mathbb{R}^+,\mu)} \right).$$

Proof First, we show the following implication (i) $\Rightarrow$ (ii). The identity operator I satisfies

$$I = E_S + E_{S^c} = E_S F_\Sigma + E_S F_{\Sigma^c} + E_{S^c},$$

we have from the orthogonality of E_S and E_{S^c}

$$\begin{aligned} \|f - E_S F_\Sigma f\|^2_{L^2(\mathbb{R}^+,\mu)} &= \|E_S F_{\Sigma^c} f + E_{S^c} f\|^2_{L^2(\mathbb{R}^+,\mu)} \\ &= \|E_S F_{\Sigma^c} f\|^2_{L^2(\mathbb{R}^+,\mu)} + \|E_{S^c} f\|^2_{L^2(\mathbb{R}^+,\mu)}. \end{aligned}$$

It follows by $\|E_S\| = 1$ that

$$\|f - E_S F_\Sigma f\|_{L^2(\mathbb{R}^+,\mu)} \le \left(\|F_{\Sigma^c} f\|^2_{L^2(\mathbb{R}^+,\mu)} + \|E_{S^c} f\|^2_{L^2(\mathbb{R}^+,\mu)} \right)^{\frac{1}{2}}. \quad (9)$$

On the other hand, we have

$$\begin{aligned} \|f - E_S F_\Sigma f\|_{L^2(\mathbb{R}^+,\mu)} &\ge \|f\|_{L^2(\mathbb{R}^+,\mu)} - \|E_S F_\Sigma f\|_{L^2(\mathbb{R}^+,\mu)} \\ &\ge \|f\|_{L^2(\mathbb{R}^+,\mu)} - \|E_S F_\Sigma\|.\|f\|_{L^2(\mathbb{R}^+,\mu)}. \end{aligned}$$

It follows from inequality (9)

$$(1 - \|E_S F_\Sigma\|)\|f\|_{L^2(\mathbb{R}^+,\mu)} \le \left(\|E_{S^c} f\|^2_{L^2(\mathbb{R}^+,\mu)} + \|F_{\Sigma^c} f\|^2_{L^2(\mathbb{R}^+,\mu)} \right)^{\frac{1}{2}}. \quad (10)$$

As $\|E_S F_\Sigma\| < 1$, then we obtain the desired result.

Let us now show the second implication (ii) $\Rightarrow$ (i). We have

$$\|E_S F_\Sigma\| \le \|E_S\| \|F_\Sigma\| \le 1.$$

We suppose that $\|E_S F_\Sigma\| = 1$. Then, we can find a bandlimited sequence $f_n \in L^2(\mathbb{R}^+, \mu)$ on Σ of norm 1 (in particular $f_n = F_\Sigma f_n$) such that

$$\|E_S f_n\|_{L^2(\mathbb{R}^+,\mu)} \longrightarrow 1 \text{ as } n \longrightarrow \infty.$$

By the fact that $E_S f$ and $E_S^c f$ are orthogonal for every f, we have

$$\|E_{S^c} f_n\|^2_{L^2(\mathbb{R}^+,\mu)} = \|f_n\|^2_{L^2(\mathbb{R}^+,\mu)} - \|E_S f_n\|_{L^2(\mathbb{R}^+,\mu)} \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

which contradicts (3). □

Theorem 1 *If* $0 < \mu(S)\nu(\Sigma) < 1$ *then for all functions* $f \in L^2(\mathbb{R}^+, \mu)$ *such that supp* $\mathcal{F}(f) \subset \Sigma$

$$\|f\|_{L^2(\mathbb{R}^+,\mu)} \leq \left(1 - \sqrt{\mu(S)\nu(\Sigma)}\right)^{-1} \|f\|_{L^2(S^c,\mu)}.$$

