



Properness of Polynomial Maps with Newton Polyhedra

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Abstract

We discuss the notion of properness of a polynomial map $f : \mathbb{K}^m \rightarrow \mathbb{K}^n$, $\mathbb{K} = \mathbb{C}$ or \mathbb{R} , at a point of the target. We present a method to describe the set of non-proper points of f with respect to Newton polyhedra of f . We obtain an explicit precise description of such a set of f when f satisfies certain condition (1.5). A relative version is also given in Sect. 3. Several tricks to describe the set of non-proper points of f without the condition (1.5) is also given in Sect. 5.

Keywords Polynomial map · Proper map · Newton polyhedra

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We consider a polynomial map $f = (f^1, \dots, f^n) : \mathbb{K}^m \rightarrow \mathbb{K}^n$, defined by

$$f^j = \sum_{\mathbf{v}} c_{\mathbf{v}}^j \mathbf{x}^{\mathbf{v}}, \quad c_{\mathbf{v}}^j \in \mathbb{K}, \quad \mathbf{x}^{\mathbf{v}} = (x_1)^{v_1} \cdots (x_m)^{v_m},$$
$$\mathbf{x} = (x_1, \dots, x_m), \quad \mathbf{v} = (v_1, \dots, v_m), \quad (0.1)$$

where $\mathbb{K} = \mathbb{C}$ or \mathbb{R} . We say that a point $y_0 \in \mathbb{K}^n$ is **proper** for f (or a **proper point** of f) if, for any (algebraic) arc $\mathbf{x}(t) : \mathbb{K}^*, 0 \rightarrow \mathbb{K}^m$, $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$, the following condition holds:

$$\lim_{t \rightarrow 0} f(\mathbf{x}(t)) = y_0 \implies \lim_{t \rightarrow 0} \mathbf{x}(t) \text{ exists in } \mathbb{K}^m.$$

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Let S_f denote the set of points y_0 in \mathbb{K}^n which are not proper points of f . We say that $f : \mathbb{K}^m \rightarrow \mathbb{K}^n$ is **proper** if $S_f = \emptyset$.

In this paper, we are looking for a method to determine whether a point y_0 in \mathbb{K}^n is proper or not. The first statement is Theorem 1.3, which gives a complete description of S_f when f satisfies certain non-degeneracy condition with respect to the Newton polyhedron of f (see (1.5) in Theorem 1.3). Our approach is based on simple and careful analysis of f along arcs $x(t)$, which suggests us usefulness of using arcs to describe the set S_f , even though f is degenerate (Remark 2.6). In Sect. 3, we describe a relative version of our discussion. We present several examples to show how our method works in Sect. 4.

The set S_f was introduced by Jelonek [4], [5] and showed that it is empty or a uniruled hypersurface of \mathbb{K}^n when $\mathbb{K} = \mathbb{C}$ and $m = n$. It is thus an interesting problem to seek a method to describe S_f in several concrete examples. Chen et al. [2] have investigated the bifurcation locus of a polynomial map $\mathbb{K}^m \rightarrow \mathbb{K}^n, m \geq n$, with respect to Newton polyhedron. The bifurcation locus is the minimal locus in the target where the map is not locally trivial, and they show a subset of the bifurcation locus under their non-degeneracy condition. Jelonek and Lason [6] called S_f as the non-properness set of f and showed that it is covered by parametric curves of degree at most $d - 1$ where d is the algebraic degree of f for $\mathbb{K} = \mathbb{C}$. Their words ‘‘covered by parametric curves’’ mean that the set S_f has a ‘‘ \mathbb{C} -ruling’’. They also discuss real counterpart of their results. Recently, El Hilany [3] has investigated to describe the set S_f via the Newton polyhedra of f . He calls S_f as Jelonek set. He has introduced the notion of T -BG maps and claimed that S_f is described using only the data of f at several faces of its Newton polyhedra. Comparing with these results, our method provides much precise information on the set S_f with simple description. For example, Theorem 1.3 shows an explicit decomposition of S_f providing an explicit ruling of each component in many cases. In Sect. 3, we present a relative version of our theorem. Namely, we consider the non-properness set $S_{f|_X}$ for $f|_X : X \rightarrow \mathbb{K}^n$ where $f = (f^1, \dots, f^n) : \mathbb{K}^m \rightarrow \mathbb{K}^n$ is a certain polynomial map and $X = (f^{n-k+1}, \dots, f^n)^{-1}(c), c \in \mathbb{K}^k$. In Sect. 5, we present tricks to describe S_f for certain degenerate f .

We say some words for the definition of S_f here. We compactify f as $\bar{f} : X \rightarrow Y$ where X and Y are suitable projective manifolds. We set $X_\infty = X \setminus \mathbb{K}^m, Y_\infty = Y \setminus \mathbb{K}^n$ and we can assume that X_∞ and Y_∞ are simple normal crossing divisors. Then, the condition $y_0 \in S_f$ is equivalent to one of the following conditions.

- There exists an algebraic arc $x(t) : \mathbb{K}, 0 \rightarrow X$, such that

$$\lim_{t \rightarrow 0} x(t) \in X_\infty, \quad \text{and} \quad \lim_{t \rightarrow 0} f(x(t)) = y_0. \tag{0.2}$$

- There exists an analytic arc $x(t) : \mathbb{K}, 0 \rightarrow X$ defined near 0 with (0.2).
- There exists a sequence $\{x_k\}$ in X , such that $\lim_{k \rightarrow \infty} x_k \in X_\infty$ and $\lim_{k \rightarrow \infty} f(x_k) = y_0$.

The last condition is equivalent to the condition that y_0 is not a proper point of f as a continuous map between metric spaces. We also have

$$S_f = \bar{f}(X_\infty) \cap \mathbb{K}^n = \bar{f}(X_\infty) \cap (Y \setminus Y_\infty).$$

Since \tilde{f} is proper, the set $\tilde{f}(X_\infty)$ is closed in Y and we obtain that S_f is closed.

When $\mathbb{K} = \mathbb{C}$ and $m > n$, Noether’s normalization asserts that, for any $\mathbf{y} \in \mathbb{C}^n$, there is a linear surjection $p : \mathbf{f}^{-1}(\mathbf{y}) \rightarrow \mathbb{C}^d, 0 \leq d \leq m$, where $d = \dim_{\mathbb{C}} \mathbf{f}^{-1}(\mathbf{y})$. If $\mathbf{f}^{-1}(\mathbf{y})$ is compact, then $p(\mathbf{f}^{-1}(\mathbf{y})) = \mathbb{C}^d$ is compact and we obtain $d = 0$. Since $d \geq m - n$, we conclude that $m \leq n$. This implies that S_f is the closure of the image of \mathbf{f} whenever $m > n$. Therefore, we assume $m \leq n$ when $\mathbb{K} = \mathbb{C}$.

When $\mathbb{K} = \mathbb{C}$, Jelonek’s result asserts that S_f is Zariski closed. However, if $\mathbb{K} = \mathbb{R}$, S_f may not be Zariski closed (for example, $S_f = \{(0, y_2) \in \mathbb{R}^2 : y_2 \geq 0\}$ for $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2, (x_1, x_2) \mapsto (x_1, x_1^2 x_2^2)$).

Throughout the paper, we use the following notational convention:

$$\mathbb{K}^J = \{(x_1, \dots, x_n) \in \mathbb{K}^n : x_i = 0, i \notin J\}, \mathbb{Z}^J = \{(v_1, \dots, v_n) \in \mathbb{Z}^n : v_i = 0, i \notin J\}.$$

for a subset J of $\{1, \dots, n\}$. We set $\mathbb{Z}_{\geq 0}^J = \{(v_1, \dots, v_n) \in \mathbb{Z}^J : v_i \geq 0, i \in J\}$. We also set $(\mathbb{Z}_{\geq 0})^n = \{(v_1, \dots, v_n) \in \mathbb{Z}^n : v_i \geq 0, i = 1, \dots, n\}$. We often abbreviate $\mathbb{Z}_{\geq 0}$ as \mathbb{Z}_{\geq} following custom. We identify \mathbb{K}^n with $\mathbb{K}^J \times \mathbb{K}^{J^c}$ where $J^c = \{1, \dots, n\} \setminus J$ without notice.

