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RESEARCH CONTRIBUTION

On Partial Differential Operators Which Annihilate the Roots of the Universal Equation of Degree *k*

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Abstract
The aim of this paper is to study in details the regular holonomic $D-$module introduced in Barlet (Math Z 302 $n^0 3$: 1627–1655, 2022 arXiv:1911.09347 [math]) whose local solutions outside the polar hyper-surface $\{\Delta(\sigma).\sigma_k = 0\}$ are given by the local system generated by the power λ of the local branches of the multivalued function which is the root of the universal degree k equation $z^k + \sum_{h=1}^{k}(-1)^h\sigma_h z^{k-h} = 0$. We show that for $\lambda \in \mathbb{C}\setminus\mathbb{Z}$ this D-module is the minimal extension of the holomorphic vector bundle with an integrable meromorphic connection with a simple pole which is its restriction on the open set $\{\sigma_k\Delta(\sigma) \neq 0\}$. We then study the structure of these D-modules in the cases where $\lambda = 0, 1, -1$ which are a little more complicated, but which are sufficient to determine the structure of all these D-modules when λ is in $\mathbb{Z}$.
As an application we show how these results allow to compute, for instance, the Taylor expansion of the root near -1 of the equation:

$$z^k + \sum_{h=-1}^{k}(-1)^h\sigma_h z^{k-h} - (-1)^k = 0.$$

near $z^k - (-1)^k = 0$.

Keywords Regular holonomic system · $\mathcal{D}$-modules · Trace functions · Minimal extension

Mathematics Subject Classification 32C38 · 35Q15 · 14F10 · 35C10 · 32S40

A Milène qui a supporté courageusement deux confinements avec cette équation.

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1 Introduction

1.1 .

There are several ways to define an interesting function. Of course, the simplest one is to give its value at each point by an explicit finite formula or as a sum of an infinite series (converging somewhere at least). Another way is to give a functional equation which characterizes it. A third approach is to give a partial differential system which has our function f as its unique solution (up to normalisation).

For instance, the function $f(z) = e^z$ may be defined as

1. $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.
2. $f(z + z') = f(z).f(z')$ with $f(0) = 1$ and $f(1) = e$.
3. $\frac{\partial f}{\partial z} = f$ and $f(0) = 1$.

In general, to increase our understanding of such a function it is useful to dispose of at least two kinds of characterization as above. For instance, in the basic example of e^z the description 3. gives easily the formula 1 and also the functional equation 2. Note that the third approach will often lead to a description of the first kind via the Taylor expansion at least when we dispose of a regular holonomic system defining f which is enough simple and suitably described to allow an inductive explicite computation of the coefficients of the Taylor expansion. However, this means that we are at least able to well describe essentially all partial differential operators which annihilate f.

We shall consider, in this paper, the case of the multivalued function $z(\sigma)^\lambda$ on $\mathbb{C}^k$, with $\sigma := (\sigma_1, \dots, \sigma_k)$ and λ a complex parameter, where $z(\sigma)$ is defined as the root of the universal monic polynomial of degree k:

$$P_\sigma(z) := \sum_{h=0}^{k} (-1)^h \sigma_h z^{k-h} \quad \text{with the convention} \quad \sigma_0 \equiv 1.$$

It is well-known that the description of this function with the first approach is quite difficult (at least for $k \geq 5$). The definition given above of this multivalued function may be seen as a description of the second kind.

The aim of this paper is to give a description of the third kind which characterizes this multivalued function. More precisely we describe completely the structure of the regular holonomic $\mathcal{D}_N$-module[1] $\mathcal{D}_N \big/ \mathcal{J}_\lambda$ where $\mathcal{J}_\lambda$ is the left ideal in $\mathcal{D}_N$ which annihilates $z^\lambda(\sigma)$.

The case where λ is in $\mathbb{Z}$ is of special interest (for $\lambda = 0$ the left ideal $\mathcal{J}_0$ will be defined in a natural way inside the annihilator of the function 1) and is less simple.

Let me explain how I come to study this question.

In the article [1], we characterize the trace functions F on $N := \mathbb{C}^k$ as a solutions of a sub-holonomic $\mathcal{D}_N$-module $\mathcal{M}$ given by explicit generators in the Weyl algebra. Recall that an entire function F on N is a trace function when there exists a entire function f on $\mathbb{C}$, such that

$$F(\sigma) = \sum_{j=1}^{k} f(z_j(\sigma))$$

where $z_1(\sigma), \dots z_j(\sigma)$ are the roots of the polynomial $P_\sigma(z) := z^k + \sum_{h=1}^{k} (-1)^h \sigma_h z^{k-h}$.

Then, adding the quasi-homogeneity condition $U_0 - \lambda$, where $U_0 := \sum_{h=1}^{k} h\sigma_h \partial_h$ is the expression in $\sigma_1, \dots, \sigma_k$ of the Euler vector field $\sum_{j=1}^{k} z_j \partial_{z_j}$, to the ideal annihilating trace functions defines a (regular) holonomic $\mathcal{D}_N$-module $\mathcal{N}_\lambda$ whose local solutions are now given by linear combinations of the branches of the multi-valued function $z(\sigma)^\lambda$. Therefore, the goal of this article to understand the structure of theses regular holonomic $\mathcal{D}_N$-modules for each value of the parameter $\lambda \in \mathbb{C}$.

We give the statements of the main results in detail in Sect. 1.2 below

As an application of the structure theorem for the $\mathcal{D}$-module $\mathcal{N}_1 := \mathcal{D}_N \big/ \mathcal{J}_1$ we compute the Taylor series at the point $\sigma^0 = (0, 0, \dots, -1)$ of the holomorphic function of $\sigma_1, \dots, \sigma_k$ which gives the root of the polynomial:

$$z^k + \sum_{h=1}^{k} (-1)^h \sigma_h z^{k-h} - (-1)^k = 0$$

[1] N will be $\mathbb{C}^k$ with coordinates $\sigma_1, \dots, \sigma_k$.

which is near -1. The fact that the associated $\mathcal{D}_N$-module corresponding to $\lambda = 1$ is not simple make this computation quite complicate when we use the $\mathcal{D}_N$-module $\mathcal{N}_1$ itself. However, with the remark that $z(\sigma) - \sigma_1/k$ is a solution of the simple part $\mathcal{N}_1^{\square}$ of this $\mathcal{D}$-module deduced from the structure Theorem 4.1.9, we obtain a complete explicit computation of the Taylor series at σ^0, corresponding to the equation $z^k - (-1)^k = 0$, of the root which is near -1.

Of course, this method is valid to compute (with some more numerical complications, but without theoretical difficulty) the Taylor expansion of any (uni-valued) holomorphic branch of the multivalued function $z(\sigma)^\lambda$ near any point $\sigma^0 \in N$ for any given complex number λ.

To conclude this introduction, let me remark that we produce in this article an explicit description of the image via Riemann–Hilbert correspondence of the minimal extension of a rather involved local system defined in the complement of a hypersurface with rather complicated singularities: the discriminant.

1.2 The Results

To make the statements clear we have to precise some notations which will be used (and defined again) in the text.

Notations

1. The coordinates on $M := \mathbb{C}^k$ are $z_1, \dots, z_k$ and their elementary symmetric functions $\sigma_1, \dots, \sigma_k$ are the coordinates on $N \simeq \mathbb{C}^k$. The corresponding quotient map by the symmetric group is denoted by $quot : M \to N$.
2. D_N is the sheaf of holomorphic differential operators on N and $\partial_1, \dots, \partial_k$ are the partial derivative in $\sigma_1, \dots, \sigma_k$.
3. For each integer $p \geq -1$ we denote U_p the image by the tangent map of the quotient map $quot$ of the vector field $\sum_{j=1}^k z_j^{p+1} \frac{\partial}{\partial z_j}$.
   The vector fields U_{-1}, U_0 and U_1 on N are given by

$$U_{-1} := \sum_{h=0}^{k-1} (-1)^h (k-h)\sigma_h \partial_{h+1}, \quad U_0 := \sum_{h=1}^{k} h\sigma_h \partial_h,$$
$$U_1 := \sum_{h=1}^{k} (\sigma_1\sigma_h - (h+1)\sigma_{h+1})\partial_h.$$

   with the convention $\sigma_0 \equiv 1$ and $\sigma_{k+1} = 0$.
4. For λ a complex number, the D_N-module $\mathcal{N}_\lambda$ is the quotient of D_N by the left ideal $\mathcal{J}_\lambda$ generated by the following global sections of D_N:
   - $A_{i,j} := \partial_i\partial_j - \partial_{i+1}\partial_{j-1}$ for $i \in [1, k-1]$ and $j \in [2, k]$
   - $\mathcal{T}^m := \partial_1\partial_{m-1} + \partial_m E$ for $m \in [2, k]$, where $E := \sum_{h=1}^k \sigma_h \partial_h$
   - $U_0 - \lambda$.

5. We denote $\mathcal{O}_N(\star\sigma_k)$ the left D_N-module $\mathcal{O}_N[\sigma_k^{-1}]$.The D_N-module quotient $\mathcal{O}_N(\star\sigma_k)\big/\mathcal{O}_N$ will be denoted $\underline{H}^1_{[\sigma_k=0]}(\mathcal{O}_N)$.
6. We define $\mathcal{N}_1^*$ as the kernel of the map $\varphi_1 : \mathcal{N}_1 \to \mathcal{O}_N$ sending 1 to σ_1.
7. We define $\mathcal{N}_0^*$ as the kernel of the map $\varphi_0 : \mathcal{N}_0 \to \mathcal{O}_N$ sending 1 to 1.
8. We define $\mathcal{N}_{-1}^*$ as the kernel of the map $\varphi_{-1} : \mathcal{N}_{-1} \to \mathcal{O}_N(\star\sigma_k)$ sending 1 to σ_{k-1}/σ_k.

1.2.1 The Case $\lambda \in \mathbb{C} \setminus \mathbb{Z}$

Theorem 1.2.1 *Let $\lambda \in \mathbb{C} \setminus \mathbb{Z}$. Then, $\mathcal{N}_\lambda$ is the minimal extension of the associated meromorphic connection (see 3.3.11). Therefore, $\mathcal{N}_\lambda$ is a simple $\mathcal{D}_N$-module.*

Moreover, for each $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ we have an isomorphism $\mathcal{T}_{\lambda+1} := \square U_{-1} : \mathcal{N}_\lambda \to \mathcal{N}_{\lambda+1}$ given by right multiplication by U_{-1}.

See Theorem 3.3.11 for details.

1.2.2 The Case $\lambda \in \mathbb{N}^*$

Theorem 1.2.2 *The diagram below describes the structure of $\mathcal{N}_1$. The torsion Θ in $\mathcal{N}_1$ is generated by the class of $\partial_k U_{-1}$ and we have an isomorphism $\underline{H}^1_{[\sigma_k=0]}(\mathcal{O}_N) \to \Theta$ defined by sending $1/\sigma_k$ to $[\partial_k U_{-1}]$.*

The $\mathcal{D}_N$-modules $\Theta \simeq \underline{H}^1_{[\sigma_k=0]}(\mathcal{O}_N)$ and $\mathcal{N}_1^\square$ are simple $\mathcal{D}_N$-modules.

The following commutative diagram of left $\mathcal{D}_N$-modules has exact lines and columns where the maps i and e are defined by $i([U_{-1}]) = [U_{-1}]$ (so the map i induced by the identity of $\mathcal{N}_1$) and $e([U_{-1}]) = k$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \Theta \arrow[r] \arrow[d] & Im(\mathcal{T}_1) \arrow[r, "i"] \arrow[d] & \mathcal{O}_N[U_{-1}] \arrow[r] \arrow[d, "e", "\simeq"'] & 0 \\
0 \arrow[r] & \mathcal{N}_1^* \arrow[r] \arrow[d] & \mathcal{N}_1 \arrow[r, "\varphi_1"] \arrow[d] & \mathcal{O}_N \arrow[r] & 0 \\
 & \mathcal{N}_1^*/\Theta \arrow[r, "\theta"] \arrow[d] & \mathcal{N}_1^\square \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Moreover, for any $p \in \mathbb{N}^*$ the D_N-module $\mathcal{N}_p$ is isomorphic to $\mathcal{N}_{p+1}$ via the map $\mathcal{T}_{p+1} := \square U_{-1} : \mathcal{N}_p \to \mathcal{N}_{p+1}$ given by right multiplication by U_{-1}.

See Theorems 4.1.5 and 4.1.9.

1.2.3 The Case $\lambda = 0$

Theorem 1.2.3 *Define $\mathcal{N}_0^\square := \mathcal{D}_N U_1 \subset \mathcal{N}_0^*$. Then, $\mathcal{N}_0^\square$ is simple and isomorphic to $\mathcal{N}_1^\square$ via the map induced by the map $\square U_1 : \mathcal{N}_1 \to \mathcal{N}_0$, the quotient $\mathcal{N}_0 \big/ \mathcal{N}_0^\square$ is isomorphic to $\mathcal{O}_N(\star\sigma_k)$ and the quotient $\mathcal{N}_0 \big/ \mathcal{N}_0^*$ is isomorphic to $\mathcal{O}_N$.*

The structure of $\mathcal{N}_0$ is described by the exact sequences of D_N-modules:

$$0 \longrightarrow \mathcal{N}_0^\square \longrightarrow \mathcal{N}_0^* \longrightarrow \underline{H}^1_{[\sigma_k=0]}(\mathcal{O}_N) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{N}_0^\square \longrightarrow \mathcal{N}_0 \longrightarrow \mathcal{O}_N(\star\sigma_k) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{N}_0^\square \longrightarrow \mathcal{N}_0 \xrightarrow{\square U_{-1}} \mathcal{N}_1 \longrightarrow \mathcal{N}_1^\square \longrightarrow 0$$

See Theorem 4.2.4.

1.2.4 The Case $\lambda \in -\mathbb{N}^*$

Theorem 1.2.4 *We have the following commutative diagram of $\mathcal{D}_N$-module with exact lines and columns, where the $\mathcal{D}_N$-linear map $\varphi_{-1} : \mathcal{N}_{-1} \to \mathcal{O}_N(\star\sigma_k)$ is defined by $\varphi_{-1}(1) = \sigma_{k-1}\big/\sigma_k$:*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{N}_{-1}^* \arrow[r] \arrow[d, "="] & \mathcal{N}_{-1}^* \oplus \mathcal{O}_N U_1 \arrow[r, "\varphi_{-1}"] \arrow[d] & \mathcal{O}_N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{N}_{-1}^* \arrow[r] \arrow[d] & \mathcal{N}_{-1} \arrow[r, "\varphi_{-1}"] \arrow[d] & \mathcal{O}_N(\star\sigma_k) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathcal{Q} \arrow[r, "\chi"] \arrow[d] & \underline{H}^1_{[\sigma_k=0]}(\mathcal{O}_N) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Therefore, χ is an isomorphism. Moreover, the map $\mathscr{T}_0$ induces an isomorphism of $\mathcal{N}_{-1}^*$ onto the simple $\mathcal{D}_N$-module $\mathcal{N}_0^\square = \mathcal{D}_N U_1 \subset \mathcal{N}_0^*$. Therefore, $\mathcal{N}_{-1}^*$ is simple.

Moreover, for each $p \in -\mathbb{N}^*$ we have an isomorphism $\mathscr{T}_p : \square U_{-1} : \mathcal{N}_{p-1} \to \mathcal{N}_p$ given by right multiplication by U_{-1}.

See Theorem 4.3.8.

Section 2 is devoted to preliminaries, Sect. 3 concludes by the simplicity of $\mathcal{N}_\lambda$ for $\lambda \notin \mathbb{Z}$ and Sect. 4 studies the cases where λ is in $\mathbb{Z}$.

Section 5 gives the application and Sect. 6 is an appendix useful in the study of the characteristic variety of the D_N-modules $\mathcal{N}_\lambda$.

2 The D_N-Modules $\mathcal{W}$ and $\mathcal{M}$

Notations We fix in the sequel and integer $k \geq 2$. Let $\mathbb{C}[\sigma]\langle\partial\rangle$ be the Weyl algebra in the variables $\sigma_1, \ldots, \sigma_k$. We shall note $N = \mathbb{C}^k$ which is the target of the quotient map:

$$quot : M := \mathbb{C}^k \to \mathbb{C}^k \big/ \mathfrak{S}_k = N \simeq \mathbb{C}^k$$

by the natural action of the permutation group $\mathfrak{S}_k$ on $\mathbb{C}^k$. We shall note T_{quot} its tangent map.

Then, $\mathcal{D}_N$ denotes the sheaf of holomorphic differential operators on N and we shall use the same notations for modules on $\mathbb{C}[\sigma]\langle\partial\rangle$ and for the corresponding sheaves of $\mathcal{D}_N$-modules on N.

We denote $\Delta \in \mathbb{C}[\sigma]$ the discriminant of the polynomial $P_\sigma(z) := z^k + \sum_{h=1}^k (-1)^h \sigma_h z^{k-h}$ and $H_\Delta := \{\sigma \in N \ / \ \Delta(\sigma) = 0\}$ the corresponding hypersurface in N.

For basic results on $\mathcal{D}$-modules the reader may consult, for instance, the books [5] or [6].

2.1 The D−Module $\mathcal{W}$

In this section, we shall consider the $\mathcal{D}_N$-module $\mathcal{W} := \mathcal{D}_N \big/ \mathcal{A}$ where $\mathcal{A}$ is the left ideal sheaf in $\mathcal{D}_N$ generated by

$$A_{i,j} := \partial_i \partial_j - \partial_{i+1}\partial_{j-1} \quad \text{for} \quad i \in [1, k-1] \quad \text{and} \quad j \in [2, k] \tag{1}$$

Notations Let $\mathcal{D}_N(m)$ be the sub-sheaf of $\mathcal{D}_N$ of partial differential operators of order at most equal to m. Then, let $\mathcal{W}(m)$ be the sub-$\mathcal{O}_N$-module in $\mathcal{W}$ of the classes induced by germs in $\mathcal{D}_N(m)$.

Let $\mathcal{W}(m)$ be the sub$-\mathcal{O}_N$-module in $\mathcal{W}$ of the classes induced by germs in $\mathcal{D}_N(m)$.

As we have $\mathcal{A}(m) := \mathcal{D}_N(m) \cap \mathcal{A} = \sum_{i,j} \mathcal{D}_N(m-2)A_{i,j}$ for each $m \in \mathbb{N}$, the quotient $\mathcal{W}(m) = \mathcal{D}_N(m) \big/ \mathcal{A}(m)$ injects in $\mathcal{W}$ and we have

$$\mathcal{W} = \cup_{m \geq 0} \mathcal{W}_m.$$

Note that $\mathcal{A}(1) = 0$ so $\mathcal{W}(1) = \mathcal{D}_N(1)$.

It is clear that the characteristic variety of the $\mathcal{D}_N$-module $\mathcal{W}$ is equal to $N \times S(k)$ in the cotangent bundle $T_N^* \simeq N \times \mathbb{C}^k$ of N, where $S(k)$ is the algebraic cone in $\mathbb{C}^k$ defined by the equations

$$\eta_i \eta_j - \eta_{i+1}\eta_{j-1} = 0 \quad \forall i \in [1, k-1] \quad \text{and} \quad \forall j \in [2, k]. \tag{2}$$

We describe this two-dimensional cone and the corresponding ideal in the appendix (see Sect. 6). We shall use in the present section the following results which are proved in the appendix (Proposition 6.1.5 and Corollary 6.1.6).

Proposition 2.1.1 *Let* $L_1 := \{\eta_1 = 0\} \cap S(k)$ *and* $L_k := \{\eta_k = 0\} \cap S(k)$. *Then,* L_1 *is the line directed by the vector* $(0, \dots, 0, 1)$ *and* L_k *the line directed by the vector* $(1, 0, \dots, 0)$. *The maps* $\varphi_1 : S(k)\setminus L_1 \to \mathbb{C}^* \times \mathbb{C}$ *and* $\varphi_k : S(k)\setminus L_k \to \mathbb{C}^* \times \mathbb{C}$ *which are defined by the formulas*

$$\varphi_1(\eta) := (\eta_1, -\eta_2/\eta_1) \quad \text{and} \quad \varphi_k(\eta) := (\eta_k, -\eta_{k-1}/\eta_k) \tag{3}$$

are isomorphisms. Therefore, $S(k) \setminus \{0\}$ *is smooth and connected surface.*

Corollary 2.1.2 *The ideal of* $\mathbb{C}[\eta]$ *defined by the equations in* (2) *is prime. Moreover,* $S(k)$ *is a normal surface.*

Notation For α in $\mathbb{N}^k$ define $q := |\alpha| := \sum_{h=1}^k \alpha_h$ and $r := w(\alpha) = \sum_{h=1}^k h\alpha_h$.

Definition 2.1.3 Let P be a germ of section of $\mathcal{D}_N$. We say that P is **bi-homogeneous of type** (q, r) if we may write

$$P = \sum_{|\alpha|=q, w(\alpha)=r} a_\alpha \partial^\alpha$$

where a_α are germs of holomorphic functions in N.

It is clear that any germ P of section of $\mathcal{D}_N$ has a unique decomposition:

$$P = \sum_{q,r} P_{q,r}$$

where $P_{q,r}$ is a bi-homogeneous germ of section of $\mathcal{D}_N$ of type (q, r). Note that this sum is finite because for a given order q the corresponding type (q, r) has non zero representative only when r is in $[q, kq]$.

Lemma 2.1.4 *Let* P *be a germ of section of* $\mathcal{D}_N$ *and write the decomposition of* P *in its bi-homogeneous components as* $P = \sum_{q,r} P_{q,r}$. *Then,* P *is a germ of section in* $\mathcal{A}$ *if and only if for each type* (q, r) $P_{q,r}$ *is a germ of section in* $\mathcal{A}$.

Proof It is clear that P is in $\mathcal{A}$ when each $P_{q,r}$ is in $\mathcal{A}$. Conversely, assume that P is in $\mathcal{A}$. Then, we may write

$$P = \sum_{(i,j)} B_{i,j} A_{i,j} \quad \text{with} \quad i \in [1, k-1] \quad \text{and} \quad j \in [2, k]$$

and where $B_{i,j}$ are germs of sections of $\mathcal{D}_N$.

Write $B_{i,j} = \sum_{q,r}(B_{i,j})_{q,r}$ the decomposition of $B_{i,j}$ in its bi-homogeneous components; this gives

$$P = \sum_{q,r}\sum_{p\geq 0}\left(\sum_{i+j=p}(B_{i,j})_{q,r}A_{i,j}\right)$$

where $\sum_{i+j=p}(B_{i,j})_{q,r}A_{i,j}$ is bi-homogeneous of type $(q+2, r+p)$ for each (i,j), such that $i+j = p$. This implies that $P_{q,r}$ is equal to the sum $\sum_{i+j+s=r}(B_{i,j})_{q-2,s}A_{i,j}$. Therefore, each $P_{q,r}$ is a germ of section in $\mathcal{A}$. ■

Lemma 2.1.5 *The class of ∂^α in $\mathcal{W}$ only depends on $q := |\alpha|$ and $r := w(\alpha)$. It will be denoted $y_{q,r}$. Moreover, if $\mathcal{W}_q$ is the sub$-\mathcal{O}_N$-module of $\mathcal{W}$ generated by the $y_{q,r}$ for $r \in [q, kq]$, $\mathcal{W}_q$ is a free $\mathcal{O}_N$-module of rank $kq - q + 1$ with basis $y_{q,q}, y_{q,q+1}, \ldots, y_{q,kq}$ and, as $\mathcal{O}_N$-module, we have the direct decompositions:*

$$\mathcal{W}(m) = \oplus_{q=0}^{m}\mathcal{W}_q \quad \text{and} \quad \mathcal{W} = \oplus_{q\in\mathbb{N}}\,\mathcal{W}_q. \tag{4}$$

Remark that the action of $\mathcal{D}_N$ on $\mathcal{W}$ is defined by

$$\partial_j(y_{q,r}) = y_{q+1,r+j} \quad \forall j \in [1,k] \quad \forall q \in \mathbb{N} \quad \text{and} \quad \forall r \in [q, kq] \tag{5}$$

and that $\partial_j\mathcal{W}_q \subset \mathcal{W}_{q+1}$.

Proof The fact that the class induced by ∂^α in $\mathcal{W}$ depends only on $|\alpha|$ and $w(\alpha)$ is a direct consequence of the fact that the class induced by x^α in $\mathbb{C}[x_1,\ldots,x_k]\big/IS(k)$ only depends on $q := |\alpha|$ and $r := w(\alpha)$ (see Proposition 6.1.1 in the appendix). Then, it is clear that $y_{q,q}, y_{q,q+1}, \ldots, y_{q,kq}$ is a $\mathcal{O}_N-$basis of $\mathcal{W}_q$ looking at the symbols and using the appendix (Sect. 6) over the sheaf of $\mathbb{C}-$algebras $\mathcal{O}_N$. ■

The global polynomial solutions of the $\mathcal{D}_N$-module $\mathcal{W}$ are described by our next lemma.

Definition 2.1.6 For each $q \in \mathbb{N}$ and each $r \in [q, kq]$ define the polynomial

$$m_{q,r}(\sigma) := \sum_{|\alpha|=q,\, w(\alpha)=r}\frac{\sigma^\alpha}{\alpha!} \tag{6}$$

Lemma 2.1.7 *Any $m_{q,r} \in \mathbb{C}[\sigma_1,\ldots,\sigma_k]$ is annihilated by the left ideal $\mathcal{A}$ in $\mathcal{D}_N$ and if a polynomial $P \in \mathbb{C}[\sigma_1,\ldots,\sigma_k]$ is annihilated by $\mathcal{A}$, P is, in a unique way, a $\mathbb{C}-$linear combination of the $m_{q,r}$ for $q \geq 0$ and $r \in [q, kq]$ which gives the bi-homogeneous decomposition of $P(\partial_1,\ldots,\partial_k) \in \mathbb{C}[\partial_1,\ldots,\partial_k]$ (see Lemma 2.1.4).*

Proof First, we shall verify that each polynomial m_r^q is annihilated by each $A_{i,j}$ for all $i \in [1, k-1]$ and all $j \in [2, k]$. We have for each $(i,j) \in [1,k]^2$:

$$\partial_i\partial_j(m_{q,r})(\sigma) = \sum_{|\beta|=q-2,\, w(\beta)=r-(i+j)}\frac{\sigma^\beta}{\beta!} = m_{q-2,r-(i+j)(\sigma)}$$

because $\alpha_i \alpha_j = 0$ implies $\partial_i \partial_j \sigma^\alpha = 0$. The right hand-side above only depends on $i + j$ for q and r fixed. This is enough to conclude our verification.

