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RESEARCH CONTRIBUTION

Generalized Permutahedra and Schubert Calculus

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Abstract
We connect generalized permutahedra with Schubert calculus. Thereby, we give sufficient vanishing criteria for Schubert intersection numbers of the flag variety. Our argument utilizes recent developments in the study of Schubitopes, which are Newton polytopes of Schubert polynomials. The resulting tableau test executes in polynomial time.

1 Introduction

1.1 Background

Let $X = Flags(\mathbb{C}^n)$ be the variety of complete flags of vector spaces

$$F_\bullet : \langle 0 \rangle \subset F_1 \subset F_2 \subset \cdots \subset F_i \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n, \ \dim(F_i) = i.$$

X has a left action of GL_n, and hence also by lower triangular invertible matrices B_-. The B_--orbits X_w° are indexed by permutations w in the symmetric group S_n. Let $\leq$ denote Bruhat order. The *Schubert varieties* are the closures

$$X_w = \coprod_{v \geq w} X_v^\circ;$$

this is codimension $\ell(w) = \#\{(i,j) : 1 \leq i < j \leq n, w(i) > w(j)\}$. Thus, $X = X_{id}$ and X_{w_0} is the *Schubert point*, where $w_0 = n\ n-1\ n-2\ \cdots 2\ 1$.

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The Poincaré duals $\sigma_w := [X_w]$ form the *Schubert basis* of $H^*(X)$, the cohomology ring of X. A *Schubert problem* is a tuple $(w^{(1)}, w^{(2)}, \dots, w^{(k)}) \in S_n^k$ with $\sum_{i=1}^k \ell(w^{(i)}) = \binom{n}{2} = \dim_{\mathbb{C}}(X)$. The *Schubert intersection number* is

$$\begin{aligned} C_{w^{(1)},w^{(2)},\dots,w^{(k)}} :=& \text{multiplicity of } \sigma_{w_0} \text{ in} \prod_{i=1}^k \sigma_{w^{(i)}} \in H^*(X) \\ =& \text{number of points in} \bigcap_{i=1}^k g_i X_{\sigma^{(i)}}, \end{aligned} \tag{1}$$

where $(g_1, \dots, g_k)$ are elements of a dense open subset $\mathcal{O}$ of GL_n^k (whose existence is guaranteed by Kleiman transversality). For more on this topic, see the book [8] or the expository articles [10, 11].

Algorithms exist for computing these numbers; see, e.g., [5, 13, 15] and the references therein. It is the famous open problem of Schubert calculus to find a combinatorial counting rule that computes $C_{w^{(1)},w^{(2)},\dots,w^{(k)}}$. Such a rule would generalize the classical *Littlewood–Richardson rule* governing Schubert calculus of Grassmannians.

This paper explores a related, but not necessarily easier, open problem

Find an efficient algorithm to decide if $C_{w^{(1)},w^{(2)},\dots,w^{(k)}} = 0$.

Known algorithms to compute $C_{w^{(1)},w^{(2)},\dots,w^{(k)}}$ do not provide a solution (being inefficient). In the Grassmannian setting, neither does the Littlewood–Richardson rule, *per se*. However, the *saturation theorem* [14] permits a *polynomial-time* algorithm in that case [6, 17], by way of linear programming results. For flag varieties, criteria were found by Knutson [12] and Purbhoo [18]; no efficiency guarantees were stated.

1.2 Vanishing Criterion

Our main goal is to connect the theory of generalized permutahedra to Schubert calculus. We give a sufficient test for $C_{w^{(1)},w^{(2)},\dots,w^{(k)}} = 0$ and prove it executes in polynomial time. The starting point is a simple consideration about Schubert polynomials. However, it becomes effective due to recent developments about Newton polytopes of Schubert polynomials [1, 7, 16], as instances of generalized permutahedra.

The *Rothe diagram* of $w \in S_n$, denoted $D(w)$, is the subset of boxes of $[n] \times [n]$ given by

$$D(w) := \left\{(i, j) : 1 \le i, j \le n, \, j < w(i), i < w^{-1}(j)\right\}.$$

Let $code(w) = (c_1(w), c_2(w), \dots, c_n(w))$, where c_i counts boxes of $D(w)$ in row i. Define

$$D := D\left(w^{(1)}, \dots, w^{(k)}\right)$$

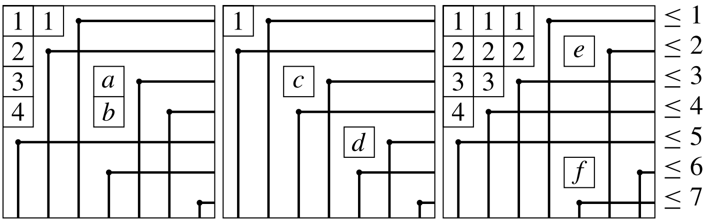
by concatenating $D(w^{(1)}), \ldots, D(w^{(k)})$, left to right. Set $Tab_{w^{(1)},\ldots,w^{(k)}}$ to be the set of fillings of D with nonnegative integers, such that:

(a) Each column is strictly increasing from top to bottom.
(b) Any label ℓ in row r satisfies $\ell \leq r$.
(c) The number of ℓ's is $n - \ell$, for $1 \leq \ell \leq n$.

The first version of our test is:

Theorem A *Let* $(w^{(1)}, \ldots, w^{(k)})$ *be a Schubert problem. If* $Tab = \emptyset$*, then* $C_{w^{(1)},w^{(2)},\ldots,w^{(k)}} = 0$*. There is an algorithm to determine emptiness in* $O(poly(k, n))$*.*

Example 1.1 Let $w^{(1)} = 3256147$, $w^{(2)} = 2143657$, $w^{(3)} = 4632175$. Below, we depict D. The numerically labelled boxes are forced by conditions (a) and (b) for any (putative) $T \in Tab$.



Condition (b) forces $e \leq 2$, $a, c \leq 3$, $b \leq 4$, $d \leq 5$, $f \leq 6$. Thus, to satisfy (c), $e = 2$ is also forced, which implies $a, c = 3$. Therefore, T has at least five 3's, violating (c) for $\ell = 3$.

