# AN OBSERVATION ABOUT CONFORMAL POINTS ON SURFACES 

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#### Abstract

We study the existence of points on a compact oriented surface at which a symmetric bilinear two-tensor field is conformal to a Riemannian metric. We give applications to the existence of conformal points of surface diffeomorphisms and vector fields.


## 1. Statement of results

1.1. Conformal points. Let $\Sigma$ be a compact, oriented surface, possibly with non-empty boundary $\partial \Sigma$. Denote by $C_{1}, \ldots, C_{n}$ the boundary components of $\Sigma$ with the induced orientation. Let $\operatorname{Sym}\left(\left(T^{*} \Sigma\right)^{\otimes 2}\right) \rightarrow \Sigma$ be the bundle of symmetric bilinear tensors on $\Sigma$. Fix a Riemannian metric $g$ on $\Sigma$, that is, a positive-definite section of $\operatorname{Sym}\left(\left(T^{*} \Sigma\right)^{\otimes 2}\right) \rightarrow \Sigma$.
Definition 1.1. We say that a section $h$ of $\operatorname{Sym}\left(\left(T^{*} \Sigma\right)^{\otimes 2}\right) \rightarrow \Sigma$ is conformal to $g$ at the point $z \in \Sigma$ if there exists $c \in \mathbb{R}$ such that $h_{z}=c g_{z}$.

Motivated by [1], the goal of this note is to study the set of points

$$
\mathcal{C}(g, h) \subset \Sigma
$$

at which $h$ is conformal to $g$ (see Theorem (1.2) in order to investigate conformal points of diffeomorphisms $F: \Sigma \rightarrow \Sigma$, in which case $h=F^{*} g$ (see Theorem 1.4 and Corollary 1.5), and of vector fields (see Corollary 1.7).

Our main observation is that $\mathcal{C}(g, h)$ is the zero-set of a section $H^{a}$ in a distinguished vector bundle $E^{a} \rightarrow \Sigma$ over the surface, which we describe now. Let $\operatorname{End}(T \Sigma) \rightarrow \Sigma$ be the bundle of endomorphisms of $T \Sigma$ and let

$$
\begin{equation*}
E^{a} \subset \operatorname{End}(T \Sigma) \tag{1.1}
\end{equation*}
$$

be the subbundle of those endomorphisms which are symmetric with respect to $g$ and have zero trace. For all $z \in \Sigma$, an element of $R \in E_{z}^{a}$ has the matrix expression

$$
\left(\begin{array}{cc}
a & b \\
b & -a
\end{array}\right) \quad a, b \in \mathbb{R}
$$

with respect to a positive, orthonormal basis of $T_{z} \Sigma$. Thus, any non-zero element $R \in E_{z}^{a}$ is, up to a positive scalar multiple, a reflection $R: T_{z} \Sigma \rightarrow T_{z} \Sigma$ along a line in $T_{z} \Sigma$. In particular, the $S^{1}$-bundle associated with $E^{a}$ is the bundle of unoriented lines in $T \Sigma$. This $S^{1}$-bundle is doubly covered by the bundle of oriented lines in $T \Sigma$ which, in turn, is the unit-tangent bundle of $\Sigma$, that is, the $S^{1}$-bundle associated with $T \Sigma \rightarrow \Sigma$. The above discussion shows that $E^{a}$ is an oriented plane bundle over $\Sigma$ with Euler number

$$
\begin{equation*}
e\left(E^{a}\right)=2 e(T \Sigma)=2 \chi(\Sigma) \tag{1.2}
\end{equation*}
$$

Given a symmetric bilinear two-tensor field $h$ over $\Sigma$, let $H$ be the section of $\operatorname{End}(T \Sigma)$ representing $h$ with respect to $g$, namely

$$
\begin{equation*}
g_{z}\left(u, H_{z} v\right)=h_{z}(u, v), \quad \forall z \in \Sigma, \forall u, v \in T_{z} \Sigma \tag{1.3}
\end{equation*}
$$

We denote by

$$
\begin{equation*}
H^{a}:=H-\frac{\operatorname{tr} H}{2} I \tag{1.4}
\end{equation*}
$$

Date: September 8, 2023.
2020 Mathematics Subject Classification. 53C18 (Primary) 57R22 (Secondary).
Key words and phrases. Conformal points, Poincaré-Hopf, line fields.
the section of $E^{a}$ corresponding to the trace-free part of $H$. Here $I$ is the section of $\operatorname{End}(T \Sigma)$ such that $I_{z}$ is the identity of $T_{z} \Sigma$ for all $z \in \Sigma$.