Note also that the uniqueness is obvious because of the uniqueness of the Taylor expansion of a polynomial.

Let now $P := \sum_\alpha c_\alpha \frac{\sigma^\alpha}{\alpha!}$ a polynomial in $\mathbb{C}[\sigma]$ which is annihilated by the left ideal $\mathcal{A}$ in $\mathcal{D}_N$. We want to show that if α and β satisfy $|\alpha| = |\beta|$ and $w(\alpha) = w(\beta)$ we have $c_\alpha = c_\beta$. It is enough to prove this equality when there exist $i \in [1, k-1]$, $j \in [2, k]$ and $\gamma \in \mathbb{N}^k$, such that $\sigma^\alpha = \sigma_i \sigma_j \sigma^\gamma$ and $\sigma^\beta = \sigma_{i+1} \sigma_{j-1} \sigma^\gamma$ by definition of the equivalence relation[2] given by $|\alpha| = |\beta|$ and $w(\alpha) = w(\beta)$. In this case the coefficient of $\sigma^\gamma/\gamma!$ in $\partial_i \partial_j P$ is c_α and in $\partial_{i+1} \partial_{j-1} P$ is c_β. Therefore, they are equal. ■

It is easy to see that an entire holomorphic function $F : N \to \mathbb{C}$ is solution of $\mathcal{W}$ if and only if its Taylor series at the origin may be written, for some choice of $c_{q,r} \in \mathbb{C}$:

$$F(\sigma) = \sum_{q,r} c_{q,r} m_{q,r}(\sigma).$$

In the same way, a holomorphic germ $f : (N, \sigma^0) \to (\mathbb{C}, z^0)$ is solution of $\mathcal{W}$ if and only if its Taylor series may be written in the form

$$f(\sigma^0 + \sigma) = \sum_{q,r} c_{q,r} m_{q,r}(\sigma)$$

with $c_{0,0} = z^0$.

2.2 The $\mathcal{D}_N$-Module $\mathcal{M}$

Definition 2.2.1 Let $m \in [2, k]$ be an integer and define the second-order differential operators in the Weyl algebra $\mathbb{C}[\sigma]\langle\partial\rangle$

$$\mathcal{T}^m := \partial_1 \partial_{m-1} + \partial_m E \quad \text{for } m \in [2, k], \text{ where } \quad E := \sum_{h=1}^{k} \sigma_h \partial_h \tag{7}$$

Then, define the left ideal $\mathcal{I}$ in $\mathcal{D}_N$ as

$$\mathcal{I} := \mathcal{A} + \sum_{m=2}^{k} \mathcal{D}_N \mathcal{T}^m \tag{8}$$

and let $\mathcal{M}$ be the $\mathcal{D}_N$-module

$$\mathcal{M} := \mathcal{D}_N / \mathcal{I} \tag{9}$$

[2] Which is noted $\alpha \natural \beta$ in Sect. 6.

We shall now recall and precise some results of [1].

Let Z be the complex (algebraic) subspace in $N \times \mathbb{C}^k$ (with coordinates $\sigma_1, \dots, \sigma_k, \eta_1, \dots, \eta_k$) defined by the ideal of $(2,2)-$minors of the matrix

$$\begin{pmatrix} \eta_1 & -l_\sigma(\eta) \\ \eta_2 & \eta_1 \\ . & . \\ . & . \\ \eta_k & \eta_{k-1} \end{pmatrix} \tag{10}$$

where $l_\sigma(\eta) := \sum_{h=1}^k \sigma_h \eta_h$. We shall note I_Z the ideal of $\mathcal{O}_{N\times\mathbb{C}^k}$ generated by these minors and by $p_* I_Z$ its direct image by the projection $p : N \times \mathbb{C}^k \to N$. For each integer $q \geq 0$ the sub-sheaf $p_* I_Z(q)$ of sections of $p_* I_Z$ which are homogeneous of degree q along the fibers of p is a coherent $\mathcal{O}_N$-module.

Proposition 2.2.2 *The complex subspace Z is reduced, globally irreducible and Z is the characteristic cycle of the $\mathcal{D}_N$-module $\mathcal{M}$.*

Proof Let $|Z|$ be the support of the sub-space Z. The fact that Z is globally irreducible is already proved in [1] Proposition 4.2.6 as Z is conic over N. This implies that Z is reduced as a complex sub-space:

Assume that I_Z is not equal to the reduced ideal $I_{|Z|}$ of the complex analytic subset $|Z|$ in $N\times\mathbb{C}^k$. By homogeneity in the variables $\eta_1, \dots, \eta_k$ there exists $q \geq 0$, such that the quotient $\mathcal{Q}(q) := I_{|Z|}(q)\big/I_Z(q)$ is not $\{0\}$ and then the coherent sheaf $p_*(\mathcal{Q}(q))$ is not $\{0\}$ on N. However, this contradicts the fact that any global section on N of $p_* I_{|Z|}(q)$ is a global section on N of $p_* I_Z(q)$ which is the content of Proposition 4.2.6 in *loc. cit.*

To complete the proof that Z is the characteristic cycle of $\mathcal{M}$ it is enough to see that the symbol of any germ P of section in $\mathcal{I}$ vanishes on $|Z|$. This is obvious by definition of I_Z. ■

The following proposition, which is a local version of Theorem 5.1.1 in [1], will be useful. Recall that the Newton polynomial $N_m \in \mathbb{C}[\sigma_1, \dots, \sigma_k]$ is the polynomial corresponding (via the standard symmetric function Theorem) to the symmetric polynomial $\sum_{j=1}^m z_j^m$.

Proposition 2.2.3 *Let $\mathcal{I}_+$ the left ideal in $\mathcal{D}_N$ of germs of differential operators P, such that $P(N_m) = 0$ for each Newton polynomial N_m, $m \in \mathbb{N}$. Then, $\mathcal{I}_+ = \mathcal{I}$*

Proof Proposition 4.1.2 in [1] already proves the inclusion $\mathcal{I} \subset \mathcal{I}_+$. To prove the other inclusion we shall argue by contradiction. Therefore, assume that at some point σ^0 in N we have $\mathcal{I}_{+,\sigma^0}\backslash\mathcal{I}_{\sigma^0} \neq \emptyset$ and let P be in $\mathcal{I}_{+,\sigma^0} \setminus \mathcal{I}_{\sigma^0}$ with minimal order say q. Thanks to Proposition 4.2.8 in *loc. cit.* we know[3] that the symbol $s(P)$ is in $p_*(I_Z)_{\sigma^0}$ thanks to the equality $I_Z = I_{|Z|}$ proved above. Therefore, there exists a germ P_1 in $\mathcal{I}_{\sigma^0} \setminus \{0\}$ with symbol $s(P_1) = s(P)$. Then, the order of $P - P_1$ is strictly less than q. But then, $P - P_1$ is in $\mathcal{I}_{+,\sigma^0}$ with order strictly less than q and then it is in $\mathcal{I}_{\sigma^0}$. Contradiction. Therefore, $\mathcal{I} = \mathcal{I}_+$. ■

[3] This proposition proves that the symbol of a non zero germ of section of $\mathcal{I}_+$ vanishes on $|Z|$.

Notations We note $\mathcal{I}(m)$ the sub-$\mathcal{O}_N$-module generated in $\mathcal{I}$ by classes induced by differential operators of order at most equal to m. Therefore, $\mathcal{I}(m) = \mathcal{I} \cap \mathcal{D}_N(m)$.

Then, we note $\mathcal{M}(m) := \mathcal{D}_N(m)\big/\mathcal{I}(m)$.

For any non zero germ of section P of $\mathcal{D}_N$ we note $s(P)$ its symbol in $\mathcal{O}_N[\eta_1, \dots, \eta_k]$. For $P = 0$, let $s(P)$ be 0.

Recall that we note $p : N \times \mathbb{C}^k \to N$ the projection.

Lemma 2.2.4 *We have* $\lim_{m\to\infty} \mathcal{M}(m) \simeq \mathcal{M}$, *where the maps* $\mathcal{M}(m) \to \mathcal{M}(m+1)$ *are induced by the obvious inclusions*

$$\mathcal{D}_N(m) \hookrightarrow \mathcal{D}_N(m+1) \quad \text{and} \quad \mathcal{I}(m) \hookrightarrow \mathcal{I}(m+1).$$

Therefore, $\mathcal{M}$ *is equal to the inductive limit* $\lim_{m\to\infty} \mathcal{M}(m)$.

Proof Beware that the maps $\mathcal{M}(m) \to \mathcal{M}(m+1)$ are not *a priori* injective.

There is an obvious map $\lim_{m\to\infty} \mathcal{M}(m) \to \mathcal{M}$ which is clearly surjective. The point is to prove injectivity. Let P be a non zero germ at some $\sigma \in N$ of order m, such that its image in $\mathcal{M}_\sigma$ is 0. Then, by definition, there exists germs $B_h, h \in [2, k]$ and $C_{p,q}, (p, q) \in [1, k]^2$ in $D_{N,\sigma}$, such that

$$P = \sum_{h=2}^{k} B_h \mathcal{T}^h + \sum_{p,q} C_{p,q} A_{p,q}.$$

Let r be the maximal order of the germs B_h and $C_{p,q}$. Then, the equality above shows that P is in $\mathcal{I}(r+2)$. Therefore, the image of P in $\lim_{m\to\infty} \mathcal{M}(m)$ is zero, as it is already 0 in $\mathcal{M}(r+2)$. ∎

Lemma 2.2.5 *Let* P *be a non zero germ of section of the sheaf* $\mathcal{I}$. *Assume that* P *has order at most* 1. *Then,* $P = 0$.

Proof Let $P = a_0 + \sum_{h=1}^{k} a_h \partial_h$. Recall that for each $h \in [1, k]$ and each $m \in \mathbb{N}$ we have (see Proposition 5.2.1 in [1]):

$$\partial_h N_m = (-1)^{h-1} m DN_{m-h} \tag{11}$$

where the polynomials

$$DN_m := \sum_{P_\sigma(x_j)=0} \frac{x_j^{m+k-1}}{P'_\sigma(x_j)}$$

vanish for $m \in [-k+1, -1]$ and $DN_0 = 1$.

Then, the equality $\mathcal{I}_+ = \mathcal{I}$ proved in Proposition 2.2.3 implies that for each integer m, we have

$$a_0 N_m + \sum_{h=1}^{k} a_h (-1)^{h-1} m DN_{m-h} = 0, \forall m \in \mathbb{N}$$

For $m = 0$ this gives $a_0 = 0$; if we have $a_0 = a_1 = \cdots = a_p = 0$ for some $p \in [0, k-1]$ then $P[N_{p+1}] = 0$ gives $\sum_{h=p+1}^{k} a_h(-1)^h(p+1)DN_{p+1-h} = a_{p+1}(p+1)DN_0 = 0$ and then $a_{p+1} = 0$. Therefore, $P = 0$. ■

Notation Define the vector fields on N:

$U_0 := \sum_{h=1}^{k} h\sigma_h\partial_h$ and $U_{-1} := \sum_{h=0}^{k-1}(k-h)\sigma_h\partial_{h+1}$ with the convention $\sigma_0 \equiv 1$.

Lemma 2.2.6 *Let $q \geq 2$ be an integer, $\alpha \in \mathbb{N}^k$, such that $|\alpha| = q-2$ and let m be an integer in the interval $[2, k]$. The class induced by $\partial^\alpha \mathcal{T}^m$ in $\mathcal{W}$ only depends on the integers q and $r := w(\alpha) + m$. This class is given by the formula (with the convention $\sigma_0 \equiv 1$)*

$$\left[\partial^\alpha \mathcal{T}^m\right] = \sum_{h=0}^{k} \sigma_h y_{q,r+h} + (q-1)y_{q,r}, \tag{12}$$

where $y_{q,r}$ is the class induced by ∂^γ in $\mathcal{W}$ for any $\gamma \in \mathbb{N}^k$, such that $|\gamma| = q$ and $w(\gamma) = r$ (see Lemma 2.1.5).

Let λ be a complex number and let $\beta \in \mathbb{N}^k$, such that $|\beta| = q-1$ and $w(\beta) = r$. The class induced by $\partial^\beta(U_0 - \lambda)$ in $\mathcal{W}$ only depends on λ and on the integers q and r. This class is given by

$$\left[\partial^\beta(U_0 - \lambda)\right] = \sum_{h=1}^{k} h\sigma_h y_{q,r+h} + (r-\lambda)y_{q-1,r}. \tag{13}$$

In addition, the class induced by $\partial^\beta U_{-1}$ in $\mathcal{W}$, again for $|\beta| = q-1$ and $w(\beta) = r$, only depends on the integers q and r. This class is given by

$$\left[\partial^\beta U_{-1}\right] = \sum_{h=0}^{k}(k-h)\sigma_h y_{q,r+h+1} + (k(q-1)-r)y_{q-1,r+1} \tag{14}$$

where, for $r = k(q-1)$, the last term in (14) is equal to 0 by convention.

Proof By definition $\mathcal{T}^m = \partial_1\partial_{m-1} + \sum_{h=1}^{k} \sigma_h\partial_h\partial_m + \partial_m$ which implies

$$\partial^\alpha \mathcal{T}^m = \partial^\alpha\partial_1\partial_{m-1} + \sum_{h=1}^{k} \sigma_h\partial_h\partial_m\partial^\alpha + (q-1)\partial^\alpha\partial_m$$

as we have $\partial^\alpha\sigma_h\partial_h = \sigma_h\partial_h\partial^\alpha + \alpha_h\partial^\alpha$ for any $\alpha \in \mathbb{N}^k$ and any $h \in [1, k]$. Now, formula (12) follows from Lemma 2.1.5, proving our first assertion.

As $U_0 := \sum_{h=1}^{k} h\sigma_h\partial_h$ we have

$$\partial^\beta(U_0 - \lambda) = \sum_{h=1}^{k} h\partial^\beta\sigma_h\partial_h - \lambda.\partial^\beta = \sum_{h=1}^{k} h.\sigma_h\partial_h\partial^\beta + (w(\beta) - \lambda)\partial^\beta$$

which gives Formula (13) using Lemma 2.1.5, and this proves our second assertion. The third one is analogous using the fact that $U_{-1} = \sum_{h=0}^{k-1}(k-h)\sigma_h\partial_{h+1}$ with the convention $\sigma_0 \equiv 1$ and the equalities:

$$\partial^\beta \sigma_h \partial_{h+1} = \sigma_h \partial^\beta \partial_{h+1} + \beta_h \partial^{\beta+1_{h+1}-1_h}$$

$$\sum_{h=0}^{k-1}(k-h)\beta_h = k((q-1)-\beta_k) - (w(\beta) - k\beta_k) = k(q-1) - r$$

with the convention $\beta_0 = 0$ and the fact that $\partial^{\beta+1_{h+1}-1_h}$ induces $y_{q-1,r+1}$. ■

Notations

1. Let $V_q \subset \mathcal{W}_q$ be the $\mathcal{O}_N-$sub-module with basis $y_{q,r}$ for $r \in [k(q-1)+1, kq]$. Remark that $V_0 = \mathcal{W}_0 = \mathcal{W}(0) = \mathcal{O}_N$ and $V_1 = \mathcal{W}_1 = \oplus_{h=1}^k \mathcal{O}_N.\partial_h$.
2. Let $L_q : \mathcal{W}_q \to \mathcal{M}(q)$ be the map induced by restriction to $\mathcal{W}_q$ of the quotient map $\mathcal{W}(q) \to \mathcal{M}(q)$ and $l_q : V_q \to \mathcal{M}(q)$ its restriction to V_q.

Lemma 2.2.7 *Fix an integer $q \geq 0$. Then, for any $Y \in \mathcal{W}_q$ there exists $X \in V_q$, such that $L_q(Y - X)$ is in $\mathcal{M}(q-1)$, with the convention $\mathcal{M}(-1) = \{0\}$.*

Proof Remark that for $Y = y_{q,r}$ with $r \in [k(q-1)+1, kq]$ we may choose $X = Y$. Therefore, it is enough to prove the lemma for Y in the sub-module with basis $y_{q,r}$ with $r \in [q, k(q-1)]$.

Note that for $q = 0$ and for $q = 1$ there is nothing more to prove.

For each $q \geq 2$ and $r \in [q, k(q-1)]$ there exists $m \in [2, k]$, such that $r - m$ is in $[q-2, k(q-2)]$, because the addition map $(s, m) \to s + m$ is surjective[4] from $[q-2, k.(q-2)] \times [2, k]$ to $[q, k(q-1)]$. Therefore, there exists $\alpha \in \mathbb{N}^k$, such that $|\alpha| = q - 2$ and $w(\alpha) = r - m$. Then

$$\partial^\alpha \mathcal{T}^m = y_{q,r} + (q-1)y_{q-1,r} + \sum_{h=1}^k \sigma_h y_{q,r+h}$$

and the class induced by $y_{q,r}$ in $\mathcal{M}(q)$ is, modulo $\mathcal{M}(q-1)$, in the sub$-\mathcal{O}_N$-module of $\mathcal{M}(q)$ induced by the images of classes of $y_{q,r'}$ with $r' > r$. By a descending induction on $r \in [q, k(q-1)]$ we see that, modulo $\mathcal{M}(q-1)$, the image of $\mathcal{W}_q$ by L_q is equal to $L_q(V_q)$. This implies our statement by induction on q. ■

Note that the previous lemma shows that $l_q(V_q) = L_q(\mathcal{W}_q)$ for each $q \geq 0$.

Proposition 2.2.8 *For any $q \in \mathbb{N}$, there is a natural isomorphism of $\mathcal{O}_N$-modules*

$$\Lambda_q := \oplus_{p=0}^q l_p : \oplus_{p=0}^q V_p \longrightarrow \mathcal{M}(q) \tag{15}$$

which is compatible with the natural map $\oplus_{p=0}^q V_p \hookrightarrow \oplus_{p=0}^{q+1} V_p$ and the natural map $\mathcal{M}(q) \to \mathcal{M}(q+1)$.

[4] For $r \in [q, k(q-2)+2]$ take $s = r-2$ and $m = 2$, for $r = k(q-2)+j$ with $j \in [2, k]$ take $s = r - j$ and $m = j$.

Proof For $q = 0$ we have $\mathcal{M}(0) = V_0 = \mathcal{O}_N.y_{0,0}$ where $y_{0,0} = 1$. Therefore, Λ_0 is an isomorphism. For $q = 1$, Lemma 2.2.5 shows that the map Λ_1 is injective. As it is surjective (we have $V_0 = \mathcal{W}_0$ and $V_1 = \mathcal{W}_1$) the assertion is clear.

Assume that we have proved that Λ_{q-1} is an isomorphism of $\mathcal{O}_N$-modules for some $q \geq 2$. We shall prove that Λ_q is also an isomorphism.

Consider $Y := \sum_{p=0}^{q} Y_p$ with $Y_p \in V_p$ for each $p \in [0, q]$, which is in the kernel of Λ_q. If $Y_q = 0$ the induction hypothesis allows to conclude that $Y = 0$.

So assume that $Y_q \neq 0$. As Y_q is induced[5] by a differential operator of the form $\sum_{j=1}^{k} b_j \partial_k^{q-1}\partial_j$, with $b_j \in \mathcal{O}_N$ for $j \in [1, k]$, we may choose a differential operator $P \in \mathcal{I}$ of order q which induces Y, such that its symbol is equal to $\eta_k^{q-1} \sum_{j=1}^{k} b_j \eta_j$. This symbol vanishes on Z, and, as η_k does not vanish on any non empty open set on Z, we conclude that $\sum_{j=1}^{k} b_j \eta_j$ vanishes on Z. The injectivity of Λ_1 implies that $b_1 = \cdots = b_k = 0$ showing that $Y_q = 0$ and this contradicts our hypothesis. Therefore, Λ_q is injective.

We have already noticed that Lemma 2.2.7 implies the surjectivity of Λ_q for $q \geq 2$. Therefore, the proof is complete. ∎

Corollary 2.2.9 *The $\mathcal{D}_N$-module $\mathcal{M}$ has no $\mathcal{O}_N$-torsion.*

Proof This an easy consequence of the previous proposition giving that each $\mathcal{M}(q)$ is a free $\mathcal{O}_N$-module, because for any $\sigma \in N$, a non zero torsion germ in $\mathcal{M}_\sigma$ has to come from a non zero torsion element in $\mathcal{M}(q)_\sigma$ for some q large enough (may be much more larger than the order of the germ in $D_{N,\sigma}$ inducing this class in $\mathcal{M}_\sigma$) thanks to Lemma 2.2.4. ∎

2.3 On Quotients of $\mathcal{M}$

We shall use the description of the characteristic variety of $\mathcal{M}$ to examine the holonomic quotients of $\mathcal{M}$ supported by an irreducible complex subset of N.

Proposition 2.3.1 *Let Q be a holonomic quotient of $\mathcal{M}$ which is supported by an analytic subset S of N with empty interior in N. Then, S is a hyper-surface and S is contained in $\{\sigma_k = 0\} \cup \{\Delta(\sigma) = 0\}$.*

Proof Let S_0 be an irreducible component of S, the support of a holonomic quotient Q of $\mathcal{M}$. Let $d \geq 1$ be the co-dimension of S_0. Then, near the generic point in S_0 the co-normal sheaf of S_0 is a rank d vector bundle over S_0 which is contained in Z. As the fibres of Z over N have pure dimension 1 we have $d \leq 1$ and then $d = 1$ and S_0 is a hyper-surface in N. Then, S is also a hyper-surface in N.

Let now S_0 be an irreducible component of S which is not contained in $\{\Delta(\sigma) = 0\}$. Then, near the generic point in S_0 the quotient map $quot : M \to N$ is an étale cover and this shows that $\mathcal{M}$ locally is isomorphic to the quotient of $D_{\mathbb{C}^k}$ by the let ideal with generators $\frac{\partial^2}{\partial z_i \partial z_j}$ for $i \neq j$ in $[1, k]$. Therefore, the characteristic variety of $\mathcal{M}$ is locally isomorphic to $C := \cup_{j=1}^{k} N \times \{\mathbb{C}.e_j\}$ where e_j is the $j-$th vector in

[5] Note that for each $r \in [k(q-1)+1, kq]$, $r = k(q-1)+j$, then $y_{q,r}$ is induced in W_q by $\partial_k^{q-1}\partial_j$.

the canonical basis of $\mathbb{C}^k$. If an irreducible hyper-surface has its co-normal bundle contained in C, it has to be equal to the co-normal of one of the hyperplanes $\{z_j = 0\}$. This means that S is contained in $\{\sigma_k = 0\}$.

But any hyper-surface contained in $\{\Delta = 0\}$ is equal to $\{\Delta = 0\}$. Therefore, the only possible irreducible components of the support of Q are $\{\sigma_k = 0\}$ or $\{\Delta = 0\}$. ∎

We shall use the following immediate corollary of this proposition:

Corollary 2.3.2 *Let Q be a coherent holonomic quotient of $\mathcal{M}$ which is supported in a closed analytic subset S in N with empty interior in N. If Q vanishes near the generic points of $\{\sigma_k = 0\} \cup \{\Delta = 0\}$, then $Q = \{0\}$.* ∎

Let $k \geq 2$. We shall study the $\mathcal{D}_N$-module $\mathcal{M}$ near the generic point of the hyper-surface $\{\Delta = 0\}$ in N.

Let $z_1^0, z_3^0, \dots, z_k^0$ be $(k-1)$ distinct points in $\mathbb{C}$ and let $r > 0$ a real number small enough in order that the discs $D_1, D_3, \dots, D_k$ with respective centers $z_1^0, z_3^0, \dots, z_k^0$ and radius r are two by two disjoint. Let $\mathcal{U}_0 := D_1 \times D_1 \times \prod_{j=3}^k D_j$ and $\mathcal{V}$ (equal to $D_1 \times D_2$ for $k = 2$) the image of $\mathcal{U}_0$ by the quotient map $quot : \mathbb{C}^k \to \mathbb{C}^k \big/ \mathfrak{S}_k = N$. Note $\mathcal{U} := q^{-1}(\mathcal{V})$. Then, q induces an isomorphism of $\mathcal{U}_0 \big/ \mathfrak{S}_2$ onto $\mathcal{V}$.

Remark that for each $\sigma \in \mathcal{V}$ we have exactly two roots $z_1(\sigma), z_2(\sigma)$ distinct or not which are in D_1 and for each $j \in [3, k]$ we have exactly one (simple) root $z_j(\sigma)$ in D_j. We have the following holomorphic map on $\mathcal{V}$:

1. The map $\tau = (\tau_1, \tau_2) : \mathcal{V} \to \mathbb{C}^2$ given by $\tau_1(\sigma) := z_1(\sigma) + z_2(\sigma)$ and $\tau_2 := z_1(\sigma)z_2(\sigma)$ where $z_1(\sigma)$ and $z_2(\sigma)$ are the roots of P_σ which are in D_1.
2. For each $j \in [3, k]$ the map $z_j : \mathcal{V} \to D_j$ given by the unique (simple) root of P_σ in D_j.

To be completely clear, these holomorphic maps are defined on $\mathcal{V}$ by the following integral formulas:

$$\tau_1(\sigma) := \frac{1}{2i\pi} \int_{\partial D_1} \frac{\zeta P'_\sigma(\zeta) d\zeta}{P_\sigma(\zeta)}$$

$$2\tau_2(\sigma) = \tau_1^2 - \nu_2(\sigma) \quad \text{where} \quad \nu_2(\sigma) := \frac{1}{2i\pi} \int_{\partial D_1} \frac{\zeta^2 P'_\sigma(\zeta) d\zeta}{P_\sigma(\zeta)}$$

$$\text{and for } j \in [3, k] \quad z_j(\sigma) := \frac{1}{2i\pi} \int_{\partial D_j} \frac{\zeta P'_\sigma(\zeta) d\zeta}{P_\sigma(\zeta)}$$

The following lemma is obvious:

Lemma 2.3.3 *The holomorphic map $\Phi : \mathcal{V} \to \mathbb{C}^2 \times \prod_{j=3}^k D_j$ given by $(\tau_1, \tau_2, z_3, \dots, z_k)$, is an isomorphism of $\mathcal{V}$ onto the open set $\mathcal{V}_1 := D_{1,2} \times \prod_{j=3}^k D_j$ where we define $D_{1,2} := (D_1 \times D_1) \big/ \mathfrak{S}_2$ as the image of $D_1 \times D_1$ by the quotient map by the action of the permutation group $\mathfrak{S}_2$.* ∎

In the sequel, we shall use the coordinate system on $\mathcal{V}$ given by the holomorphic functions $\tau_1, \tau_2, z_3, \dots, z_k$ on $\mathcal{V}$.