Our idea (see Sect. 4) uses that $C_{w^{(1)},w^{(2)},w^{(3)}} = 0$ if $\mathfrak{S}_{w_0} = x_1^6 x_2^5 x_3^4 x_4^3 x_5^2 x_6$ does not appear in the product of Schubert polynomials $\mathfrak{S}_{w^{(1)}} \mathfrak{S}_{w^{(2)}} \mathfrak{S}_{w^{(3)}}$, combined with an argument that the rule of Theorem A permits an efficient check of this vanishing condition. □

1.3 Organization

Section 2 discusses generalized permutahedra; we derive facts we will use. Section 3 reviews the subfamily of Schubitopes. In Sect. 4, we state Theorem B, an "asymmetric" version of Theorem A; it is a stronger test (see Proposition 4.6). Theorem C gives linear inequalities necessary for $C_{w^{(1)},\ldots,w^{(k)}} > 0$. Theorems A, B, C, and Proposition 4.6 are proved together, as they follow from the same reasoning. In Sect. 5, we compare with the vanishing criteria of [12, 18]. We show examples that our test captures but are not captured by those criteria, and conversely.

2 Newton Polytopes of Products

If f is an element of a polynomial ring whose variables are indexed by some set I, the *support* of f is the lattice point set in $\mathbb{R}^I$ consisting of the exponent vectors of the monomials that have nonzero coefficient in f. The *Newton polytope* $\text{Newton}(f) \subseteq \mathbb{R}^I$

is the convex hull of the support of f. A polynomial f has *saturated Newton polytope* (SNP) if every lattice point in Newton(f) is a vector in the support of f [16].

The *standard permutahedron* is the polytope in $\mathbb{R}^n$ whose vertices consist of all permutations of the entries of the vector $(0, 1, \ldots, n-1)$. A *generalized permutahedron* is a deformation of the standard permutahedron obtained by translating the vertices in such a way that all edge directions and orientations are preserved (edges are allowed to degenerate to points). Generalized permutahedra are uniquely parameterized by *submodular functions* (see [2, Theorem 12.3] for several equivalent definitions). These are maps

$$z : 2^{[n]} \to \mathbb{R},$$

such that $z_\emptyset = 0$ and

$$z_I + z_J \geq z_{I\cup J} + z_{I\cap J} \text{ for all } I, J \subseteq [n].$$

Given z, the associated generalized permutahedron is given by

$$P(z) = \left\{ t \in \mathbb{R}^n : \sum_{i\in I} t_i \leq z_I \text{ for } I \neq [n], \text{ and } \sum_{i=1}^{n} t_i = z_{[n]} \right\}.$$

The vertices of generalized permutahedra have been determined.

Proposition 2.1 [21, Corollary 44.3a] *Let $P(z)$ be a generalized permutahedron in $\mathbb{R}^n$. The vertices of $P(z)$ are $\{v(w) : w \in S_n\}$ where $v(w) = (v_1, \ldots, v_n) \in \mathbb{R}^n$ is defined by*

$$v_{w_k} = z_{\{w_1,\ldots,w_k\}} - z_{\{w_1,\ldots,w_{k-1}\}}. \tag{2}$$

It is well known that the class of generalized permutahedra is closed under Minkowski sums (see for instance [3, Lemma 2.2]). We provide a proof for completeness. One can also easily see closure under Minkowski sums using that generalized permutahedra are exactly the polytopes whose normal fans are refined by the braid arrangement [2, Theorem 12.3].

Lemma 2.2 *If $P(z)$ and $P(z')$ are generalized permutahedra, then*

$$P(z) + P(z') = P(z + z').$$

Proof Clearly, $P(z) + P(z') \subseteq P(z + z')$. For the opposite containment, let q be a vertex of $P(z+z')$. By Proposition 2.1, write q in the form $q = v(w)$ for some $w \in S_n$. Let p and p' be the vertices of $P(z)$ and $P(z')$, respectively, corresponding to w. By (2), $q = p + p' \in P(z) + P(z')$. Convexity implies $P(z + z') \subseteq P(z) + P(z')$. □

It follows easily from [21, Theorem 46.2] that whenever z and z' are integer-valued, $P(z) \cap P(z')$ is either empty or an integral polytope (all vertices are lattice points). This is used to prove that *integer polymatroids* [21, Chapter 44] satisfy a generalization of the *integer decomposition property*. We state and prove (for convenience) the special case that applies to generalized permutahedra:

Theorem 2.3 [21, Corollary 46.2c] *If $P(z)$ and $P(z')$ are integral generalized permutahedra in $\mathbb{R}^n$, then*

$$(P(z) \cap \mathbb{Z}^n) + (P(z') \cap \mathbb{Z}^n) = (P(z) + P(z')) \cap \mathbb{Z}^n.$$

Proof Let $r \in (P(z)+P(z')) \cap \mathbb{Z}^n$. Set $Q = r + (-1)P(z')$. Clearly, Q is a generalized permutahedron (by the deformation description). Also, note that $r = p + p'$ for some $p \in P(z)$ and $p' \in P(z')$, so $p \in P \cap Q$ and $P \cap Q \neq \emptyset$. Since both r and z' are integral, Q is an integral polytope. Thus, $P \cap Q$ contains an integer point q. By definition of Q, the lattice point $r - q$ is in $P(z')$. Finally, we have

$$r = q + (r - q) \in (P(z) \cap \mathbb{Z}^n) + (P(z') \cap \mathbb{Z}^n).$$

□

Therefore, in the realm of generalized permutahedra, SNP carries through products.