Proof A straightforward computation shows that $E_S F_\Sigma$ is an integral operator with kernel

$$N(t, x) = \chi_S(t)\mathcal{F}^{-1}(\chi_\Sigma \varphi_\lambda(t))(x).$$

Indeed, we have

$$\begin{aligned} E_S F_\Sigma f(t) &= \chi_S(t) \int_{\mathbb{R}^+} \chi_\Sigma(\xi)\mathcal{F}(f)(\xi)\varphi_\lambda(t)\mathrm{d}\nu(\xi) \\ &= \chi_S(t) \int_{\mathbb{R}^+} \chi_\Sigma(\xi)\varphi_\lambda(t) \left(\int_{\mathbb{R}^+} f(x)\varphi_\lambda(x)\mathrm{d}\mu(x)\right) \mathrm{d}\nu(\xi) \\ &= \int_{\mathbb{R}^+} f(x)N(t, x)\mathrm{d}\mu(x), \end{aligned}$$

where

$$N(t, x) = \chi_S(t) \int_{\mathbb{R}^+} \chi_\Sigma(\xi)\varphi_\lambda(t)\varphi_\lambda(x)\mathrm{d}\nu(\xi).$$

Since $\nu(\Sigma) < \infty$ and φ_λ is bounded, then for all $t \in \mathbb{R}^+$, $\chi_\Sigma\varphi_\lambda(t) \in L^2(\mathbb{R}^+, \nu)$. Then, $E_S F_\Sigma$ is an integral operator with Kernel

$$N(t, x) = \chi_S(t)\mathcal{F}^{-1}(\chi_\Sigma \varphi_\lambda(t))(x).$$

As $\|E_S F_\Sigma\|_{\mathrm{HS}} = \|N\|_{L^2(\mathbb{R}^+\times\mathbb{R}^+,\mu\otimes\mu)}$, it follows from Plancherel's theorem (2) that

$$\begin{aligned} \|E_S F_\Sigma\|^2_{\mathrm{HS}} &= \int_{\mathbb{R}^+} |\chi_S(t)|^2 \left(\int_{\mathbb{R}^+} |\mathcal{F}^{-1}(\chi_\Sigma\varphi_\lambda(t))(x)|^2 \mathrm{d}\mu(\xi)\right) \mathrm{d}\mu(t) \\ &= \int_{\mathbb{R}^+} \chi_S(t) \int_{\mathbb{R}^+} \chi_\Sigma(\xi)|\varphi_\lambda(t)|^2 \mathrm{d}\nu(\xi)\mathrm{d}\mu(t). \end{aligned}$$

We can deduce from $|\varphi_\lambda(t)| < 1$ that

$$\|E_S F_\Sigma\| \leq \|E_S F_\Sigma\|_{\mathrm{HS}} \leq \sqrt{\mu(S)\succeq(\Sigma)}. \tag{11}$$

Since $\mu(S)\nu(\Sigma) < 1$, then we have from inequality (11) and Lemma 2

$$\|f\|^2_{L^2(\mathbb{R}^+,\mu)} \leq \left(1 - \sqrt{\mu(S).\nu(\Sigma)}\right)^{-2} \left(\|E_{S^c} f\|^2_{L^2(\mathbb{R}^+,\mu)} + \|F_{\Sigma^c} f\|^2_{L^2(\mathbb{R}^+,\mu)}\right).$$

Since supp $\mathcal{F}(f) \subset \Sigma$, it follows from Plancherel's theorem (2) that

$$\|F_{\Sigma^c} f\|^2_{L^2(\mathbb{R}^+,\mu)} = \int_{\Sigma^c} |\mathcal{F}(\xi)|^2 \mathrm{d}\nu(\xi) = \|\mathcal{F}(f)\|^2_{L^2(\Sigma^c,\nu)} = 0,$$

which shows the desired result. □

We are now in position to prove our main result for the Lions transform.

Theorem 2 *Let S and Σ be a pair of measurable subsets of $\mathbb{R}^+$ with $0 < \mu(S)$, $\nu(\Sigma) < \infty$, then the pair (S, Σ) is strong annihilating pair.*

Proof Let $f \in L^2(\mathbb{R}^+, \mu)$ be a nonzero function such that

$$\text{supp } f \subset S \text{ and supp } \mathcal{F}(f) \subset \Sigma.$$

Using the Cauchy–Schwartz inequality, we have

$$\|f\|^2_{L^1(\mathbb{R}^+,\mu)} \leq \mu(\text{supp } f)\|f\|^2_{L^2(\mathbb{R}^+,\mu)}.$$