Define on $\mathcal{V} \simeq \mathcal{V}_1 = D_{1,2} \times \prod_{j=3}^k D_j$ the following partial differential operators in the coordinate system described above:

- $T^2 := \partial_{\tau_1}^2 + \tau_1 \partial_{\tau_1} \partial_{\tau_2} + \tau_2 \partial_{\tau_2}^2 + \partial_{\tau_2}$
- $B_{i,j} := \partial^2 / \partial z_i \partial z_j$ for $3 \leq i < j \leq k$
- $C_{1,j} := \partial^2 / \partial \tau_1 \partial z_j$ for $j \in [3, k]$
- $C_{2,j} := \partial^2 / \partial \tau_2 \partial z_j$ for $j \in [3, k]$
- $V_0 := \tau_1 \partial_{\tau_1} + 2\tau_2 \partial_{\tau_2} + \sum_{j=3}^k z_j \partial_{z_j}$
- $V_{-1} := 2\partial_{\tau_1} + \tau_1 . \partial_{\tau_2} + \sum_{j=3}^k \partial_{z_j}$.

Proposition 2.3.4 *The isomorphism of change of coordinates Φ on $\mathcal{V}$ given by $\sigma \mapsto (\tau_1, \tau_2, z_3, \dots, z_k)$ has the following properties:*

(i) *The image of ideal $\mathcal{I}$ of $\mathcal{D}_N$ restricted to $\mathcal{V}$ by the isomorphism Φ is the left ideal generated by T^2, $B_{i,j}$, $C_{1,j}$ and $C_{2,j}$ in $D_{\mathcal{V}_1}$.*

(ii) *The vector field U_0 is sent to V_0 and the vector field U_{-1} is sent to V_{-1} by this isomorphism.*

Proof We shall use the local version of Theorem 5.1.1 in [1] which is given in Proposition 2.2.3 above.

For $\Omega \subset \mathcal{V}$ it is easy to see that the Fréchet space of trace functions admits as a dense subset the finite $\mathbb{C}-$linear combinations of the Newton functions $\nu_m, m \in \mathbb{N}$ of $z_1(\sigma), z_2(\sigma)$ and of the functions $z_j^m(\sigma), m \in \mathbb{N}$ for each $j \in [3, k]$. From the case $k = 2$ for which the left ideal $\mathcal{I}$ is generated by T^2 and the fact that each $B_{i,j}, C_{1,j}$ and $C_{2,j}$ kill each ν_m and each z_j^m, we conclude that $\mathcal{I}$ contains the left ideal generated by T^2, the $B_{i,j}$, the $C_{1,j}$ and the $C_{2,j}$.

Conversely, if P is in $\mathcal{I}$ it has to kill any ν_m and each z_j^m, $\forall j \in [3, k]$. Therefore, P has no order 0 term. Modulo the ideal generated by the $B_{i,j}$, the $C_{1,j}$ and the $C_{2,j}$ we may assume that we can write

$$P = P_0 + \sum_{m=1}^{N} \sum_{j=3}^{k} g_{j,m} \partial_{z_j}^m$$

where P_0 is a differential operator in τ_1, τ_2 with no order 0 term, and with holomorphic dependence in $z_3, \dots, z_k$ (but no derivation in these variables) and where $g_{j,m}$ are holomorphic functions on $\mathcal{V}$. Applying P to z_j^N, with $j \in [3, k]$, gives that

$$\sum_{m=1}^{N} g_{j,m} \frac{N!}{(N-m)!} z_j^{N-m} = 0$$

and then $g_{j,m} = 0$ for each $m \in [1, N]$ and each $j \in [3, k]$, because the $g_{j,m}$ are holomorphic functions of (τ_1, τ_2). Then, $P = P_0$ and $P(\nu_m) = 0$ implies that P_0 is in the left ideal generated by T^2 in the $\mathcal{O}_{\mathcal{V}_1}$-algebra generated by $\frac{\partial}{\partial\tau_1}$ and $\frac{\partial}{\partial\tau_2}$. Then, P_0 and also P are in our ideal and (i) is proved.

The verification of (ii) is easy and left to the reader. ∎

Lemma 2.3.5 *For $k = 2$ we have for each $n \in \mathbb{N}^*$*

$$(T^2 - 2n\partial_2)\Delta^n = \Delta^n T^2 + 2n(2n+1)\Delta^{n-1}. \tag{16}$$

Proof Recall that we have $E := \sigma_1\partial_1 + \sigma_2\partial_2$ and $T^2 = \partial_1^2 + \partial_2 E$ and that $\Delta = \sigma_1^2 - 4\sigma_2$. Therefore, we have

$$\begin{aligned}
&\partial_1\Delta = \Delta\partial_1 + 2\sigma_1\\
&[\sigma_1\partial_1, \Delta] = 2\sigma_1^2\\
&\partial_2.\Delta = \Delta\partial_2 - 4\\
&[\sigma_2\partial_2, \Delta] = -4\sigma_2\\
&[E, \Delta] = 2\sigma_1^2 - 4\sigma_2
\end{aligned}$$

and

$$\begin{aligned}
&[\partial_2 E, \Delta] = \partial_2(\Delta E + 2\sigma_1^2 - 4\sigma_2) - (\partial_2\Delta + 4).E = 2\Delta\partial_2 - 4\sigma_1\partial_1 - 4\\
&\partial_1^2\Delta = \partial_1(\Delta\partial_1 + 2\sigma_1) = (\Delta\partial_1 + 2\sigma_1)\partial_1 + 2\sigma_1\partial_1 + 2 = \Delta\partial_1^2 + 4\sigma_1\partial_1 + 2\\
&\left[T^2, \Delta\right] = \left[\partial_1^2, \Delta\right] + [\partial_2 E, \Delta]\\
&T^2.\Delta = \Delta T^2 + 4\sigma_1\partial_1 + 2 + 2\Delta\partial_2 - 4\sigma_1\partial_1 - 4 = \Delta T^2 + 2\partial_2\Delta + 8 + 2 - 4\\
&(T^2 - 2\partial_2)\Delta = \Delta T^2 + 6
\end{aligned}$$

which proves (16) for $n = 1$.

Assume now that we have proved the formula (16) for $n \geq 1$. Then, we have, using that $\Delta^n\partial_2 = \partial_2\Delta^n + 4n\Delta^{n-1}$

$$\begin{aligned}
(T^2 - 2n\partial_2)\Delta^{n+1} &= \Delta^n T^2\Delta + 2n(2n+1)\Delta^n\\
(T^2 - 2n\partial_2)\Delta^{n+1} &= \Delta^{n+1}T^2 + \Delta^n 2\partial_2.\Delta + 6\Delta^n + 2n(2n+1)\Delta^n\\
(T^2 - 2n\partial_2)\Delta^{n+1} &= \Delta^{n+1}T^2 + 2\partial_2\Delta^{n+1} + 8n\Delta^n + 6\Delta^n + 2n(2n+1)\Delta^n\\
(T^2 - 2(n+1)\partial_2)\Delta^{n+1} &= \Delta^{n+1}T^2 + 2(n+1)(2n+3)\Delta^n
\end{aligned}$$

because $2n(2n+1) + 8n + 6 = 2(n+1)(2n+3)$. ∎

Theorem 2.3.6 *Let Q be a coherent $\mathcal{D}_N$-module which is a quotient of $\mathcal{M}$ and which is supported by $\{\Delta = 0\}$. Then, $Q = \{0\}$. Moreover, any holonomic quotient of $\mathcal{M}$ has no $\Delta-$torsion.*

Proof Thanks to Corollary 2.3.2 it is enough to prove that such a quotient Q is zero near the generic points of $\{\Delta = 0\}$. Therefore, assume that Q is such a non zero quotient. Using now the result of Proposition 2.3.4, the $\mathcal{D}$-module Q is given near the generic points of $\{\Delta = 0\}$ by the quotient of $\mathcal{D}$ by a left ideal $\mathcal{K}$ which contains T^2. Then, there exists an integer $n > 0$, such that Δ^n belongs to $\mathcal{K}$. Then, Lemma 2.3.5 implies that $\mathcal{K}$ contains Δ^{n-1}. Then, by a descending induction on n we obtain that 1 is in $\mathcal{K}$ and this contradicts the non vanishing of Q.

The characteristic variety of a holonomic quotient of $\mathcal{M}$ which is supported in codimension ≥ 1 in N is contained in the characterisc variety of $\mathcal{M}$ so is contained in the union of $N \times \{0\}$ with the co-normal to $\{\sigma_k = 0\}$ and $\{\Delta = 0\}$ thanks to Proposition 2.3.1. However, Lemma 2.3.5 implies that near the generic point of $\{\Delta = 0\}$ a torsion element in such a quotient vanishes. Therefore, the torsion submodule of a holonomic quotient of $\mathcal{M}$ cannot have the co-normal of $\{\Delta = 0\}$ in its characteristic variety. Then, such a quotient has no $\Delta-$torsion. ■

2.4 Action of $sl_2(\mathbb{C})$ on $\mathcal{M}$

Let B be the sub-$\mathbb{C}-$algebra of the Weyl algebra $\mathbb{C}[\sigma]\langle\partial\rangle$ generated by the vector fields U_p, $p \geq -1$, where U_p is the vector field on N defined as the image by the differential T_{quot} of the quotient map:

$$quot : M := \mathbb{C}^k \longrightarrow N := \mathbb{C}^k \big/ \mathfrak{S}_k \simeq \mathbb{C}^k$$

of the vector field $\sum_{j=1}^{k} z_j^{p+1} \frac{\partial}{\partial z_j}$.

Theorem 2.4.1 *For each $p \geq -1$, we have $\mathcal{I}U_p \subset \mathcal{I}$. Then, the right action of B on $\mathcal{D}_N$ induces a morphism of algebras between B and the algebra of left $\mathcal{D}_N$-linear endomorphisms of $\mathcal{M}$.*

Moreover, the right action of B on $\mathcal{M}$ satisfies $[U_p, U_q] = (q-p)U_{p+q}, \forall p, q \geq -1$.

Proof It will be enough to show that for each integer $p \geq -1$ we have the inclusion $\mathcal{I}U_p \subset \mathcal{I}$. If it is not difficult to prove such an inclusion for $p = -1$ or $p = 0$ by a direct computation of the commutators of U_p with the generators of $\mathcal{I}$, it seems rather difficult to do it for p large, because the coordinates of U_p in the $\mathbb{C}[\sigma]$ basis $\partial_1, \dots, \partial_k$ of the polynomial vector fields on N seems more and more complicated. Therefore, we shall use the local version of Theorem 5.1.1 in [1] given in Proposition 2.2.3.

Let $P \in \mathcal{I}$, $p \geq -1$ an integer and $m \in \mathbb{N}$. Using the formula $U_p[N_m] = mN_{m+p}$ which is easy to verify on M, we get

$$P[U_p[N_m]] = P[mN_{m+p}] = 0$$

when P annihilates any Newton polynomial. Then, PU_p also annihilates any Newton polynomial and thanks to Proposition 2.2.3 we conclude that PU_p belongs to $\mathcal{I}$ proving the first assertion.

The verification of the commutation formula

$$U_pU_q - U_qU_p = (q-p)U_{p+q}$$

is easy and left to the reader. ■

Remark The commutation relations

$$[U_0, U_{-1}] = -U_{-1}, \quad [U_0, U_1] = U_1 \quad \text{and} \quad [U_1, U_{-1}] = 2U_0$$

which are easy to check in M, show that the Lie algebra $\mathcal{L}$ generated by the U_p (with the commutators given by the formula (17)) contains a sub-Lie algebra isomorphic to $sl_2(\mathbb{C})$. The formula (17) shows that $\mathcal{L}$ acts on $\mathcal{M}$ and then induces a structure of $sl_2(\mathbb{C})-$module on $\mathcal{M}$.

3 The $\mathcal{D}_N$-Modules $\mathcal{N}_\lambda$

3.1 Homothety and Translation

Notations Let λ a complex number. We define the left ideal

$$\mathcal{J}_\lambda := \mathcal{I} + \mathcal{D}_N(U_0 - \lambda)$$

in $\mathcal{D}_N$ and let $\mathcal{N}_\lambda$ be the quotient $\mathcal{D}_N\big/\mathcal{J}_\lambda$. We shall denote by $q_\lambda : \mathcal{M} \to \mathcal{N}_\lambda$ the quotient map.

We shall denote, respectively, by $\mathscr{H}_\lambda$ and $\mathscr{T}$ the endomorphisms of left $\mathcal{D}_N$-modules on $\mathcal{M}$ induced, respectively, by the right multiplications by $U_0 - \lambda$ and U_{-1} (see the Theorem 2.4.1). They satisfy the commutation relation (see *loc. cit.*)

$$\mathscr{H}_\lambda \circ \mathscr{T} - \mathscr{T} \circ \mathscr{H}_\lambda = -\mathscr{T}$$

for each $\lambda \in \mathbb{C}$ and $\mathcal{N}_\lambda$ is, by definition, the co-kernel of $\mathscr{H}_\lambda$.

As $\mathcal{I}.U_{-1} \subset \mathcal{I}$, writing this relation in the form $\mathscr{H}_{\lambda-1} \circ \mathscr{T} = \mathscr{T} \circ \mathscr{H}_\lambda$ we see that the right multiplication by U_{-1} induces a left $\mathcal{D}_N$-modules morphism

$$\mathscr{T}_\lambda : \mathcal{N}_{\lambda-1} \to \mathcal{N}_\lambda$$

for each λ.

Proposition 3.1.1 *For each $\lambda \in \mathbb{C}$ we have an exact sequence of left $\mathcal{D}_N$-modules on N*

$$0 \to \mathcal{M} \xrightarrow{\mathscr{H}_\lambda} \mathcal{M} \xrightarrow{q_\lambda} \mathcal{N}_\lambda \to 0 \tag{17}$$

where q_λ is the obvious quotient map.

Proof The quotient map q_λ is surjective by definition, so the point is to prove that the kernel of q_λ is isomorphic to $\mathcal{M}$.

This kernel is obviously given by

$$\mathcal{J}_\lambda \big/ \mathcal{I} \simeq \big(\mathcal{I} + D_N(U_0 - \lambda)\big) \big/ \mathcal{I} \simeq D_N(U_0 - \lambda) \big/ \mathcal{I} \cap D_N(U_0 - \lambda) \tag{18}$$

The proof will be an easy consequence of the following lemma.

Lemma 3.1.2 *Let P be a germ in $D_{N,\sigma}$ for some $\sigma \in N$ such that $P(U_0 - \lambda)$ is in $\mathcal{I}_\sigma$. Then, P is in $\mathcal{I}_\sigma$. Therefore, $\mathcal{I}_\sigma \cap D_{N,\sigma}(U_0 - \lambda) = \mathcal{I}_\sigma(U_0 - \lambda)$.*

Proof Assume that the lemma is wrong. Then, let P_0 in $D_{N,\sigma}$ having minimal order among germs P in $D_{N,\sigma}$ satisfying the following properties

1. $P(U_0 - \lambda)$ is in $\mathcal{I}_\sigma \cap D_{N,\sigma}(U_0 - \lambda)$;
2. P is not in $\mathcal{I}_\sigma$.

Let π be the symbol of P_0 and let g be the symbol of U_0. We have $\pi g \in p_*(I_Z)_\sigma$. However, we know that g does not vanish on any non empty open set of Z, because $\{g = 0\} \cap Z$ has pure co-dimension 1 in Z (see Lemma 2.2.5 above). Then, π vanishes on $(V \times \mathbb{C}^k) \cap Z$ where V is a neighborhood of σ in N and, as we have proved that Z is reduced and is the characteristic cycle of $\mathcal{M}$, their exists a germ P_1 in $\mathcal{I}_\sigma$ with symbol equal to π. Then, $(P_0 - P_1)(U_0 - \lambda)$ satisfies again the properties 1 and 2 and is of order strictly less than the order of P_0. Therefore, $P_0 - P_1$ is in $\mathcal{I}_\sigma$ and this contradicts the fact that we assumed that P_0 is not in $\mathcal{I}_\sigma$. ■

End of proof of 3.1.1 The previous lemma shows that for each $\lambda \in \mathbb{C}$

$$\mathcal{I} \cap D_N(U_0 - \lambda) = \mathcal{I}(U_0 - \lambda).$$

Therefore, the right multiplication by $U_0 - \lambda$ induces an isomorphism of left $\mathcal{D}_N$-modules

$$\mathcal{M} \to D_N(U_0 - \lambda) \big/ \mathcal{I}(U_0 - \lambda)$$

and the kernel of q_λ is isomorphic to $\mathcal{M}$ by the inverse of this isomorphism. ■

Definition 3.1.3 Define the $\mathcal{D}_N$-module $\mathfrak{N}$ as the quotient $\mathcal{D}_N \big/ (\mathcal{I} + \mathcal{D}_N U_{-1})$. For each $\lambda \in \mathbb{C}$ then define $\mathscr{T}_\lambda : \mathcal{N}_\lambda \to \mathcal{N}_{\lambda+1}$ as the $\mathcal{D}_N$-linear map induced by $\mathscr{T}$.

Lemma 3.1.4 *For each $\lambda \in \mathbb{C}$ the co-kernel of the $\mathcal{D}_N$-linear map $\mathscr{T}_{\lambda+1}$ is naturally isomorphic to the co-kernel of the $\mathcal{D}_N$-linear map $\tilde{\mathscr{H}}_{\lambda+1} : \mathfrak{N} \to \mathfrak{N}$ induced by $\mathscr{H}_{\lambda+1}$ and there is also a natural isomorphism of $\mathcal{D}_N$-modules between the kernels of $\tilde{\mathscr{H}}_{\lambda+1}$ and $\mathscr{T}_{\lambda+1}$.*

Proof Consider the commutative diagram of left $\mathcal{D}_N$-modules with exact lines and columns:

$$
\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
 & & & & Ker(\tilde{\mathscr{H}}_{\lambda+1}) \arrow[d] & \\
 & & \mathcal{M} \arrow[r, "\mathscr{T}"] \arrow[d, "\mathscr{H}_\lambda"] & \mathcal{M} \arrow[r] \arrow[d, "\mathscr{H}_{\lambda+1}"] & \mathfrak{N} \arrow[r] \arrow[d, "\tilde{\mathscr{H}}_{\lambda+1}"] & 0 \\
 & & \mathcal{M} \arrow[r, "\mathscr{T}"] \arrow[d] & \mathcal{M} \arrow[r] \arrow[d] & \mathfrak{N} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Ker(\mathscr{T}_{\lambda+1}) \arrow[r] \arrow[uuurrr, bend left] & \mathcal{N}_\lambda \arrow[r, "\mathscr{T}_{\lambda+1}"] \arrow[d] & \mathcal{N}_{\lambda+1} \arrow[r] \arrow[d] & \mathcal{N}^{\square}_{\lambda+1} \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}
$$

where $\mathcal{N}^{\square}_{\lambda+1}$ is, by definition, the co-kernel of $\mathscr{T}_{\lambda+1} : \mathcal{N}_\lambda \to \mathcal{N}_{\lambda+1}$. By a simple diagram chasing it is easy to see that $\mathcal{N}^{\square}_{\lambda+1}$ is also the co-kernel of $\tilde{\mathscr{H}}_{\lambda+1} : \mathfrak{N} \to \mathfrak{N}$.

A elementary diagram chasing gives also the isomorphism between kernels of $\tilde{\mathscr{H}}_{\lambda+1}$ and $\mathscr{T}_{\lambda+1}$. ■

We shall prove now that for $\lambda \neq 0, 1$ the map $\mathscr{T}_\lambda$ is an isomorphism of left $\mathcal{D}_N$-modules. This implies $\mathcal{N}^{\square}_\lambda = \{0\}$ for $\lambda \neq 0, 1$.

Lemma 3.1.5 *Let $\mathscr{G}_\lambda : \mathcal{N}_{\lambda+1} \to \mathcal{N}_\lambda$ the left $\mathcal{D}_N$-linear map given by right multiplication by U_1. Then, we have for each $\lambda \in \mathbb{C}$*

$$\mathscr{T}_\lambda \circ \mathscr{G}_{\lambda-1} = \lambda(\lambda - 1) \quad \text{on} \quad \mathcal{N}_\lambda \tag{A}$$

$$\mathscr{G}_{\lambda-1} \circ \mathscr{T}_\lambda = \lambda(\lambda - 1) \quad \text{on} \quad \mathcal{N}_{\lambda-1} \tag{B}$$

Therefore, for $\lambda \neq 0, 1$ the left $\mathcal{D}_N$-linear map $\mathscr{T}_\lambda$ is an isomorphism.

Proof We shall use the same argument than in the proof of Theorem 2.4.1 to prove the formulas

$$U_1 U_{-1} = U_0(U_0 - 1) \quad \text{modulo} \quad \mathcal{I} \quad \text{and} \quad U_{-1} U_1 = U_0(U_0 + 1) \quad \text{modulo} \quad \mathcal{I} \tag{19}$$

For each $m \in \mathbb{N}$ we have:

$$\begin{aligned} U_1 U_{-1}[N_m] &= U_1[m N_{m-1}] = m(m-1)N_m \\ U_0(U_0-1)[N_m] &= U_0[(m-1)N_m] = m(m-1)N_m \qquad \text{and also} \\ U_{-1}U_1[N_m] &= U_{-1}[m N_{m+1}] = (m+1)m N_m \\ U_0(U_0+1)[N_m] &= U_0[(m+1)N_m] = m(m+1)N_m \end{aligned}$$

and this implies Formulas (19).

These give (A) and (B) and the conclusion follows. ■

The following important result shows that adding to the ideal $\mathcal{A}$ the invariance by translation and the homogeneity 1, that is to say considering the left ideal in $\mathcal{D}_N$: $\mathcal{A}+\mathcal{D}_N U_{-1}+\mathcal{D}_N(U_0-1)$, we recover the ideal $\mathcal{J}_1+\mathcal{D}_N U_{-1}$ and $\mathcal{D}_N\big/(\mathcal{J}_1+\mathcal{D}_N U_{-1})$ is the co-kernel of the map $\mathscr{T}_1 : \mathcal{N}_0 \to \mathcal{N}_1$. Therefore, as a corollary, we shall obtain the equality

$$\mathcal{N}_1^{\square} = \mathcal{D}_N\big/\mathcal{A} + \mathcal{D}_N U_{-1} + \mathcal{D}_N(U_0-1). \tag{20}$$

Proposition 3.1.6 *For $h \in [2,k]$ we have the equality*

$$\partial_h(U_0-1) + \partial_{h-1}U_{-1} = k\mathcal{T}^h + \sum_{q=1}^{k-1}(k-q)\sigma_q A_{h-1,q+1} \tag{E_h}$$

and for $h = 1$ the equality

$$-\partial_1(U_0-1) + EU_{-1} = \sum_{q=1}^{k-1}(k-q)\sigma_q \mathcal{T}^{q+1}. \tag{E_1}$$

Proof Recall first that, if we put $E := \sum_{h=1}^{k} h\sigma_h\partial_h$ then for any $m \in [2,k]$ we have

$$\mathcal{T}^m = \partial_1\partial_{m-1} + \partial_m E = \partial_1\partial_{m-1} + E\partial_m + \partial_m.$$

For $h \in [2,k]$ we have

$$\begin{aligned} \partial_h U_0 + \partial_{h-1}U_{-1} &= \sum_{q=1}^{k} q\sigma_q\partial_q\partial_h + h\partial_h + \sum_{q=0}^{k-1}(k-q)\sigma_q\partial_{q+1}\partial_{h-1} + (k-h+1)\partial_h \\ &= \sum_{q=1}^{k} q\sigma_q\partial_q\partial_h + \sum_{q=1}^{k}(k-q)\sigma_q A_{h-1,q+1} + k\partial_1\partial_{h-1} \\ &\quad + \sum_{q=1}^{k}(k-q)\sigma_q\partial_q\partial_h + (k+1)\partial_h \end{aligned}$$

$$= kE\partial_h + k\partial_1\partial_{h-1} + k\partial_h + \partial_h + \sum_{q=1}^{k-1}(k-q)\sigma_q A_{h-1,q+1}$$

$$= k\mathcal{T}^h + \partial_h + \sum_{q=1}^{k-1}(k-q)\sigma_q A_{h-1,q+1}$$

which is (E_h).

For $h = 1$ let us compute $\sum_{q=1}^{k-1}(k-q).\sigma_q.\mathcal{T}^{q+1}$:

$$\sum_{q=1}^{k-1}(k-q)\sigma_q\mathcal{T}^{q+1} = \sum_{q=1}^{k-1}(k-q)\sigma_q\left(\partial_1\partial_q + \partial_{q+1}E\right)$$

$$\sum_{q=1}^{k-1}(k-q)\sigma_q\mathcal{T}^{q+1} = \left(\sum_{q=1}^{k-1}(k-q)\sigma_q\partial_q\right)\partial_1 + \sum_{q=1}^{k-1}(k-q)\sigma_q\partial_{q+1}E$$

$$\sum_{q=1}^{k-1}(k-q)\sigma_q\mathcal{T}^{q+1} = \big(\sum_{q=1}^{k-1}(k-q)\sigma_q\partial_q\big)\partial_1 + (U_{-1} - k\partial_1)E$$

$$= k(E - \sigma_k\partial_k)\partial_1 - (U_0 - k\sigma_k\partial_k)\partial_1 + (U_{-1} - k\partial_1)E$$

$$= kE\partial_1 - k\partial_1 E - U_0\partial_1 + U_{-1}E$$

$$= EU_{-1} - \partial_1(U_0 - 1)$$

using the commutation relations $[U_{-1}, E] = k\partial_1$, $[E, \partial_1] = -\partial_1$ and $[U_0, \partial_1] = -\partial_1$. Therefore, we obtain the equality (E_1). ■

Remarks

1. An interesting way to look at these relations is to compare them with the minors of the $(k+1, 2)$ matrix

$$\begin{pmatrix} \mathscr{T} & -(\mathscr{H}_0 - 1) \\ \partial_1 & -\mathscr{E} \\ \partial_2 & \partial_1 \\ . & . \\ . & . \\ . & . \\ \partial_k & \partial_{k-1} \end{pmatrix}$$

where, by definition, $\mathscr{E}$ is the right product by E in the Weyl algebra $\mathbb{C}[\sigma]\langle\eta\rangle$. The relations (E_h), $h \in [1, k]$ may also be seen as the fact that

$$\begin{pmatrix} \partial_1 \\ \partial_2 \\ . \\ . \\ . \\ \partial_k \end{pmatrix}(U_0 - 1) + \begin{pmatrix} -E \\ \partial_1 \\ . \\ . \\ . \\ \partial_{k-1} \end{pmatrix} U_{-1}$$

is a global section of $\mathcal{I}^k \subset \mathcal{D}_N^k$.

