

Higher dimensional versions of theorems of Euler and Fuss

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Abstract

We present higher dimensional versions of the classical results of Euler and Fuss, both of which are special cases of the celebrated Poncelet porism. Our results concern polytopes, specifically simplices, parallelotopes and cross polytopes, inscribed in a given ellipsoid and circumscribed to another. The statements and proofs use the language of linear algebra. Without loss, one of the ellipsoids is the unit sphere and the other one is also centered at the origin. Let A be the positive symmetric matrix taking the outer ellipsoid to the inner one. If $\text{tr } A = 1$, there exists a bijection between the orthogonal group $O(n)$ and the set of such labeled simplices. Similarly, if $\text{tr } A^2 = 1$, there are families of parallelotopes and of cross polytopes, also indexed by $O(n)$.

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1 Introduction

We recall the classic geometric Poncelet porism [1, 3, 5]. Consider two disjoint ellipses $E_{\text{out}}, E_{\text{in}} \subset \mathbb{R}^2$ with E_{in} contained in $\text{conv}(E_{\text{out}}) \subset \mathbb{R}^2$, the convex hull of E_{out} . A polygon with k vertices $P_0, P_1, P_2, \dots, P_k = P_0$ fits tightly between E_{out} and E_{in} if its vertices belong to E_{out} and its sides are tangent to E_{in} . We implicitly assume $P_{i+2} \neq P_i$ (for all i). Alternatively, we say the polygon is *tight*. Figure 1 shows two pairs of ellipses: in the first example several triangles ($k = 3$) fit tightly; in the second, several quadrilaterals ($k = 4$) also fit tightly.

Theorem 1 (Poncelet porism). *If the pair $E_{\text{out}}, E_{\text{in}}$ admits a tight polygon with k vertices then any point $Q_0 \in E_{\text{out}}$ is a vertex of a tight polygon with k vertices.*

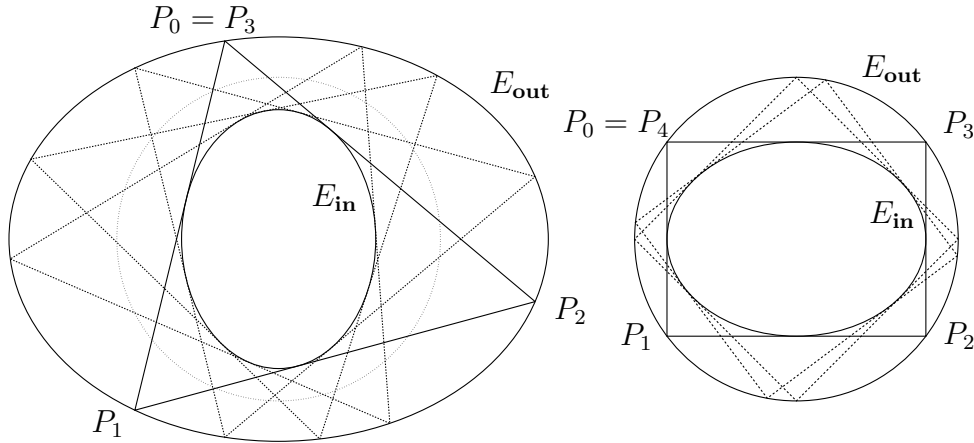


Figure 1: Poncelet porism, $k = 3$ and $k = 4$

A little projective geometry [2] shows that the general case above follows from the special case when E_{out} and E_{in} are both circles. The special cases $k = 3$ and $k = 4$ of Theorem 1 (for circles) were proved earlier by Euler and his student Fuss, respectively [4]. If E_{out} and E_{in} are circles of radii R and r , respectively, and the distance between the two centers is d , then there exist tight triangles ($k = 3$) or quadrilaterals ($k = 4$) if and only if

$$\frac{1}{(R-d)^{k-2}} + \frac{1}{(R+d)^{k-2}} = \frac{1}{r^{k-2}}, \quad k \in \{3, 4\}. \quad (1)$$

Projective geometry also implies that the general case follows from another special case, with E_{out} and E_{in} being ellipses centered at the origin, as in Figure 1. We state higher dimensional versions of the theorems of Euler and Fuss in this context, starting with Fuss.

Consider two disjoint ellipsoids $E_{\text{out}}, E_{\text{in}} \subset \mathbb{R}^n$ centered at the origin with $E_{\text{in}} \subset \text{conv}(E_{\text{out}})$. By applying a linear transformation, we assume that $E_{\text{out}} = \mathbb{S}^{n-1}$, the unit sphere. Clearly, there exists a unique positive symmetric matrix A with $AE_{\text{out}} = E_{\text{in}}$.

A closed, convex polytope $P \subset \mathbb{R}^n$ fits between E_{out} and E_{in} if $E_{\text{in}} \subset P \subset \text{conv}(E_{\text{out}})$. The polytope P is *inscribed* in E_{out} if all its vertices belong to E_{out} and P is *circumscribed* to E_{in} if all its hyperfaces are tangent to E_{in} . It fits *tightly* between E_{out} and E_{in} if it is inscribed in E_{out} and circumscribed to E_{in} . We usually consider *labeled* polytopes, for which the vertices are indexed.

A *centrally symmetric parallelotope* in \mathbb{R}^n is a convex polytope with 2^n vertices of the form $\pm v_1 \pm v_2 \pm \dots \pm v_n$ where the vectors v_1, \dots, v_n form a basis. Thus, for $n = 2$ the polytope is a parallelogram and for $n = 3$, a parallelepiped. A *label* for a parallelotope is a family $(v_k)_{1 \leq k \leq n}$ of vectors as above so that each vertex in turn is labeled by a sequence of signs. A centrally symmetric parallelotope is *orthogonal* if the basis (v_k) is orthogonal. As we shall see, if a parallelotope

is inscribed in the unit sphere then it is necessarily centrally symmetric and orthogonal. For E_{out} and E_{in} as above, let $\mathcal{P}_t(E_{\text{in}}, E_{\text{out}})$ be the set of all labeled parallelotopes fitting tightly between E_{out} and E_{in} .

As usual, $O(n)$ is the real orthogonal group. Define the map

$$\phi : \mathcal{P}_t(E_{\text{in}}, E_{\text{out}}) \rightarrow O(n)$$

taking a parallelotope with label (v_k) to the matrix $Q \in O(n)$ whose columns are obtained from the basis (v_k) by Gram-Schmidt orthonormalization.