Proposition 2.4 *If $f, g \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ have SNP and* $\mathrm{Newton}(f)$, $\mathrm{Newton}(g)$ *are generalized permutahedra, then*

(i) $\mathrm{Newton}(fg)$ *is a generalized permutahedron;*
(ii) fg *has SNP.*

Proof For any polynomials f and g, $\mathrm{Newton}(fg) = \mathrm{Newton}(f) + \mathrm{Newton}(g)$. Statement (i) follows from Lemma 2.2. Statement (ii) follows from Lemma 2.2 and Theorem 2.3. □

3 Schubitopes and an Integer Linear Program

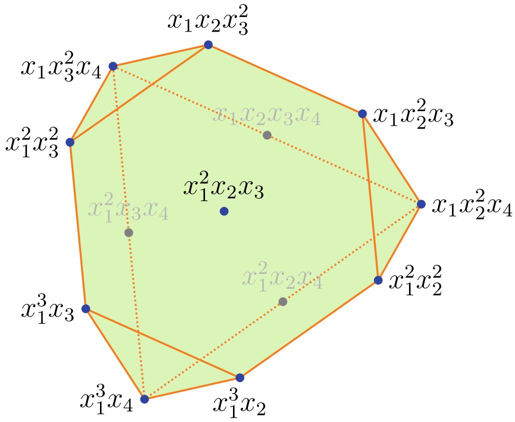
We are interested in a particular family of generalized permutahedra. For an arbitrary subset $D \subseteq [n] \times [m]$, the *Schubitope* $\mathcal{S}_D$ was defined by C. Monical, N. Tokcan, and the second author [16] (for squares $[n] \times [n]$ instead of rectangles $[n] \times [m]$, but the difference is negligible).

Fix $S \subseteq [n]$ and a column $c \in [m]$. Let $\omega_{c,S}(D)$ be formed by reading c from top to bottom and recording

- (if $(r, c) \notin D$ and $r \in S$,
-) if $(r, c) \in D$ and $r \notin S$, and
- $\star$ if $(r, c) \in D$ and $r \in S$.

Let

$$\theta_D^c(S) = \#\text{paired } (\,)\text{'s in} \, \omega_{c,S}(D) + \#\star\text{'s in} \, \omega_{c,S}(D).$$



$$\begin{aligned} \alpha_1 &\le 3\\ \alpha_2 &\le 2\\ \alpha_3 &\le 2\\ \alpha_4 &\le 1\\ \alpha_1+\alpha_2+\alpha_3 &\le 4\\ \alpha_1+\alpha_2+\alpha_4 &\le 4\\ \alpha_1+\alpha_3+\alpha_4 &\le 4\\ \alpha_2+\alpha_3+\alpha_4 &\le 3\\ \alpha_1+\alpha_2+\alpha_3+\alpha_4 &= 4 \end{aligned}$$

Fig. 1 $\mathcal{S}_{D(21543)} = \mathrm{Newton}(\mathfrak{S}_{21543})$ and a minimal set of defining inequalities

Set $\theta_D(S) = \sum_{c\in[m]} \theta_D^c(S)$. Define the *Schubitope* as

$$\mathcal{S}_D = \left\{ (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n_{\ge 0} : \sum_{i=1}^{n} \alpha_i = \#D \text{ and } \sum_{i\in S} \alpha_i \le \theta_D(S) \text{ for all } S \subset [n] \right\}.$$

Example 3.1 (*cf.* [16, Section 1]) Let $w = 21543$. The Schubert polynomial of w is

$$\mathfrak{S}_w = x_1^3x_2 + x_1^3x_3 + x_1^3x_4 + x_1^2x_2^2 + x_1^2x_3^2 + 2x_1^2x_2x_3 + x_1^2x_2x_4 + x_1^2x_3x_4 + x_1x_2x_3^2 + x_1x_2^2x_3 + x_1x_2^2x_4 + x_1x_3^2x_4 + x_1x_2x_3x_4.$$

As stated in Theorem 4.3, $\mathcal{S}_{D(w)} = \mathrm{Newton}(\mathfrak{S}_w)$. This generalized permutahedron and a minimal set of defining inequalities are shown in Fig. 1. □

Given a diagram D and any point α, we wish to efficiently determine whether $\alpha \in \mathcal{S}_D$. However, $\mathcal{S}_D$ is described by exponentially many inequalities. A way around this is to work instead with the polytope $\mathcal{P}(D,\alpha)$ introduced by A. Adve, C. Robichaux, and the second author in [1], which is able to detect membership in $\mathcal{S}_D$.

Given $D \subseteq [n] \times [m]$ and $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n_{\ge 0}$. Let

$$\mathcal{P}(D,\alpha) \subseteq \mathbb{R}^{n\times m}$$

be the polytope whose points

$$(\alpha_{ij})_{1\le i\le n, 1\le j\le m} = (\alpha_{11},\dots,\alpha_{n1},\dots,\alpha_{1m},\dots,\alpha_{nm})$$

satisfy the inequalities (I),(II),(III) below.

(I) Column-injectivity: For all $i, j \in [n]$,

$$0 \le \alpha_{ij} \le 1.$$

(II) Content: For all $i \in [n]$,

$$\sum_{j=1}^{n} \alpha_{ij} = \alpha_i.$$

(III) Row bounds: For all $s, j \in [n]$,

$$\sum_{i=1}^{s} \alpha_{ij} \geq \#\{(i, j) \in D : i \leq s\}.$$

Define $Tab(D, \alpha)$ to be the set of fillings of D with nonnegative integers, such that

(a) Each column is strictly increasing from top to bottom.
(b) Any label ℓ in row r satisfies $\ell \leq r$.
(c) The number of ℓ's is α_ℓ.

Theorem 3.2 [1, Theorem 1.3] *Suppose $D \subseteq [n] \times [m]$. Then*

$$\alpha \in \mathcal{S}_D \cap \mathbb{Z}^n \iff Tab(D, \alpha) \neq \emptyset.$$

The map $f : Tab(D, \alpha) \to \mathcal{P}(D, \alpha)$, that sets $\alpha_{ij} = 1$ if the label i appears in column j of D, and set $\alpha_{ij} = 0$ otherwise, is a bijection. Therefore, $Tab(D, \alpha) \neq \emptyset$ if and only if $\alpha_1 + \cdots + \alpha_n = \#D$ and $\mathcal{P}(D, \alpha) \cap \mathbb{Z}^{n \times m} \neq \emptyset$.

Theorem 3.3 [1, Theorem 2.2.7] *Let $D \subseteq [n] \times [m]$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ with $\alpha_1 + \cdots + \alpha_n = \#D$. Then, $\mathcal{P}(D, \alpha) \cap \mathbb{Z}^{n \times m} \neq \emptyset$ if and only if $\mathcal{P}(D, \alpha) \neq \emptyset$.*

The above two theorems, combined with the ellipsoid method and/or interior point methods in linear programming, imply:

Corollary 3.4 [1] *Deciding if $\alpha \in \mathcal{S}_D$, or equivalently, if $Tab(D, \alpha) = \emptyset$, can be determined in $O(poly(n, m))$-time.*

As explained in [1], using the codes of $w^{(i)}$ as the encoding of the decision problem, or "compressing" D, one can reduce the upper bound on the complexity. We will not pursue these technical improvements here.