2. Let g and γ be the symbols of U_0 and U_{-1}, respectively. Looking at the symbols in formulas (E_h), $h \in [1, k]$, we obtain (recall that $l_\sigma(\eta)$ is the symbol of E):

$$g(\sigma, \eta)\begin{pmatrix}\eta_1\\ \eta_2\\ \dots\\ \dots\\ \eta_k\end{pmatrix} + \gamma(\sigma, \eta)\begin{pmatrix}-l_\sigma(\eta)\\ \eta_1\\ \eta_2\\ \dots\\ \eta_{k-1}\end{pmatrix} = 0 \quad \text{on} \quad Z \tag{F}$$

3. For $\lambda \neq 0$ the sheaf of solutions[6] of $\mathcal{N}_\lambda$ near the generic point in N is the rank k local system with basis z_j^λ. This is consequence of the the fact that any local trace function[7] F which satisfies $(U_0 - \lambda)[F] = 0$ is the trace of a homogeneous function of degree λ.
4. Thanks to Lemma 3.1.5, the map induced by $\mathcal{T}_\lambda$ on solutions

$$Sol^0(\mathcal{N}_\lambda) \to Sol^0(\mathcal{N}_{\lambda-1})$$

sends z_j^λ to $\lambda z_j^{\lambda-1}$. This is clearly an isomorphism for $\lambda \neq 0, 1$.

3.2 Characteristic Varieties

Recall that for a differential operator $P \in \mathcal{D}_N$ we note $s(P)$ its symbol which is a section of the sheaf $\mathcal{O}_N[\eta]$ of homogeneous polynomials in $\eta := (\eta_1, \dots, \eta_k)$.

Proposition 3.2.1 *Let $\mathcal{I}$ be a coherent left ideal in $\mathcal{D}_N$ such that its characteristic ideal I_Z is the reduced ideal of an analytic subset Z in $N \times \mathbb{C}^k$. Let $U \in \Gamma(N, \mathcal{D}_N)$ be a differential operator of order q, such that its symbol u does not vanish on any non empty open set in Z. Assume that $\mathcal{I}U \subset \mathcal{I}$. Then, the characteristic ideal of $\mathcal{I} + \mathcal{D}_N U$ is equal to $I_Z + \mathcal{O}_{N\times\mathbb{C}^k} u$.*

Moreover, for any $\sigma \in N$ and any germ at σ of order $q + r$: $Q = P + BU$ where $P \in \mathcal{I}_\sigma$ and $B \in D_{N,\sigma}$, there exists $P_1 \in \mathcal{I}_\sigma$ of order at most $q + r$ and $B_1 \in D_{N,\sigma}$ of order at most r such that $Q = P_1 + B_1 U$.

Proof First, assume that there exists $P \in \mathcal{I}_\sigma$ and $B \in D_{N,\sigma}$, such that the symbol of $Q := P + BU$ is not in $I_Z + (u)$. Then, consider such a couple (P_0, B_0) with B_0 of order b minimal among all such couples. Then, P_0 and B_0U have the same order, because when their orders are different we have $s(Q) = s(P_0)$ or $s(Q) = s(B_0)u$ contradicting the fact that $s(Q)$ is not in $I_Z + (u)$.

In addition, if P_0 and B_0U have equal orders which is the order of Q, we have the equality $s(Q) = s(P_0) + s(B_0)u$ contradicting our assumption.

Therefore, the only case left is when P_0 and B_0U have the same order $b_0 + q$ which is strictly bigger than the order of Q. In this case we have $s(P_0) + s(B_0)u = 0$ which

6 We mean here $Sol^0(\mathcal{N}_\lambda) := \underline{Hom}_{\mathcal{D}_N}(\mathcal{N}_\lambda, \mathcal{O}_N)$.

7 See Introduction or [1].

implies that $s(B_0)u$ vanishes on Z. However, our hypothesis on u implies then that $s(B_0)$ vanishes on Z. As I_Z is reduced and is the characteristic ideal of $\mathcal{I}$ we may find a germ $B \in \mathcal{I}_\sigma$, such that $s(B) = s(B_0)$. Then, write

$$Q = P_0 + B_0U = P_0 + BU + (B_0 - B)U.$$

Since B is in $\mathcal{I}_\sigma$ and $\mathcal{I}U \subset \mathcal{I}$ we have $P_1 = P_0 + BU$ in $\mathcal{I}_\sigma$ and the order of $B_0 - B_1$ is strictly less than b. This contradicts the minimality of b and proves our first assertion.

Assume now that $Q = P + BU$ has order $q + r$ and that B has order $r + s$ with $s \geq 1$. If the order of P and BU are not equal then either P or BU is of order $q + r$ and P and BU have orders at most $q + r$ we are done.

So we may assume that P and BU have the same order $q + r + s$ with $s \geq 1$. Then, the previous considerations will produce $B' \in \mathcal{I}_\sigma$ with $s(B') = s(B)$ and then $P_1 := P + B'U$ and $B_1 := B - B'$ give that $Q = P_1 + B_1U$ with $P_1 \in \mathcal{I}_\sigma$ and $B_1 \in D_{N,\sigma}$ of order at most $q + r + s - 1$. By a descending induction on s this completes our proof, because when B_1 has order at most r the order of P_1 is at most $q + r$, because we assume that Q has order $q + r$. ∎

The following two corollaries are immediate applications of the previous proposition, using Proposition 2.2.2 and Theorem 2.4.1 which allow to verify that needed hypotheses.

Corollary 3.2.2 *The characteristic cycle of $\mathcal{N}_\lambda$, $\forall \lambda \in \mathbb{C}$, is the cycle associated to the ideal $I_Z + (g)$ in $\mathcal{O}_N[\eta]$ where g is the symbol of U_0.*

Also the characteristic cycle of $\tilde{\mathcal{M}}$ is the cycle associated to the ideal $I_Z + (\gamma)$ in $\mathcal{O}_N[\eta]$ where γ is the symbol of U_{-1}. ∎

Corollary 3.2.3 *Let $\mathcal{I}$ the left ideal in $\mathcal{D}_N$ that we introduced in Definition 2.2.1 and let $U := U_0 - \lambda$. Then, for any non zero germ $Q \in \mathcal{I} + \mathcal{D}_N U$ of order $q + 1$ there exist a germ $P \in \mathcal{I}$ of order at most $q + 1$ and a germ $B \in \mathcal{D}_N$ of order at most q such that $Q = P + BU$.* ∎

Remark

1. We shall be interested mainly by the special case of Corollary 3.2.3.
   Define for each $q \geq 0$

   $$\mathcal{J}_\lambda(q + 1) = \mathcal{I}(q + 1) + \mathcal{D}_N(q)(U_0 - \lambda).$$

   Then, this corollary gives, for each $\lambda \in \mathbb{C}$ and for each $q \in \mathbb{N}^*$ the equality $\mathcal{J}_\lambda \cap \mathcal{D}_N(q) = \mathcal{J}_\lambda(q)$. This implies that the natural map

   $$\mathcal{N}_\lambda(q) \to \mathcal{N}_\lambda \tag{21}$$

   is injective
2. Note that $\mathcal{J}_\lambda(0) := \mathcal{I}(0) = \{0\}$ as no non zero differential operator of order 0 annihilates the Newton polynomials (in fact $N_0 := k$ is enough !)

3. Also the fact that $\mathcal{I}(1) = \{0\}$ (see Lemma 2.2.5) implies the equality

$$\mathcal{J}_\lambda(1) = \mathcal{O}_N(U_0 - \lambda).$$

The irreducible component X. Let $H_\Delta := \{\Delta(\sigma) = 0\}$ in N. At the generic point σ of this hyper-surface, the polynomial P_σ has exactly one double root $\varphi(\sigma)$ and $\varphi : H_\Delta \dashrightarrow \mathbb{C}$ is a meromorphic function which is locally bounded on H_Δ. Then, define the meromorphic map

$$\Phi : H_\Delta \dashrightarrow \mathbb{P}_{k-1}$$

by letting $\Phi_h(\sigma) = (-\varphi(\sigma))^{k-h}$ for $h \in [1, k]$ in homogeneous coordinates. Let $X \subset N \times \mathbb{C}^k$ be the $N-$relative cone over the graph of the meromorphic map Φ. This is a $k-$dimensional irreducible subset in $H_\Delta \times \mathbb{C}^k$ and its fiber at the generic point in H_Δ is the line directed by the vector $\Phi_h(\sigma), h \in [1, k]$.

We shall consider the following sub-spaces in $N \times \mathbb{P}_{k-1}$ (where $s(P)$ is the symbol of P)

$$\mathbb{P}(Z) := \{(\sigma, \eta) \in N \times \mathbb{P}_{k-1} \ / \ s(P)(\sigma, \eta) = 0 \quad \forall P \in \mathcal{I} \setminus \{0\}\}$$

$$\mathbb{P}(X) := \left\{(\sigma, \eta) \in \mathbb{P}(Z) \ / \ \gamma(\sigma, \eta) := \sum_{h=0}^{k-1} (k-h).\sigma_h.\eta_{h+1} = 0\right\}$$

$$\mathbb{P}(Y) := \left\{(\sigma, \eta) \in \mathbb{P}(Z) \ / \ g(\sigma, \eta) := \sum_{h=1}^{k} h.\sigma_h.\eta_h = 0\right\}.$$

The next proposition will justify our notations in proving that $\mathbb{P}(X)$ is the graph of the meromorphic map Φ !

Proposition 3.2.4 *The subspace $\mathbb{P}(Z)$ is a complex sub-manifold of dimension k which is a $k-$branched covering of N via the natural projection $N \times \mathbb{P}_{k-1} \to N$. The subspace $\mathbb{P}(X)$ is reduced and equal to the irreducible component of*

$$\mathbb{P}(Z) \cap \big(H_\Delta \times \mathbb{P}_{k-1}\big)$$

which is the graph of the meromorphic map $\Phi : H_\Delta \to \mathbb{P}_{k-1}$ defined above, and $\mathbb{P}(Y)$ is the sum (as a cycle) of $\mathbb{P}(X)$ with the reduced hyper-surface in $\mathbb{P}(Z)$ defined by the (reduced) divisor $\{\eta_{k-1} = 0\}$ in $\mathbb{P}(Z)$.

Proof First remark that if (σ, η) is in Z and satisfies $\eta_k = 0$ then we have $\eta = 0$. Therefore, $\mathbb{P}(Z)$ is contained in the open set $\Omega_k := \{\eta_k \neq 0\}$ and, on this open set which is isomorphic to $N \times \mathbb{C}^{k-1}$, we may use the coordinates $\sigma_1, \dots, \sigma_k, \eta_1/\eta_k, \dots, \eta_{k-2}/\eta_k$ and $z := -\eta_{k-1}/\eta_k$.

Lemma 3.2.5 *We have an isomorphism*

$$\varphi_k : \mathbb{P}(Z) \to \mathbb{C}^k \quad \text{given by} \quad (\sigma, \eta) \mapsto (\sigma_1, \dots, \sigma_{k-1}, z).$$

Proof Remark first that the vanishing of the (2, 2) minors which give the equations of Z (see Formula (10) before Proposition 2.2.2) implies, assuming $\eta_k \neq 0$, that:

$$\eta_h/\eta_k = (-z)^{k-h} \quad \text{for } h \in [1, k-1]$$

and also, as the symbol of $\mathcal{T}^k$ is equal to $\eta_1\eta_{k-1} + \eta_k l_\sigma(\eta)$, that $l_\sigma(\eta)/\eta_k = -(-z)^k$. But then,

$$l_\sigma(\eta)/\eta_k = \sum_{h=1}^{k} (-1)^{k-h}\sigma_h z^{k-h} = (-1)^k(P_\sigma(z) - z^k)$$

shows that $P_\sigma(z) = 0$ on $\mathbb{P}(Z)$.

Let us show that the holomorphic map $\psi_k : \mathbb{C}^k \to \mathbb{P}(Z)$ given by

$$\eta_h = (-z)^{k-h} \quad \text{for } h \in [1, k] \quad \text{and} \quad \sigma_k = -\sum_{h=0}^{k-1} (-1)^{k-h}\sigma_h z^{k-h}$$

with the convention $\sigma_0 \equiv 1$ gives an inverse to φ_k.

First, we shall verify that ψ_k takes its values in $\mathbb{P}(Z)$. Note that the definition of σ_k implies $P_\sigma(z) = 0$. We have for $(\sigma, \eta) = \psi_k(\sigma', z)$ the equality:

$$l_\sigma(\eta) = \sum_{h=1}^{k} \sigma_h\eta_h = \sum_{h=1}^{k} (-1)^{k-h}\sigma_h z^{k-h} = -(-z)^k$$

Then, we have to verify that the vectors $(\eta_1, \dots, \eta_{k-1}, 1)$ and $((-z)^k, \eta_1, \dots, \eta_{k-1})$ are co-linear. This is clear as the second one is $\eta_{k-1} = (-z)\eta_k$-times the first one (see again Formula (10)).

To complete the proof, it is enough to check that $\psi_k \circ \varphi_k$ and $\varphi_k \circ \psi_k$ are the identity maps. This is easy verification is left to the reader. ∎

End of proof of 3.2.4. In this chart we have

$$g(\sigma, \eta)/\eta_k = \sum_{h=1}^{k} h\sigma_h\eta_h/\eta_k = \sum_{h=1}^{k} (-1)^{k-h} h\sigma_h z^{k-h}$$

$$g(\sigma, \eta)/\eta_k = (-1)^{k+1}\left(\sum_{h=1}^{k} (-1)^h (k-h)\sigma_h z^{k-h} - k\sum_{h=1}^{k} (-1)^h\sigma_h z^{k-h}\right)$$

and this gives

$$g(\sigma, \eta)/\eta_k = (-1)^{k+1}\Big(zP'_\sigma(z) - kz^k - k(P_\sigma(z) - z^k)\Big) = (-1)^{k+1} zP'_\sigma(z).$$

We have also

$$\gamma(\sigma,\eta)/\eta_k = \sum_{h=0}^{k-1}(-1)^{k-h-1}(k-h)\sigma_h z^{k-h-1} = (-1)^{k+1}P'_\sigma(z).$$

Therefore, $g = z\gamma$ in this chart,[8] and the ideal generated by g and γ in $\mathbb{P}(Z)$ is generated by γ which defined the hyper-surface $\mathbb{P}(X)$.

However, on this hyper-surface we have $P_\sigma(z) = 0$ and $P'_\sigma(z) = 0$, so z is a double root of P_σ. This implies that $\Delta(\sigma) = 0$ for (σ,η) in the analytic subset $|\mathbb{P}(X)|$.

On a Zariski dense open set in H_Δ the unique double root of P_σ is equal to $\varphi(\sigma)$ which is given by $z = -\eta_{k-1}/\eta_k$ when $(\sigma,\eta) \in |\mathbb{P}(X)|$. Therefore, $|\mathbb{P}(X)|$ contains the graph of the meromorphic map Φ. Moreover, as the projection $\mathbb{P}(Z) \to N$ is clearly a branched covering (of degree k) and over the generic point in H_Δ there exists an unique root of multiplicity 2 for P_σ, $\mathbb{P}(X)$ has generic degree 1 over H_Δ. In addition,, because $P''_\sigma(z)$ does not vanish at the generic point in $\mathbb{P}(X)$ (which has to be over the generic point of H_Δ) implies that the hyper-surface $\mathbb{P}(X)$ of $\mathbb{P}(Z)$ is reduced. This is enough to conclude that $\mathbb{P}(X)$ is equal to the graph of Φ.

The previous computation shows also that $\mathbb{P}(Y)$ is the sum of $\mathbb{P}(X)$ with the divisor defined by $\{z = 0\}$ in $\mathbb{P}(Z)$ which is a smooth and reduced hyper-surface given by the equation $\eta_{k-1} = 0$ in $\mathbb{P}(Z)$. ■

The determination of the characteristic cycles of the holonomic $\mathcal{D}_N$-modules $\mathfrak{N}$ and $\mathcal{N}_\lambda$ is an easy consequence of the previous proposition thanks to Proposition 2.2.3.

Corollary 3.2.6 *The characteristic cycle of the $\mathcal{D}_N$-module $\mathfrak{N}$ is equal to $\mathbb{P}(X)$. For each complex number λ the characteristic cycle of the $\mathcal{D}_N$-module $\mathcal{N}_\lambda$ is equal to $\mathbb{P}(Y) = \mathbb{P}(X) + \big(\mathbb{P}(Z) \cap \{\eta_{k-1} = 0\}\big)$.* ■

Remarks

1. The intersection $\mathbb{P}(Z) \cap \{\eta_{k-1} = 0\}$ is equal to $N \times [v]$ where v is the point $(0,\dots,0,1) \in \mathbb{P}_{k-1}$ and this intersection is the projectivization of the co-normal to the hyper-surface $\{\sigma_k = 0\}$.
2. At the set-theoretical level we have

$$Z \cap \{\gamma = 0\} = X \cup (N \times \{0\}) \quad \text{and}$$
$$Z \cap \{g = 0\} = X \cup \Big(\{\sigma_k = 0\} \times \{\eta_1 = \eta_2 = \cdots = \eta_{k-1} = 0\}\Big) \cup (N \times \{0\}).$$

3. Despite the previous results, $g(\sigma,\eta)$ does not belongs to the ideal of $\mathbb{C}[\sigma,\eta]$ generated by I_Z and $\gamma(\sigma,\eta)$ at the generic point in $N \times \{0\}$. This is consequence of the fact that I_Z does not contain a non zero element in $\mathbb{C}[\sigma,\eta]$ which is homogeneous of degree 1 in η, using Corollary 3.2.3.

The following lemma will be useful later on

Lemma 3.2.7 *Assume that $f\partial_k^n U_{-1}$ is in $\mathcal{J}_{\lambda,\sigma}$ for some $f \in \mathcal{O}_{N,\sigma}$, some integer $n \geq 1$ and some $\lambda \in \mathbb{C}$. Then, f is in $\sigma_k\mathcal{O}_{N,\sigma}$.*

[8] Compare withe Formula (F) at the end of Paragraph 3.1.

Proof The fact that $f\partial_k^n U_{-1}$ is in $\mathcal{J}_{\lambda,\sigma}$ implies that $f\eta_k^n\gamma$ vanishes on the characteristic variety of the $\mathcal{D}_N$-module $\mathcal{N}_\lambda$. Therefore, thanks to Corollary 3.2.6 $f.\eta_k^n\gamma$ vanishes on C, the co-normal bundle of the hyper-surface $\{\sigma_k = 0\}$. However, η_k and γ do not vanish on any non empty open set in C:

this is clear for η_k and the restriction of γ to C is equal to $\sigma_{k-1}\eta_k$ and σ_{k-1} also does not vanish on any non empty open set in C. Therefore, $f \in \mathcal{O}_{N,\sigma}$ has to vanish on C and we conclude that f is in $\sigma_k\mathcal{O}_{N,\sigma}$. ∎

3.3 The Case $\lambda \notin \mathbb{N}$

Notation For each $\lambda \in \mathbb{C}$ and each $q \geq 0$ we shall note

$$\mathcal{J}_\lambda(q+1) := \mathcal{I}(q+1) + \mathcal{D}_N(q)(U_0 - \lambda)$$

and

$$\mathcal{N}_\lambda(q+1) := \mathcal{D}_N(q+1)\big/\mathcal{J}_\lambda(q+1).$$

For $q = 0$ we note $\mathcal{J}_\lambda(0) := \mathcal{I}(0)$ and $\mathcal{N}_\lambda(0) := \mathcal{O}_N\big/\mathcal{J}_\lambda(0)$.

The goal of this paragraph is to prove the following theorem.

Theorem 3.3.1 *For $\lambda \in \mathbb{C} \setminus \mathbb{N}^*$ the $\mathcal{D}_N$-module $\mathcal{N}_\lambda$ has no $\mathcal{O}_N$-torsion.*

Proof This result is a direct consequence of Proposition 3.3.5, thanks to the injectivity for each $q \geq 0$ of the natural map $\mathcal{N}_\lambda(q) \to \mathcal{N}_\lambda$ (see Remark 1 following Corollary 3.2.3). ∎

Definition 3.3.2 For any $\lambda \in \mathbb{C} \setminus \mathbb{N}^*$, for any integer $q \geq 2$ and for any integer $r \in [q, k(q-1)]$ define the following elements in $\mathcal{W}_q$ (see Formulas (12) and (13) in Lemma 2.2.6)

$$\theta_{q,r} := (r-\lambda)\partial^\alpha\mathcal{T}^m] - (q-1)\left[\partial^\beta(U_0-\lambda)\right] \tag{22}$$

$$\text{so} \quad \theta_{q,r} = \sum_{h=0}^{k}(r-\lambda-(q-1)h)\sigma_h y_{q,r+h}. \tag{23}$$

where in Formula (23) we assume that $\alpha \in \mathbb{N}^k$ and $m \in [2,k]$ satisfy $|\alpha| = q-2$ and $w(\alpha) = r-m$, and that $\beta \in \mathbb{N}^k$ satisfies $|\beta| = q-1$ and $w(\beta) = r$.

Corollary 3.3.3 *For any integer $q \geq 1$ the kernel of the quotient map*

$$l_q : \mathcal{W}(q) \to \mathcal{N}_\lambda(q)$$

is equal to the sub-$\mathcal{O}_N$-module generated by $U_0 - \lambda \in \mathcal{W}(1)$ and the elements $\theta_{p,r}$, for each $p \in [2,q]$, and for each $r \in [p,(k-1)p]$.

Proof We have to prove that if a non zero differential operator P of order $p \leq q$ is in $\mathcal{J}_\lambda$ then it may be written as $Q + B(U_0 - \lambda)$ with $Q \in \mathcal{I}$ of order at most p (or $Q = 0$) and B of order at most $p - 1$ (or $B = 0$). When $p \geq 2$ this is precisely the statement proved in Proposition 3.2.1. For $p \leq 1$ the only P which are in $\mathcal{J}_\lambda(1)$ are in $\mathcal{O}_N(U_0 - \lambda)$ thanks to Remarks 2 and 3 following Corollary 3.2.3. ∎

Lemma 3.3.4 *Let λ be in $\mathbb{C} \setminus \mathbb{N}^*$; for each integer $q \geq 2$ the elements $\theta_{q,r}$ and $y_{q,s}$, with $r \in [q, k(q-1)]$ and $s \in [k(q-1)+1, kq]$ form a $\mathcal{O}_N$−basis of $\mathcal{W}_q$.*

Proof Let $\mathcal{W}_{q,p}$ be the $\mathcal{O}_N$-module of $\mathcal{W}_q$ with basis the $y_{q,r}$ for $r \geq p+1$. Then, we have for $r \in [q, k(q-1)]$

$$\theta_{q,r} \in (r - \lambda) y_{q,r} + \mathcal{W}_{q,r+1}$$

so the determinant of the $k(q-1) - q + 1 + k = kq - q + 1$ vectors $\theta_{q,r}, y_{q,s}$ in the basis $\big(y_{q,r}, r \in [q, kq]\big)$ of $\mathcal{W}_q$ is upper triangular and is equal to $\prod_{r=q}^{k(q-1)}(r - \lambda)$ which is in $\mathbb{C}^*$ as soon as λ is not in the subset $[q, k(q-1)]$ of $\mathbb{N}^*$. ∎

Proposition 3.3.5 *Let $q \geq 1$ be an integer and assume that λ is not an integer in $[0, k(q-1)]$. Let $L_q : \mathcal{W}_q \to \mathcal{N}_\lambda(q)$ be the restriction to $\mathcal{W}_q$ of quotient map l_q. This $\mathcal{O}_N$-linear map is surjective and its kernel is the sub-module of $\mathcal{W}_q$ with basis the θ_r^q for $r \in [q, k(q-1)]$. Therefore, $\mathcal{N}_\lambda(q)$ is a free $\mathcal{O}_N$-module of rank k.*

Proof Remark first that for $q = 1$ the result is clear as for $\lambda \neq 0$ we have

$$\mathcal{N}_\lambda(1) = \oplus_{h=1}^k \mathcal{O}_N \partial_h$$

thanks to Remark 3 following Corollary 3.2.3 and $\mathcal{W}_1 = \oplus_{h=1}^k \mathcal{O}_N . y_{1,h}$ with $L_1(y_{1,h}) = [\partial_h]$. Therefore, we may assume that $q \geq 2$.

We shall prove first that $\mathcal{N}_\lambda(q)$ is equal to the image of L_q by induction on $q \geq 2$.

Assume that $q = 2$. Then, the image of

$$\partial_j (U_0 - \lambda) - (j - \lambda)\partial_j = \sum_{h=1}^k h\sigma_h y_{2,h+j} \in \mathcal{W}_2$$

by L_2 is the class of $-(j - \lambda)\partial_j$ in $\mathcal{N}_\lambda$. Therefore, the image of L_2 contains the classes of $\partial_1, \dots, \partial_k$ as λ is not in $[1, k]$ and also contains the class of 1 as we assume $\lambda \neq 0$ and as the equality $\lambda = \sum_{h=1}^k h\sigma_h \partial_h$ holds in $\mathcal{N}_\lambda$. However, the image of L_2 contains obviously the classes of ∂^α for any multi-index $\alpha \in \mathbb{N}^k$, $|\alpha| = 2$. Therefore, our assertion is proved for $q = 2$.