As f has support of finite measure, hence f belongs in $L^1(\mathbb{R}^+, \mu)$. From Lemma 1 , $\mathcal{F}(f)$ is analytic in the open strip $\{\xi : |\xi| < \rho\}$. This contradicts that $\mathcal{F}(f)$ has support of finite measure. Then, (S, Σ) is weak annihilating pair for the Lions transform. According to [[12], I.1.3.2.A, p. 90], if $E_S F_\Sigma$ is compact (in particular if $E_S F_\Sigma$ is Hilbert–Schmidt), then if the pair (S, Σ) is weakly annihilating, it is also strongly annihilating. □

3 The Donoho–Stark's Uncertainty Principle

The classical uncertainty principle says that if a function $f(t)$ is essentially zero outside an interval of length Δt and its Fourier transform $\widehat{f}(w)$ is essentially zero outside an interval of length Δw, then

$$\Delta t \cdot \Delta w \geq 1.$$

In this section, we will prove a quantitative uncertainty inequality like (5) about the essential supports of a nonzero function $f \in L^2(\mathbb{R}^+, \mu)$ and its Lions transform. The first such inequality for the usual Fourier transform was obtained by Donoho–Stark [7].

Theorem 3 *Let Σ, $S \subset \mathbb{R}^+$ be a pair of measurable subsets and let $\varepsilon_S, \varepsilon_\Sigma > 0$ such that $\varepsilon_S^2 + \varepsilon_\Sigma^2 < 1$. Let $f \in L^2(\mathbb{R}^+, \mu)$ be a nonzero function. If f is ε_S-concentrated on S and ε_Σ-concentrated on Σ for the Lions transform, then*

$$\mu(S)\nu(\Sigma) \geq \left(1 - \sqrt{\varepsilon_S^2 + \varepsilon_\Sigma^2}\right)^2.$$

Proof The result follows from inequalities (10) and (11). □

Often the uncertainty principle is used to show that certain things are impossible, such as determining the momentum and position of a particle simultaneously or measuring the "instantaneous frequency" of a signal. In the following, we present an example where the generalized uncertainty principle shows something. Unexpectedly is possible the recovery of a signal or image despite significant amounts of missing information.

The following example is prototypical. A signal f is transmitted to a receiver who knows that f is bandlimited on Σ for the Lions transform, meaning that f is synthesized using only frequencies on Σ; equivalently $f = F_\Sigma f$. Suppose that the observation of f is corrupted by a noise $n \in L^2(\mathbb{R}^+, \mu)$ (which is nonetheless assumed to be small) and unregistered values on S. Thus, the observable function r satisfies

$$r(x) = \begin{cases} f(x) + n(x), & x \in S^c; \\ 0, & x \in S. \end{cases}$$

Here, we have assumed without loss of generality that $n = 0$ on S. Equivalently,

$$r = (I - E_S)f + n.$$

We say that f can be stably reconstructed from r, if there exists a linear operator K and a constant C such that

$$\|f - Kr\|_{L^2(\mathbb{R}^+,\mu)} \leq C\|n\|_{L^2(\mathbb{R}^+,\mu)}. \tag{12}$$

The estimate (12) shows that the noise n is at most amplified by a factor C.

Corollary 1 *If S and Σ are arbitrary measurable sets of $\mathbb{R}^+$ with $0 < \mu(S)\nu(\Sigma) < 1$. If f is bandlimited on Σ then f can be stably reconstructed from r. The constant C in Eq. (12) is not larger than $\left(1 - \sqrt{\mu(S)\nu(\Sigma)}\right)^{-1}$.*

Proof If $\mu(S)\nu(\Sigma) < 1$, using (11), $\|E_S F_\Sigma\| < 1$. Hence, $I - E_S F_\Sigma$ is invertible. Let

$$K = (I - E_S F_\Sigma)^{-1}.$$

Since f is bandlimited on Σ, then $(I - E_S)f = (I - E_S F_\Sigma)f$. Therefore,

$$f - Kr = f - K((I - E_S)f + n)$$

$$
\begin{aligned}
&= f - K(I - E_S F_\Sigma) f - Kn \\
&= f - (I - E_S F_\Sigma)(I - E_S F_\Sigma)^{-1} f - Kn \\
&= 0 - Kn.
\end{aligned}
$$