Theorem 2. *Let $E_{\text{out}} = \mathbb{S}^{n-1}$ and $E_{\text{in}} = AE_{\text{out}}$, as above. The set $\mathcal{P}_t(E_{\text{in}}, E_{\text{out}})$ is nonempty if and only if $\text{tr}(A^2) = 1$. In this case, the map $\phi : \mathcal{P}_t(E_{\text{in}}, E_{\text{out}}) \rightarrow O(n)$ is a diffeomorphism.*

The *dual* or *polar* of a bounded convex set $X \subset \mathbb{R}^n$ with $0 \in X$ is

$$\tilde{X} = \{v \in \mathbb{R}^n \mid \forall w \in X, \langle v, w \rangle \geq -1\}.$$

For instance, the dual of a centrally symmetric parallelotope is a centrally symmetric cross polytope (see Remark 2.4). We shall discuss duality further in Section 4.

For E_{out} and E_{in} as above, let $\mathcal{C}_t(E_{\text{in}}, E_{\text{out}})$ be the set of all labeled centrally symmetric cross polytopes fitting tightly between E_{out} and E_{in} . The map $\phi : \mathcal{C}_t(E_{\text{in}}, E_{\text{out}}) \rightarrow O(n)$ is defined similarly to the parallelotope situation (see Section 2). Duality applied to Theorem 2 above gives us the following similar result:

Corollary 1.1. *Let $E_{\text{in}} = \mathbb{S}^{n-1}$ and $E_{\text{out}} = A^{-1}E_{\text{in}}$. The set $\mathcal{C}_t(E_{\text{in}}, E_{\text{out}})$ is nonempty if and only if $\text{tr}(A^2) = 1$. Then, the map $\phi : \mathcal{C}_t(E_{\text{in}}, E_{\text{out}}) \rightarrow O(n)$ is a diffeomorphism.*

We now extend Euler's theorem to *simplices* in \mathbb{R}^n : convex polytopes with $n + 1$ vertices and nonempty interior. For ellipsoids E_{out} and E_{in} as above, let $\mathcal{S}_f(E_{\text{in}}, E_{\text{out}})$ (resp. $\mathcal{S}_t(E_{\text{in}}, E_{\text{out}})$) be the set of labeled simplices with vertices $v_1, \dots, v_n, v_{n+1} \in \text{conv}(E_{\text{out}})$ fitting (resp. fitting tightly) between E_{out} and E_{in} , so that $\mathcal{S}_t(E_{\text{in}}, E_{\text{out}}) \subseteq \mathcal{S}_f(E_{\text{in}}, E_{\text{out}})$.

Define the map

$$\phi : \mathcal{S}_f(E_{\text{in}}, E_{\text{out}}) \rightarrow O(n)$$

taking a simplex $S \in \mathcal{S}_f(E_{\text{in}}, E_{\text{out}})$ with vertices v_1, \dots, v_n, v_{n+1} to the matrix $Q \in O(n)$ whose columns are obtained from the basis v_1, \dots, v_n by Gram-Schmidt orthonormalization.

Theorem 3. *Let $E_{\text{in}} = \mathbb{S}^{n-1} \subset \mathbb{R}^n$ be the unit sphere. Let $E_{\text{out}} \subset \mathbb{R}^n$ be a nondegenerate ellipsoid centered at the origin. Let A be the unique positive definite real symmetric matrix such that $AE_{\text{out}} = E_{\text{in}}$.*

- (i) If $\text{tr}(A) > 1$, no simplex fits between E_{in} and E_{out} : $\mathcal{S}_f(E_{\text{in}}, E_{\text{out}}) = \emptyset$.
- (ii) If $\text{tr}(A) = 1$, every fitting simplex fits tightly: $\mathcal{S}_f(E_{\text{in}}, E_{\text{out}}) = \mathcal{S}_t(E_{\text{in}}, E_{\text{out}})$. The map $\phi : \mathcal{S}_f(E_{\text{in}}, E_{\text{out}}) \rightarrow O(n)$ is a diffeomorphism.
- (iii) If $\text{tr}(A) < 1$, $\mathcal{S}_t(E_{\text{in}}, E_{\text{out}}) \neq \mathcal{S}_f(E_{\text{in}}, E_{\text{out}})$. The map $\phi : \mathcal{S}_f(E_{\text{in}}, E_{\text{out}}) \rightarrow O(n)$ is surjective and not injective.

Remark 1.2. In the situation of item (iii) above, the cases $n = 2$ and $n \geq 3$ behave differently. For $n = 2$, $\mathcal{S}_t(E_{\text{in}}, E_{\text{out}}) = \emptyset$: this follows from the case $k = 3$ of Poncelet porism. If $n \geq 3$, the restriction $\phi : \mathcal{S}_t(E_{\text{in}}, E_{\text{out}}) \rightarrow O(n)$ is still surjective and not injective. We shall present illustrative examples but not the cumbersome computations. \diamond

Our extensions of the classical results of Euler and Fuss to higher dimensions provide a simple counterpart to Equation (1) in terms of traces. This was possible because of our choice to work with centrally symmetric ellipsoids.

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2 Fuss and parallelotopes

Lemma 2.1. *If a centrally symmetric parallelotope is inscribed in a sphere then it is orthogonal.*

Proof. Notice first that for $n = 2$ the lemma says that a parallelogram inscribed in a circle is a rectangle: this follows from the fact that both diagonals have the same length.

For the general case, consider a centrally symmetric parallelotope with vertices $\pm w_1 \pm \dots \pm w_n$ inscribed in a sphere E_{out} of radius R . Consider i and j with $1 \leq i < j \leq n$: the four vertices $\tilde{w} \pm w_i \pm w_j$, $\tilde{w} = -w_i - w_j + \sum_k w_k$, are inscribed in the intersection of a 2-dimensional plane with the sphere, which is of course a circle. The case $n = 2$ implies $w_i \perp w_j$. \square

Remark 2.2. For an alternative proof, use the notation above and notice that $|\tilde{w} \pm w_i \pm w_j| = R$ for any choice of signs. We then have

$$\langle \tilde{w} \pm w_j, w_i \rangle = \frac{1}{4} \left(|\tilde{w} \pm w_j + w_i|^2 - |\tilde{w} \pm w_j - w_i|^2 \right) = \frac{R^2 - R^2}{4} = 0$$

and therefore $\langle w_i, w_j \rangle = 0$. \diamond

Proof of Theorem 2. We first prove that if $\mathcal{P}_t(E_{\text{in}}, E_{\text{out}}) \neq \emptyset$ then $\text{tr}(A^2) = 1$. Take $P \in \mathcal{P}_t(E_{\text{in}}, E_{\text{out}})$. From Lemma 2.1, P is an orthogonal parallelotope. The edges of P are parallel to q_k , where (q_k) is an orthonormal basis. The map ϕ takes P to $Q \in O(n)$ with columns (q_k) .