1 Newton Polyhedra

Let $\Delta(\mathbf{f}^j)$ denote Newton polyhedron of \mathbf{f}^j , the convex hull of the set $\{\mathbf{v} : c_{\mathbf{v}}^j \neq 0\}$, under the notation in (0.1). For $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$, we define

$$d_j(\mathbf{p}) = -\min\{\langle \mathbf{p}, \mathbf{v} \rangle : \mathbf{v} \in \Delta(\mathbf{f}^j)\}, \tag{1.1}$$

$$\gamma_j(\mathbf{p}) = \{\mathbf{v} \in \Delta(\mathbf{f}^j) : \langle \mathbf{p}, \mathbf{v} \rangle = -d_j(\mathbf{p})\}. \tag{1.2}$$

We call $\gamma_j(\mathbf{p})$ the face of $\Delta(\mathbf{f}^j)$ supported by \mathbf{p} .

We say $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$ which is a face of $\boldsymbol{\Delta}(\mathbf{f}) = (\Delta(\mathbf{f}^1), \dots, \Delta(\mathbf{f}^n))$ if there exist $\mathbf{p} \in \mathbb{Z}^m$, so that γ_j is a face of $\Delta(\mathbf{f}^j)$ supported by \mathbf{p} . We denote

$$\boldsymbol{\gamma}(\mathbf{p}) = (\gamma_1(\mathbf{p}), \dots, \gamma_n(\mathbf{p})).$$

When we need to mention \mathbf{f} explicitly, we denote them by $\boldsymbol{\gamma}(\mathbf{f}; \mathbf{p}), \gamma_j(\mathbf{f}; \mathbf{p})$, and so on. We consider Minkowski sum $\Delta(\mathbf{f}) = \Delta(\mathbf{f}^1) + \dots + \Delta(\mathbf{f}^n)$ and its dual fan Δ^* , which we identify with the set of polyhedral cones. Note that $\gamma(\mathbf{p}) = \gamma_1(\mathbf{p}) + \dots + \gamma_n(\mathbf{p})$ is a face of $\Delta(\mathbf{f})$. We denote

$$\mathbf{f}_{\boldsymbol{\gamma}} = (f_{\gamma_1}^1, \dots, f_{\gamma_n}^n) \text{ where } f_{\gamma_j}^j = \sum_{\mathbf{v} \in \gamma_j} c_{\mathbf{v}}^j x^{\mathbf{v}}.$$

Lemma 1.1 $\boldsymbol{\gamma}(\mathbf{p}) = \boldsymbol{\gamma}(\mathbf{q}) \iff \gamma(\mathbf{p}) = \gamma(\mathbf{q})$.

Proof “ \implies ” part is clear, since “ $\boldsymbol{\gamma}(\mathbf{p}) = \boldsymbol{\gamma}(\mathbf{q}) \iff \gamma_j(\mathbf{p}) = \gamma_j(\mathbf{q}) (j = 1, \dots, n)$ ”.

Take $\mathbf{v} \in \gamma(\mathbf{q})$, so that $\mathbf{v} = \mathbf{v}_1 + \dots + \mathbf{v}_n$ for $\mathbf{v}_j \in \gamma_j(\mathbf{q})$. Since $\gamma_j(\mathbf{q}) \subset \Delta(f^j)$, we have $-d_j(\mathbf{p}) \leq \langle \mathbf{p}, \mathbf{v}_j \rangle$. If we assume $\gamma(\mathbf{q}) \subset \gamma(\mathbf{p})$, we then have

$$-\sum_{j=1}^n d_j(\mathbf{p}) \leq \sum_{j=1}^n \langle \mathbf{p}, \mathbf{v}_j \rangle = \langle \mathbf{p}, \mathbf{v} \rangle = -\sum_{j=1}^n d_j(\mathbf{p}),$$

and $\langle \mathbf{p}, \mathbf{v}_j \rangle = -d_j(\mathbf{p})$, that is, $\mathbf{v}_j \in \gamma_j(\mathbf{p})$. We conclude $\gamma_j(\mathbf{q}) \subset \gamma_j(\mathbf{p})$. By symmetry, we complete the proof of “ \Leftarrow ”. □

Compositing f with a translation of the target, the set S_f is changed by its translation. Without loss of generality, we thus can assume the following condition:

$$f^j (j = 1, \dots, n) \text{ are non-constant polynomials with non-zero constant terms.} \tag{1.3}$$

Throughout the paper, we assume the condition (1.3) unless otherwise stated.

The condition (1.3) implies that $d_j(\mathbf{p}) \geq 0$ and equality holds if $\mathbf{p} \in (\mathbb{Z}_{\geq 0})^n$. For a face $\boldsymbol{\gamma}$ of $\Delta(f)$, we set $J_{\boldsymbol{\gamma}} = \{j : 0 \notin \gamma_j\}$. We remark that

$$d_j(\mathbf{p}) > 0 \iff j \in J_{\boldsymbol{\gamma}} \text{ for } \mathbf{p} \text{ with } \boldsymbol{\gamma}(\mathbf{p}) = \boldsymbol{\gamma}. \tag{1.4}$$

Definition 1.2 We say a face $\boldsymbol{\gamma}$ of $\Delta(f)$ is **non-coordinate** if there is $\mathbf{p} \in \mathbb{Z}^m \setminus (\mathbb{Z}_{\geq 0})^m$, so that $\boldsymbol{\gamma} = \boldsymbol{\gamma}(\mathbf{p})$. Let $\Delta_{\text{nc}}(f)$ denote the set of non-coordinate faces of $\Delta(f)$.

For a polynomial map $\mathbf{g} = (g^1, \dots, g^r) : \mathbb{K}^m \rightarrow \mathbb{K}^r$, we set

$$\begin{aligned} Z(\mathbf{g}) &= \{\mathbf{x} \in (\mathbb{K}^*)^m : g^j(\mathbf{x}) = 0 (j = 1, \dots, r)\}, \\ \Sigma(\mathbf{g}) &= \{\mathbf{x} \in (\mathbb{K}^*)^m : \text{rank Jac}(\mathbf{g})(\mathbf{x}) < r\}, \end{aligned}$$

where $\text{Jac}(\mathbf{g}) = (\partial_{x_i} g^j)_{i=1, \dots, m; j=1, \dots, r}$. Remark that the codimension of $Z(\mathbf{g}) \setminus \Sigma(\mathbf{g})$ is r .

Theorem 1.3 Assume that f is a polynomial map with (1.3) and

$$Z(f_{\boldsymbol{\gamma}}^J) \setminus \Sigma(f_{\boldsymbol{\gamma}}^J) \text{ is dense in } Z(f_{\boldsymbol{\gamma}}^J) \tag{1.5}$$

for any $\boldsymbol{\gamma} \in \Delta_{\text{nc}}(f)$ where $J = J_{\boldsymbol{\gamma}}$. We have

$$S_f = \bigcup_{\boldsymbol{\gamma} \in \Delta_{\text{nc}}(f)} S_{\boldsymbol{\gamma}}(f),$$

where $S_{\boldsymbol{\gamma}}(f) = f_{\boldsymbol{\gamma}}^{J^c}(Z(f_{\boldsymbol{\gamma}}^J)) \times \mathbb{K}^J$, $J = J_{\boldsymbol{\gamma}}$, and $f_{\boldsymbol{\gamma}}^{J^c} : (\mathbb{K}^*)^m \rightarrow \mathbb{K}^{J^c}$ is the map defined by $\mathbf{x} \mapsto (f_{\gamma_j}^j(\mathbf{x}))_{j \notin J}$.

We often say that $Z(f_{\boldsymbol{\gamma}}^J)$ has the dense nonsingular locus if the condition (1.5) holds.

When $J = \{1, \dots, n\}$, we have $f_{\mathbf{y}}^{J^c} : (\mathbb{K}^*)^m \rightarrow \mathbb{K}^{J^c}$ is a constant map, since \mathbb{K}^\emptyset is a one-point set.

Remark 1.4 Chen et al. [2] said that f is non-degenerate if $Z(f_{\mathbf{y}}^J) \cap \Sigma(f_{\mathbf{y}}^J) = \emptyset$ for all $\mathbf{y} \in \Delta_{nc}(f)$. This implies (1.5) for all $\mathbf{y} \in \Delta_{nc}(f)$. However, our condition (1.5) is weaker than their non-degeneracy condition.

Remark 1.5 If $J_{\mathbf{y}} = \{1, \dots, n\}$, the condition (1.5) implies that $Z_{\mathbf{y}} = Z(f_{\mathbf{y}}^{J_{\mathbf{y}}})$ is empty. In fact, if we take a nonsingular point $\mathbf{x} \in Z_{\mathbf{y}}$, then the condition (1.5) implies that $Z_{\mathbf{y}}$ is of codimension n at \mathbf{x} . This implies that \mathbf{x} is isolated in $Z_{\mathbf{y}}$. However, this is impossible, since $f_{\mathbf{y}_j}^j$ is weighted homogeneous with respect to the weight \mathbf{p} .