4 Schubert Polynomials and Schubitopes

4.1 Schubert Polynomials

Our reference for *Schubert polynomials* is [15]. They are recursively defined; the initial condition is that for $w_0 \in S_n$

$$\mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

The *divided difference operator* on polynomials in $Pol := \mathbb{Z}[x_1, x_2, \ldots]$ is

$$\partial_i : Pol \to Pol,\ f \mapsto \frac{f(\ldots, x_i, x_{i+1}, \ldots) - f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}.$$

If $w \neq w_0$, let i satisfy $w(i) < w(i+1)$, then $\mathfrak{S}_w := \partial_i \mathfrak{S}_{ws_i}$. Since the divided difference operators satisfy the braid relations

$$\partial_i \partial_j = \partial_j \partial_i \text{ for } |i-j| \geq 2;\quad \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1},$$

it follows that $\mathfrak{S}_w$ only depends on w, and not the choices of i in the recursion.

Schubert polynomials are stable under the inclusion of $S_n \hookrightarrow S_{n+1}$ that sends w to w with $n+1$ appended. Thus, one unambiguously defines $\mathfrak{S}_w$ for $w \in S_\infty = \bigcup_{n \geq 1} S_n$. The set of Schubert polynomials $\{\mathfrak{S}_w : w \in S_\infty\}$ forms a $\mathbb{Z}$-linear basis of Pol.

Borel's isomorphism [8, Chapter 9; Prop. 3] asserts

$$H^*(X) \cong \mathbb{Q}[x_1, \ldots, x_n]/I^{S_n} \text{ where } I^{S_n} = \langle e_d(x_1, \ldots, x_n) : 1 \leq d \leq n \rangle,$$

and

$$e_d(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_d \leq n} x_{i_1} x_{i_2} \cdots x_{i_d}$$

is the dth *elementary symmetric polynomial*. Under this isomorphism

$$\sigma_w \mapsto \mathfrak{S}_w + I^{S_n}. \tag{3}$$

One has the *polynomial* identity

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_{w \in S_\infty} C_{u,v}^w \mathfrak{S}_w \in Pol.$$

Define $C_{w^{(1)},\ldots,w^{(k-1)}}^{w^{(k)}}$ to be the multiplicity of $\sigma_{w^{(k)}}$ in $\prod_{i=1}^{k-1} \sigma_{w^{(i)}} \in H^*(X)$, which we also write with the coefficient operator as $[\sigma_{w^{(k)}}] \prod_{i=1}^{k-1} \sigma_{w^{(i)}}$.

Lemma 4.1 *$C_{w^{(1)},\ldots,w^{(k-1)}}^{w^{(k)}} = C_{w^{(1)},\ldots,w^{(k-1)},w_0 w^{(k)}}$. Also, $C_{w^{(1)},\ldots,w^{(k-1)}}^{w^{(k)}} = [\mathfrak{S}_{w^{(k)}}] \prod_{i=1}^{k-1} \mathfrak{S}_{w^{(i)}}$. In particular, $C_{u,v}^w = C_{u,v,w_0 w}$.*

Proof Duality in Schubert calculus (see, e.g., [15, Proposition 3.6.11]) states that if $\ell(u) + \ell(v) = \binom{n}{2}$, then

$$\sigma_u \smile \sigma_v = \begin{cases} \sigma_{w_0} & \text{if } v = w_0 u \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$\prod_{i=1}^{k-1}\sigma_{w^{(i)}} = C^{w^{(k)}}_{w^1,\ldots,w^{(k-1)}}\sigma_{w^{(k)}} + \sum_{w\in S_n, w\neq w^{(k)}} C^{w}_{w^{(1)},\ldots,w^{(k-1)}}\sigma_w.$$

Multiply both sides by $\sigma_{w_0 w^{(k)}}$ and apply duality. Then, use (1) to obtain the first statement. The second assertion follows from (3). The final claim is merely the $k = 3$ case. □

Lemma 4.2 *If* $(w^{(1)}, \ldots, w^{(k)})$ *is a Schubert problem, then*

$$C_{w^{(1)},\ldots,w^{(k)}} = [x_1^{n-1}x_2^{n-2}\cdots x_{n-1}]\prod_{i=1}^{k}\mathfrak{S}_{w^{(i)}}.$$

Proof This follows from (1), (3), and $\mathfrak{S}_{w_0} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$. □

4.2 Schubitopes are Newton Polytopes

This result from work of Fink, Mészáros, and the first author [7] proves conjectures of [16]:

Theorem 4.3 [7, Theorems 7,10] $\mathfrak{S}_w$ *has SNP, and* $\mathrm{Newton}(\mathfrak{S}_w) = \mathcal{S}_{D(w)}$ *is a generalized permutahedron.*

Proposition 4.4 $f = \prod_{i=1}^{k-1}\mathfrak{S}_{w^{(i)}}$ *has SNP. In addition*

$$\mathrm{Newton}(f) = \sum_{i=1}^{k-1}\mathcal{S}_{D(w^{(i)})} \textit{ (Minkowski sum).} \tag{4}$$

Proof This follows from combining Theorem 4.3 with Proposition 2.4. □

By the same argument as Proposition 4, any product of *key polynomials* (see, e.g., [20]) with Schubert polynomials is SNP, and has a similarly described Newton polytope.