Let $2\ell_k$ be the length of the edge in the direction q_k so that the vertices of P are $\pm\ell_1q_1 \pm \dots \pm \ell_kq_k \pm \dots \pm \ell_nq_n$; the faces are the hyperplanes $H_{k,\pm}$ of equations $\langle \cdot, q_k \rangle = \pm\ell_k$. By Pythagoras we have $\sum_k \ell_k^2 = 1$. Let v_k be the point of tangency between E_{in} and $H_{k,+}$. By definition of A , v_k has the form $v_k = Au_k$, $u_k \in E_{\text{out}}$. We determine u_k and v_k .

The vector u_k maximizes $\langle q_k, Ax \rangle$ with the restriction $|x| = 1$. Since A is symmetric, $\langle q_k, Ax \rangle = \langle x, Aq_k \rangle$ and then

$$u_k = x = \frac{Aq_k}{|Aq_k|}, \quad v_k = Au_k = \frac{A^2q_k}{|Aq_k|}.$$

Since $\langle v_k, q_k \rangle = \ell_k$, we have

$$\ell_k = \frac{\langle A^2q_k, q_k \rangle}{|Aq_k|} = \frac{\langle Aq_k, Aq_k \rangle}{|Aq_k|} = |Aq_k|.$$

We compute the trace in the basis (q_k) , thus proving the first claim:

$$\text{tr}(A^2) = \sum_k \langle A^2q_k, q_k \rangle = \sum_k \langle Aq_k, Aq_k \rangle = \sum_k |Aq_k|^2 = \sum_k \ell_k^2 = 1.$$

We are left with proving that the map $\phi : \mathcal{P}_t(E_{\text{in}}, E_{\text{out}}) \rightarrow O(n)$ constructed above is invertible. Take $Q \in O(n)$ with columns (q_k) . Consider the parallelotope P with vertices $\pm\ell_1q_1 \pm \dots \pm \ell_kq_k \pm \dots \pm \ell_nq_n$ where each $\ell_k > 0$ is chosen so that the ellipsoid E_{in} is tangent to the hyperplane $H_{k,+}$ of equation $\langle \cdot, q_k \rangle = \ell_k$. We prove that $P \in \mathcal{P}_t(E_{\text{in}}, E_{\text{out}})$ and $\phi(P) = Q$. Clearly, P is inscribed in a sphere RE_{out} of radius $R > 0$, $R^2 = \sum_k \ell_k^2$. We need to verify that $R = 1$.

Indeed, consider a scaled parallelotope $\hat{P} = (1/R)P$, inscribed in E_{out} and circumscribed to $\widehat{E}_{\text{in}} = (1/R)E_{\text{in}}$. The matrix $\hat{A} = (1/R)A$ satisfies $\hat{A}E_{\text{out}} = \widehat{E}_{\text{in}}$ and therefore, from the previous paragraph, $\text{tr}(\hat{A}^2) = 1$. But $\text{tr}(\hat{A}^2) = R^{-2}$, so that $R = 1$, as desired. The fact that $\phi(P) = Q$ is obvious. \square

Remark 2.3. The hypothesis of polytopes being centrally symmetric parallelotopes is essential. Indeed, for dimension $n = 3$ consider parameters $r > 0$ and $s \geq 1$ and construct the convex polyhedron P with 8 vertices: $(\pm rs, \pm rs^{-1}, r)$, $(\pm rs^{-1}, \pm rs, -r)$. For $s = 1$, P is a cube. In general, this polyhedron has 6 faces: two rectangles in the planes $z = \pm r$ and four trapezoids, as shown in Figure 2.

A simple computation verifies that P is inscribed in the sphere E_{out} of radius $R = r\sqrt{s^2 + 1 + s^{-2}}$; P is also circumscribed to the sphere E_{in} of radius r . The positive symmetric matrix A with $AE_{\text{out}} = E_{\text{in}}$ is therefore $A = (r/R)I$, with trace $\text{tr}(A) = 3/\sqrt{s^2 + 1 + s^{-2}}$. By adjusting the value of s , $\text{tr}(A)$ can assume any

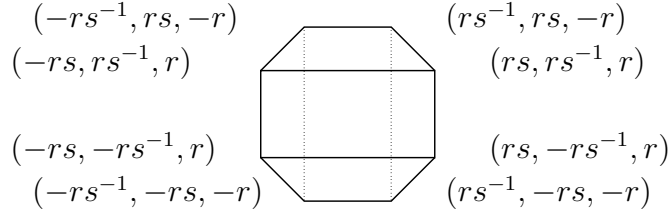


Figure 2: A convex polytope which is both inscribable and circumscribable.

value in the interval $(0, 1)$. Similar constructions are possible in higher dimension ($n > 3$) but not in the plane ($n = 2$). The situation is reminiscent of Remark 1.2. \diamond

Remark 2.4. We provide details concerning Corollary 1.1. A *centrally symmetric cross polytope* in \mathbb{R}^n is a convex polytope with non empty interior and vertices $\pm v_1, \dots, \pm v_n$. For $n = 2$, a cross polytope is a parallelogram; for $n = 3$, an octahedron.

It is easy to verify that the dual of a centrally symmetric parallelotope is a centrally symmetric cross polytope. Thus, Corollary 1.1 follows from Theorem 2 via duality, and vice versa. Define the map $\phi : \mathcal{C}_t(E_{\text{in}}, E_{\text{out}}) \rightarrow O(n)$ taking a cross polytope with vertices $\pm v_k$ to $Q \in O(n)$ with $Qe_k = v_k/|v_k|$: the map ϕ is a diffeomorphism. \diamond

3 Euler and simplices

As in the statement of Theorem 3, A is the unique positive definite real symmetric matrix such that $A(E_{\text{out}}) = E_{\text{in}} = \mathbb{S}^{n-1}$. We then have

$$E_{\text{in}} = \{v \in \mathbb{R}^n \mid \langle v, v \rangle = 1\}, \quad E_{\text{out}} = \{v \in \mathbb{R}^n \mid \langle Av, Av \rangle = 1\}.$$

Bases are understood to be families such as $(v_i)_{1 \leq i \leq n}$: labeling is important.

Proposition 3.1. *If $\mathcal{S}_f(E_{\text{in}}, E_{\text{out}}) \neq \emptyset$ then $\text{tr } A \leq 1$. Moreover, if $\text{tr } A = 1$, then $\mathcal{S}_f(E_{\text{in}}, E_{\text{out}}) = \mathcal{S}_t(E_{\text{in}}, E_{\text{out}})$.*

Thus, there are no fitting simplices if $\text{tr } A > 1$. For $\text{tr } A = 1$, a fitting simplex is tight. Part (i) and the first claim in part (ii) of Theorem 3 follow from the proposition.