Assume now that $q \geq 3$ and that our assertion is proved for $q - 1$. Remark that the image of L_q contains obviously the classes of ∂^α for each $\alpha \in \mathbb{N}^k$, $|\alpha| = q$. We shall use now the following easy formula:

- For any $r \in [q-1, k(q-1)]$ and any $j \in [1, k]$ we have in $\mathcal{N}_\lambda$ the equality

$$\partial_j L_{q-1}(y_{q-1,r}) = L_q(\partial_j y_{q-1,r}) = L_q(y_{q,r+j}).$$

For any $\beta \in \mathbb{N}^k \setminus \{0\}$ with $|\beta| \leq q-1$ we may find $j \in [1,k]$ and $\gamma \in \mathbb{N}^k$, such that $\partial^\beta = \partial_j \partial^\gamma$. By our inductive assumption there exists $x \in \mathcal{W}_{q-1}$, such that $L_{q-1}(x) = \partial^\gamma$. Then, $\partial_j x$ is in $\mathcal{W}_q$ and thanks to the formula above we have

$$L_q(\partial_j x) = \partial_j L_{q-1}(x) = \partial_j.\partial^\gamma = \partial^\beta \quad \text{in} \quad \mathcal{N}_\lambda.$$

Again, we conclude that the class of 1 in $\mathcal{N}_\lambda(q)$ is in the image of L_q using $\lambda \neq 0$ and the equality $\lambda = \sum_{h=1}^k h\sigma_h \partial_h$ which holds in $\mathcal{N}_\lambda(q)$. This complete the proof of our first statement.

However, it is clear that $\theta_{q,r}$ for $r \in [q, k(q-1)]$ are in the kernel of L_q. Therefore, the $\mathcal{O}_N$-free rank k module with basis $\big(y_{q,r}, r \in [k(q-1)+1, kq]\big)$ is surjective via L_q onto $\mathcal{N}_\lambda(q)$. The next lemma completes the proof, as we already know that $\mathcal{N}_\lambda(1)$ is a $\mathcal{O}_N$-free rank k sub-module of $\mathcal{N}_\lambda(q)$ with basis $\partial_1, \ldots, \partial_k$. ∎

Lemma 3.3.6 *Let A be an integral commutative ring and let M be a A-module. Assume that there exists a surjective A-linear map $p : A^k \to M$ and an injective A-linear map $i : A^k \to M$. Then, p is an isomorphism.*

Proof Let $j : A^k \to A^k$ be a $A-$linear map, such that $j \circ p = i$. Therefore, j is injective and the co-kernel C of j is a torsion module. Let $q : A^k \to C$ be the quotient map and let K be the kernel of p. The restriction of q to K is injective, because if $x \in K$ satisfies $q(x) = 0$ then $x = j(y)$ for some $y \in A^k$ and then $i(y) = p(j(y)) = p(x) = 0$, which implies $y = 0$ and $x = 0$. Therefore, K is a sub-module of C and then K is a $A-$torsion module. However, as $K \subset A^k$ we have $K = 0$ and so p is an isomorphism. ∎

$$\begin{tikzcd}
0 & & & & \\
& C \arrow[ul] & & & \\
0 \arrow[r] & K \arrow[r] \arrow[u] & A^k \arrow[r, "p"] \arrow[ul, "q"'] & M \arrow[r] & 0 \\
& 0 \arrow[u] & & A^k \arrow[ul, "j"] \arrow[u, "i"'] & \\
& & & 0 \arrow[u] & 0 \arrow[ul]
\end{tikzcd}$$

Lemma 3.3.7 *For $\lambda \notin \mathbb{N}$ we have $\sigma_k \Delta(\sigma)\mathcal{N}_\lambda(2) \subset \mathcal{N}_\lambda(1)$.*

The proof will be a simple consequence of the following lemma.

Lemma 3.3.8 *Let $y := (y_2, \dots, y_{2k})$ be in $\mathbb{C}[\sigma]^{2k-1}$ and consider the $\mathbb{C}[\sigma]-$linear system $(2k-1, 2k-1)$ on $\mathbb{C}[\sigma]^{2k-1}$ given by the following $\mathbb{C}[\sigma]-$linear forms:*

$$L_q(y) := \sum_{h=0}^{k} h\sigma_h y_{q+h} \quad \text{for} \quad q \in [1, k]$$
$$\Lambda_r(y) := \sum_{h=0}^{k} \sigma_h y_{r+h} \quad \text{for} \quad r \in [2, k]$$

Then, the determinant of this linear system is equal to $\sigma_k \Delta(\sigma)$ where $\Delta(\sigma)$ is the discriminant of the polynomial $P_\sigma(z) := z^k + \sum_{h=1}^{k}(-1)^h \sigma_h z^{k-h}$.

Proof Remark first that $\Delta(\sigma)$ is also the discriminant of the polynomial (see the computation below):

$$\tilde{P}_\sigma(z) := \sum_{h=0}^{k} \sigma_h z^{k-h}.$$

Then, remark also that the resultant of the polynomials $\tilde{P}_\sigma(z)$ and $k\tilde{P}_\sigma(z) - z(\tilde{P}_\sigma)'(z)$ coincides with the determinant of the $(2k-1, 2k-1)$ $\mathbb{C}[\sigma]-$linear system defined in the statement of the lemma. Therefore, it is enough to compute this resultant. It is given by

$$\begin{aligned} R(\sigma) &= \prod_{\tilde{P}_\sigma(z_j)=0} \big(k\tilde{P}_\sigma(z_j) - z_j(\tilde{P}_\sigma)'(z_j)\big) \\ &= \sigma_k \prod_{P_\sigma(-z_j)=0} (-1)^{k-1} P'_\sigma(-z_j) = \sigma_k \Delta(\sigma) \end{aligned}$$

as $\tilde{P}_\sigma(-z) = (-1)^k P_\sigma(z)$ implies $\tilde{P}'_\sigma(-z) = (-1)^{k-1} P'_\sigma(z)$. This conclude the proof ■

proof of 3.3.7 It is enough to prove that for each $(p, q) \in [1, k]^2$ there exist polynomials $a^p_{h,q}(\lambda)$ in $\mathbb{C}[\sigma, \lambda]$ (in fact affine in λ), such that

$$\sigma_k \Delta(\sigma)\partial_p\partial_q - \sum_{h=1}^{k} a^p_{h,q}(\lambda)\partial_h \in \mathcal{J}_\lambda.$$

For $m \in [2, k]$ we have

$$\mathcal{T}^m = y_{2,m} + \sum_{h=1}^{k} \sigma_h y_{2,m+h} + y_{1,m} \in \mathcal{I} \subset \mathcal{J}_\lambda$$

and for $q \in [1, k]$:

$$\partial_q(U_0 - \lambda) = \sum_{h=1}^{k} h\sigma_h y_{2,q+h} + (q-\lambda)y_{1,q} \in \mathcal{J}_\lambda.$$

This gives $(2k-1)$ $\mathbb{C}[\sigma]-$linear relations between the basis elements $y_{2,r}$, $r \in [2, 2k]$ of $\mathcal{W}_2$ modulo $L_2^{-1}(\mathcal{N}_\lambda(1))$. However, the determinant of these $2k-1$ vectors in the basis $y_{2,r}$ of $\mathcal{W}_2$ is equal to $\sigma_k\Delta(\sigma)$ thanks to the previous lemma. The conclusion follows, as we know that $L_2 : \mathcal{W}_2 \to \mathcal{N}_\lambda(2)$ is surjective for $\lambda \notin \mathbb{N}$. ■

Lemma 3.3.9 *Assume that on a Stein open set U in N the equality of the sheaves $\mathcal{N}_\lambda(2)_{|U} = \mathcal{N}_\lambda(1)_{|U}$ is true for some $\lambda \in \mathbb{C}$. Then, we have*

$$(\mathcal{N}_\lambda)_{|U} = \mathcal{N}_\lambda(1)_{|U}.$$

Proof It is enough to prove the equality $\mathcal{N}_\lambda(q)_{|U} = \mathcal{N}_\lambda(1)_{|U}$ for any $q \geq 2$, because we know that $\mathcal{N}_\lambda = \cup_{q\geq 0}\mathcal{N}_\lambda(q)$. As this is true for $q = 2$ by assumption, we shall prove this equality by induction on $q \geq 2$. Therefore, assume that this equality is proved for some $q \geq 2$ and we shall prove it for $q + 1$.

Let $\alpha \in \mathbb{N}^k$, such that $|\alpha| = q+1$ and write $\partial^\alpha = \partial_p\partial^\beta$ for some $p \in [1, k]$ and some $\beta \in \mathbb{N}^k$ with $|\beta| = q$. By the inductive assumption we may write $\partial^\beta = \sum_{h=1}^k b_h\partial_h$ in $\mathcal{N}_\lambda(q)$ with $b_h \in \mathcal{O}(U)$, because we know that $\mathcal{N}_\lambda(1) = \oplus_{h=1}^k\mathcal{O}_N\partial_h$ on N. Then, we obtain that $\partial_p\partial^\beta$ is in $\mathcal{N}_\lambda(2)_{|U} = \mathcal{N}_\lambda(1)_{|U}$, concluding our induction. ■

Corollary 3.3.10 *For each $\lambda \in \mathbb{C} \setminus \mathbb{N}$ there exists a meromorphic integrable connection $\nabla_\lambda : \mathcal{O}_N^k \to \frac{1}{\sigma_k.\Delta}.\mathcal{O}_N^k \otimes \Omega_N^1$ with a simple pole on the reduced hyper-surface $\{\sigma_k\Delta(\sigma) = 0\} \subset N$, such that the restriction of $\mathcal{N}_\lambda$ to the Stein (in fact affine) open set $U := \{\sigma_k\Delta(\sigma) \neq 0\}$ is isomorphic to the $\mathcal{D}_U$-module defined by $(\mathcal{O}_N^k, \nabla_\lambda)$. Moreover, this isomorphism is the restriction of an injective $\mathcal{D}_N$-linear map*

$$\mathcal{N}_\lambda \to \big(\mathcal{O}_N^k(*\sigma_k\Delta(\sigma)), \nabla_\lambda\big).$$

Proof This is an easy consequence of the $\mathcal{O}_N$ isomorphism $\mathcal{N}_\lambda(1) \to \oplus_{h=1}^k\mathcal{O}\partial_h$ and previous Lemmas 3.3.7 and 3.3.9. ■

We shall conclude this section by the following theorem.

Theorem 3.3.11 *Let $\lambda \in \mathbb{C}\setminus\mathbb{Z}$. Then, $\mathcal{N}_\lambda$ is the minimal extension of the meromorphic connection given by $(\mathcal{N}_\lambda(1), \nabla_\lambda)$. Therefore, $\mathcal{N}_\lambda$ is a simple $\mathcal{D}_N$-module.*

Proof To see that $\mathcal{N}_\lambda$ is the minimal extension of the simple pole meromorphic connection $\big(\mathcal{N}_\lambda(1), \nabla_\lambda\big)$ it is enough to prove that $\mathcal{N}_\lambda$ has no torsion, and this is given by Proposition 3.3.5, and no *co-torsion*, that is to say that there is no non trivial coherent left ideal $\mathcal{K}$ in $\mathcal{D}_N$ containing $\mathcal{J}_\lambda$ and generically equal to $\mathcal{J}_\lambda$ on N. Such an ideal defines a holonomic quotient Q of $\mathcal{N}_\lambda$ which is supported in a closed analytic subset S of N with empty interior in N. As $\mathcal{N}_\lambda$ is a quotient of $\mathcal{M}$, we may apply Corollary 2.3.2

and so it is enough to show that near the generic points of $\{\sigma_k \Delta(\sigma) = 0\}$ such an ideal $\mathcal{K}$ is equal to $\mathcal{J}_\lambda$ or to $\mathcal{D}_N$.

Near the generic point of $\{\sigma_k = 0\}$ we have $\Delta \neq 0$ and we may use a local isomorphism of N given by a holomorphic section of the quotient map

$$quot : M = \mathbb{C}^k \to \mathbb{C}^k \big/ \mathfrak{S}_k = N.$$

Via such an isomorphism $\mathcal{N}_\lambda$ is the quotient of $D_{\mathbb{C}^k}$ by the left ideal generated by the $\frac{\partial^2}{\partial z_i \partial z_j}$ for $i \neq j \in [1, k]$ and $\sum_{j=1}^k z_j \frac{\partial}{\partial z_j} - \lambda$. The lemma below allows to conclude this case. For the other case, that is to say near the generic point of $\{\Delta = 0\}$, Theorem 2.3.6 completes the proof.

The fact that $\mathcal{N}_\lambda$ is a simple $\mathcal{D}_N$-module is then consequence of the irreductibility of the monodromy representation of its associated meromorphic connection. ■

Lemma 3.3.12 *Let $\mathcal{J}_\lambda$ for $\lambda \notin -\mathbb{N}^*$ be the ideal in $D_{\mathbb{C}^k}$ generated by the differential operators $\frac{\partial^2}{\partial z_i \partial z_j}$ for $1 \leq i < j \leq k$ and $\sum_{h=1}^k z_h . \frac{\partial}{\partial z_h} - \lambda$. Let assume that Q is a quotient of the $D_{\mathbb{C}^k}$−module $\mathcal{N}_\lambda := D_{\mathbb{C}^k} \big/ \mathcal{J}_\lambda$ in a neighborhood U of the point $(z_1^0, \dots, z_k^0)$ in $\mathbb{C}^k$ where $z_1^0 = 0$ and $z_i \neq z_j$ for $1 \leq i < j \leq k$, with support in $\{z_1 = 0\}$. Then, $Q = 0$.*

Proof Assume that $Q \neq 0$ Then $Q = D_U \big/ \mathcal{K}$ where $\mathcal{K}$ is a left ideal in D_U, such that $\mathcal{J}_\lambda \subsetneq \mathcal{K} \subsetneq D$. Then, restricting the open neighborhood U of z^0 if necessary, there exists a positive integer n, such that z_1^n belongs[9] to $\mathcal{K}$. Then, we have

$$\frac{\partial}{\partial z_1} z_1^n = n z_1^{n-1} + z_1^n \frac{\partial}{\partial z_1} \in \mathcal{K} \quad \text{so writing this as} \tag{a}$$

$$n z_1^{n-1} + z_1^{n-1} \left(\sum_{h=1}^k z_h \frac{\partial}{\partial z_h} - \lambda \right) + \lambda . z_1^{n-1} - z_1^{n-1} \left(\sum_{h=2}^k z_h \frac{\partial}{\partial z_h} \right) \in \mathcal{K} \quad \text{and then}$$

$$(n + \lambda) z_1^{n-1} - \sum_{h=2}^k z_h z_1^{n-1} \frac{\partial}{\partial z_h} \in \mathcal{K}$$

as $\sum_{h=1}^k z_h \frac{\partial}{\partial z_h} - \lambda \in \mathcal{J}_\lambda \subset \mathcal{K}$ on U. However, $z_1^n \in \mathcal{K}$ implies also, for each $j \in [2, k]$:

$$\frac{\partial^2}{\partial z_1 \partial z_j} z_1^n = n z_1^{n-1} \frac{\partial}{\partial z_j} + z_1^n \frac{\partial^2}{\partial z_1 \partial z_j} \in \mathcal{K} \quad \text{which implies} \tag{b}$$

$$n z_1^{n-1} \frac{\partial}{\partial z_j} \in \mathcal{K} \quad \forall j \in [2, k]$$

again as $\mathcal{J}_\lambda \subset \mathcal{K}$. Combining (a) and (b) we conclude that z_1^{n-1} belongs to $\mathcal{K}$, as we assume $n > 0$ and $\lambda \notin \mathbb{N}^*$.

[9] The class of 1 in Q is of z_1−torsion!

By a descending induction on n we conclude that 1 belongs to $\mathcal{K}$ which contradicts our assumption that Q is not 0. ∎

Remark Note that the $\mathcal{D}_N$-linear map

$$\varphi_{-1} : \mathcal{N}_{-1} \to \underline{H}^1_{[\sigma_k=0]}(\mathcal{O}_N) := \mathcal{O}_N[\sigma_k^{-1}]\big/\mathcal{O}_N$$

defined by $\varphi_{-1}(1) := \sigma_{k-1}/\sigma_k$ is surjective, because $\varphi_{-1}(\partial_{k-1}) = 1/\sigma_k$. This shows that for $p = -1$ the sheaf $\mathcal{N}_{-1}$ has a non zero quotient supported by $\{\sigma_k = 0\}$. Then, using the isomorphism $\mathscr{T}_\lambda : \mathcal{N}_{\lambda-1} \to \mathcal{N}_\lambda$ for $\lambda \in -\mathbb{N}^*$ to deduce the case $\lambda - 1$ from the case λ for each $\lambda \in -\mathbb{N}^*$, we see that the sheaf $\mathcal{N}_{-p}$ has a non zero quotient supported by $\{\sigma_k = 0\}$ for any $p \in \mathbb{N}^*$.

4 The $\mathcal{D}_N$-Modules $\mathcal{N}_p$, $p \in \mathbb{Z}$

4.1 Structure of $\mathcal{N}_p$, $p \geq 1$

The first important remark is that, thanks to Lemma 3.1.5, it is enough to determine the structure of $\mathcal{N}_1$ as for each $p \geq 2$ the $\mathcal{D}_N$-module $\mathcal{N}_p$ is isomorphic to $\mathcal{N}_1$ via the right multiplication by U_1^{p-1}.

4.1.1 Minimality of $\mathcal{N}_1^\square$

Recall that $\mathcal{N}_1^\square$ is the co-kernel of the left $\mathcal{D}_N$-linear map $\mathscr{T}_1 : \mathcal{N}_0 \to \mathcal{N}_1$ defined by the right multiplication by U_{-1}.

Thanks to formulas E_h, $h \in [2, k]$ (see Proposition 3.1.6) we obtain that $\mathcal{N}_1^\square$ is the quotient of $\mathcal{D}_N$ by the left ideal $\mathcal{A} + \mathcal{D}_N(U_0 - 1) + \mathcal{D}_N U_{-1}$, because these formulas imply that the partial differential operators $\mathcal{T}^m$, $m \in [2, k]$ are contained in the ideal $\mathcal{A} + \mathcal{D}_N(U_0 - 1) + \mathcal{D}_N U_{-1}$ and we have $\mathcal{J}_1 = \mathcal{I} + \mathcal{D}_N(U_0 - 1)$ by definition (see Formula (8) for the definition of $\mathcal{I}$ and the beginning of Paragraph 3.1 for the definition of the ideal $\mathcal{J}_\lambda$).

We shall note $\mathcal{N}_1^\square(q) := \mathcal{D}_N(q)\big/\big(\mathcal{J}_1 \cap \mathcal{D}_N(q)\big)$ for each integer $q \geq 0$.

Proposition 4.1.1 *For each q the natural map $\mathcal{N}_1^\square(q) \to \mathcal{N}_1^\square$ is injective.*

Proof The proof will use Proposition 3.2.1 two times: the first time for the left ideal $\mathcal{A}$ and with $U := U_{-1}$ and the second time for the left ideal $\mathcal{A} + \mathcal{D}_N U_{-1}$ and with $U := U_0 - 1$. This will give the equalities

$$\begin{aligned}
\big(\mathcal{A} + \mathcal{D}_N U_{-1}\big) \cap \mathcal{D}_N(q) &= \mathcal{A}(q) + \mathcal{D}_N(q-1)U_{-1} \quad \text{and} \\
\big(\mathcal{A} + \mathcal{D}_N U_{-1} + \mathcal{D}_N(U_0 - 1)\big) \cap \mathcal{D}_N(q) &= \mathcal{A}(q) + \mathcal{D}_N(q-1)U_{-1} \\
&\quad + \mathcal{D}_N(q-1)(U_0 - 1).
\end{aligned}$$

This will conclude the proof.

To apply Proposition 3.2.1 we have to show that the following properties hold:

(i) The coherence of $\mathcal{A}$ and of $\mathcal{A} + \mathcal{D}_N U_{-1}$.
(ii) The fact that the characteristic ideals of $\mathcal{A}$ and of $\mathcal{A} + \mathcal{D}_N U_{-1}$ are reduced.
(iii) The inclusions $\mathcal{A}U_{-1} \subset \mathcal{A}$ and $\big(\mathcal{A} + \mathcal{D}_N U_{-1})(U_0 - 1)\big) \subset \mathcal{A} + \mathcal{D}_N U_{-1}$.
(iv) The symbol of U_{-1} does not vanish on any non empty open set of the characteristic variety of $\mathcal{D}_N\big/\mathcal{A}$.
(v) The symbol of $U_0 - 1$ does not vanish on any non empty open set of the characteristic variety of $\mathcal{D}_N\big/\mathcal{A} + \mathcal{D}_N U_{-1}$.

The point (i) is clear.

The characteristic ideal of $\mathcal{A}$ is the pull-back by the projection $p_2 : N \times \mathbb{C}^k \to \mathbb{C}^k$ of the ideal of the reduced ideal $IS(k)$ of the surface $S(k)$ (see Corollary 6.1.6 in the appendix).

The point (ii) is completed by the following lemma:

Lemma 4.1.2 *Let $\gamma(\sigma, \eta) := \sum_{h=0}(k-h)\sigma_h\eta_{h+1}$ and $g(\sigma, \eta) := \sum_{h=1}^k h\sigma_h\eta_h$. Then, defined the following ideals in $\mathcal{O}_N[\eta]$, where $I_1 := (p_2)^*(IS(k))$:*

$$I_2 := I_1 + (\gamma) \quad \text{and} \quad I_3 := I_2 + (g).$$

Then, I_2 is reduced and g does not vanish on any non empty open set of the analytic subset $(N \times S(k)) \cap \{\gamma = 0\}$.

Proof To see that I_2 is reduced, as $N \times S(k)$ is normal, it is enough to prove that $\{\gamma = 0\}$ defined a reduced and irreducible hyper-surface in $N \times S(k)$. Looking at the chart on the dense open set $\eta_k \neq 0$ of $N \times S(k)$ which is given by the map $(\sigma, \eta) \mapsto (\sigma, -\eta_{k-1}/\eta_k, \eta_k) \in N \times \mathbb{C} \times \mathbb{C}^*$ (see Paragraph 3.2) we find that γ is given in this chart by

$$\gamma(\sigma, \eta) = (-1)^{k-1}P'_\sigma(z)\eta_k \quad \text{where} \quad z := -\eta_{k-1}/\eta_k$$

using the fact that $\eta_h = (-z)^{k-h}\eta_k$ in this chart. This gives the fact that $\{\gamma = 0\}$ is reduced and irreducible in $N \times S(k)$.

The computation of g in the same chart gives that

$$g(\sigma, \eta) = (-1)^k zP'_\sigma(z)\eta_k - (-1)^k kP_\sigma(z)\eta_k.$$

and this proves that g does not vanishes identically on any non zero open set in $(N \times S(k)) \cap \{\gamma = 0\}$, because

$$(N \times S(k)) \cap \{\gamma = 0\} \cap \{g = 0\} \subset Z \cap \{\gamma = 0\}$$

which has dimension k, so co-dimension 2 in $N \times S(k)$. ■

End of proof of 4.1.1 The point $iii)$ is consequence of the following easy formulas:

$$A_{p,q}U_{-1} = U_{-1}A_{p,q} - (k-p-1)A_{p+1,q} - (k-q)A_{p,q+1}$$
$$A_{p,q}U_0 = U_0A_{p,q} - (p+q)A_{p,q}$$

$$U_{-1}(U_0 - 1) = U_0 U_{-1}.$$

The points (iv) and (v) are obvious, because a non zero germ of section of $\mathcal{O}_N[\eta]$ which is homogeneous of degree 1 in η does not vanishes of $N \times S(k)$. ■

Recall that in $\mathcal{W}(q) := \oplus_{p=0}^{q} \mathcal{W}_p$ we have, for each $\beta \in \mathbb{N}^k$ with $|\beta| = q - 1$ and $w(\beta) = r - 1$ (compare with Formulas (13) and (14), but here $w(\beta) = r - 1$)

$$\left[\partial^\beta (U_0 - 1)\right] = \sum_{h=1}^{k} h\sigma_h y_{q,r+h-1} + (r-2) y_{q-1,r-1}. \tag{24}$$

and

$$\left[\partial^\beta U_{-1}\right] = \sum_{h=0}^{k} (k-h)\sigma_h y_{q,r+h} + (k(q-1) - r + 1) y_{q-1,r} \tag{25}$$

Now, note β^+ a multi-index with $|\beta^+| = q-1$ and $w(\beta^+) = r$, when $r \neq k(q-1)+1$ and $\beta^+ = 0$ for $r = k(q-1)+1$.

Then for $r \neq k(q-1)+1$ we have

$$\partial^{\beta^+}(U_0 - 1) = \sum_{h=0}^{k} h\sigma_h y_{q,r+h} + (r-1) y_{q-1,r}$$

with the convention $\sigma_0 \equiv 1$ and $\partial^{\beta^+}(U_0 - 1) = 0$ for $r = k(q-1)+1$.

Then define for $q \geq 2$ and $r \in [q, k(q-1)]$ the following elements in $\mathcal{W}_q$:

$$\tilde{\theta}_{q,r} := (r-1)\partial^\beta U_{-1} - (k(q-1) - r + 1)\partial^{\beta^+}(U_0 - 1) \tag{26}$$

This gives

$$\tilde{\theta}_{q,r} = k \sum_{h=0}^{k} \big((r-1) - h(q-1)\big)\sigma_h y_{q,r+h} \tag{27}$$

Remark that for $r = k(q-1)+1$ and $h = k$ the vector $y_{q,r+k}$ is not defined in $\mathcal{W}_q$ and we cannot use the formula (27) to define $\tilde{\theta}_{q,k(q-1)+1}$. But with our convention $\partial^{\beta^+}(U_0 - 1) = 0$ for $r = k(q-1)+1$, we define the vector

$$\tilde{\theta}_{q,k(q-1)+1} := k(q-1)\partial_k^{q-1} U_{-1} = k(q-1) \sum_{h=0}^{k-1} (k-h)\sigma_h . y_{q,k(q-1)+1+h} \tag{28}$$

which is in $\mathcal{W}_q$.

Then, for $q \geq 2$, let $\tilde{\Theta}_q \subset \mathcal{W}_q$ be the sub-$\mathcal{O}_N$-module generated by the elements $\tilde{\theta}_{q,r}, r \in [q, k(q-1]+1]$. Of course $\tilde{\Theta}_q$ is in the kernel of the $\mathcal{O}_N-$linear map

$$L_q : \mathcal{W}_q \to \mathcal{N}_1^{\square}(q)$$

induced by the quotient map $\mathcal{W} \to \mathcal{N}_1^{\square}$.