Corollary 4.5 *If* $\alpha \in \mathbb{Z}^n_{\geq 0}$, *then*

$$[x^\alpha]\prod_{i=1}^{k-1}\mathfrak{S}_{w^{(i)}} \neq 0 \iff \alpha \in \sum_{i=1}^{k-1}\mathcal{S}_{D(w^{(i)})}.$$

Proof Let $f = \prod_{i=1}^{k-1}\mathfrak{S}_{w^{(i)}}$. If $[x^\alpha]f \neq 0$, then $\alpha \in \mathrm{Newton}(f)$. Now, apply (4). Conversely, by (4), $\alpha \in \mathrm{Newton}(f)$. By Proposition 4.4, f has SNP. Hence, $[x^\alpha]f \neq 0$. □

4.3 The Asymmetric Version of Theorem A

Let $D' := D(w^{(1)}, \ldots, w^{(k-1)})$ and let $Tab' := Tab'_{w^1,\ldots,w^{(k)}}$ be the set of fillings of D' with nonnegative integers, such that:

(a) Each column is strictly increasing from top to bottom.
(b) Any label ℓ in row r satisfies $\ell \le r$.
(c) The number of ℓ's is $c_\ell(w^{(k)})$.

Theorem B *Let $(w^{(1)}, \ldots, w_0 w^{(k)})$ be a Schubert problem. If $Tab' = \emptyset$, then $C^{w^{(k)}}_{w^{(1)},w^{(2)},\ldots,w^{(k-1)}} = 0$. There is an algorithm to determine emptiness in $O(poly(k, n))$.*

Proposition 4.6 *If Theorem A's test shows $C_{w^{(1)},\ldots,w^{(k-1)},w_0 w^{(k)}} = 0$, then Theorem B's also shows $C^{w^{(k)}}_{w^{(1)},\ldots,w^{(k-1)}} = 0$.*

Example 4.7 The converse of Proposition 4.6 is false. That is, Theorem B provides a strictly stronger test than Theorem A. For example

$$\mathfrak{S}_{4123}\mathfrak{S}_{1342} = x_1^4 x_3 + x_1^4 x_2 + x_1^3 x_2 x_3$$

avoids $code(4312) = 3200$ as an exponent vector, proving $C^w_{u,v} = C^{4312}_{4123,1342} = 0$. However

$$\mathfrak{S}_u \mathfrak{S}_v \mathfrak{S}_{w_0 w} = x_1^4 x_2^2 + x_1^4 x_3^2 + 3x_1^4 x_2 x_3 + x_1^3 x_2 x_3^2 + \underline{x_1^3 x_2^2 x_3} + x_1^5 x_3 + x_1^5 x_2$$

implies $Tab \neq \emptyset$, and hence, Theorem A does not show $C_{u,v,w_0 w} = C_{4123,1342,1243} = 0$. □

4.4 The Schubitope Inequalities and Schubert Calculus

The Schubitope inequalities provide necessary conditions for nonvanishing of a Schubert intersection number.

Theorem C *If $C_{w^{(1)},w^{(2)},\ldots,w^{(k)}} > 0$, then $(n-1, n-2, \ldots, 2, 1)$ must satisfy the Schubitope inequalities defining $\mathcal{S}_D$ where $D = D(w^{(1)}, \ldots, w^{(k)})$. Similarly, if $C^{w^{(k)}}_{w^{(1)},\ldots,w^{(k-1)}} > 0$, then $code(w^{(k)})$ must satisfy the Schubitope inequalities defining $\mathcal{S}_{D'}$ where $D' = D(w^{(1)}, \ldots, w^{(k-1)})$.*

Let

$$s_\lambda(x_1, \ldots, x_k) = \sum_T x^T$$

be the *Schur polynomial* of λ, where the sum is over semistandard Young tableaux of shape λ filled using $\{1, 2, \ldots, k\}$ and $x^T = \prod_{i=1}^k x_i^{\#i \in T}$. Then

$$s_\lambda(x_1, \ldots, x_k) s_\mu(x_1, \ldots, x_k) = \sum_\nu c^\nu_{\lambda,\mu} s_\nu(x_1, \ldots, x_k),$$

where $c^{\nu}_{\lambda,\mu}$ is the *Littlewood–Richardson coefficient*. By the proof of [16, Proposition 2.9]

$$x^{\nu} \in s_{\lambda}s_{\mu} \text{ if and only if } \nu \in \text{Newton}(s_{\lambda+\mu}) = \mathcal{P}_{\lambda+\mu}(\text{ the permutahedron for } \lambda+\mu). \tag{5}$$

By Rado's theorem [19, Theorem 1], this means $\nu \leq_{\text{Dom}} \lambda + \mu$ (dominance order). That is

$$c^{\nu}_{\lambda,\mu} > 0 \implies \sum_{i=1}^{t} \nu_i \leq \sum_{j=1}^{t} \lambda_j + \sum_{k=1}^{t} \nu_k, \text{ for } t \geq 1.$$

These are instances of the famous *Horn's inequalities*; see the survey [9]. Those are generalized in the "Levi-movable" case of X in work of P. Belkale-S. Kumar [4]. Our methods are in the same vein. Hence, we speculate that Theorem C is a first glimpse of putative linear inequalities that control $C_{w^{(1)},\ldots,w^{(k)}} > 0$. We hope to study this further in a sequel.

4.5 Proof of Theorems A, B, C and Proposition 4.6

We combine the proofs of these four results, since they all stem from the same reasoning.

We prove Theorem B first. It is known (*e.g.*, follows from [15, Theorem 2.5.1]) that

$$[x^{code(w)}]\mathfrak{S}_w \neq 0. \tag{6}$$

Hence

$$\left[x^{code(w^{(k)})}\right]\prod_{i=1}^{k-1}\mathfrak{S}_{w^{(i)}} = 0 \Rightarrow C^{w^{(k)}}_{w^{(1)},w^{(2)},\ldots,w^{(k-1)}} = 0.$$

By one direction of Corollary 4.5

$$\left[x^{code(w^{(k)})}\right]\prod_{i=1}^{k-1}\mathfrak{S}_{w^{(i)}} = 0 \impliedby code(w^{(k)}) \notin \text{Newton}\left(\prod_{i=1}^{k-1}\mathfrak{S}_{w^{(i)}}\right) = \sum_{i=1}^{k-1}\mathcal{S}_{D(w^{(i)})}. \tag{7}$$

By Theorem 4.3, each $\mathcal{S}_{D(w^{(i)})}$ is a generalized permutahedron. Hence, by Lemma 2.2

$$\text{Newton}\left(\prod_{i=1}^{k-1}\mathfrak{S}_{w^{(i)}}\right) = \mathcal{S}_{D'}.$$

Now, we may apply Theorem 3.2 in the special case that $D = D'$ and $\alpha = code(w^{(k)})$ to obtain the second sentence of the theorem. The final sentence follows from Corollary 3.4. This completes the proof of Theorem B.