Proof of Proposition 3.1. Take $S \in \mathcal{S}_f(E_{\text{in}}, E_{\text{out}})$ with vertices $(v_i)_{1 \leq i \leq n+1}$. In particular, $\text{conv}(\{v_1, \dots, v_{n+1}\})$ contains the origin in its interior. A hyperface $F_i \subset S$ is the convex closure of the vertices v_j , $j \neq i$, and belongs to a hyperplane H_i . Take $w_i \in \mathbb{S}^{n-1}$ the closest point to H_i and $t_i \geq 1$ such that $t_i w_i \in H_i$:

$$H_i = \{q \in \mathbb{R}^n \mid \langle q, t_i w_i \rangle = \langle t_i w_i, t_i w_i \rangle = t_i^2\}.$$

We must then have $\langle v_i, w_j \rangle = t_i \geq 1$ for $i \neq j$; on the other hand, $\langle v_i, w_i \rangle < 0$, otherwise 0 would not belong to the interior of $\text{conv}(\{v_1, \dots, v_{n+1}\})$.

Define extended vectors $\hat{v}_i = (v_i, 1)$, $\hat{w}_j = (w_j, -t_j) \in \mathbb{R}^{n+1}$. For $i \neq j$ we have $\langle \hat{v}_i, \hat{w}_j \rangle = 0$; also, $\langle \hat{v}_i, \hat{w}_i \rangle < -t_i \leq -1$. We show that the families $(\hat{v}_i)_{1 \leq i \leq n+1}$ and $(\hat{w}_i)_{1 \leq i \leq n+1}$ form bases of \mathbb{R}^{n+1} . Indeed, if $\sum_i c_i \hat{v}_i = 0$, the inner product with \hat{w}_j gives $c_j \langle \hat{v}_j, \hat{w}_j \rangle = 0$, so that $c_j = 0$. The same argument applies to (\hat{w}_i) . Said differently, (\hat{v}_i) and (\hat{w}_i) are biorthogonal bases.

We phrase these properties in matrix notation. Let \hat{V} and \hat{W} be $(n+1) \times (n+1)$ matrices with columns given by (\hat{v}_i) and (\hat{w}_i) respectively. We then have

$$\hat{W}^T \hat{V} = D = \text{diag}(\langle \hat{v}_1, \hat{w}_1 \rangle, \dots, \langle \hat{v}_{n+1}, \hat{w}_{n+1} \rangle) = \text{diag}(-t_1, \dots, -t_{n+1}).$$

We have $D \leq -I$ in the sense that $\langle u, (D + I)u \rangle \leq 0$ for all $u \in \mathbb{R}^{n+1}$. The trace of a real $(n+1) \times (n+1)$ matrix X is

$$\text{tr } X = \text{tr } \hat{W}^T X (\hat{W}^T)^{-1} = \text{tr } \hat{W}^T X \hat{V} D^{-1} = \sum_i \frac{\langle \hat{w}_i, X \hat{v}_i \rangle}{(-t_i)}. \quad (2)$$

Let $A_- = A \oplus (-1)$ be the $(n+1) \times (n+1)$ matrix obtained from A by adding a final row and column of zeroes and an entry equal to -1 in position $(n+1, n+1)$. From Equation (2) for $X = A_-$,

$$\text{tr } A_- = \sum_i \frac{\langle \hat{w}_i, A_- \hat{v}_i \rangle}{(-t_i)} = \sum_i \frac{\langle w_i, Av_i \rangle + t_i}{(-t_i)}. \quad (3)$$

As $w_i \in \mathbb{S}^{n-1}$, $t_i \geq 1$ and $Av_i \in \text{conv}(E_{\text{in}})$, the numerators are greater or equal to zero. Thus $\text{tr } A_- = \text{tr } A - 1 \leq 0$.

Now suppose $\text{tr } A = 1$ and $S \in \mathcal{S}_f(E_{\text{in}}, E_{\text{out}})$. From the computations above, $\text{tr } A = 1$ if and only if the numerators in Equation (3) are equal to zero, that is, if and only if $\langle w_i, Av_i \rangle = -t_i$ for all i . As $w_i \in \mathbb{S}^{n-1}$, and $Av_i \in \text{conv}(E_{\text{in}})$, we have (by Cauchy-Schwartz) $\langle w_i, Av_i \rangle \geq -1$ with equality if and only if $Av_i = -w_i$. Since $t_i \geq 1$ we must have $t_i = 1$ and $Av_i = -w_i$. Thus, the hyperplane H_i containing the face F_i is tangent to E_{in} : S is circumscribed to E_{in} . Moreover, $w_i = -Av_i$ implies $v_i \in E_{\text{out}}$, i.e., S is inscribed in E_{out} , proving that $S \in \mathcal{S}_t(E_{\text{in}}, E_{\text{out}})$, and therefore that $\mathcal{S}_t(E_{\text{in}}, E_{\text{out}}) = \mathcal{S}_f(E_{\text{in}}, E_{\text{out}})$. \square

Remark 3.2. Take $n = 3$ and $a, b, c > 1$. Let

$$E_{\text{out}} = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}, \quad A = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{pmatrix}.$$

For $\tilde{w}_1 = -(x, y, z) \in E_{\text{in}} = \mathbb{S}^2$, consider the tetrahedron S with vertices $v_1 = -A^{-1}\tilde{w}_1 = (ax, by, cz) \in E_{\text{out}}$, $v_2 = (ax, -by, -cz)$, $v_3 = (-ax, by, -cz)$ and $v_4 = (-ax, -by, cz)$, so that S is inscribed in E_{out} . These four vertices are

alternating vertices of a rectangular parallelepiped with edges parallel to the axes. Following the construction in the proof of Proposition 3.1, and setting

$$F(x, y, z) = (ax)^{-2} + (by)^{-2} + (cz)^{-2},$$

we have $t_1 = t_2 = t_3 = t_4$ and

$$w_1 = -\frac{1}{t_1 F(x, y, z)} \left(\frac{1}{ax}, \frac{1}{by}, \frac{1}{cz} \right), \quad F(x, y, z) = t_1^{-2}.$$

Thus, S is circumscribed to a sphere of radius t_1 : it fits (resp. tightly) between E_{in} and E_{out} if and only if $F(x, y, z) \leq 1$ (resp. $F(x, y, z) = 1$).