For $q = 1$ define $\tilde{\Theta}_1 := \mathcal{O}_N U_{-1}$ and $V_1 := \oplus_{h=2}^{k} \mathcal{O}_N[\partial_h]$ (where $y_{1,h} := [\partial_h]$ in $\mathcal{W}_1$).

Lemma 4.1.3 *For each $q \geq 1$ we have a direct sum decomposition $\mathcal{W}_q = \tilde{\Theta}_q \oplus V_q$ where V_q is the $\mathcal{O}_N-$sub-module with basis $y_{q,r}$ with $r \in [k(q-1)+2, kq]$.*

Proof For $q = 1$ our assertion is clear. For $q \geq 2$ (and so $r \geq 2$) the difference $\tilde{\theta}_{q,r} - k(r-1)y_{q,r}$ is a $\mathbb{C}[\sigma]-$linear combination of the $y_{q,s}$ for $s \geq r+1$, so the matrix of the vectors $\tilde{\theta}_{q,r}$ for $r \in [q, k(q-1)+1]$ and $y_{q,s}, s \in [k(q-1)+2, kq]$ is triangular in the basis $y_{q,t}, t \in [q, kq]$, of $\mathcal{W}_q$ with determinant equal to

$$k^{(k-1)(q-1)+1} \prod_{r=q}^{k(q-1)+1} (r-1) = k^{(k-1)(q-1)+1} \frac{(k(q-1))!}{(q-2)!}$$

which is a positive integer. ∎

Lemma 4.1.4 *For each $q \geq 1$ the map $l_q : V_q \to \mathcal{N}_1^{\square}(q)$ induced by L_q is bijective.*

Proof We shall prove this lemma by induction on $q \geq 1$. First remark that the map $l_1 : V_1 \to \mathcal{N}_1^{\square}(1)$ is surjective (in fact an isomorphism of free rank $(k-1)$ $\mathcal{O}_N$-modules), because $1 = \sum_{h=1}^{k} h\sigma_h\partial_h$ and $k\partial_1 = -\sum_{h=1}^{k-1}(k-h)\sigma_h\partial_{h+1}$ in $\mathcal{N}_1^{\square}$.

So let $q \geq 2$ and assume that $l_{q-1} : V_{q-1} \to \mathcal{N}_1^{\square}(q-1)$ is surjective. Then, $\mathcal{N}_1^{\square}(q-1)$ is contained in the image of L_q, because for $r \in [k(q-2)+2, k(q-1)]$ the relation (26) shows that the image of $y_{q-1,r}$ by l_{q-1} is in the image of L_q.

Then remark that L_q induces a surjective map onto the quotient $\mathcal{N}_1^{\square}(q)/\mathcal{N}_1^{\square}(q-1)$ and that $\tilde{\Theta}_q$ is in the kernel of L_q. Therefore, l_q is surjective on $\mathcal{N}_1^{\square}(q)$. Therefore, we have a surjective map l_q of the rank $k-1$ free $\mathcal{O}_N$-module V_q onto $\mathcal{N}_1^{\square}(q)$ and an injective map of the rank $k-1$ free $\mathcal{O}_N$-module $\mathcal{N}_1^{\square}(1)$ into $\mathcal{N}_1^{\square}(q)$. Then, Lemma 3.3.6 gives that l_q is bijective. ∎

Theorem 4.1.5 *The restriction of $\mathcal{N}_1^{\square}(1)$ to the Zariski open set $\{\Delta(\sigma) \neq 0\}$ is a rank $(k-1)$ free*[10] *$\mathcal{O}_N$-module with a simple pole meromorphic connection along $\{\Delta = 0\}$ given by the inclusion $\Delta(\sigma)\mathcal{N}_1^{\square}(2) \subset \mathcal{N}_1^{\square}(1)$ (see Lemma 4.1.6 below). Its sheaf of horizontal sections is locally generated by $z_i - z_j$ where $z_h, h \in [1, k]$, are local branches of the multivalued function $z(\sigma)$ defined by $P_\sigma(z(\sigma)) \equiv 0$. The $\mathcal{D}_N$-module $\mathcal{N}_1^{\square}$ is the minimal extension on N of this vector bundle with its integrable regular meromorphic connection. Therefore, it is a simple $\mathcal{D}_N$-module.*

[10] Isomorphic to $\oplus_{h=2}^{k} \mathcal{O}_N \partial_h$.

Proof Lemma 4.1.4 gives that the map $l_q : V_q \to \mathcal{N}_1^{\square}(q)$ is a isomorphism of $\mathcal{O}_N$-modules for each $q \geq 1$ and Proposition 4.1.1 implies that $\mathcal{N}_1^{\square}$ is the union of the sheaves $\mathcal{N}_1^{\square}(q), q \geq 1$. Therefore, the $\mathcal{D}_N$-module $\mathcal{N}_1^{\square}$ has no $\mathcal{O}_N$-torsion.

Thanks to Lemma 4.1.6 below we have the inclusion $\Delta\mathcal{N}_1^{\square}(2) \subset \mathcal{N}_1^{\square}(1)$. This implies that $\mathcal{N}_1^{\square}(1) \simeq \mathcal{O}_N^{k-1}$ has an integrable meromorphic connection ∇_1 with a simple pole along $\{\Delta(\sigma) = 0\}$ on N. The fact that $\mathcal{K}_1 = \mathcal{J}_1 + \mathcal{D}_N U_{-1}$ implies that the horizontal sections of $\mathcal{N}_1^{\square}(1)$ are trace functions (see [1]) which are homogeneous of degree 1 and killed by U_{-1}. Therefore, they are $\mathbb{C}-$linear combinations of $z_1(\sigma), \dots, z_k(\sigma)$, the local branches of the multivalued function $z(\sigma)$ on N defined by $P_\sigma(z(\sigma)) = 0$.

The condition for $\sum_{h=1}^k a_h z_h(\sigma), a_h \in \mathbb{C}$, to be killed by $U_{-1} \simeq \sum_{h=1}^k \frac{\partial}{\partial z_h}$ is given by $\sum_{h=1}^k a_h = 0$ and then the horizontal sections are linear combinations of the differences $z_i - z_j,\ i, j \in [1, k]$. A basis of horizontal sections is given, for instance, by $z_2(\sigma) - \sigma_1/k, \dots, z_k(\sigma) - \sigma_1/k$ (note that $\sum_{j=1}^k (z_j(\sigma) - \sigma_1/k) \equiv 0$).

The $\mathcal{D}_N$-module $\mathcal{N}_1^{\square}$ has neither $\mathcal{O}_N$-torsion nor $\mathcal{O}_N$-co-torsion, because its characteristic variety is the union of $N \times \{0\}$ and X (X is defined in Sect. 3.2) and, thanks to Theorem 2.3.6, it has neither Δ-torsion nor Δ-co-torsion as a quotient of $\mathcal{M}$. Therefore, $\mathcal{N}_1^{\square}$ is the minimal extension of the meromorphic connection $\left(\mathcal{N}_1^{\square}(1), \nabla_1\right)$ and it is a simple $\mathcal{D}_N$-module, because the monodromy representation of the local system of horizontal sections of $(\mathcal{N}_1^{\square}(1), \nabla_1)$ is irreducible. ■

Lemma 4.1.6 *We have* $\Delta\mathcal{N}_1^{\square}(2) \subset \mathcal{N}_1^{\square}(1)$.

Proof In $\mathcal{W}_2 \big/ \mathcal{W}_1$ the $2k-1$ vectors induced by $\partial_j(U_0 - 1), j \in [2, k]$ and $\partial_h U_{-1}, h \in [1, k]$ are given in the basis $y_{2,r}, r \in [2, 2k]$ of this free $\mathcal{O}_N$-module by the relations:

$$A_j := \partial_j(U_0 - 1) = \sum_{p=1}^{k} p\sigma_p y_{2,j+p}$$

$$B_h := \partial_h U_{-1} = \sum_{p=0}^{k-1} (k-p)\sigma_p y_{2,h+p+1}$$

with the convention $\sigma_0 = 1$.

Put $\tilde{P}_\sigma(z) := \sum_{p=0}^k \sigma_p z^{k-p}$ and $y_{2,k+p} = z^{k-p}$.

Then $B_k = \tilde{P}'_\sigma(z)$ and $A_k = z\tilde{P}'_\sigma(z) - k\tilde{P}_\sigma(z)$. Therefore, the resultant of A_k and B_k is equal to $(-k)^{k-1}\Delta(\sigma)$. The determinant of the vectors $A_j, j \in [2, k]$ and $B_h, h \in [1, k]$ in the basis $y_{2,r}, r \in [2, 2.k]$ of $\mathcal{W}_2 \simeq \mathcal{W}_2 \big/ \mathcal{W}_1$ is then equal to $(-k)^{k-1}\Delta(\sigma)$ (compare with Lemma 3.3.8). ■

4.1.2 The Structure Theorem for $\mathcal{N}_p, p \geq 1$

We first examine the case $p = 1$. As already explained in the beginning of this section this will be enough to describe the structure of $\mathcal{N}_p$ for any $p \in \mathbb{N}^*$.

The torsion sub-module of $\mathcal{N}_1$ is described by the following result. Remark that we already know from Theorem 4.1.5 that the torsion sub-module of $\mathcal{N}_1$ is contained in the image of $\mathscr{T}_1 : \mathcal{N}_0 \to \mathcal{N}_1$ as $\mathcal{N}_1^{\square} = \mathcal{N}_1 \big/ Im(\mathscr{T}_1)$ has no torsion.

Proposition 4.1.7 *There exists a injective $\mathcal{D}_N$-linear map $\chi : \underline{H}^1_{[\sigma_k=0]}(\mathcal{O}_N) \to \mathcal{N}_1$ which sends the class $[1/\sigma_k]$ in $\underline{H}^1_{[\sigma_k=0]}(\mathcal{O}_N)$ to the class $[\partial_k U_{-1}]$ in $\mathcal{N}_1$. Its image is the torsion sub-module Θ of $\mathcal{N}_1$.*

Proof Note first that $\underline{H}^1_{[\sigma_k=0]}(\mathcal{O}_N)$ is given by $\mathcal{D}_N \big/ \big(\sum_{h=1}^{k-1} \mathcal{D}_N \partial_h + \mathcal{D}_N \sigma_k \big)$ as the annihilator of $[1/\sigma_k]$ is generated by $\partial_h, h \in [1, k-1]$ and σ_k. To show that χ exists it is enough to show that $\partial_h, h \in [1, k-1]$ and σ_k annihilate the class $[\partial_k U_{-1}]$ in $\mathcal{N}_1$. The fact that $\partial_h[\partial_k U_{-1}] = 0$ in $\mathcal{N}_1$ for $h \in [1, k-1]$ is a direct consequence of Formulas $(E_h), h \in [2, k]$ which give $[\partial_h U_{-1}] = 0$ in $\mathcal{N}_1$. Then, Formula (E_1) gives the vanishing of the class of $EU_{-1} = \sum_{h=1}^k \sigma_h \partial_h U_{-1}$ in $\mathcal{N}_1$. Therefore, we obtain that $\sigma_k[\partial_k U_{-1}]$ vanishes in $\mathcal{N}_1$ and χ is well defined. Moreover, as $\underline{H}^1_{[\sigma_k=0]}(\mathcal{O})$ is a $\mathcal{D}_N$-module with support in $\{\sigma_k = 0\}$, its image is contained in the torsion sub-module in $\mathcal{N}_1$.

Note that we know that the torsion in $\mathcal{N}_1$ is only σ_k-torsion thanks to Corollary 3.2.6 and Theorem 2.3.6.

To prove the injectivity of χ, assume that the kernel of χ is not 0 and consider an element $K := \sum_{p=1}^m f_p \partial_k^p [1/\sigma_k]$ in this kernel with $f_p \in \mathcal{O}_N \big/ (\sigma_k)$ and with m minimal. Then, we have $0 = \chi(K) = [\sum_{p=1}^m f_p \partial_k^{p+1} U_{-1}]$ in $\mathcal{N}_1$. Therefore, $f_m \eta_k^{m+1} \gamma$ is the symbol of an element in $\mathcal{J}_1$. Then, Lemma 3.2.7 implies that f_m is in $\sigma_k \mathcal{O}_N$ contradicting the minimality of m. Therefore, χ is injective.

To complete the proof we have to show that if P induces a torsion class in $\mathcal{N}_1$ then there exists $Q \in \mathcal{D}_N$, such that $P - Q\partial_k U_{-1}$ is in $\mathcal{J}_1$. As we already know (because $\mathcal{N}_1^{\square}$ has no torsion) that there exists $P_1 \in \mathcal{D}_N$, such that $\mathscr{T}_1(P_1) = [P_1 U_{-1}] = [P]$ in $\mathcal{N}_1$ and as we know that $\partial_h U_{-1} = 0$ for each $h \in [1, k-1]$ we may assume that P_1 is in $\mathcal{O}_N[\partial_k]$. However, $\partial_k^n U_{-1}$ is torsion in $\mathcal{N}_1$ for $n \geq 1$, because $\partial_k U_{-1}$ is torsion (see above). Therefore, the only point to prove is that if $f U_{-1}$ is torsion in $\mathcal{N}_1$ for some $f \in \mathcal{O}_N$ then $f = 0$. This a consequence of the following lemma. ■

Lemma 4.1.8 *The class of U_{-1} is not in the σ_k-torsion of $\mathcal{N}_1$.*

Proof Assume that $\sigma_k^n U_{-1}$ is in $\mathcal{J}_1$ for some $n \in \mathbb{N}$. Then, choose n minimal with this property and compute

$$\partial_k \sigma_k^n U_{-1} = n\sigma_k^{n-1} U_{-1} + \sigma_k^n \partial_k U_{-1} \in \mathcal{J}_1.$$

As $\sigma_k \partial_k U_{-1}$ is in $\mathcal{J}_1$ (see above) we obtain that $n = 0$ by minimality of n. However, U_{-1} is not in $\mathcal{J}_1$, because its symbol $\gamma(\sigma, \eta)$ restricted to the co-normal C to the hyper-surface $\{\sigma_k = 0\}$ is equal to σ_{k-1} which does not vanish identically on C. In addition, C is a component of the characteristic variety of $\mathcal{N}_1$ (see Paragraph 3.2). This concludes the proof. ■

Theorem 4.1.9 *The diagram below describes the structure of $\mathcal{N}_1$, where Θ is the torsion sub-module of $\mathcal{N}_1$, where $\varphi_1 : \mathcal{N}_1 \to \mathcal{O}_N$ is the $\mathcal{D}_N$-linear map defined by*

$\varphi_1(1) = \sigma_1$ and where the isomorphism $\chi : \underline{H}^1_{[\sigma_k=0]}(\mathcal{O}_N) \simeq \Theta$ is defined by sending $1/\sigma_k$ to $[\partial_k U_{-1}]$.

The $\mathcal{D}_N$-modules $\Theta \simeq \underline{H}^1_{[\sigma_k=0]}(\mathcal{O}_N)$ and $\mathcal{N}_1^\square$ are simple $\mathcal{D}_N$-modules.

Moreover, we have the direct sum decomposition of left $\mathcal{D}_N$-modules:

$$\mathcal{N}_1\big/\Theta = Im(\mathcal{T}_1)\big/\Theta \oplus \mathcal{N}_1^*\big/\Theta = \mathcal{O}_N[U_{-1}] \oplus \mathcal{N}_1^\square.$$

The following commutative diagram of left $\mathcal{D}_N$-modules has exact lines and columns where the maps i and e are defined by $i([U_{-1}]) = [U_{-1}]$ and $e([U_{-1}]) = k$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \Theta \arrow[r] \arrow[d] & Im(\mathcal{T}_1) \arrow[r, "i"] \arrow[d] & \mathcal{O}_N[U_{-1}] \arrow[r] \arrow[d, "e", "\simeq"'] & 0 \\
0 \arrow[r] & \mathcal{N}_1^* \arrow[r] \arrow[d] & \mathcal{N}_1 \arrow[r, "\varphi_1"] \arrow[d] & \mathcal{O}_N \arrow[r] & 0 \\
 & \mathcal{N}_1^*/\Theta \arrow[r, "\theta"] \arrow[d] & \mathcal{N}_1^\square \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Proof Note first that the quotient by the torsion sub-module Θ (which is the image of $\mathcal{D}_N\partial_k U_{-1}$ in $\mathcal{N}_1$; see Proposition 4.1.7) of the image of $\mathcal{N}_0$ (equal to $D_N U_{-1}$ in $\mathcal{N}_1$) is isomorphic to $\mathcal{O}_N$, because its generator $[U_{-1}]$ is killed by $\partial_h, \forall h \in [1, k]$ (see Formulas (E_h), $h \in [1, k]$) and this quotient has no torsion, because Θ is also the torsion sub-module of $Im(\mathcal{T}_1)$. This gives the exactness of the first line. The exactness of the second line and of the columns are clear.

Note also that $\varphi_1(U_{-1}) = k$, so the upper right square commutes. The commutations of the other squares are obvious.

To show that the map θ is well defined and is an isomorphism is a simple exercise in diagram chasing which is left to the reader.

The direct sum decomposition of $\mathcal{N}_1\big/\Theta$ is given by the left $\mathcal{D}_N$-linear map

$$r : \mathcal{N}_1\big/\Theta \to Im(\mathcal{T}_1)\big/\Theta$$

constructed as follows:

Note first that $\varphi_1(U_{-1}) = k$. For $[P] \in \mathcal{N}_1$ let $f := \varphi_1([P])$. Then, we define

$$r([P]) := \big[(f/k)U_{-1}\big] \in Im(\mathcal{T}_1)\big/\Theta.$$

As Θ is in the kernel of φ_1, this map is well defined on $\mathcal{N}_1\big/\Theta$ and $[P] - r([P])$ is in $ker(\varphi_1) = \mathcal{N}_1^*$ and defines a class in $\mathcal{N}_1^*\big/\Theta$. Remark that Lemma 4.1.8 shows that the kernel of r is equal to $\mathcal{N}_1^*\big/\Theta$, because φ_1 is injective on $\mathcal{O}_N[U_{-1}]$. This gives the desired splitting, as r induces the identity on $Im(\mathscr{T}_1)\big/\Theta$. ■

4.2 The Structure of $\mathcal{N}_0$

Define $\mathcal{N}_0^*$ as the kernel of the $\mathcal{D}_N$-linear map $\varphi_0 : \mathcal{N}_0 \to \mathcal{O}_N$ given by $\varphi_0(1) = 1$. The sub-module $\mathcal{N}_0^*$ is generated by $\partial_1, \dots, \partial_k$. We shall show that it contains a copy of $\mathcal{O}_N$.

Note that the $\mathcal{D}_N$-module $\mathcal{N}_0$ has no $\mathcal{O}_N$-torsion thanks to Theorem 3.3.1.

Proposition 4.2.1 *The kernel of the $\mathcal{D}_N$-linear map $\mathscr{T}_1 : \mathcal{N}_0 \to \mathcal{N}_1$ given by the right multiplication by U_{-1} is $\mathcal{D}_N U_1$ which is contained in $\mathcal{N}_0^*$ and the quotient $\mathcal{N}_0^*\big/\mathcal{D}_N U_1$ is isomorphic to the $\mathcal{D}_N$-module $\underline{H}^1_{[\sigma_k=0]}(\mathcal{O}_N)$.*

The proof of this proposition will used the following results from [1] Proposition 5.2.1.

Proposition 4.2.2 *For each $m \in \mathbb{Z}, m \geq -k+1$ and for each $\sigma \in N$, such that $\Delta(\sigma) \neq 0$ define*

$$DN_m(\sigma) := \sum_{P_\sigma(x_j)=0} \frac{x_j^{m+k-1}}{P'_\sigma(x_j)}. \tag{29}$$

Each DN_m is the restriction to the open set $\{\Delta(\sigma) \neq 0\}$ of a polynomial of (pure) weight m in $\mathbb{C}[\sigma_1, \dots, \sigma_k]$ and the following properties are satisfied:

(i) *For $m \in [-k+1, -1]$, $DN_m = 0$ and $DN_0 = 1$.*
(ii) *For each $m \geq 1$, $\sum_{h=0}^{k}(-1)^h \sigma_h DN_{m-h} = 0$ with the convention $\sigma_0 \equiv 1$.*
(iii) *For each $h \in [1, k]$ and each $m \geq 0$ we have*

$$\partial_h N_m = (-1)^{h-1} m DN_{m-h}.$$ ■

We shall use also the following lemma.

Lemma 4.2.3 *For any $h \in [2, k]$ we have*

$$\partial_h U_1 + \partial_{h-1}(U_0 + 1) \in \mathcal{I}. \tag{F_h}$$

Moreover, we have also

$$\partial_1 U_1 - E(U_0 + 1) \in \mathcal{I}. \tag{F_1}$$

Therefore, we have $\partial_h U_1 = -\partial_h$ for $h \in [2, k]$ and $\partial_1 U_1 = E$ in $\mathcal{N}_0^*$.

Remark Note that Formulas (F_h) for any $h \in [1, k]$ give that $\partial_h U_1 = 0$ in $\mathcal{N}_{-1}$.

Proof Thanks to the characterization of trace functions given in [1] it is enough to prove that for each $m \in \mathbb{N}$ we have $\partial_h U_1[N_m] = -\partial_{h-1}(U_0+1)[N_m]$ for $h \in [2,k]$ and $\partial_1 U_1(N_m) = E(U_0+1)[N_m]$ for all $m \in \mathbb{N}$. This is consequence of the following formulas which use the results of [1] recalled in Proposition 4.2.2 and the equality $U_p[N_m] = mN_{m+p}$ which is valid for each $m \in \mathbb{N}$ and each integer $p \geq -1$, because U_p is the image by the tangent map to the quotient map $quot : M = \mathbb{C}^k \to \mathbb{C}^k\big/\mathfrak{S}_k = N$ of the vector field $\sum_{j=1}^k z_j^{p+1}\frac{\partial}{\partial z_j}$:

$$\partial_h U_1[N_m] = \partial_h[m.N_{m+1}] = (-1)^{h-1}m(m+1)DN_{m+1-h} \quad \forall h \in [1,k] \quad \forall m \in \mathbb{N}$$

$$\partial_{h-1}(U_0+1)[N_m] = \partial_{h-1}[(m+1)N_m] = (-1)^h m(m+1)DN_{m-h+1} \quad \forall h \in [2,k]$$

proving Formulas (F_h) for $h \in [2,k]$.

Now $\sigma_h\partial_h[N_m] = (-1)^{h-1}m\sigma_h DN_{m-h}$ gives

$$E(U_0+1)[N_m] = m(m+1)\sum_{h=1}^{k}(-1)^{h-1}\sigma_h DN_{m-h} = m(m+1)DN_m$$

because for $m \geq 1$ we have $\sum_{h=0}^k(-1)^h\sigma_h DN_{m-h} = 0$ (see ii) above) and also $E[1] = 0$ for $m = 0$. However, for $h = 1$ we have

$$\partial_1 U_1[N_m] = \partial_1[mN_{m+1}] = m(m+1)DN_m \quad \forall m \in \mathbb{N}.$$

This gives Formula (F_1). ■

Proof of Proposition 4.2.1 Remark first that U_1 is in $\mathcal{N}_0^*$ and, thanks to the previous lemma, that $\mathcal{D}_N U_1$ contains $\partial_1, \dots, \partial_{k-1}$ and $\sigma_k\partial_k$. Define the sub-$\mathcal{D}_N$-module $S := \sum_{h=1}^{k-1}\mathcal{D}_N\partial_h$. Then, we have a natural surjective $\mathcal{D}_N$-linear map $\alpha : S + \mathcal{D}_N\partial_k\big/(S + \mathcal{D}_N\sigma_k\partial_k) \to \mathcal{N}_0^*\big/\mathcal{D}_N U_1$. However, we have

$$S + \mathcal{D}_N\partial_k\big/(S + \mathcal{D}_N\sigma_k\partial_k) \simeq \mathcal{D}_N\partial_k\big/(S \cap \mathcal{D}_N\sigma_k\partial_k) \simeq \mathcal{D}_N\big/(S + \mathcal{D}_N\sigma_k)$$

thanks to the equality $S \cap \mathcal{D}_N\sigma_k\partial_k = (S + \mathcal{D}_N\sigma_k)\partial_k$. Moreover, the $\mathcal{D}_N$-module

$$\mathcal{D}_N\big/(S + \mathcal{D}_N\sigma_k) \simeq \underline{H}^1_{[\sigma_k=0]}(\mathcal{O}_N)$$

is simple, so α must be an isomorphism. ■

Theorem 4.2.4 *Define $\mathcal{N}_0^\square := \mathcal{D}_N U_1 \subset \mathcal{N}_0^*$. Then, $\mathcal{N}_0^\square$ is simple and isomorphic to $\mathcal{N}_1^\square$ via the map induced by the map $\square U_1 : \mathcal{N}_1 \to \mathcal{N}_0$, and the quotient $\mathcal{N}_0\big/\mathcal{N}_0^\square$ is isomorphic to $\mathcal{O}_N(\star\sigma_k)$.*

Proof The only point which is not already proved above is the fact that right multiplication by U_1, $\square U_1 : \mathcal{N}_1 \to \mathcal{N}_0$ has its image in $\mathcal{N}_0^*$ and induces an isomorphism of $\mathcal{N}_1^\square$ to $\mathcal{N}_0^\square = \mathcal{D}_N U_1 \subset \mathcal{N}_0^*$. However, U_1 is in $\mathcal{N}_0^*$ so the first assertion is clear. This map vanishes on the image of $\mathscr{T}_1$, because $U_{-1}U_1 = (U_0+1)U_0$ modulo $\mathcal{I}$ (and also $\mathcal{I}U_1 \subset \mathcal{I}$ see Theorem 2.4.1), so that right multiplication by U_1 induces a map which is clearly surjective. As $\mathcal{N}_1^{\square}$ is simple, this surjective map is an isomorphism.