Thus, we conclude that

$$
z \in \mathcal{C}(g, h) \quad \Longleftrightarrow \quad H_{z}^{a}=0
$$

From this relationship we see that, generically, $h$ has only finitely many conformal points and all of them lie in the interior of $\Sigma$. In this case, we can use the Poincaré-Hopf Theorem for unoriented line fields on oriented surfaces with boundary to algebraically count conformal points. To give the precise statement, let us introduce some notation under the assumption that $\mathcal{C}(g, h)$ is finite and $\mathcal{C}(g, h) \subset \Sigma \backslash \partial \Sigma$. For each $z \in \mathcal{C}(g, h)$, we define

$$
\operatorname{ind}_{(g, h)}(z) \in \mathbb{Z}
$$

as the index of $z$ seen as a zero of the section $H^{a}$ of $E^{a} \rightarrow \Sigma$. We count the elements in $\mathcal{C}(g, h)$ algebraically via the integer

$$
\begin{equation*}
[\mathcal{C}(g, h)]:=\sum_{z \in \mathcal{C}(g, h)} \operatorname{ind}_{(g, h)}(z) \in \mathbb{Z} \tag{1.5}
\end{equation*}
$$

Moreover, for every boundary component $C_{i}$ of $\Sigma$, with $i=1, \ldots, n$, we define

$$
\begin{equation*}
w_{i}(g, h) \in \mathbb{Z} \tag{1.6}
\end{equation*}
$$

as the winding number of the section $\left.H^{a}\right|_{C_{i}}$ with respect to $R^{i} \in E^{a}$, where $R^{i}(z)$ is the reflection along the line $T_{z} \partial \Sigma \subset T_{z} \Sigma$ for $z \in C_{i}$.
Theorem 1.2. Let $g$ be a Riemannian metric on a compact, oriented surface $\Sigma$. Then the following two statements hold.
(1) For any symmetric bilinear two-tensor field $h$ over $\Sigma$ such that $\mathcal{C}(g, h)$ is finite and $\mathcal{C}(g, h) \subset \Sigma \backslash \partial \Sigma$, the equality

$$
\begin{equation*}
[\mathcal{C}(g, h)]=2 \chi(\Sigma)+\sum_{i=1}^{n} w_{i}(g, h) \tag{1.7}
\end{equation*}
$$

holds, where $\chi(\Sigma)$ denotes the Euler characteristic of $\Sigma$.
(2) Let $\mathcal{C} \subset \Sigma \backslash \partial \Sigma$ be a finite set of points, $\iota: \mathcal{C} \rightarrow \mathbb{Z}$ an arbitrary function, and $w_{1}, \ldots, w_{n} \in \mathbb{Z}$ arbitrary integers satisfying

$$
\begin{equation*}
\sum_{z \in \mathcal{C}} \iota(z)=2 \chi(\Sigma)+\sum_{i=1}^{n} w_{i} . \tag{1.8}
\end{equation*}
$$

Then there exists a symmetric bilinear two-tensor field $h$ over $\Sigma$ such that $\mathcal{C}=\mathcal{C}(g, h)$, $\iota(z)=\operatorname{ind}_{(g, h)}(z)$ for all $z \in \mathcal{C}$ and $w_{i}=w_{i}(g, h)$ for all $i=1, \ldots, n$.
Remark 1.3. For the convenience of the reader, we give the short proof of Theorem 1.2 in Section 22 although this can be deduced from the literature. For statement (1), we refer to [10], [12] and [5] which deal with the Poincaré-Hopf Theorem for oriented line fields on surfaces with boundary and to [7] III.2.2], [9], 8] and [3] which deal with the Poincaré-Hopf theorem for unoriented line fields on surfaces without boundary. For statement (2), we refer to the Extension Theorem in [6, p. 145]. Finally, we notice that, passing to the orientation double cover, Theorem 1.2 also holds for non-orientable surfaces.