In order to study the function F , it suffices to consider the octant $x, y, z \geq 0$: F goes to infinity at the boundary and has a unique critical point, a global minimum, at $x^2 = a^{-1}/\text{tr } A$, $y^2 = b^{-1}/\text{tr } A$, $z^2 = c^{-1}/\text{tr } A$. Thus, the minimum of F is $(\text{tr } A)^2$. If $\text{tr } A > 1$, no tetrahedron in this family fits, consistently with Proposition 3.1. If $\text{tr } A = 1$ we construct a unique tetrahedron which fits, and fits tightly. If $\text{tr } A < 1$, there exists a closed disk of values of (x, y, z) in the first octant for which the tetrahedron fits; on the boundary of the disk, the tetrahedron fits tightly, consistently with Remark 1.2.

Given E_{out} with $\text{tr } A < 1$, there exist similar tight tetrahedra in other positions; the algebra for such examples is far more complicated. \diamond

4 Constructing simplices

In this section we complete the proof of Theorem 3. More concretely, we construct the map $\phi^{-1} : O(n) \rightarrow \mathcal{S}_f(E_{\text{in}}, E_{\text{out}})$ when $\text{tr}(A) = 1$.

For a tight simplex S , let $(v_i)_{1 \leq i \leq n+1}$ denote its vertices and $(w_i)_{1 \leq i \leq n+1}$ the family of points of tangency of its hyperfaces with E_{in} . Set $\tilde{v}_i = -A^{-1}w_i$ and $\tilde{w}_i = -Av_i$. The simplex \tilde{S} with vertices (\tilde{v}_i) is another tight simplex for the same pair of ellipsoids, as follows from expanding the corresponding algebraic expressions. The points of tangency to the hyperfaces of \tilde{S} are (\tilde{w}_i) . We call \tilde{S} the *dual* of S ; S is *self-dual* if $\tilde{S} = S$. Figure 3 shows an example with $n = 2$: a self-dual triangle $S \in \mathcal{S}_t(E_{\text{in}}, E_{\text{out}})$. We have $\langle v_i, w_j \rangle = 1$ for $i \neq j$, which in turn gives that v_i is equidistant to all points w_j , since $w_j \in E_{\text{in}} = \mathbb{S}^{n-1}$, a simple geometric fact.

Remark 4.1. Recall that in Remark 3.2, S has vertices $(\pm ax, \pm by, \pm cz)$ with an even number of negative signs, where $(x, y, z) \in E_{\text{in}} = \mathbb{S}^2$. In the notation of Proposition 3.1, the tetrahedron S is tight if and only if $t_1 = 1$. In this case, the dual \tilde{S} has vertices

$$\tilde{v}_i = \left(\pm \frac{1}{x}, \pm \frac{1}{y}, \pm \frac{1}{z} \right),$$

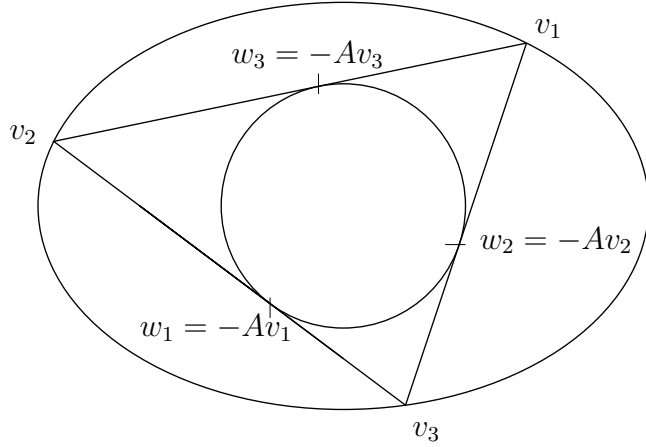


Figure 3: A self-dual triangle

again with an even number of negative signs. Thus, S is self-dual precisely when $\text{tr } A = 1$, consistently with Corollary 4.2 below. \diamond

Corollary 4.2. *If $\text{tr } A = 1$ then any simplex S which fits between E_{in} and E_{out} is self-dual.*

Proof. This is the content of the last paragraph of the proof of Proposition 3.1: if $\text{tr } A = 1$ then $w_i = -Av_i = \tilde{w}_i$ (for all i). \square

Given a tight simplex $S \in \mathcal{S}_t(E_{\text{in}}, E_{\text{out}})$, its vertices v_i must satisfy $\|Av_i\| = 1$. From Corollary 4.2, the hyperfaces of S are tangent to E_{in} at $w_i = -Av_i$, so that $\langle -Av_i, v_j \rangle = 1$, for $i \neq j$. We embed \mathbb{R}^n into \mathbb{R}^{n+1} with an extra final coordinate: given $v \in \mathbb{R}^n$, we *lift* it to obtain $\hat{v} = (v, 1)$. We denote the lifted hyperplane by $\hat{\mathbb{R}}^n \subset \mathbb{R}^{n+1}$. In particular, we have vectors $\hat{v}_i = (v_i, 1) \in \hat{\mathbb{R}}^n$ in the lifted ellipsoid $\hat{E}_{\text{out}} = E_{\text{out}} \times \{1\} \subset \hat{\mathbb{R}}^n$. Set $\hat{A} = A \oplus (+1)$, similar to A_- in the proof of Proposition 3.1, but with $(\hat{A})_{n+1, n+1} = +1$. The symmetric positive definite matrix \hat{A} induces an inner product $\langle\langle \cdot, \cdot \rangle\rangle$ in \mathbb{R}^{n+1} . The lifted vectors \hat{v}_i ($1 \leq i \leq n+1$) form an orthogonal basis: indeed, for $i \neq j$ we have

$$\langle\langle \hat{v}_i, \hat{v}_j \rangle\rangle = \langle Av_i, v_j \rangle + 1 = 0.$$

The following result is a reformulation of parts of Proposition 3.1.

Corollary 4.3. *Assume that $(\hat{v}_i)_{1 \leq i \leq n+1}$ is an orthogonal basis of \mathbb{R}^{n+1} under $\langle\langle \cdot, \cdot \rangle\rangle$. If $\hat{v}_i \in \hat{E}_{\text{out}}$ for all i then $\text{tr } A = 1$. If $\hat{v}_i \in \hat{E}_{\text{out}}$ for all $i \leq n$ and $\hat{v}_{n+1} \in \text{conv}(\hat{E}_{\text{out}}) \setminus \hat{E}_{\text{out}}$ then $\text{tr } A < 1$.*

Proof. As in the proof of Proposition 3.1, write $\hat{v}_i = (v_i, 1) \in \hat{E}_{\text{out}}$, $v_i \in E_{\text{out}}$, $w_i = -Av_i \in E_{\text{in}}$ and $\hat{w}_i = (w_i, 1)$. Take $A_- = A \oplus (-1)$ so that $A_- \hat{v}_i = -\hat{w}_i$.