So that

$$
\begin{aligned}
\|f - Kr\|_{L^2(\mathbb{R}^+,\mu)} &= \|Kn\|_{L^2(\mathbb{R}^+,\mu)} \\
&\leq \|(I - E_S F_\Sigma)^{-1}\| \|n\|_{L^2(\mathbb{R}^+,\mu)} \\
&\leq \sum_{k=0}^{\infty} \|E_S F_\Sigma\|^k \|n\|_{L^2(\mathbb{R}^+,\mu)} \\
&\leq \sum_{k=0}^{\infty} (\mu(S)\nu(\Sigma))^{\frac{k}{2}} \|n\|_{L^2(\mathbb{R}^+,\mu)} \\
&= \left(1 - \sqrt{\mu(S)\nu(\Sigma)}\right)^{-1} \|n\|_{L^2(\mathbb{R}^+,\mu)}.
\end{aligned}
$$

The constant C in Eq. (12) is, therefore, not larger than $\left(1 - \sqrt{\mu(S)\nu(\Sigma)}\right)^{-1}$. □

The identity

$$
K = (I - E_S F_\Sigma)^{-1} = \sum_{k=0}^{\infty} (E_S F_\Sigma)^k
$$

suggests an algorithm for computing Kr. Put

$$
f^{(n)} = \sum_{k=0}^{n} (E_S F_\Sigma)^k r,
$$

then

$$
f^{(0)} = r, \;\; f^{(n+1)} = r + E_S F_\Sigma f^{(n)} \text{ and } f^{(n)} \rightarrow Kr \; as \; n \rightarrow \infty.
$$

As f is bandlimited on Σ we deduce that

$$
f^{(n+1)} - f = E_S F_\Sigma (f^{(n)} - f). \tag{13}
$$

Algorithms of this type have applied to a lot of problems in signal recovery (see for examples [13, 16]).

Theorem 4 *If S and Σ are arbitrary measurable sets of $\mathbb{R}^+$. If the pair (S, Σ) is strongly annihilating for the Lions transform, then f can be stably reconstructed from r. The constant C in Eq. (12) is not larger than $\left(1 - \sqrt{\mu(S)\nu(\Sigma)}\right)^{-1}$.*

Proof The result follows immediately from the proof of Theorem 1 and from Lemma 2. □

Under the hypotheses of Theorem 4 and from equality (13), the following error estimate holds:

$$\|f - f^{(n)}\|_{L^2(\mathbb{R}^+,\mu)} \leq \|E_S F_\Sigma\| \|f - r\|_{L^2(\mathbb{R}^+,\mu)}.$$

4 Quantitative Uncertainty Principle for Lions Transform

Many uncertainty principles have already been proved for the Fourier transform on L^p space for $1 < p \leq 2$ (see [18]). In this section, we shall investigate the case where f and $\mathcal{F}(f)$ are close to zero outside measurable sets. Here, the notion of 'close to zero' is formulated as follows.
We say that $\mathcal{F}(f)$ is ε_Σ-concentrated on Σ if and only if

$$\|\mathcal{F}(f) - \mathcal{F}(F_\Sigma f)\|_{L^q(\mathbb{R}^+,\nu)} \leq \varepsilon_\Sigma \|\mathcal{F}(f)\|_{L^q(\mathbb{R}^+,\nu)}. \tag{14}$$

By Riesz's interpolation theorem [19], we deduce that for every $1 \leq p \leq 2$ and for every $f \in L^p(\mathbb{R}^+, \mu)$ the function $\mathcal{F}(f)$ belongs to the space $L^q(\mathbb{R}^+, \nu)$, $q = p/(p-1)$, and

$$\|\mathcal{F}(f)\|_{L^q(\mathbb{R}^+,\nu)} \leq \|f\|_{L^p(\mathbb{R}^+,\mu)}. \tag{15}$$

Theorem 5 *If* $\nu(\Sigma) < \infty$ *and* $f \in L^p(\mathbb{R}^+, \mu)$, $1 \leq p \leq 2$,

$$F_\Sigma f(x) = \int_\Sigma \mathcal{F}(f)(\xi)\varphi_{-\xi}(x)\mathrm{d}\nu(\xi).$$