We also remark that $Z_{\mathbf{y}} = (\mathbb{K}^*)^m$ when $J_{\mathbf{y}} = \emptyset$.

Corollary 1.6 A polynomial map f with (1.3) is proper, if for any $\mathbf{y} \in \Delta_{nc}(f)$ none of γ_j , $j = 1, \dots, n$, contains the origin and $Z(f_{\mathbf{y}})$ has a dense nonsingular locus for any $\mathbf{y} \in \Delta_{nc}(f)$.

Remark 1.7 When $k = \dim \gamma_J = \sum_{j \in J} \dim \gamma_j$, $J = J_{\mathbf{y}}$, $f_{\mathbf{y}}^J$ is a system of polynomials of k Laurent monomials of \mathbf{x} and $Z_{\mathbf{y}} = Z(f_{\mathbf{y}}^{J_{\mathbf{y}}})$ is isomorphic to $X \times (\mathbb{K}^*)^{n-k}$ for some algebraic variety X in $(\mathbb{K}^*)^k$. If $j \in J_{\mathbf{y}}$, $d_j = 0$ and $f^j(x)$ ($j \in J$) is invariant under the natural \mathbb{K}^* -action(s). Thus, $f^J(Z_{\mathbf{y}}) = f^J(X \times \{(1, \dots, 1)\})$. When $f_{\mathbf{y}}^J$ is complete intersection, we have that

$$\dim f_{\mathbf{y}}^{J^c}(Z_{\mathbf{y}}) = \dim X - \dim F = k - \#J - \dim F,$$

where F is a suitable fiber of $f_{\mathbf{y}}^{J^c} : X \rightarrow \mathbb{K}^{J^c}$. We thus have $\dim S_{\mathbf{y}} = k - \dim F$. When $\mathbb{K} = \mathbb{C}$, S_f is a hypersurface and $\bigcup_{\mathbf{y} \in \Delta_{nc}(f)} S_{\mathbf{y}}$ should be the union of the closures of $S_{\mathbf{y}}$ with $\dim \mathbf{y} = n - 1$ (and $\dim F = 0$).

To prove Theorem 1.3, we actually show the following.

Theorem 1.8 If a polynomial map $f : \mathbb{K}^m \rightarrow \mathbb{K}^n$ satisfies (1.3), then

$$\bigcup_{\mathbf{y} \in \Delta_{nc}(f)} S'_{\mathbf{y}}(f) \subset S_f \subset \bigcup_{\mathbf{y} \in \Delta_{nc}(f)} S_{\mathbf{y}}(f), \tag{1.6}$$

where $S'_{\mathbf{y}}(f) = f_{\mathbf{y}}^{J^c}(\overline{Z(f_{\mathbf{y}}^J) \setminus \Sigma(f_{\mathbf{y}}^J)}) \times \mathbb{K}^J$, $J = J_{\mathbf{y}}$. Here, \overline{Z} denotes the closure of a set Z .

Remark 1.9 Assume that $m = n = 2$. Take $\mathbf{y} = (\gamma_1, \gamma_2) \in \Delta_{nc}(f)$ for a non-degenerate map $f : \mathbb{K}^2 \rightarrow \mathbb{K}^2$. We assume that $\mathbf{y} = \mathbf{y}(\mathbf{p})$, $\mathbf{p} = (p_1, p_2)$ with $p_1 < 0$ and $p_2 > 0$.

- If $0 \in \gamma_1$ and $0 \in \gamma_2$, then $f_{\gamma_1}^1$ and $f_{\gamma_2}^2$ are polynomial of a monomial $u = x_1^{q_1} x_2^{q_2}$. We denote them as $g_1(u)$ and $g_2(u)$. The defining equation of $S_{\mathbf{y}}$ is the resultant of $g_1 - y_1$ and $g_2 - y_2$ where (y_1, y_2) is a coordinate system of the target.
- If $0 \notin \gamma_1$ and $0 \in \gamma_2$, then we can write $f_{\gamma_1}^1 = x_1^p g_1(u)$ and $f_{\gamma_2}^2 = g_2(u)$ with $u = x_1^{q_1} x_2^{q_2}$ similarly. The defining equation of $S_{\mathbf{y}}$ is the resultant of g_1 and $g_2 - y_2$.

2 Proof of Theorem 1.8

We are going to evaluate $\mathbf{f}(x) = (f^1(x), \dots, f^n(x))$ along a curve $\mathbf{x}(t)$ defined by

$$\mathbf{x}(t) = (t^{p_1} v^1(t), \dots, t^{p_m} v^m(t)) \quad \text{where } \mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m, \tag{2.1}$$

$$\mathbf{v}(t) = (v^1(t), \dots, v^m(t)) = \sum_{i=0}^{\infty} \mathbf{v}_i t^i, \quad \mathbf{v}_i = (v_i^1, \dots, v_i^m), \quad \mathbf{v}_0 \in (\mathbb{K}^*)^m. \tag{2.2}$$

We denote $\mathcal{A}(\mathbf{p})$ the set of such arcs. It is clear that

$$\lim_{t \rightarrow 0} \mathbf{x}(t) = \infty \iff \mathbf{p} \notin (\mathbb{Z}_{\geq 0})^m. \tag{2.3}$$

We have an obvious decomposition $S_{\mathbf{f}} = \bigcup_{\mathbf{p}} S_{\mathbf{f}}(\mathbf{p})$, where

$$S_{\mathbf{f}}(\mathbf{p}) = \{\mathbf{y} \in \mathbb{K}^n : \exists \mathbf{x}(t) \in \mathcal{A}(\mathbf{p}), \lim_{t \rightarrow 0} \mathbf{x}(t) = \infty, \lim_{t \rightarrow 0} \mathbf{f}(\mathbf{x}(t)) = \mathbf{y}\}. \tag{2.4}$$

We have $S_{\mathbf{f}}(\mathbf{p}) = \emptyset$ if $\mathbf{p} \in (\mathbb{Z}_{\geq 0})^m$ by (2.3).

Remark 2.1 Observe that the arcs having several components being identically zero are not in $\mathcal{A}(\mathbf{p})$. However, this does not affect to detect $S_{\mathbf{f}}$. Adding the terms t^l , $l \gg 1$, to such components does not affect the conditions for $S_{\mathbf{f}}$ and we can restrict our attention to $\mathcal{A}(\mathbf{p})$.

Lemma 2.2 For $\mathbf{p} \notin (\mathbb{Z}_{\geq 0})^m$, we have $S_{\mathbf{f}}(\mathbf{p}) \subset S_{\mathbf{y}}(\mathbf{p})(\mathbf{f})$.

Proof We express $f^j(\mathbf{x}(t))$ as

$$f^j(\mathbf{x}(t)) = t^{-d_j} (\hat{f}_0^j + \hat{f}_1^j t + \dots + \hat{f}_{d_j-1}^j t^{d_j-1} + \hat{f}_{d_j}^j t^{d_j} + o(t^{d_j})), \tag{2.5}$$

where $d_j = d_j(\mathbf{p}) \geq 0$. We have $\hat{f}_0^j = f_{\gamma_j}^j(\mathbf{v}_0)$ where $\gamma_j = \gamma_j(\mathbf{p})$. Setting $f^j(x) = \sum_{\mathbf{v}} c_{\mathbf{v}}^j \mathbf{x}^{\mathbf{v}}$, more precisely, we have

$$f^j(\mathbf{x}(t)) = t^{-d_j(\mathbf{p})} \sum_{\mathbf{v}} c_{\mathbf{v}}^j t^{(\mathbf{p}, \mathbf{v}) + d_j(\mathbf{p})} \mathbf{v}(t)^{\mathbf{v}}. \tag{2.6}$$

If $\mathbf{y} \in S_{\mathbf{f}}(\mathbf{p})$, then there exists an arc $\mathbf{x}(t) \in \mathcal{A}(\mathbf{p})$, so that

$$\lim_{t \rightarrow 0} \mathbf{x}(t) = \infty, \quad \text{and} \quad \lim_{t \rightarrow 0} \mathbf{f}(\mathbf{x}(t)) = \mathbf{y}.$$

Using the notation in (2.5) and $J = J_{\mathbf{y}}(p)$, we have

$$\hat{f}_0^j = \hat{f}_1^j = \dots = \hat{f}_{d_j-1}^j = 0 \quad (j \in J),$$

and $f_{\gamma_j(p)}^j(\mathbf{v}_0) = 0$ ($j \in J$). This implies that J^c component of \mathbf{y} is given by $f_{\mathbf{y}}^{J^c}(\mathbf{v}_0)$, and we complete the proof. \square

Remark 2.3 If $\dim \gamma_j(p) = 0$ for some $j \in J$, then $\hat{f}_0^j \neq 0$, and $S_{\mathbf{y}}(p)$ is empty.