The proof of Theorem A is the same, except that we use Lemma 4.2.

Theorem C follows from the above arguments, combined with Theorems 3.2 and 4.3.

Finally, we turn to Proposition 4.6. We prove the contrapositive. Suppose Theorem B's test is inconclusive, that is

$$\left[x^{code(w^{(k)})}\right] \mathfrak{S}_{w^{(1)}} \cdots \mathfrak{S}_{w^{(k-1)}} \neq 0. \tag{8}$$

Claim 4.8 *If $w \in S_n$, then $code(w) + code(w_0 w) = (n-1, n-2, \ldots, 3, 2, 1, 0)$.*

Proof of Claim 4.8 By definition of $D(w)$

$$c_r(w) = (w(r) - 1) - \#\{i < r : w(i) < w(r)\}.$$

On the other hand

$$\begin{aligned} c_r(w_0 w) &= (w_0 w(r) - 1) - \#\{i < r : w_0 w(i) < w_0 w(r)\} \\ &= ((n + 1 - w(r)) - 1) - \#\{i < r : w(r) < w(i)\}. \end{aligned}$$

Hence, $c_r(w) + c_r(w_0 w) = n - r$, as desired. □

By (6) and (8) combined

$$\begin{aligned} &\left[x_1^{n-1} x_2^{n-2} \cdots x_{n-1}\right] \left(\mathfrak{S}_{w^{(1)}} \cdots \mathfrak{S}_{w^{(k-1)}}\right) \mathfrak{S}_{w_0 w^{(k)}} \\ &= \left[x_1^{n-1} x_2^{n-2} \cdots x_{n-1}\right] \left(x^{code(w^{(k)})} + \cdots\right) \left(x^{code(w_0 w^{(k)})} + \cdots\right) \neq 0, \end{aligned}$$

where inequality is by Claim 4.8. Thus, Theorem A's test is inconclusive. □

4.6 A Flexible Version of the Asymmetric Test

The condition (c) in defining Tab' can be replaced by the exponent vector of any monomial in $\mathfrak{S}_{w^{(k)}}$. Unfortunately, the number of such exponent vectors is potentially large. Instead, one can sample points from $\mathcal{S}_{D(w)}$ as follows. Construct the Rothe diagram $D(w)$. Fix a column c of $D(w)$. Suppose the boxes of $D(w)$ in that column are in rows $r_1, r_2, \ldots, r_z$. Find integers $1 \leq x_1 < x_2 < \ldots < x_z$, such that $x_j \leq r_j$. Repeat for every column c. The result is an element of $Tab(D(w), \alpha)$ for some α. (Thus, one can create a randomized version of Theorem B.)

It is possible that, even with choice, no exponent vector exhibits nonvanishing:

Example 4.9 $C^{451623}_{231645,231645} = 0$. Now

$$\mathfrak{S}_{451623} = x_1^3 x_2^3 x_4^2 + x_1^3 x_2^3 x_3 x_4 + x_1^3 x_2^3 x_3^2.$$

Here, $code(451623) = 3302$. One can check that

$$[x^{code(451623)}]\mathfrak{S}^2_{231645} > 0, [x_1^3x_2^3x_4^2]\mathfrak{S}^2_{231645} > 0, \text{ and } [x_1^3x_2^3x_3x_4]\mathfrak{S}^2_{231645} > 0.$$

Thus, Theorem B's test is inconclusive using any choice of monomial from $\mathfrak{S}_{451623}$. □

Individual monomials have no geometric meaning in Schubert calculus. Thus, our tests *seem* inherently combinatorial, as opposed to being avatars of the geometry.

Remark 4.10 Textbook linear programming results implying efficiency of Theorems A and B offer an additional benefit. There is a short certificate when Tab or Tab' is empty. This follows from standard reasoning using Farkas' lemma.

Theorem C provides an alternative certification method. Recording one Schubitope inequality defining $\mathcal{S}_D$ for which $(n, n-1, \ldots, 2, 1)$ fails proves $C_{w^{(1)},\ldots,w^{(k)}} = 0$. (A similar statement holds about $\mathcal{S}_{D'}$.)

5 Comparisons to Other Vanishing Tests

We compare our tests to three non-*ad hoc* vanishing tests. There are examples where our method is successful where the others are not, and *vice versa*.

5.1 Bruhat Order

Bruhat order on S_n is (combinatorially) defined as the reflexive and transitive closure of the covering relations $u \leq ut_{ij}$ if $\ell(ut_{ij}) = \ell(u) + 1$, where t_{ij} is the transposition interchanging i and j. There exist efficient tests to determine $u \leq v$, such as the Ehresmann *tableau criterion* [15, Proposition 2.2.11]. The following is well known; we include a proof, since we do not know where it exactly appears in the literature:

Fact 5.1 *(Bruhat vanishing test)* $C_{w^{(1)},\ldots,w^{(k)}} = 0$ *if* $w^{(i)} \not\leq w_0w^{(j)}$ *for some* i, j.

Proof We prove the case $k = 3$; the general case is similar. Say $u \not\leq w_0w$ but $C_{u,v,w} > 0$. By Lemma 4.1, $C_{u,v}^{w_0w} = C_{u,v,w} > 0$. *Monk's formula* [15, Theorem 2.7.1] states that if $z \in S_n$

$$\sigma_z \smile \sigma_{t_{m,m+1}} = \sum \sigma_{zt_{jk}} \in H^*(X); \tag{9}$$

the sum is over all $j \leq m < k$, such that $\ell(zt_{jk}) = \ell(w) + 1$ and $zt_{jk} \in S_n$. Suppose $s_m := t_{m,m+1}$ and $v = s_{m_1}s_{m_2}\cdots s_{m_{\ell(v)}}$ is a reduced expression for v. By (9), for some $\alpha \in \mathbb{Z}_{>0}$

$$\prod_{i=1}^{\ell(v)} \sigma_{s_{m_i}} = \alpha\, \sigma_v + (\text{positive sum of Schubert classes}). \tag{10}$$