We have the following exact sequences of $\mathcal{D}_N$-modules which describe the structure of $\mathcal{N}_0$ and $\mathcal{N}_0^*$:

$$0 \longrightarrow \mathcal{N}_0^{\square} \longrightarrow \mathcal{N}_0^* \longrightarrow \underline{H}^1_{[\sigma_k=0]}(\mathcal{O}_N) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{N}_0^{\square} \longrightarrow \mathcal{N}_0 \longrightarrow \mathcal{O}_N(\star\sigma_k) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{N}_0^{\square} \longrightarrow \mathcal{N}_0 \xrightarrow{\square U_{-1}} \mathcal{N}_1 \longrightarrow \mathcal{N}_1^{\square} \longrightarrow 0$$

where the map $\square U_1 : \mathcal{N}_1 \to \mathcal{N}_0$ given by right multiplication by U_1 induces an isomorphism $\mathcal{N}_1^{\square} \to \mathcal{N}_0^{\square}$, showing that $\mathcal{N}_0^{\square}$ is a simple $\mathcal{D}_N$-module. ■

Note that the local horizontal basis of $\mathcal{N}_0^{\square}$ on the open set $\{\Delta(\sigma)\sigma_k \neq 0\}$ is (locally) generated by the $(Log\, z_i - Log\, z_j)$ and their images by the isomorphism induced by U_1 are the $(z_i - z_j)$ which generates a local horizontal basis of $\mathcal{N}_1^{\square}$ on the open set $\{\Delta(\sigma)\sigma_k \neq 0\}$.

4.3 The Structure of $\mathcal{N}_p$ for $p \in -\mathbb{N}^*$

Again it is enough, thanks to Lemma 3.1.5, to describe the structure of $\mathcal{N}_{-1}$. Define $\mathcal{N}_{-1}^*$ as the kernel of the $\mathcal{D}_N$-linear map

$$\varphi_{-1} : \mathcal{N}_{-1} \to \mathcal{O}_N(\star\sigma_k)$$

which is given by $\varphi_{-1}(1) = \sigma_{k-1}/\sigma_k$. This map is well defined, because the meromorphic function $\sigma_{k-1}/\sigma_k = \sum_{j=1}^k 1/z_j$ is a local trace function of the open set $\{\sigma_k \neq 0\}$ and so it is killed by $\mathcal{I}$ everywhere as the $\mathcal{D}_N$-module $\mathcal{O}_N(*\sigma_k)$ has no torsion. Moreover, we have, still on the open set $\{\sigma_k \neq 0\}$:

$$U_0(\sigma_{k-1}/\sigma_k) = (k-1)\sigma_{k-1}/\sigma_k - k\sigma_{k-1}/\sigma_k = -\sigma_{k-1}/\sigma_k.$$

So $U_0 + 1$ is also in the annihilator of σ_{k-1}/σ_k in $\mathcal{O}_N(\star\sigma_k)$. Therefore, the map φ_{-1} is well defined. It is surjective, because $\varphi_{-1}(\partial_{k-1}) = 1/\sigma_k$.

Lemma 4.3.1 *The symbol of U_1 does not vanish identically on X for each integer $k \geq 2$.*

Proof Recall that X is defined in Sect. 3.2. We have $\eta_h = (-z)^{k-h}\eta_k$ on X where $z = -\eta_{k-1}/\eta_k$ is the double root of P_σ at the generic point of H_Δ (recall that X is the closure of the graph of the meromorphic map $H_\Delta \dashrightarrow \mathbb{C}$ given by the double root of P_σ at the generic point of H_Δ). As $U_1 = \sum_{h=1}^k (\sigma_1\sigma_h - (h+1)\sigma_{h+1})\partial_h$ (with the convention $\sigma_{k+1} = 0$), we obtain

$$
\begin{aligned}
s(U_1) &= (-1)^k \eta_k \sum_{h=1}^{k} (\sigma_1\sigma_h - (h+1)\sigma_{h+1})(-1)^h z^{k-h} \\
&= (-1)^k \eta_k \sum_{h=1}^{k} (-1)^h \sigma_1 \sigma_h z^{k-h} + (-1)^k \eta_k \sum_{p=2}^{k} (-1)^{p-1}(k-p)\sigma_p z^{k-p+1} + \\
&\quad - (-1)^k k\eta_k \sum_{p=2}^{k} (-1)^{p-1} \sigma_p z^{k-p+1} \\
s(U_1) &= (-1)^k \eta_k \sigma_1 \big(P_\sigma(z) - z^k\big) + (-1)^k \eta_k z^2 \big(P'_\sigma(z) - kz^{k-1} + (k-1)\sigma_1 z^{k-2}\big) \\
&\quad - (-1)^k k\eta_k z \big(P_\sigma(z) - z^k + \sigma_1 z^{k-1}\big) \\
&= -(-1)^k \eta_k \sigma_1 z^k - (-1)^k k\eta_k z^{k+1} + (-1)^k (k-1)\eta_k \sigma_1 z^k \\
&\quad + (-1)^k k\eta_k z^{k+1} - (-1)^k k\eta_k \sigma_1 z^k \\
&= -2(-1)^k \eta_k \sigma_1 z^k
\end{aligned}
$$

as $P_\sigma(z) = P'_\sigma(z) = 0$ on X. ∎

So U_1 is not zero in any $\mathcal{N}_\lambda$ for any $\lambda \in \mathbb{C}$.

Proposition 4.3.2 *The kernel of the $\mathcal{D}_N$-linear map $\mathscr{T}_0 : \mathcal{N}_{-1} \to \mathcal{N}_0$ given by right multiplication by U_{-1} is equal to $\mathcal{D}_N U_1 = \mathcal{O}_N U_1$ in $\mathcal{N}_{-1}$.*

Proof Recall that Formulas $(F_h), h \in [1, k]$ show the equality $\mathcal{D}_N U_1 = \mathcal{O}_N U_1$ in $\mathcal{N}_{-1}$ (see Remark following Lemma 4.2.3). Moreover, we know that $\mathcal{N}_{-1}$ has no $\mathcal{O}_N$-torsion, thanks to Theorem 3.3.1, so $\mathcal{D}_N U_1$ is a sub-module of $\mathcal{N}_{-1}$ which is isomorphic to $\mathcal{O}_N$ as U_1 is not zero in $\mathcal{N}_{-1}$, because its symbol does not vanish on X (see Lemma 4.3.1 above).

The end of the proof of this proposition will use the following lemmas:

Lemma 4.3.3 *Let a and b be holomorphic function on an open set U in N, such that the function $a\gamma - bg$ is a section on $U \times \mathbb{C}^k$ which vanishes on $Z \cap (U \times \mathbb{C}^k)$. Then, a and b vanishes identically on U*

Proof The first remark is that we have $a\gamma = bg$ on $U \times \mathbb{C}^k$, because the sheaf $p_*(I_Z)$ has no non zero section which homogeneous of degree 1 in $\eta_1, \dots, \eta_k$ (see Lemma 2.2.5). Then, looking at the coefficients of η_1 and η_2 in the equality $a\gamma = bg$ gives

$$
ka = \sigma_1 b \quad \text{and} \quad (k-1)\sigma_1 a = 2\sigma_2 b \quad \text{and so} \quad (k-1)\sigma_1^2 b = 2k\sigma_2.b
$$

which implies $b \equiv 0$ and then $a \equiv 0$ on U. ∎

Lemma 4.3.4 *Let $P \in \mathcal{D}_N$, such that $PU_{-1} = AU_0 + Q$ with $A \in \mathcal{D}_N$ and $Q \in \mathcal{I}$. Then, A is unique modulo $\mathcal{I}$.*

Proof We have to show that $AU_0 \in \mathcal{I}$ implies that A is in $\mathcal{I}$.

If this is not true, let $A \in \mathcal{D}_N \setminus \mathcal{I}$ be of minimal order, such that $AU_0 \in \mathcal{I}$. We have $s(A)g \in I_Z$ and, as g is generically $\neq 0$ on Z and I_Z is prime (so reduced), there exists $A_1 \in \mathcal{I}$ with $A - A_1$ of order strictly less than the order of A. Then, $(A - A_1)U_0$ is in $\mathcal{I}$.

This contradicts the minimality of A, since $A - A_1$ cannot be in $\mathcal{I}$. ■

Lemma 4.3.5 *There exists a natural $\mathcal{D}_N$-linear map $\psi : Ker(\mathscr{T}_0) \to \mathcal{N}_1$ given by $\psi(P) = [A]$ when $PU_{-1} = AU_0$ modulo $\mathcal{I}$.*

Proof First recall that the right multiplication by U_{-1} induces a $\mathcal{D}_N$-linear map $\mathscr{T}_0 : \mathcal{N}_{-1} \to \mathcal{N}_0$, because we have $\mathcal{I}U_{-1} \subset \mathcal{I}$ and the relation $(U_0 + 1)U_{-1} = U_{-1}U_0$. If $P \in \mathcal{D}_N$ induces a germ of section of $Ker(\mathscr{T}_0)$ then the previous lemma shows that if we write $PU_{-1} = AU_0 + Q$ with $Q \in \mathcal{I}$, the image of the germ A in $\mathcal{D}_N\big/\mathcal{I}$ is well defined. Then, we have a $\mathcal{D}_N$-linear map $Ker(\mathscr{T}_0) \to \mathcal{D}_N\big/\mathcal{I} = \mathcal{M}$ and after composition by the quotient maps $\mathcal{D}_N\big/\mathcal{I} \to \mathcal{N}_1$ we obtain the desired map. ■

End of proof of Proposition 4.3.2 First remark that U_1 is sent to 0 in $\mathcal{N}_1$ by the map ψ because of the relation $U_1U_{-1} = (U_0 - 1)U_0 \quad \text{modulo} \quad \mathcal{I}$.

So ψ composed with the quotient map $\mathcal{N}_1 \to \mathcal{N}_1^{\square}$ induces a map

$$\tilde{\psi} : Ker(\mathscr{T}_0)\big/\mathcal{O}_N U_1 \to \mathcal{N}_1^{\square}.$$

We shall prove that this map is injective and not surjective. As $\mathcal{N}_1^{\square}$ is simple, this will prove that $Ker(\mathscr{T}_0) = \mathcal{O}_N U_1$ completing the proof of Proposition 4.3.2.

We shall first prove the injectivity of $\tilde{\psi}$, so the fact that if $[P] \in Ker(\mathscr{T}_0)$ satisfies $\psi(P) = [A]$ with $[A] = 0$ in $\mathcal{N}_1^{\square}$ then $[P]$ is a germ of section of the sub-sheaf $\mathcal{D}_N U_1$ of $Ker(\mathscr{T}_0)$.

Let $P \in \mathcal{D}_N$ of minimal order, such that the class of $[P]$ in $Ker(\mathscr{T}_0)\big/\mathcal{D}_N U_1$ is not zero and satisfies $\tilde{\psi}([P]) = 0$. Then, we have

$$(P + X(U_0 + 1) + Q_0)U_{-1} = A_0U_0 + Q_1 \quad \text{with} \quad Q_0, Q_1 \in \mathcal{I} \quad \text{and } X \in \mathcal{D}_N.$$

Then, thanks to the relations $(U_0 + 1)U_{-1} = U_{-1}U_0$ modulo $\mathcal{I}$ we obtain

$$PU_{-1} = AU_0 + Q_2 \quad \text{with } Q_2 := Q_1 - Q_0U_{-1} \in \mathcal{I} \quad \text{and } A = A_0 - XU_{-1}.$$

Then, our hypothesis implies that there exist $R, S, Q_3 \in \mathcal{D}_N$ with $Q_3 \in \mathcal{I}$, such that $A = R(U_0 - 1) + SU_{-1} + Q_3$. Therefore

$$(P - S(U_0 + 1) - RU_1)U_{-1} = Q_3U_0 \quad \text{modulo } \mathcal{I}$$

and Q_3U_0 is again in $\mathcal{I}$. So looking at the symbols we find

$$s(P - S(U_0 + 1) - RU_1)\gamma \in I_Z.$$

As γ is generically $\neq 0$ on Z and I_Z is prime (then reduced) we conclude that there exists $P_1 \in \mathcal{I}$ with symbol $s(P_1) = s(P - S(U_0 + 1) - RU_1)$. Therefore, the order of $P - P_1 - S(U_0 + 1) - RU_1$ is strictly less than the order of P but the class of $P - P_1 - S(U_0 + 1) - RU_1$ in $Ker(\mathscr{T}_0)\big/\mathcal{D}_N U_1$ is the same than the class induced by P. This contradict the minimality of the order of P; so $Ker(\mathscr{T}_0) = \mathcal{D}_N U_1$ and the map $\tilde{\psi}$ is injective.

To conclude it is now enough to prove that $\tilde{\psi}$ is not surjective, as explained above. Therefore, assume that there exists $P \in \mathcal{D}_N$ with $PU_{-1} = AU_0 + Q$ with $Q \in \mathcal{I}$ and $[A - 1] = 0$ in $\mathcal{N}_1^{\square}$. This would implies that $PU_{-1} = (1 + T(U_0 - 1) + YU_{-1})U_0$ modulo $\mathcal{I}$ and so we obtain the equality

$$(P - Y(U_0 + 1) - TU_1)U_{-1} = U_0 \quad \text{modulo} \quad \mathcal{I}.$$

So looking at the symbols restricted to Z this gives:

$$s\big(P - Y(U_0 + 1) - TU_1\big)\gamma = g$$

in $\mathcal{O}_Z$. By homogeneity in η this implies that $f := s\big(P - Y(U_0 + 1) - TU_1\big)$ is the pull-back of a holomorphic function on an open set in N and this gives a contradiction thanks to Lemma 4.3.3. ■

Proposition 4.3.6 *Let $\mathcal{N}_{-1}^{\square}$ be the sub-$\mathcal{D}_N$-module of $\mathcal{N}_{-1}^{*}$ which is generated by $\partial_1, \ldots, \partial_{k-2}$. Then, $\mathscr{T}_0$ sends $\mathcal{N}_{-1}^{\square}$ onto $\mathcal{N}_0^{\square}$ and induces an isomorphism between theses two simple $\mathcal{D}_N$-modules.*

Proof As we know that $\mathcal{N}_0^{\square}$ is equal to $\mathcal{D}_N U_1 \subset \mathcal{N}_0^{*}$, we first check that the generators of $\mathcal{N}_{-1}^{\square}$ have their images by $\mathscr{T}_0$ in $\mathcal{D}_N U_1$.

For $h \in [1, k - 2]$ Formulas (E_{h+2}) and (F_{h+1}) imply

$$\partial_{h+2}U_1 + \partial_{h+1}(U_0 + 1) + \partial_h U_{-1} + \partial_{h+1}(U_0 - 1) \in \mathcal{I}$$

which implies $\partial_h U_{-1} = -\partial_{h+2}U_1$ in $\mathcal{N}_0$.

Note that, as $\mathcal{N}_1^{\square}$ is obviously generated[11] by $\partial_1, \ldots, \partial_{k-2}$ its image by the right multiplication by U_1 in $\mathcal{N}_0^{*}$ is generated by $\partial_h U_1$, $h \in [1, k - 2]$ giving a direct proof of the surjectivity of $\mathscr{T}_0 : \mathcal{N}_{-1}^{\square} \to \mathcal{D}_N U_1 = \mathcal{N}_0^{\square}$.

The injectivity of this map is clear thanks to Proposition 4.3.2 and the fact that $\varphi_{-1}(U_1) = -k$ which implies that the sub-modules $\mathcal{D}_N U_1 = \mathcal{O}_N U_1$ and $Ker(\mathscr{T}_0)$ of $\mathcal{N}_{-1}$ have an intersection reduced to $\{0\}$. ■

Proposition 4.3.7 *The sub-module $\mathcal{N}_{-1}^{\square}$ defined in the previous proposition is equal to $\mathcal{N}_{-1}^{*}$.*

[11] In fact knowing that it is simple, it is generated by any non zero element in it.

Proof By definition $\mathcal{N}^*_{-1}$ is the kernel of the map $\varphi_{-1} : \mathcal{N}_{-1} \to \mathcal{O}_N(*\sigma_k)$ which sends [1] to σ_{k-1}/σ_k. Then, $\mathcal{N}^*_{-1}$ is generated by the annihilator of σ_{k-1}/σ_k in $\mathcal{O}_N(*\sigma_k)$. Therefore, $\mathcal{N}^*_{-1}$ is generated by the class in $\mathcal{N}_1$ of

$$\partial_1, \ldots, \partial_{k-2}, \partial^2_{k-1}, \sigma_{k-1}\partial_{k-1} - 1, \sigma_k\partial_k + 1.$$

We already know that $\partial_1, \ldots, \partial_{k-2}$ are in $\mathcal{N}^{\square}_{-1}$ for $h \in [1, k-2]$ by definition of $\mathcal{N}^{\square}_{-1}$. Moreover, we have $\partial^2_{k-1} = \partial_k\partial_{k-2}$ modulo $\mathcal{I}$ gives that $\partial^2_{k-1}U_{-1} = \partial_k\partial_{k-2}U_{-1}$ belongs to $\mathcal{N}^{\square}_0$. Then, Proposition 4.3.6 implies that ∂^2_{k-1} is in $\mathcal{N}^{\square}_{-1} + Ker(\mathcal{T}_0)$. Therefore, write $\partial^2_{k-1} = e + fU_1$ with $e \in \mathcal{N}^{\square}_{-1}$ and $f \in \mathcal{O}_N$ using Proposition 4.3.2.

Then $\varphi_{-1}(\partial^2_{k-1}) = 0$ implies $\varphi(fU_1) = -kf = 0$, because e is in $\mathcal{N}^{\square}_{-1} \subset \mathcal{N}^*_{-1}$. Therefore, $f = 0$ and ∂^2_{k-1} is in $\mathcal{N}^{\square}_{-1}$.

So it is enough to prove that $a := \sigma_{k-1}\partial_{k-1} - 1$ and $b = \sigma_k\partial_k + 1$ are in $\mathcal{N}^{\square}_{-1}$ to complete the proof.

Formula (E_1) gives $EU_{-1} = -\partial_1$ in $\mathcal{N}_0$ and Formula (F_2) gives $\partial_2 U_1 = -\partial_1$ in $\mathcal{N}_0$. This implies that $\mathcal{T}_0(E) = \partial_2 U_1 \in \mathcal{N}^{\square}_0$. This implies that E is in $\mathcal{N}^{\square}_{-1} + Ker(\mathcal{T}_0)$.

So write $E = e + fU_1$ with $e \in \mathcal{N}^{\square}_{-1}$ and $f \in \mathcal{O}_N$ using Proposition 4.3.2.

Now

$$\varphi_{-1}(E) = E[\sigma_{k-1}/\sigma_k] = 0 \quad \text{and} \quad \mathcal{N}^{\square}_{-1} \subset \mathcal{N}^*_1 = ker(\varphi_{-1}).$$

Therefore, $\varphi_{-1}(fU_1) = f\varphi_{-1}(U_1) = -kf = 0$. This implies $E = e$ is in $\mathcal{N}^{\square}_{-1}$.

But $a + b = E$ modulo $\mathcal{N}^{\square}_{-1}$. Therefore, $a + b$ belongs to $\mathcal{N}^{\square}_{-1}$.

We have also in $\mathcal{N}_{-1}$:

$$0 = U_0 + 1 = (k-1)(\sigma_{k-1}\partial_{k-1} - 1) + k(\sigma_k\partial_k + 1) \quad \text{modulo } \mathcal{N}^{\square}_{-1}$$

and this gives $(k-1)a + kb \in \mathcal{N}^{\square}_{-1}$, concluding the proof. ∎

Theorem 4.3.8 *We have the following commutative diagram of $\mathcal{D}_N$-module with exact lines and columns, where the $\mathcal{D}_N$-linear map $\varphi_{-1} : \mathcal{N}_{-1} \to \mathcal{O}_N(\star\sigma_k)$ is defined by $\varphi_{-1}(1) = \sigma_{k-1}\big/\sigma_k$:*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{N}^*_{-1} \arrow[r] \arrow[d, "="] & \mathcal{N}^*_{-1} \oplus \mathcal{O}_N U_1 \arrow[r, "\varphi_{-1}"] \arrow[d] & \mathcal{O}_N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{N}^*_{-1} \arrow[r] \arrow[d] & \mathcal{N}_{-1} \arrow[r, "\varphi_{-1}"] \arrow[d] & \mathcal{O}_N(\star\sigma_k) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathcal{Q} \arrow[r, "\chi"] \arrow[d] & \underline{H}^1_{[\sigma_k=0]}(\mathcal{O}_N) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
$$

Therefore, χ is an isomorphism. Moreover, the map $\mathscr{T}_0$ induces an isomorphism of $\mathcal{N}^*_{-1}$ onto the simple $\mathcal{D}_N$-module $\mathcal{N}_0^\square = \mathcal{D}_N U_1 \subset \mathcal{N}_0^*$.

Proof The exactness of the first line is consequence of the equality $\varphi_{-1}(U_1) = -k$. The exactness of the second line is consequence of the surjectivity of φ_{-1} which is consequence of the equality $\varphi_{-1}(\partial_{k-1}) = 1/\sigma_k$.

As $\mathcal{Q}$ is the obvious quotient the injectivity of the induced map χ is easily obtained by a diagram chasing. ■

The local solutions of $\mathcal{N}_{-1}$ are the $1/z_j$, $j \in [1, k]$ and the local solutions of $\mathcal{N}^*_{-1}$ are the $1/z_j - 1/z_h$ which generate the linear combinations of the $1/z_j$ which are killed by $U_1 = \sum_{j=1}^k z_j^2 \frac{\partial}{\partial z_j}$.

Conclusion For each integer $p \geq 2$ define

$$
\mathcal{N}_p^\square := \mathcal{D}_N \big/ \mathcal{I} + \mathcal{D}_N(U_0 - p) + \mathcal{D}_N U_{-1}^p
$$

and $\mathcal{N}^*_{-p} := ker(\varphi_{-p})$, where $\varphi_{-p} : \mathcal{N}_{-p} \to \mathcal{O}_N(*\sigma_k)$ is given by $\varphi_{-p}(1) = U_{-1}^{p-1}[\sigma_{k-1}/\sigma_k]$. Then, we have the chain of isomorphisms:

$$
\begin{tikzcd}
\ldots \mathcal{N}^*_{-p-1} \arrow[r, bend right, "\mathscr{T}_{-p}"'] & \mathcal{N}^*_{-p} \ldots \arrow[l, bend right, "\square U_1"'] \arrow[r, bend right, "\mathscr{T}_{-1}"'] & \mathcal{N}^*_{-1} \arrow[l, bend right, "\square U_1"'] \arrow[r, bend right, "\mathscr{T}_0"'] & \mathcal{N}_0^\square & \mathcal{N}_1^\square \arrow[l, bend right, "\square U_1"'] \arrow[r, bend right, "\mathscr{T}_2"'] & \mathcal{N}_2^\square \ldots \arrow[l, bend right, "\square U_1"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
 \arrow[r, bend right, "\mathscr{T}_{p-1}"'] & \mathcal{N}_p^\square \arrow[l, bend right, "\square U_1"'] \arrow[r, bend right, "\mathscr{T}_p"'] & \mathcal{N}_{p+1}^\square \ldots \arrow[l, bend right, "\square U_1"']
\end{tikzcd}
$$

where $\mathscr{T}_p := \square U_{-1}$ is given by right multiplication by U_{-1}.

4.4 Some Higher Order Solutions of $\mathcal{N}_p$ for $p \in \mathbb{N}$

Let $N := \mathbb{C}^k$ with coordinates $\sigma_1, \dots, \sigma_k$ and note D_N the sheaf of (holomorphic) differential operators on N and $\mathcal{D}b_N^{p,q}$ the sheaf of $(p, q)-$currents on N.

Recall that $\mathcal{D}b_N^{p,q}$ is a left D_N-module and that we have the following theorem due to M. Kashiwara (see [7])

Theorem 4.4.1 *For any regular holonomic D_N-module $\mathcal{N}$ and any integer $j \geq 1$ we have*

$$Ext^j_{D_N}(\mathcal{N}, \mathcal{D}b_N^{0,p}) = 0.$$

Note that the case $p \geq 1$ is an obvious consequence of the case $p = 0$ as $\mathcal{D}b_N^{0,p}$ is the direct sum of C_k^p copies of $\mathcal{D}b_N^{0,0}$ as a left D_N-module.

Corollary 4.4.2 *For any regular holonomic D_N-module $\mathcal{N}$ and any integer $j \geq 0$ we have a natural isomorphism of sheaves of $\mathbb{C}-$vector spaces*

$$Sol^j(\mathcal{N}) := Ext^j_{D_N}(\mathcal{N}, \mathcal{O}_N) \simeq H^j\big((Hom_{D_N}(\mathcal{N}, \mathcal{D}b_N^{0,\bullet}), \bar{\partial}^\bullet)\big).$$

For instance, if $\mathcal{N} := D_N\big/\mathcal{J}$ is a regular holonomic system (where $\mathcal{J}$ is a coherent left ideal in D_N), we have a natural isomorphism of sheaves of complex vector spaces, for each j:

$$Sol^j(D_N\big/\mathcal{J}) \simeq \{T \in \mathcal{D}b_N^{0,j} / \,/ \mathcal{J}.T = 0,\ \bar{\partial}T = 0\}\Big/\bar{\partial}\big(\{T \in \mathcal{D}b_N^{0,j-1} / \,/ \mathcal{J}.T = 0\}\big).$$

Proof As the Dolbeault–Grothendieck complex $(\mathcal{D}b_N^{0,\bullet}, \bar{\partial}^\bullet)$ is a resolution of $\mathcal{O}_N$ by D_N-modules for which the functor

$$\mathcal{N} \mapsto Hom_{D_N}(\mathcal{N}, -)$$

is exact, thanks to the previous theorem, the conclusion follows by degeneracy of the spectral sequence. ■

Proposition 4.4.3 *Let σ^0 be a point the hypersurface $\{\sigma_k = 0\}$ in N and let d be the multiplicity of the root 0 in P_{σ^0}. Let U be a small open neighborhood of σ^0 in N on which there exists a holomorphic map $f : U \to Sym^d(\mathbb{C})$ whose value at $\sigma \in U$ is the $d-$tuple of roots of P_σ which are near by 0.*[12]

Then define for $q \in \mathbb{N}$ the distribution on U (given by a locally integrable function)

$$T_q(\sigma) = \sum_{j=1}^{d} z_j^q Log|z_j|^2 \quad \text{where } [z_1, \dots, z_d] = f(\sigma) \tag{30}$$

[12] To be more precise, let D be an open disc with center 0 in $\mathbb{C}$, such that $\bar{D}$ contains only the root 0 of P_{σ^0}. Then, choose U small enough, such that for all $\sigma \in U$ the polynomial P_σ has exactly d roots in D.