Lemma 2.4 If $\mathbf{y} = \mathbf{y}(p) \in \Delta_{nc}(f)$, then

$$f_{\mathbf{y}}^{J^c}(Z(f_{\mathbf{y}}^J) \setminus \Sigma(f_{\mathbf{y}}^J)) \times \mathbb{K}^J \subset S_f(p) \text{ where } J = J_{\mathbf{y}}(p).$$

Proof Take $p \in \mathbb{Z}^n \setminus (\mathbb{Z}_{\geq})^n$ and consider a curve defined by (2.1). We compare (2.5) with (2.6) substituting by (2.2) and taking modulo t^{i+1} . Remarking that the terms concerning \mathbf{v}_i in \hat{f}_i^j depend on the terms in $f_{\gamma_j}^j$ only, we obtain that

$$\hat{f}_l^j = (df_{\gamma_j}^j)_{\mathbf{v}_0}(\mathbf{v}_l) + r_l^j(\mathbf{v}_0, \dots, \mathbf{v}_{l-1}) \quad (l = 1, 2, \dots), \tag{2.7}$$

where $r_l^j(\mathbf{v}_0, \dots, \mathbf{v}_{l-1})$ is a suitable polynomial of $\mathbf{v}_0, \dots, \mathbf{v}_{l-1}$.

Take a point $\mathbf{v}_0 \in Z(f_{\mathbf{y}}^J) \setminus \Sigma(f_{\mathbf{y}}^J)$ where $J = J_{\mathbf{y}}$. Let $(a_k^j)_{j \in J; k \geq 1}$ be any sequence. Suppose we have already taken $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{l-1}$, so that

$$\hat{f}_k^j = a_k^j \quad (1 \leq k < l, j \in J).$$

By (2.7), there exists \mathbf{v}_l , so that $\hat{f}_l^j = a_l^j$ for $j \in J$, whenever $\text{Jac}((f_{\gamma_j}^j)_{j \in J})$ is of full rank at \mathbf{v}_0 . Choose $(a_i^j)_{j \in J; i \geq 0}$, so that $a_i^j = 0$ ($0 \leq i < d_j$). Then, the corresponding curve $\mathbf{x}(t)$ has the following property:

$$\lim_{t \rightarrow 0} f^j(\mathbf{x}(t)) = \begin{cases} f_{\gamma_j}^j(\mathbf{v}_0) & (j \notin J), \\ a_{d_j}^j & (j \in J). \end{cases}$$

Since one can choose $a_{d_j}^j$ arbitrary, we conclude that $f_{\mathbf{y}}^{J^c}(Z(f_{\mathbf{y}}^J) \setminus \Sigma(f_{\mathbf{y}}^J)) \times \mathbb{K}^J \subset S_f(p)$. \square

In the situation above, we have

Corollary 2.5 $S'_{\mathbf{y}}(f) \subset S_f$ for $\mathbf{y} \in \Delta_{nc}(f)$.

Proof We obtain

$$S'_{\mathbf{y}}(f) = f_{\mathbf{y}}^{J^c}(\overline{Z(f_{\mathbf{y}}^J) \setminus \Sigma(f_{\mathbf{y}}^J)}) \times \mathbb{K}^J \subset \overline{f_{\mathbf{y}}^{J^c}(Z(f_{\mathbf{y}}^J) \setminus \Sigma(f_{\mathbf{y}}^J)) \times \mathbb{K}^J} \subset \overline{S_f} = S_f,$$

since S_f is closed. \square

Remark 2.6 In the case that f does not satisfy (1.5), we would proceed further analysis using higher order differentials of composite maps. Actually, in the expression (2.5), we have

$$\hat{f}_l^j = \sum_{k=0}^l \sum_{i_1+2i_2+\dots+ki_k=k} \frac{1}{i_1! \dots i_k!} (d^{i_1+\dots+i_k} f_{l-k}^j)_{\mathbf{v}_0} (\overbrace{\mathbf{v}_1, \dots, \mathbf{v}_1}^{i_1}, \dots, \overbrace{\mathbf{v}_k, \dots, \mathbf{v}_k}^{i_k}),$$

where

$$f^j(\mathbf{x}) = f_0^j(\mathbf{x}) + f_1^j(\mathbf{x}) + \dots + f_{e_j}^j(\mathbf{x}) \text{ with} \\ f_k^j(t^{p_1}x_1, \dots, t^{p_m}x_m) = t^{-d_j+k} f_k^j(\mathbf{x}). \tag{2.8}$$

Here, we use the notation in (2.1), (2.2), and $d^k g$ denotes the symmetric multilinear form defined by k th-order differential of g . The first few of \hat{f}_l^j are as follows:

$$\begin{aligned} \hat{f}_0^j &= f_0^j(\mathbf{v}_0), \\ \hat{f}_1^j &= f_1^j(\mathbf{v}_0) + (df_0^j)_{\mathbf{v}_0}(\mathbf{v}_1), \\ \hat{f}_2^j &= f_2^j(\mathbf{v}_0) + (df_0^j)_{\mathbf{v}_0}(\mathbf{v}_2) + (df_1^j)_{\mathbf{v}_0}(\mathbf{v}_1) + \frac{1}{2}(d^2 f_0^j)_{\mathbf{v}_0}(\mathbf{v}_1, \mathbf{v}_1), \\ \hat{f}_3^j &= f_3^j(\mathbf{v}_0) + (df_0^j)_{\mathbf{v}_0}(\mathbf{v}_3) + (df_1^j)_{\mathbf{v}_0}(\mathbf{v}_2) + (df_2^j)_{\mathbf{v}_0}(\mathbf{v}_1) \tag{2.9} \\ &\quad + (d^2 f_0^j)_{\mathbf{v}_0}(\mathbf{v}_1, \mathbf{v}_2) + \frac{1}{2}(d^2 f_1^j)_{\mathbf{v}_0}(\mathbf{v}_1, \mathbf{v}_1) + \frac{1}{6}(d^3 f_0^j)_{\mathbf{v}_0}(\mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_1). \tag{2.10} \end{aligned}$$

The set $S_f(\mathbf{p})$ is described by eliminating $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots$ from the following system:

$$0 = \hat{f}_l^j \ (l = 0, 1, 2, \dots, d_j - 1), \quad y_j = \hat{f}_{d_j}^j \ (j = 1, \dots, n),$$

where (y_1, \dots, y_n) denotes the coordinate system of the target.

3 Relative Version

The definition of non-properness set has an obvious generalization for a polynomial map $f : X \rightarrow Y$ between algebraic varieties X and Y defined over \mathbb{K} . We say f is not proper at $y_0 \in Y$ if there exist an arc $\mathbf{x}(t) : \mathbb{K}^*, 0 \rightarrow X$, so that

$$\lim_{t \rightarrow 0} \mathbf{x}(t) \text{ does not exist, and } \lim_{t \rightarrow 0} f(\mathbf{x}(t)) = y_0.$$

We denote by S_f the set of non-proper points of $f : X \rightarrow Y$.