We discuss now two situations where the set $\mathcal{C}(g, h)$ naturally appears.
1.2. Carathéodory's conjecture. First, let us consider a smooth embedding $\rho: S^{2} \rightarrow \mathbb{R}^{3}$. Here $\Sigma=S^{2}$ and we take $g^{\rho}$ and $h^{\rho}$ to be the first and the second fundamental form of the embedding $\rho$, respectively, with respect to the ambient Euclidean metric. The elements of $\mathcal{C}\left(g^{\rho}, h^{\rho}\right)$ are the so-called umbilical points, namely points at which the two principal curvatures of the embedding coincide. In this case, (1.7) yields the well-known result that $\left[\mathcal{C}\left(g^{\rho}, h^{\rho}\right)\right]=4$, namely that the algebraic count of umbilical points is equal to four. For example, when $\rho$ is an ellipsoid of revolution, $\mathcal{C}\left(g^{\rho}, h^{\rho}\right)$ consists exactly of the two poles, both having index two. In general, it is natural to ask which further conditions must the points $z \in \mathcal{C}\left(g^{\rho}, h^{\rho}\right)$ and their indices satisfy
besides $\left[\mathcal{C}\left(g^{\rho}, h^{\rho}\right)\right]=4$. For instance, Carathéodory's conjecture [4 13] asserts that convexity of the embedding $\rho$ implies $\operatorname{ind}(z) \leq 2$ for all $z \in \mathcal{C}\left(g^{\rho}, h^{\rho}\right)$, and, in particular, entails that $\mathcal{C}\left(g^{\rho}, h^{\rho}\right)$ always contains at least two points.
1.3. Conformal points of a diffeomorphism. The second situation in which $\mathcal{C}(g, h)$ naturally appears is when $h=F^{*} g$, where $F: \Sigma \rightarrow \Sigma$ is any orientation-preserving diffeomorphism of $\Sigma$. In this case, $\mathcal{C}\left(g, F^{*} g\right)$ is the set of so-called conformal points of $F$ (with respect to $g$ ). Assuming that $\mathcal{C}\left(g, F^{*} g\right)$ is finite and $\mathcal{C}\left(g, F^{*} g\right) \subset \Sigma \backslash \partial \Sigma$, we are going to give a formula for $w_{i}\left(g, F^{*} g\right)$ in terms of the behavior of $F$ at the boundary. In order to state the result, for $i=1, \ldots, n$ let $\nu_{i}: C_{i} \rightarrow T \Sigma$ be the outward normal at the boundary component $C_{i}$ and $\tau_{i}: C_{i} \rightarrow T \Sigma$ be the unit vector tangent to $C_{i}$ in the positive direction. The pair ( $\nu_{i}, \tau_{i}$ ) then forms a positive orthonormal frame for $g$ along $C_{i}$. We trivialize $\left.T \Sigma\right|_{\partial \Sigma}=\sqcup_{i} C_{i} \times \mathbb{R}^{2}$ using $\left(\nu_{i}, \tau_{i}\right)$ at $C_{i}, i=1, \ldots, n$. Since $F$ maps boundary components to boundary components (not necessarily the same) we can express $\mathrm{d} F$ in this trivialization as

$$
\left.\mathrm{d} F\right|_{C_{i}}=: N_{i}=c_{i}\left(\begin{array}{ll}
a_{i} & 0  \tag{1.9}\\
b_{i} & 1
\end{array}\right) .
$$

Here $a_{i}, c_{i}: C_{i} \rightarrow(0, \infty), b_{i}: C_{i} \rightarrow \mathbb{R}$ and $\left(a_{i}, b_{i}\right)$ is never equal to ( 1,0 ) since $F$ has no conformal point on $C_{i}$ by assumption.

Theorem 1.4. For all $i=1, \ldots, n$ we have the equality

$$
\begin{equation*}
w_{i}\left(g, F^{*} g\right)=w\left(a_{i}-1, b_{i}\right), \tag{1.10}
\end{equation*}
$$

where $w\left(a_{i}-1, b_{i}\right)$ is the winding number of the curve $\left(a_{i}-1, b_{i}\right): C_{i} \cong S^{1} \rightarrow \mathbb{R}^{2} \backslash\{0\}$ around the origin.