Then, the current $\bar{\partial} T_q$ defines a section on U of the sheaf $Sol^1(\mathcal{N}_q)$, such that its germ at a point σ^0 is non zero in $Sol^1(\mathcal{N}_q)_{\sigma^0}$.

Proof Let $pr : H \to \mathbb{C}$ and $\pi : H \to N$ are the projections, where

$$H := \{(\sigma, z) \in N \times \mathbb{C} \ / \ P_\sigma(z) = 0\}.$$

We may assume that the open set $pr(\pi^{-1}(U))$ is the disjoint union of D with an open set Ω in $\mathbb{C}$. Then, if we define the locally integrable function $f : D \cup \Omega$ as $f(z) = z^q Log|z|^2$ on D and $f \equiv 0$ on Ω we have $T_q(\sigma) = \pi_*(f)(\sigma) = \sum_{j=1}^k f(z_j)$ where $z_1, \dots, z_k$ are the roots of P_σ. It is then easy to verify that $\mathcal{I}T_q = 0$ and that $(U_0 - q)T_q = N_q(\sigma)$ the q-th Newton function of the $d-$tuple $d(\sigma)$ of roots of P_σ which are in D. Therefore, it is holomorphic on U. Then, the $(0, 1)-$current $\bar{\partial} T_q$ is $\bar{\partial}-$closed and is killed by $\mathcal{J}_q$. Then, thanks to Corollary 4.4.2 it induces a section on U of the sheaf $Sol^1(\mathcal{N}_q)$.

Fix now $\tau \in U$ and assume that the germ at τ of the previous section vanishes. Then, there exists on an open polydisc V with center τ in U and a $(0, 0)-$current S satisfying

$$\mathcal{I}S = 0, \quad (U_0 - q)S = 0 \quad \text{and} \quad \bar{\partial} S = \bar{\partial} T.$$

Then, we may write $S = T - F$ where F is holomorphic on V. However, then F satisfies also $\mathcal{I}F = 0$ and $(U_0 - q).F(\sigma) = N_q(d(\sigma))$ for all $\sigma \in V$. The first equation implies that F is a global trace function on V (up to shrink V around τ if necessary) and using Lemma 3.1.2 in [1] we see that, up to a locally constant function on $D \cup \Omega$, $(U_0 - q).F$ is the trace of a holomorphic function h define by $h(z) = z^q$ on D and 0 on Ω. However, if $F = Trace(g)$ where g is holomorphic on $D \cup \Omega$ this implies

$$z\frac{\partial g}{\partial z}(z) = h(z) + \kappa(z)$$

where κ is constant equal to k on D. Therefore, on D the meromorphic function $G := g\big/z^q$ satisfies

$$G'(z) = 1/z + \kappa/z^{q+1}.$$

This is clearly impossible for $q \geq 1$. For $q = 0$ this gives that $G = g$ is constant and so is $F = Trace(g)$. But then, $U_0 F = d$ is impossible for $d \geq 1$. This shows that at each point σ^0 of the hyper-suface $\{\sigma_k = 0\}$ in N the germ induced by $\bar{\partial} T_q$ in $Sol^1(\mathcal{N}_q)_{\sigma^0}$ is not zero. Therefore, the support of the sheaf $Sol^1(\mathcal{N}_q)$ contains this hyper-surface for each $q \in \mathbb{N}$. ■

Remark The exact sequence

$$0 \to \mathcal{M} \overset{(U_0 - q)}{\to} \mathcal{M} \to \mathcal{N}_q \to 0$$

gives a long exact sequence

$$0 \to Sol^0(\mathcal{N}_q) \to Sol^0(\mathcal{M}) \overset{U_0-q}{\to} Sol^0(\mathcal{M}) \overset{\partial}{\to} Sol^1(\mathcal{N}_q) \to Sol^1(\mathcal{M}) \to \dots$$

and it is clear that the germ at the origin of the Newton polynomial N_q in $Sol^0(\mathcal{M})$ is not in the image of $U_0 - q$. Our computation above shows that the image of the germ of N_q at the point 0 is mapped by the connector ∂ to the germ in $Sol^1(\mathcal{N}_q)_0$ which is constructed above.

The reader will find the computation of the conjugate of the D_N-modules $\mathcal{N}_\lambda$ for each $\lambda \in \mathbb{C}$ in the article [4], so the computation of the sheaves:

$$Hom_{D_N}(\mathcal{N}_\lambda, Db_N^{0,0}).$$

5 An Application

5.1 A Taylor Expansion

We shall consider now the universal monic degree k equation near the point σ^0 defined by $\sigma_1^0 = \sigma_2^0 = \dots = \sigma_{k-1}^0 = 0$ and $\sigma_k^0 = -1$. We shall denote by $z(\sigma)$ the root of $P_{\sigma^0+\sigma}(z) = 0$ which is near the (simple) root -1, for σ small enough, of the equation $P_{\sigma^0+\sigma}(z) = z^k - (-1)^k = 0$ when σ is small enough. Define

$$F(\sigma^0 + \sigma) := z(\sigma) - \sigma_1/k := \sum_{\alpha \in \mathbb{N}^k} C_\alpha \frac{\sigma^\alpha}{\alpha!} \tag{31}$$

the Taylor expansion at the point σ^0 of $z(\sigma) - \sigma_1/k$ which a solution near σ^0 of the $\mathcal{D}_N$-module $\mathcal{N}_1^\square$ (see the Theorem 4.1.5).

The reader may compare the computation below with [8].

An easy consequence of the results in the paragraph 4.1 is the following theorem.

Theorem 5.1.1 *The following differential operators annihilate the function F in a neighborhood of σ^0, where we note ∂_h for the partial derivative relative to σ_h.*

1. $A_{i,j} := \frac{\partial^2}{\partial i \partial j} - \frac{\partial^2}{\partial_{i+1}\partial_{j-1}} \quad \forall i \in [1, k-1]$ and $\forall j \in [2, k]$.
2. $\hat{U}_0 - 1 := \sum_{h=1}^k h\sigma_h\partial_h - k\partial_k - 1$.
3. $U_{-1} := \sum_{h=0}^{k-1}(k-h)\sigma_h\partial_{h+1}$ *with the convention* $\sigma_0 \equiv 1$

Proof This is consequence of the fact that F is a solution in an open neighborhood of σ^0 of the regular holonomic system $\mathcal{N}_1^\square \simeq \mathcal{D}_N \big/ \mathcal{A} + \mathcal{D}_N(U_0 - 1) + \mathcal{D}_N(U_{-1})$. Remark that the operator $\hat{U}_0$ is the expression of U_0 in the coordinates centered at σ^0. The other operators have in these coordinates the same expression than in the usual coordinates centered at the origin. ∎

Corollary 5.1.2 *The coefficients C_α is the expansion* (1) *only depend on the integers $q := |\alpha| = \sum_{h=1}^k \alpha_h$ and $w(\alpha) := \sum_{h=1}^k h\alpha_h$ so we may rewrite the expansion* (1) *with the convention $C_{q,r} = 0$ when $r \notin [q, kq]$:*

$$F(\sigma^0 + \sigma) = \sum_{q,r} C_{q,r} m_{q,r}(\sigma) \tag{32}$$

where for $q \in \mathbb{N}$ and $r \in [q, kq]$ we define the polynomial $m_{q,r} \in \mathbb{C}[\sigma]$ by the formula

$$m_{q,r}(\sigma) := \sum_{|\alpha|=q, w(\alpha)=r} \frac{\sigma^\alpha}{\alpha!}$$

Proof This is obvious consequence of the description of the holomorphic functions which are annihilated by the differential operators $A_{i,j}$ for all $i \in [1, k-1]$ and $j \in [2, k]$ (see the paragraph 2.1) which generate the left ideal $\mathcal{A}$ in $\mathcal{D}_N$. ■

Proposition 5.1.3 *We have the following formulas, with the conventions $m_{q,r} = 0$ for $r \notin [q, kq]$ (in particular for $q < 0$ or $r < 0$):*

1. $(\hat{U}_0 - 1)(m_{q,r}) = (r-1)m_{q,r} - km_{q-1,r-k} \quad \forall q \geq 0,\ \forall r \in [q, kq]$
2. $U_{-1}(m_{q,r}) = (kq - r + 1)m_{q,r-1} + km_{q-1,r-1} \quad \forall q \geq 0,\ \forall r \in [q, kq]$.

Proof The first formula is a direct consequence of the easy formulas

$$U_0(\sigma^\alpha/\alpha!) = w(\alpha)\sigma^\alpha/\alpha! \quad \text{and} \quad \partial_k(\sigma^\alpha/\alpha!) = \sigma^\beta/\beta!$$

when $\alpha_k \geq 1$, with $\beta + 1_k = \alpha$ and

$$\partial_k(\sigma^\alpha/\alpha!) = 0 \quad \text{when} \quad \alpha_k = 0.$$

The second formula is little more tricky:

For $h \in [2, k-1]$ we have

$$\sigma_h \partial_{h+1}(\sigma^\alpha/\alpha!) = \beta_h \sigma^\beta/\beta! \quad \text{when} \quad \alpha_{h+1} \geq 1$$

with $\beta + 1_{h+1} = \alpha + 1_h$, and

$$\sigma_h \partial_{h+1}(\sigma^\alpha/\alpha!) = 0 \quad \text{when} \quad \alpha_{h+1} = 0.$$

Moreover, for any β with $|\beta| = q - 1$ and $w(\beta) = r - 1$ and for each $h \in [2, k]$ there exists exactly one α if $\beta_h \neq 0$ with $\sigma_h \partial_{h+1}(\sigma^\alpha/\alpha!) = \beta_h \sigma^\beta/\beta!$, and it satisfies $|\alpha| = q$ and $w(\alpha) = r$, and no such α exists if $\beta_h = 0$. This means that that $\sigma_h \partial_{h+1}(m_{q,r})$ contains $\sigma^\beta/\beta!$ with the coefficient β_h.

For $h = 1$ the situation is simpler: $\partial_1(\sigma^\alpha/\alpha!) = \sigma^\beta/\beta!$ when $\alpha_1 \geq 1$ with $\beta + 1_1 = \alpha$, and $\partial_1(\sigma^\alpha/\alpha!) = 0$ when $\alpha_1 = 0$.

Then for each β with $|\beta| = q - 1$ and $w(\beta) = r - 1$ there exists a unique α, such that $\partial_1(\sigma^\alpha/\alpha!) = \sigma^\beta/\beta!$ and it satisfies $|\alpha| = q$ and $w(\alpha) = r$. Therefore, we conclude that

$$\begin{aligned} U_{-1}(m_{q,r}) &= \sum_{|\alpha|=q, w(\alpha)=r} \sum_{h=0}^{k} (k-h)\sigma_h \partial_{h+1}\left(\frac{\sigma^\alpha}{\alpha!}\right) \\ &= k m_{q-1,r-1} + \sum_{|\beta|=q, w(\beta)=r-1} \sum_{h=1}^{k} (k-h)\beta_h \frac{\sigma^\beta}{\beta!} \\ &= k m_{q-1,r-1} + \big(k(q-1) - (r-1)\big) m_{q,r-1} \end{aligned}$$

concluding the proof. ∎

Taking in account Eqs. 2 and 3. of Theorem 5.1.1 (Eq. 1. are used already in the Corollary 5.1.2), we obtain:

Corollary 5.1.4 *The coefficients $C_{q,r}$ of the Taylor expansion* (1) *satisfies the relations:*

$$(r-1)C_{q,r} - kC_{q+1,r+k} = 0 \quad \forall q \geq 1, \quad \forall r \in [q, kq] \tag{A}$$

$$(kq - r + 1)C_{q,r} + kC_{q+1,r} = 0 \quad \forall q \geq 1, \quad \forall r \in [q+1, kq]. \tag{B}$$

The formula (B) *gives, for each* $r \geq 2$ *and each* $s \in \mathbb{N}$*, such that* $0 \leq s \leq \frac{(k-1).r}{k}$

$$C_{r-s,r} = \frac{(-1)^s C_{r,r}}{\prod_{j=1}^{s}(r - j - (r-1)/k)} \tag{B^*}$$

Formula (A) *gives for each* $r \geq 1$*:*

$$C_{r+k,r+k} = (-1)^{k-1}\frac{r-1}{k}\prod_{p=0}^{k-2}\big(r + p - (r-1)/k\big)C_{r,r} \tag{A^*}$$

Moreover, we have

$$C_{q,r} = 0 \quad \forall q \geq 2 \text{ and } \quad \forall r \equiv 1 \text{ modulo } k, \;\; r \in [q, kq] \tag{C}$$

Proof Looking at the coefficient of $m_{q,r}$ for $q \geq 1$ and $r \in [q, kq]$ in the equality $(\hat{U}_0 - 1)[F] \equiv 0$ gives the gives (A). Looking at the coefficient of $m_{q,r-1}$ for $q \geq 1$ and $r \in [q+1, kq]$ in $U_{-1}[F] \equiv 0$ gives (B).

The formula (B^*) is a direct consequence of the formula (B) with $q := r - s$ by an easy induction on $s \in [0, \frac{(k-1).r}{k}]$.

Using formula (B^*) with $r' = r + k$ and $s = k - 1$ we obtain

$$C_{r+1,r+k} = \frac{(-1)^{k-1} C_{r+k,r+k}}{\prod_{p=0}^{k-2}(r + p - (r-1)/k)}$$

Combining this formula with the formula (A) with $q = r \geq 1$ which gives

$$C_{r+1,r+k} = \frac{r-1}{k} C_{r,r}$$

and we obtain the formula (A^*).

Formula (C) is a direct consequence of (A^*) for $r = 1$ with an induction on $a \geq 1$ when $r = 1 + ak$. ■

We shall see below that the vanishing of $C_{1+ak,1+ak}$ is also valid for $a = 0$ giving $C_{q,1+ak} = 0$ for any $q \geq 1$ and any a, such that $q \leq 1 + ak \leq kq$.

Remark It is enough to compute $C_{0,0}$ and $C_{1,h}$ for each $h \in [1, k]$ to determine all coefficients $C_{q,r}$ in (1) with $r \in [q, kq]$:

The formula (B) determines $C_{h,h}$, $h \in [2, k]$ from $C_{1,h}$ with $r = h$ and $s = h - 1$. Then, the formulas (A^*) and (C) gives $C_{r,r}$ for any $r \geq 1$. Then, the formula (B) completes the computation of $C_{q,r}$ for any $q \geq 0$ and any $r \in [q, kq]$.

Lemma 5.1.5 *We have the following values:*

$$\begin{aligned} &C_{0,0} = -1 \\ &C_{1,1} = 0 \\ &C_{1,h} = 1/k \quad \text{for } h \in [2, k]. \end{aligned}$$

Proof The value of $C_{0,0}$ is $F(\sigma^0)$ which is -1 by definition of F. The values of $C_{1,h}$ is the derivative $\partial_h(F + \sigma_1/k)(\sigma^0)$, because we have $m_{1,h} = \sigma_h$ for $h \in [1, k]$. Therefore, it is enough to make an order 1 expansion of F at σ^0 to compute the values of the $C_{1,h}$, $h \in [1, k]$. This is given by the following computation at the first order of $P_{\sigma^0+\sigma}(z(\sigma)) \equiv 0$, where we define

$$z(\sigma) = -1 + \sum_{j=1}^{k} c_j \sigma_j + o(||\sigma||)$$

which gives $c_1 = C_{1,1} + 1/k$ and $c_j := C_{1,j}$ for $j \in [2, k]$ and then:

$$\left(-1 + \sum_{h=1}^{k} c_h \sigma_h\right)^k + \sum_{j=1}^{k} (-1)^j \sigma_j \left(-1 + \sum_{h=1}^{k} c_h \sigma_h\right)^{k-j} - (-1)^k = o(||\sigma||)$$

$$\begin{aligned} &(-1)^{k-1} k \left(\sum_{h=1}^{k} c_h \sigma_h\right) + \sum_{j=1}^{k} (-1)^k \sigma_j = o(||\sigma||) \quad \text{and so} \\ &c_j = 1/k \quad \forall j \in [1, k] \end{aligned}$$

Then, $C_{1,1} = 0$ and $C_{1,j} = c_j = 1/k$ for each $j \in [2, k]$. ■

6 Appendix: The Surface $S(k)$

6.1 Study of S(k)

For $k \geq 2$ an integer and $\alpha \in \mathbb{N}^k$ define

- the **length** of α given by $|\alpha| := \sum_{h=1}^{k} \alpha_h$;
- the **weight** of α given by $w(\alpha) := \sum_{h=1}^{k} h.\alpha_h$.

We shall say that α and β are **equivalent**, noted by $\alpha \sharp \beta$, when $|\alpha| = |\beta|$ and $w(\alpha) = w(\alpha)$.

Remark that for any $\gamma \in \mathbb{N}^k$ we have $(\alpha + \gamma)\sharp(\beta + \gamma)$ if and only if $\alpha \sharp \beta$.

Let A be a $\mathbb{C}-$algebra which is commutative, unitary and integral. In the algebra $A[x_1, \dots, x_k]$ let $IS(k)$ be the ideal generated by the polynomials $x_p.x_q - x_{p+1}.x_{q-1}$ for all $p \in [1, k-1]$ and $q \in [2, k]$.

We shall say that the two monomials x^α and x^β in $A[x_1, \dots, x_k]$ are equivalent when α and β are equivalent. In this case we shall also write $x^\alpha \sharp x^\beta$.

Remark that for any $p \in [1, k-1]$ and any $q \in [2, k]$ $x_p.x_q$ is equivalent to $x_{p+1}.x_{q-1}$.

For a monomial $m := x^\alpha$ we define its length by $l(m) := |\alpha|$ and its weight $w(m) := w(\alpha)$.

Our first result is the following characterization of the elements in $IS(k)$.

Proposition 6.1.1 *Two monomials x^α and x^β in $A[x_1, \dots, x_k]$ are equivalent if and only if $x^\alpha - x^\beta$ is in $IS(k)$.*

The proof of this proposition will need a preliminary lemma and the next definition.

Definition 6.1.2 We shall say that a monomial m in $A[x_1, \dots, x_k]$ is **minimal** when it has one of the following forms:

1. there exists p, q in $\mathbb{N}$, such that $m = x_1^p x_k^q$;
2. there exists p, q in $\mathbb{N}$ and $j \in [2, k-1]$ such that $m = x_1^p x_j x_k^q$.

Remark Any monomial (minimal or not) is not in the ideal $IS(k)$, because the point $x_1 = x_2 = \cdots = x_k = 1$ is not in $|S(k)|$ the common set of zeros in A^k of the generators of $IS(k)$ and any monomial does not vanish at this point.

Lemma 6.1.3 *For each $\alpha \in \mathbb{N}^k$ there exists an unique minimal monomial $x^{\mu(\alpha)}$, such that $x^\alpha \sharp x^{\mu(\alpha)}$. Moreover, for each α $x^\alpha - x^{\mu(\alpha)}$ is in $IS(k)$*

Proof Let us begin by proving the uniqueness assertion.

We have to show that two minimal monomials which are equivalent are equal. If both are in case 1. (so $m := x_1^p x_k^q$) this is obvious as the length is equal to $l(m) = p+q$

and the weight is $w(m) = p + kq$ and then $(k-1).q = w(m) - l(m)$ proving the uniqueness of q and then of p.

If both are in case 2. let $m := x_1^p x_j x_k^q$ and $m' = x_1^{p'} x_{j'} x_k^{q'}$ then we have

$$\begin{aligned}
l(m) &= p + 1 + q = l(m') = p' + 1 + q' \quad \text{and} \\
w(m) &= p + j + kq = w(m') = p' + j' + kq' \quad \text{which imply} \\
j - j' &= (k-1)(q' - q) \quad \text{with} \quad |j - j'| \in [0, k-3]. \\
\text{So} \quad j &= j' \quad \text{and then} \quad q = q' \quad \text{and} \quad p = p'.
\end{aligned}$$

If $m = x_1^p x_j x_k^q$ and $m' = x_1^{p'} x_k^{q'}$ we have

$$\begin{aligned}
(m) &= p + 1 + q = l(m') = l(m') = p' + q' \quad \text{and} \\
w(m) &= p + j + kq = w(m') = p' + kq' \quad \text{which imply} \\
j - 1 &= (k-1)(q' - q) \quad \text{with} \quad j \in [2, k-1] \quad \text{and this is impossible.}
\end{aligned}$$

The assertion of existence is clear for $|\alpha| = 0, 1$. We shall prove the existence of $\mu(\alpha)$ by an induction on the length $|\alpha|$ of α.

Assume that the lemma is proved for all $\beta \in \mathbb{N}^k$ with length $1 \leq |\beta| < |\alpha|$. Then, write $x^\alpha = x_r x^\beta$ for some $r \in [1, k]$. By the induction hypothesis we know that there exists a minimal monomial $x^{\mu(\beta)}$ with $x^\beta \sharp x^{\mu(\beta)}$. Then, we obtain that $x^\alpha \sharp x_r x^{\mu(\beta)}$. If $x^{\mu(\beta)} = x_1^p x_k^q$, then $x_r x_1^p x_k^q$ is minimal for any choice of $r \in [1, k]$. If $x^{\mu(\beta)} = x_1^p x_j x_k^q$ then remark that we have $x_r x_j \sharp x_1 x_{r+j-1}$ for $r + j - 1 \leq k$ and $x_r x_j \sharp x_k x_{r+j-k}$ for $r + j \geq k + 1$ and this allows to conclude the induction.

Remark that if, in the induction above, we assume that $x^\beta - x^{\mu(\beta)}$ belongs to $IS(k)$ we obtain that $x^\alpha - x^{\mu(\alpha)}$ is also in $IS(k)$; for instance in the case $x^{\mu(\beta)} = x_1^p x_j x_k^q$

$$\begin{aligned}
x^\alpha - x^{\mu(\alpha)} &= x_r(x^\beta - x^{\mu(\beta)}) + (x_r x_j - x_1 x_{r+j-1}) x_1^p x_k^q \quad \text{for} \quad r + j \leq k + 1, \\
x^\alpha - x^{\mu(\alpha)} &= x_r(x^\beta - x^{\mu(\beta)}) + (x_r x_j - x_k x_{r+j-k}) x_1^p x_k^q \quad \text{for} \quad r + j \geq k + 2.
\end{aligned}$$

The other cases are analogous. ■

Proof of the proposition 6.1.1 The previous lemma gives that $x^\alpha \sharp x^\beta$ implies $x^\alpha - x^{\mu(\alpha)}$ and $x^\beta - x^{\mu(\alpha)}$ are in $IS(k)$, so also $x^\alpha - x^\beta$. Conversely, assume that $x^\alpha - x^\beta$ is in $IS(k)$. As the ideal $IS(k)$ is homogeneous (in the sense of length) if $l(\alpha) \neq l(\beta)$ we conclude that both x^α and x^β are in $IS(k)$. This contradicts the remark following Definition 6.1.2.

In a similar way the ideal $IS(k)$ is quasi-homogeneous in the sense of the weight w. Therefore, if $w(\alpha) \neq w(\beta)$ then x^α and x^β are in $IS(k)$ which is again impossible. Therefore, $x^\alpha - x^\beta$ is in $IS(k)$ implies that $\alpha \sharp \beta$. ■

Corollary 6.1.4 *For any $q \in \mathbb{N}$ and any $r \in [q, kq]$ there exists a minimal monomial $\mu_{q,r}$ (necessarily unique), such that $|\mu_{q,r}| = q$ and $w(\mu_{q,r}) = r$.*

Proof The assertion is clear for $q = 0, 1$. Therefore, let us prove it by induction on q. Therefore, let $q \geq 2$ and let $r \in [q, kq]$, and assume that we know that $\mu_{q',r'}$ exists for any $q' \leq q-1$ and any $r' \in [q', kq']$. If r is in $[q, k(q-1)+1]$, then $r-1$ is in $[q-1, k(q-1)]$ and $\mu_{q-1,r-1}$ exists. Therefore, $\mu_{q,r} := x_1 m_{q-1,r-1}$ is the solution.

If r is in $[k(q-1)+1, kq]$ then $r-k$ is in $[q-1, k(q-1)]$and, because for $q \geq 2$ we have $kq - 2k + 1 \geq q - 1$ and also $r - k \leq kq - k \leq k(q-1)$, $\mu_{q-1,r-k}$ is defined and $\mu_{q,r} := x_k \mu_{q-1,r-k}$ is the solution. ■

Proposition 6.1.5 *Let $L_1 := \{\eta_1 = 0\} \cap S(k)$ and $L_k := \{\eta_k = 0\} \cap S(k)$. Then, L_1 is the line directed by the vector $(0, \dots, 0, 1)$ and L_k the line directed by the vector $(1, 0, \dots, 0)$. The maps $\varphi_1 : S(k) \setminus L_1 \to \mathbb{C}^* \times \mathbb{C}$ and $\varphi_k : S(k) \setminus L_k \to \mathbb{C}^* \times \mathbb{C}$ which are defined by the formulas*

$$\varphi_1(\eta) := (\eta_1, -\eta_2/\eta_1) \quad \text{and} \quad \varphi_k(\eta) := (\eta_k, -\eta_{k-1}/\eta_k) \tag{33}$$

are isomorphisms. Therefore, $S(k) \setminus \{0\}$ is smooth and connected surface.

Proof of the proposition 6.1.5 Consider the holomorphic map

$$\psi_1 : \mathbb{C}^2 \to S(k) \quad (\zeta_0, \zeta_1) \mapsto x_h := \zeta_0(-\zeta_1)^{h-1} \; \forall h \in [1, k].$$

It induces the inverse to the map φ_1 on $\zeta_0 \neq 0$ and the map ψ_k defined by $x_h = (-\zeta_0)^{k-h}.\zeta_1 \quad \forall h \in [1, k]$ gives the inverse of φ_k on $\zeta_1 \neq 0$. ■

Corollary 6.1.6 *The ideal $IS(k)$ of $\mathbb{C}[x]$ is prime. Moreover, (S_k) is a normal surface.*

Proof of the corollary 6.1.6 The only point which is not a direct consequence of the previous proposition is the normality of $S(k)$. However, as the blow-up of the maximal ideal at the origin in $S(k)$ gives a desingularization of $S(k)$ with the rational curve[13] over the origin in $S(k)$. Therefore, is a rational singular point and $S(k)$ is normal. ■

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[13] The image of the map $\psi_0 : \mathbb{P}_1 \to \mathbb{P}_{k-1}$ by $\xi_h = \zeta_1^{h-1}\zeta_0^{k-h}$.