Let $\mathbf{f} = (f^1, \dots, f^n) : \mathbb{K}^m \rightarrow \mathbb{K}^n$ be a polynomial map with (1.3). Set $\mathbf{f}' = (f^1, \dots, f^{n-k})$ and $\mathbf{f}'' = (f^{n-k+1}, \dots, f^n)$. Set $\mathbf{y} = (\mathbf{y}', \mathbf{y}'')$ and $\mathbf{y}_0 = (\mathbf{y}'_0, \mathbf{y}''_0)$. In this section, we describe a generalization of the discussion above to the map

$$\mathbf{f}'|_X : X \rightarrow \mathbb{K}^{n-k} \text{ where } X = (\mathbf{f}'')^{-1}(\mathbf{y}''_0).$$

Since $f|_X = (f'|_X, y''_0)$, we identify $f|_X$ with the map $f'|_X$ via the embedding $\mathbb{K}^{n-k} \times \{y''_0\} \subset \mathbb{K}^n$. This means that we identify \mathbb{K}^{n-k} with $\mathbb{K}^{n-k} \times \{y''_0\}$, and we can identify $f^{\{1, \dots, n-k\} \setminus J}$ with $f^{\{1, \dots, n\} \setminus J}$. We call this map by f^{J^c} . For a face $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$ of $\Delta(f)$, we define $\boldsymbol{\gamma}' = (\gamma_1, \dots, \gamma_{n-k})$ and $\boldsymbol{\gamma}'' = (\gamma_{n-k+1}, \dots, \gamma_n)$. In the same way, we can identify $f^{\{1, \dots, n-k\} \setminus J}$ with $f^{\{1, \dots, n\} \setminus J}$ on the set $Z(f_{\boldsymbol{\gamma}'}, f''_{\boldsymbol{\gamma}''} - y''_0)$. So denote its image by $f_{\boldsymbol{\gamma}'}^{J^c}$ as $f_{\boldsymbol{\gamma}'}^{J^c}(Z(f_{\boldsymbol{\gamma}'}, f''_{\boldsymbol{\gamma}''} - y''_0))$. Set

$$S_{\boldsymbol{\gamma}'; \boldsymbol{\gamma}''} = f_{\boldsymbol{\gamma}'}^{J^c}(Z(f_{\boldsymbol{\gamma}'}, f''_{\boldsymbol{\gamma}''} - y''_0)) \times \mathbb{K}^J, \quad J = J_{\boldsymbol{\gamma}'} = \{j \in \{1, \dots, n-k\} : 0 \notin \gamma_j\},$$

$$\text{and } S'_{\boldsymbol{\gamma}'; \boldsymbol{\gamma}''} = f_{\boldsymbol{\gamma}'}^{J^c}(\overline{Z(f_{\boldsymbol{\gamma}'}, f''_{\boldsymbol{\gamma}''} - y''_0) \setminus \Sigma(f_{\boldsymbol{\gamma}'}, f''_{\boldsymbol{\gamma}''})}) \times \mathbb{K}^J.$$

Under the notation and assumption above, we have the following:

Theorem 3.1 *We assume that the nonsingular locus of X is dense in X and X has no component in $\{x_1 \cdots x_n = 0\}$. Then*

$$\bigcup_{\boldsymbol{\gamma} \in \Delta_{nc}(f)} S'_{\boldsymbol{\gamma}'; \boldsymbol{\gamma}''} \subset S_{f|_X} \subset \bigcup_{\boldsymbol{\gamma} \in \Delta_{nc}(f)} S_{\boldsymbol{\gamma}'; \boldsymbol{\gamma}''}. \tag{3.1}$$

If $Z(f_{\boldsymbol{\gamma}'}, f''_{\boldsymbol{\gamma}''} - y''_0)$ has dense nonsingular loci for $\boldsymbol{\gamma} \in \Delta_{nc}$, we have equalities in (3.1).

The assumption that X has no component in $\{x_1 \cdots x_n = 0\}$ comes from Remark 2.1. If there is an arc $\mathbf{x}(t)$ in $X \cap \mathbb{K}^I, I \subsetneq \{1, \dots, n\}$, with $\mathbf{x}(t) \rightarrow \infty$, and $f(\mathbf{x}(t)) \rightarrow y_0 (t \rightarrow 0)$, one can choose $\hat{\mathbf{x}}(t) \in \mathcal{A}(\mathbf{p})$ for some \mathbf{p} with $\hat{\mathbf{x}}(t) \rightarrow \infty$, and $f(\hat{\mathbf{x}}(t)) \rightarrow y_0 (t \rightarrow 0)$. However, we do not know that $\hat{\mathbf{x}}(t)$ can be chosen in X in such a case.

Proof of Theorem 3.1 First, for $\mathbf{x}(t) \in \mathcal{A}(\mathbf{p})$, we can write

$$\begin{aligned} f^j(\mathbf{x}(t)) &= t^{-d_j}(\hat{f}_0^j + \hat{f}_1^j t + \dots + \hat{f}_{d_j-1}^j t^{d_j-1} \\ &\quad + \hat{f}_{d_j}^j t^{d_j} + o(t^{d_j})) \quad (j = 1, \dots, n-k), \\ f^j(\mathbf{x}(t)) - y_0^j &= t^{-d_j}(\hat{f}_0^j + \hat{f}_1^j t + \dots + \hat{f}_{d_j-1}^j t^{d_j-1} \\ &\quad + \hat{f}_{d_j}^j t^{d_j} + o(t^{d_j})) \quad (j = n-k+1, \dots, n), \end{aligned}$$

where $d_j = d_j(\mathbf{p})$. We remark that $\mathbf{y} \in S_{f|_X}$ if and only if there is an arc $\mathbf{x}(t) : \mathbb{K}^*, 0 \rightarrow X$, so that $\lim_{t \rightarrow 0} \mathbf{x}(t) = \infty$, and that $\lim_{t \rightarrow 0} f(\mathbf{x}(t)) = \mathbf{y}$. In a similar way to the proof of Lemma 2.2, we have

$$\hat{f}_0^j = 0 \quad (j \in J_{\boldsymbol{\gamma}} \cup \{n-k+1, \dots, n\}).$$

This implies that $f_{\gamma_j(\mathbf{p})}^j(\mathbf{v}_0) = 0$ for $j \in J_{\boldsymbol{\gamma}}$, and $(f^j - y_0^j)_{\gamma_j(\mathbf{p})}(\mathbf{v}_0) = 0$ for $j = n-k+1, \dots, n$, which show the second inclusion.

By the discussion similar to the proof of Lemma 2.4, for any a^j ($j \in J$), we can construct a formal power series $\mathbf{v}(t) = (v^1(t), \dots, v^n(t))$, such that

$$f^j(t^{p_1}v^1(t), \dots, t^{p_n}v^n(t)) = a^j + o(t) \quad (j \in J), \tag{3.2}$$

$$f^j(t^{p_1}v^1(t), \dots, t^{p_n}v^n(t)) = y_0^j \quad (j = n - k + 1, \dots, n), \tag{3.3}$$

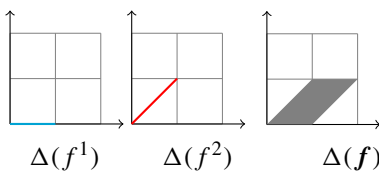
where $\mathbf{y}_0 = (y_0^1, \dots, y_0^n)$. Remark that we can reduce this system to polynomials by multiplying some power t^l . By the approximation theorem of Artin ([1]), we can take a convergent power series $\mathbf{v}(t) = (v^1(t), \dots, v^n(t))$ which satisfies (3.2) and (3.3). This completes the proof of the first inclusion. \square

If X has a component X_1 in $\{x_1 \cdots x_n = 0\}$, we could proceed a similar computation for $f|_{X_1}$ which is a polynomial map with less number of variables and obtain that $S_{f|_{X_1}} \subset S_f|_X$.

4 Examples

Example 4.1 Let us start with the simplest example $\mathbb{K}^2 \rightarrow \mathbb{K}^2$, $f(x, y) = (x, xy)$. For the assumption (1.3) we consider $f(x, y) = (c_1 + x, c_2 + xy)$ where c_1, c_2 are non-zero constants. It suffices to consider only 3 faces below thanks to Remark 2.3. Since

$$\begin{aligned} f_{\mathbf{y}(1,-1)} &= (c_1, c_2 + xy), Z(f_{\mathbf{y}(1,-1)}^{\emptyset}) = (\mathbb{K}^*)^2, \\ f_{\mathbf{y}(0,-1)} &= (c_1 + x, xy), Z(f_{\mathbf{y}(1,-1)}^{\{2\}}) = \emptyset, \\ f_{\mathbf{y}(-1,1)} &= (x, c_2 + xy), Z(f_{\mathbf{y}(1,-1)}^{\{1\}}) = \emptyset, \end{aligned}$$



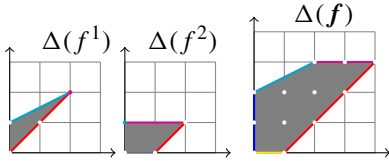
we have $S_f = \{(c_1, c_2 + xy) : (x, y) \in (\mathbb{K}^*)^2\} = \{c_1\} \times \mathbb{K}$.