If $v_i \in E_{\text{out}}$, we have $\langle v_i, A^2 v_i \rangle = 1$ and therefore $\langle\langle \hat{v}_i, A_- \hat{v}_i \rangle\rangle = -\langle\langle \hat{v}_i, \hat{w}_i \rangle\rangle = 0$. Similarly, if $v_i \in \text{conv}(E_{\text{out}}) \setminus E_{\text{out}}$ then $\langle\langle \hat{v}_i, A_- \hat{v}_i \rangle\rangle < 0$. Thus, in the first scenario, if A_- is written in the basis $(\hat{v}_i)_{1 \leq i \leq n+1}$ of \mathbb{R}^{n+1} then its diagonal entries are equal to 0 and therefore $\text{tr}(A_-) = \text{tr}(A) - 1 = 0$. In the second scenario, all diagonal entries are nonpositive and at least one of them is negative and therefore $\text{tr}(A_-) = \text{tr}(A) - 1 < 0$. \square

Lemma 4.4. *Let $(v_i)_{i \leq n}$ be a basis of \mathbb{R}^n consisting of vectors $v_i \in E_{\text{out}}$. Assume furthermore that the vectors $\hat{v}_i = (v_i, 1) \in \mathbb{R}^{n+1}$ are orthogonal with respect to $\langle\langle \cdot, \cdot \rangle\rangle$. Then there exists a unique vector $v_\star \in \mathbb{R}^n$ such that, for all $i \leq n$, $\langle\langle \hat{v}_\star, \hat{v}_i \rangle\rangle = 0$ (where $\hat{v}_\star = (v_\star, 1)$). Furthermore, $v_\star \neq 0$.*

Proof. The family $(\hat{v}_i)_{i \leq n}$ is a basis of a subspace $X \subset \mathbb{R}^{n+1}$ of codimension 1. Thus, the subspace $X^\perp \subset \mathbb{R}^{n+1}$, the orthogonal complement of X under $\langle\langle \cdot, \cdot \rangle\rangle$, is a line. Let \tilde{v} be a generator of X^\perp , and write $\tilde{v} = (v_\star, c)$ where $v_\star \in \mathbb{R}^n$ and $c \in \mathbb{R}$. If $c = 0$ we have $0 = \langle\langle \tilde{v}, \hat{v}_i \rangle\rangle = \langle A\tilde{v}, v_i \rangle$ for all $i \leq n$ and therefore $A\tilde{v} = 0$, a contradiction. If $\tilde{v} = (0, c)$ then $0 = \langle\langle (0, c), \hat{v}_i \rangle\rangle = c$, a contradiction. We may therefore assume without loss of generality that $\tilde{v} = (v_\star, 1)$ with $v_\star \neq 0$, completing the proof. \square

The following result is a kind of converse of Corollary 4.3.

Lemma 4.5. *Let $(v_i)_{i \leq n}$ be a basis of \mathbb{R}^n and $v_\star \in \mathbb{R}^n$ as in Lemma 4.4. Then $\text{tr} A = |Av_\star|^2$. In particular, if $\text{tr}(A) < 1$ then $v_\star \in \text{conv}(E_{\text{out}}) \setminus E_{\text{out}}$; if $\text{tr}(A) = 1$ then $v_\star \in E_{\text{out}}$; if $\text{tr}(A) > 1$ then $v_\star \notin E_{\text{out}}$.*

Proof. Set $v_{n+1} = v_\star$. For $i \leq n+1$, set $\hat{v}_i = (v_i, 1)$ so that $(\hat{v}_i)_{i \leq n+1}$ is an orthogonal basis. For $i \leq n$, set $w_i = -Av_i \in E_{\text{in}}$ and $t_i = 1$. Set $t_{n+1} = 1/|Av_{n+1}|$ and $w_{n+1} = -t_{n+1}Av_{n+1}$ so that $w_{n+1} \in E_{\text{in}}$ and $t_{n+1} > 0$. For $i \leq n+1$, define the hyperplane

$$H_i = \{q \in \mathbb{R}^n \mid \langle q, w_i \rangle = t_i\}.$$

From orthogonality, $i \neq j$ implies $v_i \in H_j$. Let $A_- = A \oplus (-1)$ be as in the proof of Proposition 3.1. Equation (3) still holds (with the same proof) and gives us

$$\text{tr} A - 1 = \sum_i \frac{\langle w_i, Av_i \rangle + t_i}{(-t_i)}.$$

For $i \leq n$ we have $\langle w_i, Av_i \rangle + t_i = -1 + 1 = 0$. For $i = n+1$ we have $\langle w_i, Av_i \rangle + t_i = t_{n+1}(1 - \langle Av_{n+1}, Av_{n+1} \rangle)$. We therefore have $\text{tr} A - 1 = \langle Av_{n+1}, Av_{n+1} \rangle - 1$, or, equivalently, $\text{tr} A = |Av_\star|^2$. \square

The heart of the proof of Theorem 3 is the construction of $\phi^{-1} : O(n) \rightarrow \mathcal{S}_f(E_{\text{in}}, E_{\text{out}})$ by a process similar to Gram-Schmidt which we call *adjusted orthogonalization*. We first describe this procedure, leaving the verification of certain technical aspects for Lemmas 4.6, 4.8 and 4.9.

Given an orthogonal matrix $Q \in O(n)$ with columns u_i , we obtain v_i (for $i \leq n$) by performing *adjusted orthogonalization* on the lifted vectors $\hat{u}_i = (u_i, 1)$. The procedure is illustrated in Figure 4 and is described below. In a nutshell, we start with a family (u_i) of vectors in $E_{\text{in}} \subset \mathbb{R}^n$, obtain (\hat{u}_i) with $\hat{u}_i = (u_i, 1) \in \mathbb{R}^{n+1}$, apply the procedure to define (\hat{v}_i) and finally project back, yielding the family (v_i) with $\hat{v}_i = (v_i, 1)$ and $v_i \in E_{\text{out}} \subset \mathbb{R}^n$. There will be a final extra step to obtain $v_{n+1} \in \mathbb{R}^n$, in a similar but different manner.