This formula, which will be proved in Section 3 allows us to compute $w_{i}\left(g, F^{*} g\right)$ if we understand the behavior of $F$ at points on the boundary sufficiently well. A remarkable example of this phenomenon is illustrated by the next corollary.

Corollary 1.5. If $F: \Sigma \rightarrow \Sigma$ is the identity on the boundary and preserves an area form on $\Sigma$, then

$$
\begin{equation*}
w_{i}\left(g, F^{*} g\right)=0, \quad \forall i=1, \ldots, n . \tag{1.11}
\end{equation*}
$$

It follows that for this type of diffeomorphisms

$$
\begin{equation*}
[\mathcal{C}(F)]=2 \chi(\Sigma), \tag{1.12}
\end{equation*}
$$

that is, the number of conformal points of such an $F$ is twice the Euler characteristic.
Proof. By (1.10) the assertion is equivalent to showing $w_{i}\left(a_{i}-1, b_{i}\right)=0$. Since $F$ is the identity at the boundary we conclude that $\mathrm{d} F \cdot \tau_{i}=\tau_{i}$ and thus $c_{i}=1$ in (1.9). Since $F$ preserves an area form, it follows that $\operatorname{det} N_{i}=1$, which implies that $a_{i}=1$ in (1.9). Therefore, the curve $\left(a_{i}-1, b_{i}\right)=\left(0, b_{i}\right)$ is contained in the $y$-axis and does not cross 0 . We conclude that its winding number around the origin $w\left(a_{i}-1, b_{i}\right)$ vanishes.

Remark 1.6. Equation (1.12) was proved in [1], when $\Sigma=D^{2}$, and $F$ satisfies some additional conditions, which hold, for instance, when $F$ is $C^{1}$-close to the identity,

If we linearize the property of being a conformal point for a diffeomorphism at the identity of $\Sigma$, we get a corresponding condition for conformal points of vector fields on $\Sigma$. This condition is easier phrased after reinterpreting conformality in terms of complex geometry, as we explain next.
1.4. Conformal points and complex structures. Let $\jmath$ be the complex structure associated with the Riemannian metric $g$ and the orientation of $\Sigma$. In other words, $\jmath$ yields a section of $\operatorname{End}(T \Sigma)$ such that $v$ and $J_{z} v$ form a positive, orthogonal basis of $T_{z} \Sigma$ for all $z \in \Sigma$ and all $v \in T_{z} \Sigma \backslash\{0\}$. Thus, $\jmath_{z}$ has the matrix expression

$$
\left(\begin{array}{cc}
0 & -1  \tag{1.13}\\
1 & 0
\end{array}\right)
$$

with respect to a positive, orthonormal basis of $T_{z} \Sigma$. An endomorphism $H: T_{z} \Sigma \rightarrow T_{z} \Sigma$ commutes with $J_{z}$ if and only if $H$ has the matrix expression

$$
\left(\begin{array}{cc}
a & -b \\
b & a
\end{array}\right) \quad a, b \in \mathbb{R}
$$

in such a basis. In particular, we deduce that $H$ is, up to a scalar multiple, a rotation matrix. Analogously, $H$ anticommutes with $\jmath_{z}$ if and only if $H$ has the matrix expression

$$
\left(\begin{array}{cc}
a & b \\
b & -a
\end{array}\right) \quad a, b \in \mathbb{R}
$$

in such a basis. In particular, we deduce that $E^{a}$, see (1.1), is exactly the bundle of endomorphisms anticommuting with $\jmath$. Therefore, if we denote by $E^{c} \rightarrow \Sigma$ the bundle of endomorphisms commuting with $\jmath$, we get the splitting

$$
\begin{equation*}
\operatorname{End}(T \Sigma)=E^{c} \oplus E^{a} \tag{1.14}
\end{equation*}
$$