Example 4.2 Consider the map $f(x, y) = (x^2y^2 + xy + y + 1, x^2y + y + x + 1)$. Since

$$f_{\mathbf{y}}(-1,1) = (x^2y^2 + xy + 1, x(xy + 1)), Z(f_{\mathbf{y}}^{[2]}(1,-1)) = \{xy + 1 = 0\}$$

$$f_{\mathbf{y}}(0,-1) = (x^2y^2, (x^2 + 1)y), Z(f_{\mathbf{y}}^{[1,2]}(0,-1)) = \emptyset$$

$$f_{\mathbf{y}}(1,-2) = (x^2y^2 + y, y), Z(f_{\mathbf{y}}^{[1,2]}(1,-2)) = \emptyset$$



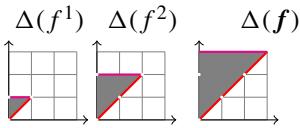
we have

$$S_f = \{x^2y^2 + xy + 1 : xy + 1 = 0\} \times \mathbb{K} = \{t^2 + t + 1 : t + 1 = 0\} \times \mathbb{K} = \{1\} \times \mathbb{K}.$$

Example 4.3 Consider the map $f(x, y) = (xy + y + 1, x^2y^2 + y^2 + xy + 1)$, we have

$$f_{\mathbf{y}}(-1,1) = (xy + 1, x^2y^2 + xy + 1), Z(f_{\mathbf{y}}^{\emptyset}(1,-1)) = (\mathbb{K}^*)^2$$

$$f_{\mathbf{y}}(0,-1) = ((x + 1)y, (x^2 + 1)y^2), Z(f_{\mathbf{y}}^{[1,2]}(0,-1)) = \emptyset$$



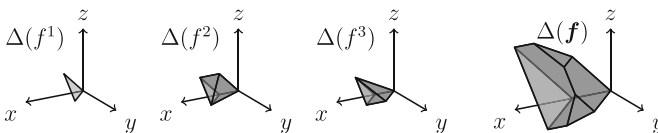
Therefore, we have

$$\begin{aligned} S_f &= \{(X, Y) : \exists(x, y) \in (\mathbb{K}^*)^2 \text{ s.t. } (X, Y) = (xy + 1, x^2y^2 + xy + 1)\} \\ &= \{(X, Y) : \exists t \in \mathbb{K}^* \text{ s.t. } (X, Y) = (t + 1, t^2 + t + 1)\} \\ &= \{Y - X^2 + X - 1 = 0\}. \end{aligned}$$

Example 4.4 Consider the map $f : \mathbb{K}^3 \rightarrow \mathbb{K}^3$ defined by

$$f(x, y, z) = (1 + xy + xz, 1 + axz + x(1 - xy + xz), 1 + bxy + x(1 + xy + xz)),$$

$a \neq 0, b \neq 0$. The Newton polyhedra look like



and we obtain the following data.

p	$f_{\mathbf{y}}(p)$	(d_1, d_2, d_3)	$J_{\mathbf{y}}(p)$	$Z_{\mathbf{y}}$
$(1, -1, -1)$	$(1 + xy + xz, 1 + axz, 1 + bxy)$	$(0, 0, 0)$	\emptyset	$(\mathbb{K}^*)^3$
$(1, -1, -2)$	$(xz, axz, 1 + bxy + x^2z)$	$(1, 1, 0)$	$\{1, 2\}$	\emptyset
$(1, -2, -1)$	$(xy, 1 - x^2y + axz, bxy)$	$(1, 0, 1)$	$\{1, 3\}$	\emptyset
$(0, -1, -1)$	$(x(y + z), x(xz - xy + az), x(xz + xy + by))$	$(1, 1, 1)$	$\{1, 2, 3\}$	\emptyset
$(-1, 1, 1)$	$(1 + xy + xz, x(1 - xy + xz), x(1 + xy + xz))$	$(0, 1, 1)$	$\{2, 3\}$	

We have

$$S_{\mathbf{y}}(1, -1, -1) = \{\exists(x, y, z) (X, Y, Z) = (1 + xy + xz, -2(1 + axz), -1 - bxy)\}$$

$$= \{abX + bY + aZ = ab - a - b\},$$

$$S_{\mathbf{y}}(-1, 1, 1) = \{\exists(x, y, z) X = 1 + xy + xz, (1 - xy + xz, 1 + xy + xz) = 0\} = \{X = 0\}.$$

We conclude that $S_f = S_{\mathbf{y}}(1, -1, -1) \cup S_{\mathbf{y}}(-1, 1, 1)$.

5 Degenerate Case

We present several tricks to handle the case when (1.5) does not hold for some $\mathbf{y} \in \Delta_{nc}(f)$.

5.1 The First Trick

Let $h : \mathbb{K}^{m+k} \rightarrow \mathbb{K}^{n+k}$ be a polynomial map with (1.5). We assume that $h^{n+i}(x) = \varphi_1(x_1, \dots, x_m) - x_{m+i}$ for $i = 1, \dots, k$. Let X be a subset of \mathbb{K}^{m+k} defined by

$$x_{m+i} = \varphi_i(x_1, \dots, x_m), \quad i = 1, \dots, k.$$

The set X is isomorphic to \mathbb{K}^m by the map defined by

$$(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, \varphi_1(x_1, \dots, x_m), \dots, \varphi_k(x_1, \dots, x_m)).$$

If $f(x_1, \dots, x_m) = h(x_1, \dots, x_m, \varphi_1(x_1, \dots, x_m), \dots, \varphi_k(x_1, \dots, x_m))$, then we have

$$S_f = S_{h|_X}$$

via the identification of \mathbb{K}^n with $\mathbb{K}^n \times \{0\}$. When h satisfies the required assumptions, one can use Theorem 3.1 to describe S_f , even though f does not satisfy (1.5) for some $\mathbf{y} \in \Delta_{nc}(f)$.

Example 5.1 Let $h : \mathbb{K}^3 \rightarrow \mathbb{K}^3$ be the map defined by

$$h(x, y, z) = (1 + x + z^2, 1 + x^2 + z^3, y^2 - x^3 - z).$$

Let X be the subset of \mathbb{K}^3 defined by $z = y^2 - x^3$. Setting $f(x, y) = h(x, y, y^2 - x^3)$, we have

$$f(x, y) = (1 + x + (y^2 - x^3)^2, 1 + x^2 + (y^2 - x^3)^3).$$

Applying Theorem 3.1, we conclude that h is proper, and thus so is f .

5.2 The Second Trick

We show another trick, which we do not use higher dimension. If $V_{\mathbf{y}} = Z(f_{\mathbf{y}}^J) \setminus \overline{Z(f_{\mathbf{y}}^J)} \setminus \Sigma(f_{\mathbf{y}}^J)$, $J = J_{\mathbf{y}}$, is not empty for some $\mathbf{y} \in \Delta_{nc}(f)$, we may have a chance to change $S_{\mathbf{y}}(f)$ (resp. $S'_{\mathbf{y}}(f)$) in (1.6) by a smaller subset of $S_{\mathbf{y}}(f)$ (resp. by a subset of $S'_{\mathbf{y}}(f)$).

Let $f : \mathbb{K}^m \rightarrow \mathbb{K}^n$ be a polynomial map with (1.3). Set $f_l^J = (f_l^j)_{j \in J}$ for $J \subset \{1, \dots, n\}$ where f_k^j is defined in (2.8). Remark that $f_0^j(x) = f_{y_j}^j(x)$. We set

$$W(\mathbf{p}) = \left\{ \mathbf{y} \in \mathbb{K}^n : \begin{array}{l} \exists \mathbf{x} \in V_{\mathbf{y}}, \mathbf{y}^{J^c} = f_{\mathbf{y}}^{J^c}(\mathbf{x}), \\ \text{rank} \begin{pmatrix} \partial_{x_i} f_0^{J'}(\mathbf{x}) \\ \partial_{x_i} f_0^{J''}(\mathbf{x}) \end{pmatrix} = \text{rank} \begin{pmatrix} \partial_{x_i} f_0^{J'}(\mathbf{x}) & f_1^{J'}(\mathbf{x}) - \mathbf{y}^{J'} \\ \partial_{x_i} f_0^{J''}(\mathbf{x}) & f_1^{J''}(\mathbf{x}) \end{pmatrix}, \end{array} \right\}$$

where $\mathbf{y} = \mathbf{y}(\mathbf{p})$, $\mathbf{y}^{J'} = (y^{j'})_{j' \in J'}$,

$$\begin{aligned} J' &= J'(\mathbf{p}) = \{j \in \{1, \dots, n\} : d_j(\mathbf{p}) = 1\}, \text{ and} \\ J'' &= J''(\mathbf{p}) = \{j \in \{1, \dots, n\} : d_j(\mathbf{p}) \geq 2\}. \end{aligned}$$

Remark that $J = J' \cup J''$ because of (1.4). Under the notations and the assumptions above, we have