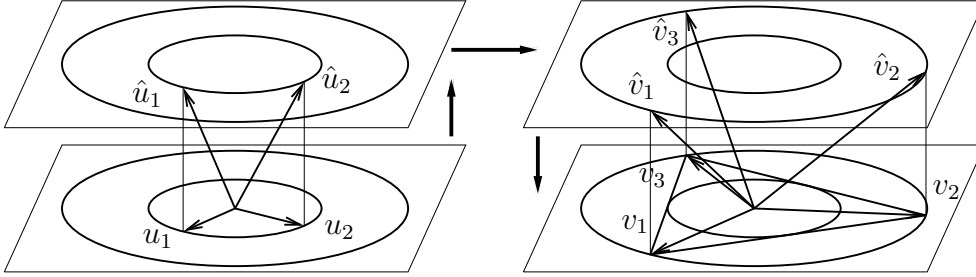


Figure 4: Adjusted orthogonalization of a basis (u_i) obtains vectors (v_i) . In this example, $\text{tr } A = 1$, $v_{n+1} \in E_{\text{out}}$ and the simplex S is tight.

By construction, the family $(\hat{v}_j)_{1 \leq j \leq n}$ obtained by adjusted orthogonalization from (\hat{u}_j) is orthogonal with respect to $\langle\langle \cdot, \cdot \rangle\rangle$ and consists of vectors $\hat{v}_j \in \hat{E}_{\text{out}}$. For $\hat{v}_j = (v_j, 1)$, the basis (v_j) induces the same flag as (u_j) : for any $i \leq n$, the two families $(u_j)_{j \leq i}$ and $(v_j)_{j \leq i}$ span the same subspace of dimension i of \mathbb{R}^n .

Set $v_1 = u_1 / \|Au_1\| \in E_{\text{out}}$ and $\hat{v}_1 = (v_1, 1) \in \hat{E}_{\text{out}}$. Given $1 < j < n$ and $\hat{v}_1, \dots, \hat{v}_{j-1} \in \hat{E}_{\text{out}}$, we show how to obtain $v_j \in E_{\text{out}} \subset \mathbb{R}^n$ and $\hat{v}_j = (v_j, 1) \in \hat{E}_{\text{out}}$. As in the usual Gram-Schmidt process, we want v_j to belong to the space spanned by u_1, \dots, u_j or, equivalently, by v_1, \dots, v_{j-1}, u_j . We also want the orthogonality requirement $\langle\langle \hat{v}_j, \hat{v}_k \rangle\rangle = 0$ (for $k < j$): this translates into v_j belonging to an affine subspace of codimension $j - 1$. Thus, if we are in general position (as we shall prove to be), the two conditions yield that $\hat{v}_j = (v_j, 1)$ belongs to a line in $\hat{\mathbb{R}}^{n+1}$. As we shall prove in Lemma 4.9, this line intersects \hat{E}_{out} in two distinct points. The vector \hat{v}_j is chosen to be the intersection point for which $v_j = pu_j - \sum_{k < j} c_k v_k$ with $p > 0$. For reference, we itemize the crucial properties of \hat{v}_j :

- (0) $\hat{v}_j = (v_j, 1)$,
- (1) $v_j = pu_j - \sum_{k < j} c_k v_k$, $p > 0$,
- (2) $\langle\langle \hat{v}_j, \hat{v}_k \rangle\rangle = 0$, $k < j$,
- (3) $\hat{v}_j \in \hat{E}_{\text{out}}$, or, equivalently, $\|Av_j\| = 1$.

After completing the procedure for $j \leq n$, there is an extra step $j = n + 1$ to obtain the vectors v_{n+1} and \hat{v}_{n+1} . Notice first that the subspace spanned by

v_1, \dots, v_n is \mathbb{R}^n , thus imposing no restriction on v_{n+1} . Assuming general position, the orthogonality requirements $\langle\langle \hat{v}_j, \hat{v}_k \rangle\rangle = 0$ (for $k \leq n$) yield a single point \hat{v}_{n+1} in the hyperplane $\hat{\mathbb{R}}^n \subset \mathbb{R}^{n+1}$. As we shall see, if $\text{tr } A = 1$ then $\hat{v}_{n+1} \in \hat{E}_{\text{out}}$. On the other hand, if $\text{tr } A < 1$ then \hat{v}_{n+1} is in the interior of \hat{E}_{out} .

If $\text{tr } A = 1$, the construction obtains the vertex family $(v_j)_{j \leq n+1}$ of a tight simplex S associated with the orthogonal matrix Q with columns $(u_j)_{j \leq n}$. The correspondence provides the inverse of $\phi : \mathcal{S}_t(E_{\text{in}}, E_{\text{out}}) \rightarrow O(n)$, as the flags associated with the two bases $(u_j)_{j \leq n}$ and $(v_j)_{j \leq n}$ coincide.

For $\text{tr } A < 1$, the same construction gives rise to a fitting simplex S for which all hyperfaces are tangent to E_{in} but the last vertex v_{n+1} is strictly inside E_{out} : the simplex is not tight. This however suffices to derive the surjectivity of the map $\phi : \mathcal{S}_t(E_{\text{in}}, E_{\text{out}}) \rightarrow O(n)$.

We now provide details. First, we describe adjusted orthogonalization in terms of projected vectors. Clearly, a linearly independent family $(u_j)_{j \leq i}$, $u_j \in \mathbb{R}^n$, lifts to a linearly independent family $(\hat{u}_j)_{j \leq i}$.

For a subspace $W \subset \mathbb{R}^n$, consider the ellipsoids $E_{\text{in}} \cap W$ and $E_{\text{out}} \cap W$. As above, there exists a unique real symmetric positive linear transformation $A_W : W \rightarrow W$ with $A_W[E_{\text{out}} \cap W] = E_{\text{in}} \cap W$. If X is a real symmetric matrix, we denote its spectrum by $\sigma(X)$, which is to be interpreted as a finite multiset of real numbers.

Lemma 4.6. *If $W_1 \subsetneq W_2$ then $\text{tr}(A_{W_1}) < \text{tr}(A_{W_2})$.*

Proof. It suffices to consider the case $\dim(W_2) = 1 + \dim(W_1) = k$. In this case, the spectra of A_{W_1} and A_{W_2} are interlaced and therefore $\text{tr}(A_{W_1}) < \text{tr}(A_{W_2})$, as desired. \square

Remark 4.7. The subspace W is not assumed to be invariant under A . Indeed, if the spectrum of A is simple there are only finitely many invariant subspaces. In particular, A_W is not to be confused with the restriction $A|_W : W \rightarrow A[W]$.