Proof Let $f \in L^p(\mathbb{R}^+, \mu)$, $1 \leq p \leq 2$ and $q = \frac{p}{p-1}$. Then, by Hölder's inequality and (15)

$$\begin{aligned}\|\mathcal{F}(f)\|_{L^1(\Sigma,\nu)} &= \int_\Sigma |\mathcal{F}(f)(x)|\mathrm{d}\nu(x)\\ &\leq (\nu(\Sigma))^{\frac{1}{p}} \|\mathcal{F}(f)\|_{L^q(\mathbb{R}^+,\nu)}\\ &\leq (\nu(\Sigma))^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^+,\nu)}\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{F}(f)\|_{L^2(\Sigma,\nu)} &= \left(\int_\Sigma |\mathcal{F}(f)(x)|^2\mathrm{d}\nu(x)\right)^{\frac{1}{2}}\\ &\leq (\nu(\Sigma))^{\frac{q-2}{2q}} \|\mathcal{F}(f)\|_{L^q(\mathbb{R}^+,\nu)}\\ &\leq (\nu(\Sigma))^{\frac{q-2}{2q}} \|f\|_{L^p(\mathbb{R}^+,\mu)}.\end{aligned}$$

Hence $\chi_\Sigma \mathcal{F}(f) \in L^1(\mathbb{R}^+, \nu) \cap L^2(\mathbb{R}^+, \nu)$ and by (7)

$$F_\Sigma f = \mathcal{F}^{-1}\left(\chi_\Sigma \mathcal{F}(f)\right).$$

□

Lemma 3 *If $f \in L^p(\mathbb{R}^+, \mu)$, $1 \leq p \leq 2$ and $q = \frac{p}{p-1}$ then*

$$\|\mathcal{F}(F_\Sigma f)\|_{L^q(\mathbb{R}^+,\nu)} \leq \|f\|_{L^p(\mathbb{R}^+,\mu)}.$$

Proof Let $f \in L^p(\mathbb{R}^+, \mu)$, $1 \leq p \leq 2$ and $q = \frac{p}{p-1}$. From (15) and (7)

$$\|\mathcal{F}(F_\Sigma f)\|_{L^q(\mathbb{R}^+,\nu)} = \left(\int_\Sigma |\mathcal{F}(f)(x)|^q \mathrm{d}\nu(x)\right)^{\frac{1}{q}} \leq \|\mathcal{F}(f)\|_{L^q(\mathbb{R}^+,\nu)} \leq \|f\|_{L^p(\mathbb{R}^+,\mu)}$$

this yields the desired result. □

Lemma 4 *Let S and Σ be measurable subsets of $\mathbb{R}^+$, if $f \in L^p(\mathbb{R}^+, \mu)$, $1 \leq p \leq 2$ and $q = \frac{p}{p-1}$ then*

$$\|\mathcal{F}(F_\Sigma E_S f)\|_{L^q(\mathbb{R}^+,\nu)} \leq (\mu(S))^{\frac{1}{q}} (\nu(\Sigma))^{\frac{1}{q}} \|f\|_{L^p(\mathbb{R}^+,\mu)}.$$

Proof Assume that $\mu(S) < \infty$ and $\nu(\Sigma) < \infty$. Let $f \in L^p(\mathbb{R}^+, \mu)$, $1 \leq p \leq 2$ and $q = \frac{p}{p-1}$. From (7)

$$\mathcal{F}(F_\Sigma E_S f) = \chi_\Sigma \mathcal{F}(E_S f)$$

thus

$$\|\mathcal{F}(F_\Sigma E_S f)\|_{L^q(\mathbb{R}^+,\nu)} = \left(\int_\Sigma |\mathcal{F}(E_S f)(x)|^q \mathrm{d}\nu(x)\right)^{\frac{1}{q}} \tag{16}$$

so

$$\mathcal{F}(E_S f)(x) = \int_S f(x)\varphi_{-\xi}(x)\mathrm{d}\mu(x)$$

since $|\varphi_\lambda(x)| \leq 1$ and by Hölder's inequality

$$\begin{aligned} |\mathcal{F}(E_S f)(x)| &\leq \int_S |f(x)|\mathrm{d}\mu(x) \\ &\leq (\mu(S))^{\frac{1}{q}} \|f\|_{L^p(\mathbb{R}^+,\mu)} \end{aligned}$$