Theorem 5.2 *If $V_{\mathbf{y}(\mathbf{p})}$, $\mathbf{p} \in \mathbb{Z}^n \setminus (\mathbb{Z}_{\geq})^n$, is not empty for $\mathbf{y}(\mathbf{p}) \in \Delta_{nc}(f)$, then*

$$S'_f(\mathbf{p}) \subset S_f(\mathbf{p}) \subset S'_{\mathbf{y}(\mathbf{p})}(f) \cup W(\mathbf{p}).$$

Moreover, we conclude that $W'(\mathbf{p}) \subset S_f(\mathbf{p})$, where

$$W'(\mathbf{p}) = \{ \mathbf{y} \in W(\mathbf{p}) : \exists \mathbf{x} \in V_{\mathbf{y}(\mathbf{p})}, \mathbf{y}^{J^c} = f_{\mathbf{y}(\mathbf{p})}^{J^c}(\mathbf{x}), \text{rank}(\partial_{x_i} f_0^{J''}(\mathbf{x}))_{i=1, \dots, m} = \#J'' \}.$$

Proof If $\mathbf{y} \in S_f(\mathbf{p})$, $\mathbf{p} \in \mathbb{Z}^n \setminus (\mathbb{Z}_{\geq})^n$, there exists $\mathbf{x}(t) \in \mathcal{A}(\mathbf{p})$ with $f(\mathbf{x}(t)) \rightarrow \mathbf{y}$ ($t \rightarrow 0$). By (2.9) and (2.10), we have

$$\begin{aligned} 0 &= f_0^j(\mathbf{v}_0) \quad (j \in J) \\ \mathbf{y}^{j'} &= f_1^{j'}(\mathbf{v}_0) + (df^{j'})_{\mathbf{v}_0}(\mathbf{v}_1) \quad (j' \in J') \\ 0 &= f_1^{j''}(\mathbf{v}_0) + (df^{j''})_{\mathbf{v}_0}(\mathbf{v}_1) \quad (j'' \in J''). \end{aligned}$$

Here, we use the expression in (2.1) and (2.2). This implies that

$$\text{rank} \begin{pmatrix} \partial_{x_i} f_0^{J'}(\mathbf{x}) \\ \partial_{x_i} f_0^{J''}(\mathbf{x}) \end{pmatrix} = \text{rank} \begin{pmatrix} \partial_{x_i} f_0^{J'}(\mathbf{x}) & f_1^{J'}(\mathbf{x}) - \mathbf{y}^{J'} \\ \partial_{x_i} f_0^{J''}(\mathbf{x}) & f_1^{J''}(\mathbf{x}) \end{pmatrix},$$

and we conclude $S_f(\mathbf{p}) \subset S'_{\mathbf{y}(\mathbf{p})}(\mathbf{f}) \cup W(\mathbf{p})$.

Now, we assume $\mathbf{y} \in W'(\mathbf{p})$. There exists $\mathbf{x} \in V_{\mathbf{y}}(\mathbf{p})$, such that

$$\text{rank} (\partial_{x_i} f_0^{J''}(\mathbf{x}))_{i=1, \dots, m} = \#J'' \tag{5.1}$$

By the discussion in the second paragraph of the proof of Lemma 2.4, we can choose $\mathbf{x}(t)$ to attain arbitrary $\hat{f}_l^{j''}$ ($j'' \in J'', l \geq 1$) whenever $\mathbf{x} \notin \Sigma(\mathbf{f}^{J''})$. This implies that $W'(\mathbf{p}) \subset S_f(\mathbf{p})$. □

We present a trick to describe $S_f(\mathbf{y}) = \bigcup_{\mathbf{p}: \mathbf{y}(\mathbf{p})=\mathbf{y}} S_f(\mathbf{p})$ where $S_f(\mathbf{p})$ is the set defined by (2.4).

Assume that \mathbf{f} does not satisfy (1.5) for some face \mathbf{y} . We take a primitive $\mathbf{p} \in \mathbb{Z}^n \setminus (\mathbb{Z}_{\geq})^n$, so that $\mathbf{y}(\mathbf{p}) = \mathbf{y}$. Here, \mathbf{p} is primitive means that the greatest common divisor of all components of \mathbf{p} is 1. Assume that there exists a rational map $\mathbb{K}^m \times \mathbb{K}^n \rightarrow \mathbb{K}^n$, $(\mathbf{x}, \mathbf{z}) \mapsto \Psi(\mathbf{x}, \mathbf{z})$ with the following properties.

- There exist a certain rational map $\mathbf{g} : \mathbb{K}^m \rightarrow \mathbb{K}^n$, possibly with points of indeterminacy, so that $\mathbf{f}(\mathbf{x}) = \Psi(\mathbf{x}, \mathbf{g}(\mathbf{x}))$, and \mathbf{g} satisfies (1.5) for the face supported by \mathbf{p} .
- The limit $\lim_{t \rightarrow 0} \Psi(\mathbf{x}(t), \mathbf{z})$ exists for $\mathbf{x}(t) \in \mathcal{A}(\mathbf{p})$. We assume that this limit depends on \mathbf{v}_0 , and denote the limit by $\Psi_{\mathbf{v}_0}(\mathbf{z})$, under the notation in (2.1) and (2.2),
- The limit $\lim_{t \rightarrow 0} \mathbf{g}(\mathbf{x}(t))$ for $\mathbf{x}(t) \in \mathcal{A}(\mathbf{p})$ exists.

Theorem 5.3 *Under the notations and assumptions above, the set $S_f(\mathbf{p})$, $\mathbf{y}(\mathbf{p}) = \mathbf{y}$, is in the image of the following map:*

$$Z(\mathbf{g}_{\mathbf{y}(\mathbf{p})}^J) \times \mathbb{K}^J \rightarrow \mathbb{K}^n, (\mathbf{x}, \mathbf{z}^J) \mapsto \Psi_{\mathbf{x}}(\mathbf{g}_{\mathbf{y}(\mathbf{p})}^{J^c}(\mathbf{x}) \times \mathbf{z}^J), J = J_{\mathbf{y}(\mathbf{g}; \mathbf{p})}.$$

Proof For $\mathbf{p} \in \mathbb{Z}^n \setminus (\mathbb{Z}_{\geq})^n$

$$\begin{aligned} S_f(\mathbf{p}) &= \{ \mathbf{y} \in \mathbb{K}^n : \exists \mathbf{x}(t) \in \mathcal{A}(\mathbf{p}), \lim_{t \rightarrow 0} \mathbf{f}(\mathbf{x}(t)) = \mathbf{y} \} \\ &= \{ \mathbf{y} \in \mathbb{K}^n : \exists \mathbf{x}(t) \in \mathcal{A}(\mathbf{p}), \lim_{t \rightarrow 0} \Psi(\mathbf{x}(t), \mathbf{g}(\mathbf{x}(t))) = \mathbf{y} \} \\ &\subset \left\{ \mathbf{y} \in \mathbb{K}^n : \exists \mathbf{v}_0 \in (\mathbb{K}^*)^m, \exists \mathbf{z} \in \mathbb{K}^n, \mathbf{y} = \Psi_{\mathbf{v}_0}(\mathbf{z}), g_{\gamma_j(\mathbf{p})}^j(\mathbf{v}_0) = \begin{cases} 0 & (j \in J) \\ z^j & (j \notin J) \end{cases} \right\} \\ &= \{ \mathbf{y} \in \mathbb{K}^n : \exists \mathbf{v}_0 \in Z(\mathbf{g}_{\mathbf{y}(\mathbf{p})}^J), \exists \mathbf{z} \in \mathbb{K}^n, \mathbf{y} = \Psi_{\mathbf{v}_0}(\mathbf{z}), g_{\gamma_j(\mathbf{p})}^j(\mathbf{v}_0) = z^j (j \notin J) \}. \end{aligned}$$

□

Since $S_f(\mathbf{y}) = \bigcup_{\mathbf{p}: \mathbf{y}(\mathbf{p})=\mathbf{y}} S_f(\mathbf{p})$, this may describe $S_f(\mathbf{y})$, as we see in the following example.