By induction using (9)

$$[\sigma_y]\,\sigma_u \prod_{i=1}^{\ell(v)} \sigma_{s_{m_i}} \neq 0 \iff y \geq u. \tag{11}$$

By the positivity of Schubert calculus, and the assumption $C_{u,v}^{w_0 w} > 0$

$$[\sigma_{w_0 w}]\,\sigma_u(\alpha\,\sigma_v + (\text{positive sum of Schubert classes})) \neq 0.$$

In view of (10), this contradicts (11). □

We give bad news first:

Example 5.2 $(u, v, w) = (1243, 1342, 3142)$ is a vanishing problem detected by Fact 5.1 since $1342 = v \not\leq w_0 w = 2413$. Our methods do not detect $C_{u,v}^{w_0 w} = C_{1243,1342}^{2413}$. Since

$$\mathfrak{S}_{1243}\mathfrak{S}_{1342} = x_2x_3^2 + x_1x_3^2 + 3x_1x_2x_3 + x_2^2x_3 + \underline{x_1x_2^2} + x_1^2x_3 + \underline{x_1^2x_2}$$

contains both monomials of $\mathfrak{S}_{w_0 w} = \mathfrak{S}_{2413} = x_1x_2^2 + x_1^2x_2$, no monomial of $\mathfrak{S}_{w_0 w}$ can be used to detect vanishing. In particular, Theorem B is inconclusive (and hence, by Proposition 4.6, the symmetric test is also inconclusive.) Since $C_{u,v,w} = C_{v,w}^{w_0 u} = C_{u,w}^{w_0 v}$, one hope that the asymmetric method shows either $C_{v,w}^{w_0 u} = C_{1342,3142}^{4312} = 0$ or $C_{u,w}^{w_0 v} = C_{1243,3142}^{4213} = 0$. Unfortunately, both attempts are similarly inconclusive. □

Example 5.3 The vanishing of the Schubert problem $(u, v, w) = (1423, 1423, 1423)$ is undetected by Fact 5.1. Now

$$\mathfrak{S}_{1423}^3 = x_2^6 + 3x_1x_2^5 + 6x_1^2x_2^4 + 7x_1^3x_2^3 + 6x_1^4x_2^2 + 3x_1^5x_2 + x_1^6$$

does not contain $\mathfrak{S}_{w_0} = \mathfrak{S}_{4321} = x_1^3x_2^2x_3$, and hence, vanishing is seen by Theorem A. □

5.2 A. Knutson's Descent Cycling

In [12], A. Knutson introduced a vanishing criterion. Recall that $u \in S_n$ has a *descent* at position i if $u(i) > u(i+1)$ and has an *ascent* at position i otherwise. That is, respectively, $us_i \leq u$ and $us_i \geq u$.

Fact 5.4 (dc triviality) *If* (u, v, w) *is a Schubert problem, such that* $us_i \geq u$, $vs_i \geq v$, $ws_i \geq w$, *then* $C_{u,v,w} = 0$.

Example 5.5 The triple $(1423, 1423, 1342)$ is dc trivial, and hence, $C_{1423,1423,1342} = 0$. Here, the asymmetric test (Theorem B) is inconclusive (again, thus by Proposition 4.6, the symmetric test is also inconclusive). Indeed, $C_{u,v}^{w_0 w} = C_{1423,1423}^{4213} = 0$ is not detected, since

$$\mathfrak{S}_{1423}^2 = x_2^4 + 2x_1x_2^3 + 3x_1^2x_2^2 + \underline{2x_1^3x_2} + x_1^4,$$

but $\mathfrak{S}_{w_0w} = \mathfrak{S}_{4213} = x_1^3x_2$. Also, $C_{u,w}^{w_0v} = 0$ and $C_{v,w}^{w_0u} = 0$ are not detected, since

$$\mathfrak{S}_{1423}\mathfrak{S}_{1342} = x_2^3x_3 + 2x_1x_2^2x_3 + 2x_1^2x_2x_3 + \underline{x_1^3x_3} + x_1x_2^3 + x_1^2x_2^2 + \underline{x_1^3x_2}.$$

Since $\mathfrak{S}_{w_0u} = \mathfrak{S}_{w_0v} = \mathfrak{S}_{4132} = x_1^3x_3 + x_1^3x_2$, no lattice point in $\mathcal{S}_{D(4132)}$ proves vanishing. □

Example 5.6 The Schubert problem (3256147, 2143657, 4632175) from Example 1.1 is not dc trivial, but $C_{3256147,2143657,4632175} = 0$, as determined by Theorem A. □

Define the *descent cycling equivalence* $\sim$ on Schubert problems by

(dc.1) $(u, v, w) \sim (us_i, v, ws_i), (u, vs_i, ws_i)$ if $us_i \geq u, vs_i \geq v, ws_i \leq w$;
(dc.2) $(u, v, w) \sim (us_i, v, ws_i), (us_i, vs_i, w)$ if $us_i \leq u, vs_i \geq v, ws_i \geq w$;
(dc.3) $(u, v, w) \sim (u, vs_i, ws_i), (us_i, vs_i, w)$ if $vs_i \leq v, us_i \geq u, ws_i \geq w$.

Therefore, $C_{u,v,w} = 0$ if (u, v, w) is $\sim$ equivalent to a dc trivial problem.

Example 5.7 As reported in [12], for $n = 6$, there is one dc equivalence class of problems (u, v, w) which vanishes, but does not contain a dc trivial triple. This is precisely the problem studied in Example 4.9, which our methods also cannot explain. □

Example 5.8 Let $(u, v, w) = (3216547, 3216547, 4261573)$ be a problem in S_7. Theorem A shows $C_{u,v,w} = 0$ (any element of Tab must contain at least seven 1's). The $\sim$ class contains 9 elements, namely

(3216574, 3261547, 4216537), (3216547, 3216574, 4261537), (3261547, 3216574, 4216537),
(3261547, 3216547, 4216573), (3216574, 3216547, 4261537), (3216547, 3216547, 4261573),
(3261574, 3216547, 4216537), (3216547, 3261574, 4216537), (3216547, 3261547, 4216573).