Then, by (16)

$$\|\mathcal{F}(F_\Sigma E_S f)\|_{L^q(\mathbb{R}^+,\nu)} \leq (\mu(S))^{\frac{1}{q}} (\nu(\Sigma))^{\frac{1}{q}} \|f\|_{L^p(\mathbb{R}^+,\mu)}.$$

Thus, the proof is complete. □

Theorem 6 *Let S and Σ be a measurable subsets of $\mathbb{R}^+$ and $f \in L^p(\mathbb{R}^+,\mu)$, $1 < p \leq 2$. If f is ε_S-concentrated to S in L^p-norm and $\mathcal{F}(f)$ is ε_Σ-concentrated to Σ in $L^q(\mathbb{R}^+,\nu)$-norm, then*

$$\|\mathcal{F}(f)\|_{L^q(\mathbb{R}^+,\nu)} \leq \frac{(\mu(S))^{\frac{1}{q}}\,(\nu(\Sigma))^{\frac{1}{q}} + \varepsilon_S}{1-\varepsilon_\Sigma}\|f\|_{L^p(\mathbb{R}^+,\mu)}$$

Proof Let $f \in L^p(\mathbb{R}^+,\mu)$, $1 \leq p \leq 2$ and $q = \frac{p}{p-1}$. From (8), (14) and Lemma 3, it follows that

$$\begin{aligned}\|\mathcal{F}(f)-\mathcal{F}(F_\Sigma E_S f)\|_{L^q(\mathbb{R}^+,\nu)} &\leq \|\mathcal{F}(f)-\mathcal{F}(F_\Sigma f)\|_{L^q(\mathbb{R}^+,\nu)}\\ &\quad+\|\mathcal{F}(F_\Sigma f)-\mathcal{F}(F_\Sigma E_S f)\|_{L^q(\mathbb{R}^+,\nu)}\\ &\leq \varepsilon_\Sigma\|\mathcal{F}(f)\|_{L^q(\mathbb{R}^+,\nu)} + \|f-E_S f\|_{L^p(\mathbb{R}^+,\mu)}\\ &\leq \varepsilon_\Sigma\|\mathcal{F}(f)\|_{L^q(\mathbb{R}^+,\nu)} + \varepsilon_S\|f\|_{L^p(\mathbb{R}^+,\mu)}.\end{aligned}$$

The triangle inequality and the Lemma 4 show that

$$\begin{aligned}\|\mathcal{F}(f)\|_{L^q(\mathbb{R}^+,\nu)} &\leq \|\mathcal{F}(F_\Sigma E_S f)\|_{L^q(\mathbb{R}^+,\nu)} + \|\mathcal{F}(f)-\mathcal{F}(F_\Sigma E_S f)\|_{L^q(\mathbb{R}^+,\nu)}\\ &\leq \left[(\mu(S))^{\frac{1}{q}}\,(\nu(\Sigma))^{\frac{1}{q}} + \varepsilon_S\right]\|f\|_{L^p(\mathbb{R}^+,\mu)} + \varepsilon_\Sigma\|\mathcal{F}(f)\|_{L^q(\mathbb{R}^+,\nu)}\end{aligned}$$

which gives the desired result. □

Next the second continuous-time uncertainty principle of concentrated type for the $L^1(\mathbb{R}^+,\mu)\cap L^p(\mathbb{R}^+,\mu)$ theory is given by the following theorem.