Example 5.4 Consider the map $f : \mathbb{K}^3 \rightarrow \mathbb{K}^3$ defined by

$$f(x_1, x_2, x_3) = (1 + x_1x_3 + x_2x_3, 1 + x_1(1 - x_1x_3 + x_2x_3), 1 + x_2(1 - x_1x_3 + x_2x_3)).$$

The map f satisfies the condition (1.5) except the face $\gamma(-1, -1, 1)$, as we see in the following data.

p	$f_{\gamma}(p)$	(d_1, d_2, d_3)	$J_{\gamma}(p)$
$(1, 1, -1)$	$(1 + x_1x_3 + x_2x_3, 1, 1)$	$(0, 0, 0)$	\emptyset
$(1, 1, -2)$	$(x_3(x_1 + x_2), 1 + x_1x_3(x_2 - x_1), 1 + x_2x_3(x_2 - x_1))$	$(1, 0, 0)$	$\{1\}$
$(-1, 0, 1)$	$(1 + x_1x_3, x_1(1 - x_1x_3), 1 + x_2 - x_1x_2x_3)$	$(0, 1, 0)$	$\{2\}$
$(0, -1, 1)$	$(1 + x_2x_3, 1 + x_1 + x_1x_2x_3, x_2(1 + x_2x_3))$	$(0, 0, 1)$	$\{3\}$
$(-1, -1, 1)$	$(1 + x_1x_3 + x_2x_3, x_1(1 - x_1x_3 + x_2x_3), x_2(1 - x_1x_3 + x_2x_3))$	$(0, 1, 1)$	$\{2, 3\}$

We easily see that

$$S_{\gamma(1,1,-1)}(f) = \{y_1 = y_3 = 1\}, \quad S_{\gamma(1,1,-2)}(f) = \{y_2 + y_3 = 2\},$$

$$S_{\gamma(-1,0,1)}(f) = \{y_1 = 2, y_3 = 1\}, \quad S_{\gamma(0,-1,1)}(f) = \{y_1 = 0, y_2 = 1\}.$$

We also have that

$$W(-1, -1, 1) = \left\{ \begin{array}{l} (y_1, y_2, y_3) : \exists(x_1, x_2, x_3) \ 1 - x_1x_3 + x_2x_3 = 0, \ y_1 = 1 + x_1x_3 + x_2x_3, \\ \text{rank} \begin{pmatrix} 1 - 2x_1x_3 + x_2x_3 & x_1x_3 & x_1(x_2 - x_1) \ 1 - y_2 \\ x_2x_3 & 1 - x_1x_3 + 2x_2x_3 & x_2(x_2 - x_1) \ 1 - y_3 \end{pmatrix} = 1 \end{array} \right\}$$

$$= \{y_1(y_2 - y_3) - 2y_2 + 2 = 0\}.$$

We will show that this coincides with $S_{\gamma(-1,-1,1)}(f)$, considering the rational map

$$\Phi : \mathbb{K}^3 \times \mathbb{K}^3 \rightarrow \mathbb{K}^3, \ (x, y) \mapsto z = (z_1, z_2, z_3) = \left(y_1, y_2, -\frac{x_2}{x_1}y_2 + y_3 \right).$$

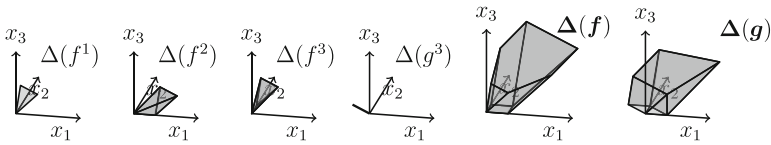
Remark that $g(x) = \Phi(x, f(x))$ defines the map

$$\mathbb{K}^* \times \mathbb{K}^2 \rightarrow \mathbb{K}^3, \ (x_1, x_2, x_3) \mapsto (1 + x_1x_3 + x_2x_3, 1 + x_1(1 - x_1x_3 + x_2x_3), 1 - \frac{x_2}{x_1}),$$

and obtain the following data:

p	$g_{\gamma}(p)$	(d_1, d_2, d_3)	$J_{\gamma}(p)$
$(-1, -1, 1)$	$(1 + x_1x_2 + x_1x_3, x_1(1 - x_1x_2 + x_2x_3), 1 - \frac{x_2}{x_1})$	$(0, 1, 0)$	$\{2\}$

The Newton polyhedra look like



As in the proof of Theorem 1.8, we conclude that

$$\begin{aligned}
 S_{\mathbf{y}}(-1, -1, 1)(\mathbf{g}) &= \left\{ (z_1, z_2, z_3) \in \mathbb{K}^3 : \begin{array}{l} \exists(x_1, x_2, x_3) \ 1 - x_1x_3 + x_2x_3 = 0 \\ (z_1, z_3) = (1 + x_1x_3 + x_2x_3, 1 - x_2/x_1) \end{array} \right\} \\
 &= \{(z_1, z_2, z_3) \in \mathbb{K}^3 : z_1z_3 = 2\}. \tag{5.2}
 \end{aligned}$$

For (x_1, x_2, x_3) in (5.2), we have

$$-\frac{x_2}{x_1} = \left(1 - \frac{x_2}{x_1}\right) - 1 = z_3 - 1 = \frac{2}{z_1} - 1 = \frac{2}{y_1} - 1.$$

Setting

$$\Psi : \mathbb{K}^3 \times \mathbb{K}^3 \rightarrow \mathbb{K}^3, \ (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{z} = \left(y_1, y_2, \frac{x_2}{x_1}y_2 + y_3\right),$$

we have $\mathbf{y} = \Psi(\mathbf{x}, \Phi(\mathbf{x}, \mathbf{y}))$, and thus, $\mathbf{f}(\mathbf{x}) = \Psi(\mathbf{x}, \mathbf{g}(\mathbf{x}))$. The set

$$\Psi(Z(\mathbf{g}_{\mathbf{y}}^2(-1, 1, 1)) \times S_{\mathbf{g}}(-1, -1, 1)) \text{ is defined by } y_1\left(\frac{2}{y_1} - 1\right)y_2 + y_3 = z_1z_3 = 2,$$

and we obtain $S_{\mathbf{f}}(\mathbf{y}(-1, -1, 1)) = \{y_1y_2 - y_1y_3 - 2y_2 + 2 = 0\}$.

We thus obtain that $S_{\mathbf{f}} = \{y_1 + y_3 = 2\} \cup \{y_1y_2 - y_1y_3 - 2y_2 + 2 = 0\}$.

Example 5.5 Let us use the first trick to handle Example 5.4. We consider the map $\mathbf{h} : \mathbb{K}^4 \rightarrow \mathbb{K}^4$ defined by

$$\mathbf{h}(x_1, x_2, x_3, x_4) = (1 + x_1x_3 + x_2x_3, 1 + x_1x_4, 1 + x_2x_4, 1 - x_1x_3 + x_2x_3 - x_4),$$

since $\mathbf{f}(x_1, x_2, x_3) = \mathbf{h}(x_1, x_2, x_3, 1 - x_1x_3 + x_2x_3)$. We analyze $\mathbf{h}(\mathbf{x}(t))$ for $\mathbf{x}(t) \in \mathcal{A}(-1, -1, 1, 1)$, because $\mathbf{p} = (-1, -1, 1, k)$, $k \geq 1$, supports a three-dimensional face of $\Delta(\mathbf{g})$ if and only if $k = 1$. Setting

$$\mathbf{x}(t) = (t^{-1}(x_0^1 + x_1^1t + \dots), t^{-1}(x_0^2 + x_1^2t + \dots), t(x_0^3 + x_1^3t + \dots), t(x_0^4 + x_1^4t + \dots)),$$

we have

$$\mathbf{h}(\mathbf{x}(t)) = (1 + x_0^1x_0^3 + x_0^2x_0^3, 1 + x_0^1x_0^4, 1 + x_0^2x_0^4, 1 - x_0^1x_0^3 - x_0^2x_0^3 - x_0^4) + o(t).$$

Assuming $\mathbf{x}(t)$ is in $X = \{\mathbf{x} \in \mathbb{K}^4 : 1 - x_1x_3 + x_2x_3 = x_4\}$, we obtain that $1 - x_0^1x_0^3 - x_0^2x_0^3 = x_0^4$. Under this condition, we eliminate x_0^1, x_0^2, x_0^3 from the system

$$(1 + x_0^1x_0^3 + x_0^2x_0^3, 1 + x_0^1x_0^4, 1 + x_0^2x_0^4, 1 - x_0^1x_0^3 - x_0^2x_0^3) = (y_1, y_2, y_3, 0),$$

we conclude that $S_{\mathbf{y}}(-1, -1, 1, 1)(\mathbf{h}) = \{2 - 2y_2 - y_1y_3 + y_1y_2 = 0\} \times \{0\}$. We thus obtain that $S_f = \{y_1 + y_3 = 2\} \cup \{y_1y_2 - y_1y_3 - 2y_2 + 2 = 0\}$.

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