None are dc trivial. □

5.3 K. Purbhoo's Root Games

K. Purbhoo's *root games* from [18] give a vanishing criteria. Fix the positive roots Φ^+ associated with GL_n to be $\alpha_{i,j} = \varepsilon_i - \varepsilon_j$ for $1 \leq i < j \leq n$, where ε_i is the ith standard basis vector. The poset P of positive roots takes the form shown in Fig. 2.

The maximal element of this poset is the highest root α_{1n}. For each i, place a token • in square α_{mn} if $w^{(i)}(m) > w^{(i)}(n)$. This is called the *initial position*. An *upper order ideal* A is an up-closed subset of P, a subset containing any roots lying above any of its members (see Fig. 2). This initial position is *doomed* if there exists an upper order ideal A, such that there are more tokens in A than $\#A$. This is [18, Theorem 3.6]:

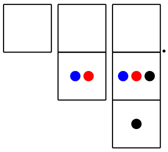
Fig. 2 The poset P of positive roots associated with GL_7

α_{12}	α_{13}	α_{14}	α_{15}	α_{16}	α_{17}
	α_{23}	α_{24}	α_{25}	α_{26}	α_{27}
		α_{34}	α_{35}	α_{36}	α_{37}
			α_{45}	α_{46}	α_{47}
				α_{56}	α_{57}
					α_{67}

Fact 5.9 *(Doomed root game) If $(w^{(1)}, \ldots, w^{(k)})$'s initial position is doomed, $C_{w^{(1)},\ldots,w^{(k)}} = 0$.*

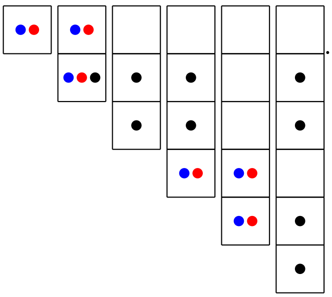
This test is quite handy. However, the number of upper order ideals for type A_{n-1} is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, which is exponential in n.

Example 5.10 The vanishing of (1423, 1423, 1342) is seen by Fact 5.9. This is doomed



As is explained in Example 5.5, our methods are inconclusive here. □

Example 5.11 Let $u = v = 3216547$ and $w = 1652473$. Below, we mark the inversions of u, v, w with •, •, • respectively



This game is not doomed, so Fact 5.9 is inconclusive here. (Descent cycling does not help either, as the equivalence class of size 9 contains no dc trivial elements.) Also, Theorem A does not succeed. However, Theorem B's test shows $C_{u,v}^{w_0 w} = C_{3216547,3216547}^{7236415} = 0$. □

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References

1. Adve, A., Robichaux, C., Yong, A.: Computational complexity, Newton polytopes, and Schubert polynomials. Sém. Lothar. Combin. 82B:Art. 52, 12 (2020)
2. Aguiar, M., Ardilla, F.: Hopf monoids and generalized permutahedra (2017). arXiv:1709.07504 [math]
3. Ardila, F., Benedetti, C., Doker, J.: Matroid polytopes and their volumes. Discrete Comput. Geom. **43**(4), 841–854 (2010)
4. Belkale, P., Kumar, S.: Eigenvalue problem and a new product in cohomology of flag varieties. Invent. Math. **166**(1), 185–228 (2006)
5. Billey, S.C.: Kostant polynomials and the cohomology ring for G/B. Duke Math. J. **96**(1), 205–224 (1999)
6. De Loera, J., McAllister, T.: On the computation of Clebsch–Gordan coefficients and the dilation effect. Exp. Math. **15**(1), 7–19 (2006)
7. Fink, A., Mészáros, K., St. Dizier, A.: Schubert polynomials as integer point transforms of generalized permutahedra. Adv. Math. **332**, 465–475 (2018)
8. Fulton, W.: Young tableaux. With applications to representation theory and geometry. London Mathematical Society Student Texts, vol. 35, pp. x+260. Cambridge University Press, Cambridge (1997)
9. Fulton, W..: Eigenvalues, invariant factors, highest weights, and Schubert calculus. Bull. Am. Math. Soc. (N.S.) **37**(3), 209–249 (2000)
10. Gillespie, M.: Variations on a theme of Schubert calculus. In: Barcelo, H., Karaali, G., Orellana, R. (eds.) Recent Trends in Algebraic Combinatorics. Association for Women in Mathematics Series, vol 16. Springer (2019)
11. Kleiman, S.L., Laksov, D.: Schubert calculus. Am. Math. Mon. **79**, 1061–1082 (1972)
12. Knutson, A.: Descent-cycling in Schubert calculus. Exp. Math. **10**(3), 345–353 (2001)
13. Knutson, A.: A Schubert calculus recurrence from the noncomplex W-action on G/B (2003). arXiv:math/0306404
14. Knutson, A., Tao, T.: The honeycomb model of $\mathrm{GL}_n(\mathbf{C})$ tensor products. I. Proof of the saturation conjecture. J. Am. Math. Soc. **12**(4), 1055–1090 (1999)
15. Manivel, L., Symmetric functions, Schubert polynomials and degeneracy loci. Translated from the 1998 French original by John R. Swallow. SMF/AMS Texts and Monographs. American Mathematical Society, Providence (2001)
16. Monical, C., Tokcan, N., Yong, A.: Newton polytopes in algebraic combinatorics. Sel. Math. (N.S.) **25**(5), 37, Paper No. 66 (2019)
17. Mulmuley, K., Narayanan, H., Sohoni, M.: Geometric complexity theory III: on deciding nonvanishing of a Littlewood–Richardson coefficient. J. Algebraic Comb. **36**(1), 103–110 (2012)
18. Purbhoo, K.: Vanishing and nonvanishing criteria in Schubert calculus. Int. Math. Res. Not. Art. ID 24590, 38 (2006)
19. Rado, R.: An inequality. J. Lond. Math. Soc. **27**, 1–6 (1952)
20. Reiner, V., Shimozono, M.: Key polynomials and a flagged Littlewood–Richardson rule. J. Comb. Theory Ser. A **70**(1), 107–143 (1995)
21. Schrijver, A.: Combinatorial optimization. Polyhedra and efficiency. Vol. B. Matroids, trees, stable sets. Chapters 39–69. Algorithms and Combinatorics, vol. 24, B, pp. i–xxxiv and 649–1217. Springer, Berlin (2003)