Theorem 7 *Let S and Σ be a measurable subsets of $\mathbb{R}^+$ and $f \in L^1(\mathbb{R}^+,\mu)\cap L^p(\mathbb{R}^+,\mu)$, $1 < p \leq 2$. If f is ε_S-concentrated to S in $L^1(\mathbb{R}^+,\mu)$-norm and $\mathcal{F}(f)$ is ε_Σ-concentrated to Σ in $L^q(\mathbb{R}^+,\nu)$-norm, then*

$$\|\mathcal{F}(f)\|_{L^q(\mathbb{R}^+,\nu)} \leq \frac{(\mu(S))^{\frac{1}{p}}\,(\nu(\Sigma))^{\frac{1}{q}}}{(1-\varepsilon_S)(1-\varepsilon_\Sigma)}\|f\|_{L^p(\mathbb{R}^+,\mu)}.$$

Proof Let $f \in L^1(\mathbb{R}^+,\mu)\cap L^p(\mathbb{R}^+,\mu)$, $1 < p \leq 2$. Since $\mathcal{F}(f)$ is ε_S-concentrated to Σ in $L^q(\mathbb{R}^+,\nu)$-norm, $q = \frac{p}{p-1}$, then

$$\begin{aligned}\|\mathcal{F}(f)\|_{L^q(\mathbb{R}^+,\nu)} &\leq \varepsilon_\Sigma\|\mathcal{F}(f)\|_{L^q(\mathbb{R}^+,\nu)} + \left(\int_\Sigma |\mathcal{F}(f)|^q \mathrm{d}\nu(x)\right)^{\frac{1}{q}}\\ &\leq \varepsilon_\Sigma\|\mathcal{F}(f)\|_{L^q(\mathbb{R}^+,\nu)} + (\nu(\Sigma))^{\frac{1}{q}}\,\|\mathcal{F}(f)\|_{L^\infty(\mathbb{R}^+,\nu)}\end{aligned}$$

thus by $|\varphi_\lambda(x)| \leq 1$, we have

$$\|\mathcal{F}(f)\|_{L^q(\mathbb{R}^+,\nu)} \leq \frac{(\nu(\Sigma))^{\frac{1}{q}}}{1-\varepsilon_\Sigma}\|f\|_{L^1(\mathbb{R}^+,\mu)}. \tag{17}$$

On the other hand, since f is ε_S-concentrated to S in L^1-norm

$$\begin{aligned}\|f\|_{L^1(\mathbb{R}^+,\mu)} &\leq \varepsilon_S\|f\|_{L^1(\mathbb{R}^+,\mu)} + \int_S |f(t)|\mathrm{d}\mu(t)\\ &\leq \varepsilon_S\|f\|_{L^1(\mathbb{R}^+,\mu)} + (\mu(S))^{\frac{1}{p}}\,\|f\|_{L^p(\mathbb{R}^+,\mu)}\end{aligned}$$

thus

$$\|f\|_{L^1(\mathbb{R}^+,\mu)} \leq \frac{(\mu(S))^{\frac{1}{p}}}{1-\varepsilon_S}\|f\|_{L^p(\mathbb{R}^+,\mu)}. \tag{18}$$

Combining (17) and (18), we obtain the result of this theorem. □

Let $E^p(F)$, $1 \leq p \leq 2$, be the set of functions $g \in L^p(\mathbb{R}^+,\mu)$ that are bandlimited to Σ i.e ($g \in E^p(F)$ implies $F_\Sigma g = g$).

We say that f is ε-bandlimited to Σ in L^p-norm if there is a $g \in E^p(F)$ with

$$\|f-g\|_{L^p(\mathbb{R}^+,\mu)} \leq \varepsilon_\Sigma\|f\|_{L^p(\mathbb{R}^+,\mu)}.$$

Then, the space $E^p(F)$ satisfies the following property.

Lemma 5 *Let S and Σ be measurable subsets of $\mathbb{R}^+$. For $g \in E^p(F)$, $1 \leq p \leq 2$,*

$$\|E_S g\|_{L^p(\mathbb{R}^+,\mu)} \leq (\mu(S))^{\frac{1}{p}}\,(\nu(\Sigma))^{\frac{1}{p}}\,\|g\|_{L^p(\mathbb{R}^+,\mu)}$$