On the other hand, A and A_W define the same inner products in W : if $v_1, v_2 \in W$ then $\langle Av_1, v_2 \rangle = \langle A_W v_1, v_2 \rangle$. The same holds for $W \oplus \mathbb{R}$: we have two expressions but the same inner product $\langle\langle \cdot, \cdot \rangle\rangle$. \diamond

Lemma 4.8. *Assume $\text{tr } A \leq 1$ and $i \leq n$. Consider a family $(\hat{v}_j)_{j \leq i}$ of vectors in \hat{E}_{out} . If this family is orthogonal for the inner product $\langle\langle \cdot, \cdot \rangle\rangle$ then it projects to a linearly independent family $(v_j)_{j \leq i}$.*

Proof. Let $W \subseteq \mathbb{R}^n$ be the subspace generated by $(v_j)_{j \leq i}$. If this family is orthogonal for the inner product $\langle\langle \cdot, \cdot \rangle\rangle$ it follows from Remark 4.7 that the family is also orthogonal in $W \oplus \mathbb{R}$ under the inner product $\langle\langle \cdot, \cdot \rangle\rangle_W$ defined by A_W . If the family V is not linearly independent, apply Corollary 4.3 to W , to deduce that $\text{tr}(A_W) = 1$. Apply Lemma 4.6 with $W_1 = W$ and $W_2 = \mathbb{R}^n$ to deduce that $\text{tr}(A) > 1$, a contradiction. \square

Lemma 4.9. *Assume $\text{tr} A \leq 1$ and $i < n$. Let $X_i \subset X_{i+1} \subseteq \mathbb{R}^n$ be subspaces of dimensions i and $i + 1$. Let $(v_j)_{j \leq i}$ be a family of vectors in $E_{\text{out}} \cap X_i$ and $\hat{v}_j = (v_j, 1)$. Assume that $(\hat{v}_j)_{j \leq i}$ is orthogonal with respect to $\langle\langle \cdot, \cdot \rangle\rangle$. Then there exist precisely two vectors $v \in E_{\text{out}} \cap X_{i+1}$ such that $\hat{v} = (v, 1)$ is $\langle\langle \cdot, \cdot \rangle\rangle$ -orthogonal to all \hat{v}_j , $j \leq i$. For either vector v , taking $v_{i+1} = v$ yields a basis of X_{i+1} , and the two bases have opposite orientations.*

Proof. From Lemma 4.8, $(v_j)_{j \leq i}$ is a basis of X_i . Apply Lemma 4.4 to the restriction to X_i to obtain $v_\star \in X_i$. From Lemmas 4.5 and 4.6, v_\star is interior to E_{out} . Thus, the affine subspace of X_{i+1} defined by the i equations $\langle \cdot, Av_i \rangle = -1$ is a line containing v_\star and therefore crossing $E_{\text{out}} \cap X_{i+1}$ transversally in exactly two points. These are our desired points. There is one in each connected component of $X_{i+1} \setminus X_i$, completing the proof. \square

Proof of Theorem 3. Recall that item (i) and the first claim in item (ii) have already been proved in Section 3: we just completed the proof of item (ii) by constructing $\phi^{-1} : O(n) \rightarrow \mathcal{S}_t(E_{\text{in}}, E_{\text{out}})$ when $\text{tr}(A) = 1$.

If $\text{tr}(A) < 1$, consider an auxiliary ellipsoid sE_{out} for $s \leq 1$. The corresponding positive symmetric matrix is $\tilde{A} = s^{-1}A$ so that $\text{tr}(\tilde{A}) = s^{-1}\text{tr}(A) \leq 1$ for $s \in [1/\text{tr}(A), 1]$. For each such value of s , given $Q \in O(n)$ we obtain distinct simplices S_s with $S_s \in \mathcal{S}_f(E_{\text{in}}, E_{\text{out}}) \setminus \mathcal{S}_t(E_{\text{in}}, E_{\text{out}})$ and $\phi(S_s) = Q$, completing the proof of item (iii). \square

5 Icosahedron and dodecahedron

Theorems 2 and 3 comprise higher dimensional generalizations for $k = 3$ and $k = 4$ of Poncelet porism. It is natural to ask whether similar versions of Poncelet porism hold for other values of k . One tempting idea is to look at other regular polytopes, such as dodecahedra and icosahedra (recall that one is the dual of the other). The icosahedron between two ellipsoids does not appear to satisfy a similar porism: we present a failed attempt.

We consider a special situation: take E_{out} to be the unit sphere \mathbb{S}^{n-1} and E_{in} to be the smaller ellipsoid with equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1.$$

A convex polytope is *combinatorially regular* if it has the same combinatorial structure as the regular polytopes. Thus, simplices, parallelotopes and cross polytopes are examples of combinatorially regular polytopes. We consider convex combinatorially regular centrally symmetric icosahedra inscribed in E_{out} and circumscribed to E_{in} . We further simplify the problem by considering icosahedra in two special positions, chosen so as to exploit symmetry.

We first consider icosahedra for which the points $(0, 0, \pm 1)$ are vertices. The other 10 vertices form two regular pentagons on planes parallel to the xy plane. Up to rotation, the z coordinate $z_{\text{penta}} \in (0, 1)$ for the top pentagon is the only degree of freedom in the construction. Up to symmetry, there are two faces which should be tangent to E_{in} ; it can be verified that in an open interval a and b are smooth functions of z_{penta} . Applying the implicit function theorem, for any value of b between, say, 0.6 and 0.9, there appears to exist a unique value of a for which the construction works. Thus, a is a smooth function of b : $a = \alpha_1(b)$.

We next consider icosahedra with a pair of opposite faces parallel to the xy plane, like a solid icosahedron sitting on a table. More precisely, two opposite faces of the icosahedron are equilateral triangles in the planes $z = \pm b$. We may assume that the vertices of the top face are $(s, 0, b)$, $(-s/2, \pm s\sqrt{3}/2, b)$, where $s = \sqrt{1 - b^2}$. The other six vertices form two other equilateral triangles (not faces) contained in the planes $z = \pm c$ (where $c \in (0, 1)$ is another real parameter). We may assume that the six vertices are $\pm(-s', 0, c)$, $\pm(s'/2, \pm s'\sqrt{3}/2, c)$. Apart from the top and bottom faces, there are, up to symmetry, two classes of faces: the 6 faces in the first class share an edge with the top or bottom faces, the 12 faces in the other class have three vertices with three different values of z . At this point we have two degrees of freedom (a and c) and two equations, each equation verifying the tangency between the ellipsoid E_{in} and the faces in one of the two classes. Again by transversality, given b there exists a unique pair (a, c) for which the construction works. In other words, a is a smooth function of b : $a = \alpha_2(b)$. Numerics indicate, however, that the two functions α_1 and α_2 are quite different.

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