Furthermore, as $\jmath$-complex line bundle we can write

$$
E^{a} \cong T \otimes \overline{T^{*}}, \quad T:=T^{(1,0)} \Sigma
$$

where $T^{(1,0)} \Sigma$ is the holomorphic tangent bundle of $\Sigma$ and $\overline{T^{*}}$ denotes the conjugate of the dual bundle of $T$. With this identification, a local section of $E^{a}$ is given by $\frac{\partial}{\partial z} \otimes \mathrm{~d} \bar{z}$ where $z$ is a local holomorphic coordinate compatible with $\jmath$. Thus, the Euler number of $E^{a}$ as real oriented plane bundle coincides with its Chern number as complex line bundle. Using that $c_{1}(T)=\chi(\Sigma)$, this gives another derivation of (1.2) by computing

$$
c_{1}\left(E^{a}\right)=c_{1}\left(T \otimes \overline{T^{*}}\right)=c_{1}(T)-c_{1}\left(T^{*}\right)=c_{1}(T)+c_{1}(T)=2 c_{1}(T)=2 \chi(\Sigma)
$$

Finally, let us assume that $z$ is a conformal point of an orientation-preserving diffeomorphism $F: \Sigma \rightarrow \Sigma$. Then,

$$
\begin{equation*}
\left(F^{*} g\right)_{z}=c g_{z} \text { for some } c>0 \tag{1.15}
\end{equation*}
$$

If we denote by $M$ the matrix representation of $\mathrm{d}_{z} F$ with respect to positive, orthonormal bases of $T_{z} \Sigma$ and $T_{F(z)} \Sigma$, then (1.15) can be rewritten as

$$
M^{T} M=c I
$$

This condition is equivalent to saying that $M$ is, up to a scalar multiple, a rotation matrix. Since $\jmath_{z}$ and $\jmath_{F(z)}$ are represented by the matrix (1.13), we see that $\mathrm{d}_{z} F \jmath_{z}=\jmath_{F(z)} \mathrm{d}_{z} F$. We conclude that $z$ is a conformal point of $F$ if and only if $F$ is $\jmath$-holomorphic at $z$ with respect to $\jmath$-holomorphic coordinates around $z$ and $F(z)$.
1.5. Conformal points of vector fields. Let $f$ be a vector field on $\Sigma$. Let $F_{t}: \Sigma \rightarrow \Sigma$ be the time- $t$ map of the flow of $f$. Suppose that $z \in \Sigma$ is a point such that $F_{t}(z) \in \mathcal{C}\left(g,\left(F_{t}\right)^{*} g\right)$ for all $t$ close to zero. In particular, $F_{t}$ is $\jmath$-holomorphic at $z$ in a local $\jmath$-holomorphic chart for all small $t$. Taking the derivative in $t$ at $t=0$, we conclude that the vector field $f$ is $\jmath$-holomorphic at $z$. In other words $\bar{\partial}_{j} f$ is a section of $E^{a}$ which vanishes at $z$. Here, $\bar{\partial}_{j}$ denotes the Cauchy-Riemann operator sending sections of the holomorphic tangent bundle $T=T^{(1,0)} \Sigma$ to sections of $T \otimes \overline{T^{*}} \cong E^{a}$, and $f$ is identified with its image under the isomorphism

$$
T \Sigma \rightarrow T^{\mathbb{C}} \Sigma \cong T^{(1,0)} \Sigma \oplus T^{(0,1)} \Sigma \rightarrow T^{(1,0)} \Sigma
$$

where $T^{\mathbb{C}} \Sigma$ is the complexification of $T \Sigma$.
Let $\mathcal{C}(\jmath, f)$ be the set of zeros of $\bar{\partial}_{\jmath} f$. If $\mathcal{C}(\jmath, f)$ is finite and $\mathcal{C}(\jmath, f) \subset \Sigma \backslash \partial \Sigma$, then we can associate an index $\operatorname{ind}_{(\jmath, f)}(z)$ to each $z \in \mathcal{C}(\jmath, f)$ and a winding number $w_{i}(\jmath, f)$ representing the relative winding number of $\bar{\partial}_{j} f$ with respect to the canonical section $R_{i}$ along $C_{i}$ for every $i=1, \ldots, n$. Defining the algebraic count

$$
[\mathcal{C}(\jmath, f)]:=\sum_{z \in \mathcal{C}(\jmath, f)} \operatorname{ind}_{(\jmath, f)}(z),
$$

we get the following consequence of Theorem 1.2(1).