Proof If $\mu(S)=\infty$ or $\nu(\Sigma)=\infty$ the inequality is clear. Assume that $\mu(S)<\infty$ and $\nu(\Sigma)<\infty$. For $g \in E^p(F)$, $1 \leq p \leq 2$ from Theorem 5

$$\begin{aligned}|g(t)| &= \int_\Sigma \varphi_{-t}(x)\mathcal{F}(g)(x)\mathrm{d}\nu(x)\\ &\leq \int_\Sigma |\mathcal{F}(g)(x)|\mathrm{d}\nu(x)\end{aligned}$$

by $|\varphi_\lambda(x)| \leq 1$ and Hölder inequality

$$|g(t)| \leq (\nu(\Sigma))^{\frac{1}{p}}\,\|\mathcal{F}(g)\|_{L^q(\mathbb{R}^+,\nu)} \leq (\nu(\Sigma))^{\frac{1}{p}}\,\|g\|_{L^p(\mathbb{R}^+,\mu)},\ q=\frac{p}{p-1},$$

Hence,

$$\|E_S g\|_{L^p(\mathbb{R}^+,\mu)} = \left(\int_S |g(t)|^p\mathrm{d}\mu(t)\right)^{\frac{1}{p}} \leq (\mu(S))^{\frac{1}{p}}\,(\nu(\Sigma))^{\frac{1}{p}}\,\|g\|_{L^p(\mathbb{R}^+,\mu)}$$

which yields the result. □

Theorem 8 *Let S and Σ be measurable subsets of $\mathbb{R}^+$ and $f \in L^p(\mathbb{R}^+, \mu)$, $1 \leq p \leq 2$. If f is ε_Σ-bandlimited to Σ in $L^p(\mathbb{R}^+, \mu)$-norm then*

$$\|E_S g\|_{L^p(\mathbb{R}^+,\mu)} \leq \left[(1+\varepsilon_\Sigma)(\mu(S))^{\frac{1}{p}}(\nu(\Sigma))^{\frac{1}{p}} + \varepsilon_\Sigma\right]\|f\|_{L^p(\mathbb{R}^+,\mu)}.$$

Proof Let $f \in L^p(\mathbb{R}^+, \mu)$, $1 \leq p \leq 2$. Since f is ε_Σ-bandlimited to Σ in $L^p(\mathbb{R}^+, \mu)$-norm by definition there is a $g \in E(F)$ with $\|f - g\|_{L^p(\mathbb{R}^+,\mu)} \leq \varepsilon_\Sigma \|f\|_{L^p(\mathbb{R}^+,\mu)}$. For this g, we have

$$\|E_S f\|_{L^p(\mathbb{R}^+,\mu)} \leq \|E_S g\|_{L^p(\mathbb{R}^+,\mu)} + \|E_S(f-g)\|_{L^p(\mathbb{R}^+,\mu)} \leq \|E_S g\|_{L^p(\mathbb{R}^+,\mu)} + \varepsilon_\Sigma \|f\|_{L^p(\mathbb{R}^+,\mu)}.$$

Then, by Lemma 5 and the fact that $\|g\|_{L^p(\mathbb{R}^+,\mu)} \leq (1+\varepsilon_\Sigma)\|f\|_{L^p(\mathbb{R}^+,\mu)}$, we get the result. □

Corollary 2 *Let S and Σ be measurable subsets of $\mathbb{R}^+$ and $f \in L^p$, $1 \leq p \leq 2$. If f is ε_S-concentrated to S and ε_Σ-bandlimited to Σ in L^p-norm then*

$$\frac{1-\varepsilon_S-\varepsilon_\Sigma}{1+\varepsilon_\Sigma} \leq (\mu(S))^{\frac{1}{p}}(\nu(\Sigma))^{\frac{1}{p}}.$$

Proof Let $f \in L^p(\mathbb{R}^+, \mu)$, $1 \leq p \leq 2$. Since f is ε_S-concentrated to S in L^p-norm then by (8)

$$\|f\|_{L^p(\mathbb{R}^+,\mu)} \leq \varepsilon_S \|f\|_{L^p(\mathbb{R}^+,\mu)} + \|E_S f\|_{L^p(\mathbb{R}^+,\mu)}.$$

Thus,

$$\|f\|_{L^p(\mathbb{R}^+,\mu)} \leq \frac{1}{1-\varepsilon_S}\|E_S f\|_{L^p(\mathbb{R}^+,\mu)}.$$

By Lemma 5 and Theorem 8, we deduce the desired inequality. □

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