Corollary 1.7. Let $\jmath$ be a complex structure on a compact surface $\Sigma$ and $f$ a vector field on $\Sigma$ such that $\mathcal{C}(\jmath, f)$ is finite and $\mathcal{C}(\jmath, f) \subset \Sigma \backslash \partial \Sigma$. Then the equation

$$
[\mathcal{C}(\jmath, f)]=2 \chi(\Sigma)+\sum_{i=1}^{n} w_{i}(\jmath, f)
$$

holds.
1.6. An open question. Given any Riemannian metric $g$ on $\Sigma$ and diffeomorphism $F: \Sigma \rightarrow \Sigma$, it is interesting to ask which further restrictions must the points $z$ of $\mathcal{C}\left(g, F^{*} g\right)$, their indices $\operatorname{ind}_{\left(g, F^{*} g\right)}(z)$ and the numbers $w_{i}\left(g, F^{*} g\right)$ satisfy besides equation (1.7). This question is related to the uniformization theorem for compact surfaces with boundary via Theorem 1.2, (2). For instance, given any two metrics $g$ and $h$ on $\Sigma=S^{2}$ or $\Sigma=D^{2}$, we can find a diffeomorphism $F: \Sigma \rightarrow \Sigma$ such that $F^{*} g$ and $h$ are conformal at every point [11, Theorem 1]. Thus, $\mathcal{C}\left(g, F^{*} g\right)=\mathcal{C}(g, h)$, $\operatorname{ind}_{\left(g, F^{*} g\right)}(z)=\operatorname{ind}_{(g, h)}(z)$ for every $z$ in this set, and $w_{i}\left(g, F^{*} g\right)=w_{i}(g, h)$ for all $i=1, \ldots, n$. As a consequence of Theorem 1.2 , (2), there are no further restrictions in this case.

On the other hand, on a general surface $\Sigma$ there are metrics $g$ and $h$ such that $h$ and $F^{*} g$ are not conformal at all points, no matter how we choose the diffeomorphism $F$. The easiest examples where this happens is when $\Sigma=\mathbb{T}^{2}$, or when $\Sigma=D^{2}$ and we require in addition the diffeomorphism $F$ to be the identity at the boundary. For instance, on $\mathbb{T}^{2}$ conformal classes of metrics $g$ are classified by lattices $\Gamma$ in $\mathbb{C}$, up to Euclidean isometries and homotheties, where $g$ is the Riemannian metric on $\mathbb{T}^{2}=\mathbb{C} / \Gamma$ induced by the Euclidean metric on $\mathbb{C}$. To get an example on the disc, let us identify $D^{2}$ with the unit Euclidean disc in $\mathbb{C}$. Let $g$ be the Euclidean metric on $D^{2}$. Recall that the group of diffeomorphisms $\varphi: D^{2} \rightarrow D^{2}$ such that $g$ and $\varphi^{*} g$ are conformal at all points consists of the Möbius transformations preserving $D^{2}$. Consider $G: D^{2} \rightarrow D^{2}$ to be any diffeomorphism such that $\left.G\right|_{\partial D^{2}} \neq\left.\varphi\right|_{\partial D^{2}}$ for all $\varphi$. Such a $G$ surely exists since if $\varphi$ is not the identity, then $\varphi$ can have at most two fixed points on the boundary. If we define $h:=G^{*} g$, then there is no diffeomorphism $F: D^{2} \rightarrow D^{2}$ which is identity at the boundary and such that $F^{*} h$ and $g$ are conformal at every point. Indeed, if such an $F$ exists, then $(G \circ F)^{*} g=F^{*} G^{*} g=F^{*} h$ is conformal to $g$ at all points, which means that $F \circ G=\varphi$ for some Möbius transformation $\varphi$ preserving the disc. Since $F$ is the identity at the boundary, this would imply that $G=\varphi$ on the boundary. A contradiction.

Thus, in the case of $\mathbb{T}^{2}$ and of $D^{2}$, it is meaningful to ask if there is a metric $g$ and a diffeomorphism $F$ (being the identity on the boundary in the case of $D^{2}$ ) such that $\mathcal{C}\left(g, F^{*} g\right)$ is empty. If one can find a vector field $f$ (vanishing on the boundary in the case of $D^{2}$ ) such that $\mathcal{C}(\jmath, f)=\varnothing$, then $\mathcal{C}\left(g, F_{t}^{*} g\right)=\varnothing$ for small $t \neq 0$, as well, where $F_{t}$ is the time- $t$ map of the flow of $f$.

In the case of $\Sigma=\mathbb{T}^{2}$, we can readily find such a vector field for all conformal classes of complex structures. Indeed, let $\mathbb{T}^{2}=\mathbb{C} / \Gamma$ where $\Gamma$ is a lattice in $\mathbb{C}$ and let $\jmath$ be the complex structure on $\Sigma$ induced by that on $\mathbb{C}$. Up to Euclidean isometries and homotheties, we can assume that $\Gamma$ is generated by $1, \tau \in \mathbb{C}$, where $\tau=a+i b$ with $b>0$. Consider the vector field which in a global holomorphic trivialization of $T^{(1,0)} \Sigma$ is written as $f(z)=e^{\frac{2 \pi i}{b} \operatorname{Im} z}$. Notice that $f$ is well-defined since it is invariant under translations by 1 and $\tau$. Moreover,

$$
\bar{\partial}_{\jmath} f(z)=\frac{\partial}{\partial \bar{z}} e^{\frac{\pi}{b}(z-\bar{z})}=-\frac{\pi}{b} f(z)
$$

which is nowhere vanishing.
However, we do not know if such a vector field $f$ exists on $D^{2}$. Since vector fields on $D^{2}$ correspond to functions in a global trivialization of $T^{(1,0)} D^{2}$, we have the following open question.

Question 1.8. Does there exist a smooth function $f: D^{2} \rightarrow \mathbb{C}$ satisfying the following two conditions?
(1) $\forall z \in D^{2}, \quad \frac{\partial f}{\partial \bar{z}}(z) \neq 0$.
(2) $\forall z \in \partial D^{2}, \quad f(z)=0$.
1.7. Plan of the paper. Theorem 1.2 is proven in Section 2. Theorem 1.4 is proven in Section 3.
1.8. Acknowledgments. P.A. and G.B. are partially supported by the Deutsche Forschungsgemeinschaft under Germany's Excellence Strategy EXC2181/1-390900948 (the Heidelberg STRUCTURES Excellence Cluster), the Collaborative Research Center SFB/TRR 191-281071066 (Symplectic Structures in Geometry, Algebra and Dynamics), and the Research Training Group RTG 2229-281869850 (Asymptotic Invariants and Limits of Groups and Spaces). G.B. warmly thanks Thomas Rot for stimulating discussions around the topics of this paper.

## 2. Proof of Theorem 1.2

We prove Theorem 1.2(1). Let $h$ be a symmetric bilinear two-tensor field over $\Sigma$ such that $\mathcal{C}(g, h)$ is finite and $\mathcal{C}(g, h) \subset \Sigma \backslash \partial \Sigma$. Recall the definition of $H$ and $H^{a}$ from (1.3) and (1.4).

If $\Sigma$ has no boundary, then $[\mathcal{C}(g, h)]=e\left(E^{a}\right)=2 \chi(\Sigma)$ by the Poincaré-Hopf Theorem for oriented plane bundles [2, Theorem 11.17]. If $\Sigma$ has boundary, let $\hat{\Sigma}$ be the closed, oriented surface that we obtain from $\Sigma$ by gluing a disc $D_{1}, \ldots, D_{n}$ along each boundary component $C_{1}, \ldots, C_{n}$. The gluing maps $D^{2} \rightarrow D_{i}$ have the Euclidean disc

$$
D^{2}=\left\{(x, y) \in \mathbb{R}^{2} \mid x^{2}+y^{2} \leq 1\right\}
$$

as domain and send the boundary $\partial D^{2}$ traversed in the positive sense to $\bar{C}_{i}$, that is, to $C_{i}$ traversed in the negative sense. In this way, the gluing maps are positively oriented with respect to the orientation on $\hat{\Sigma}$.

We let $\hat{g}$ be any extension of $g$ to $\Sigma$ as a Riemannian metric. On the bundle $\left.E^{a}\right|_{D_{i}}$ we choose a nowhere vanishing section $M^{i}$ defined as the reflection along the direction of $\partial_{x} \in T D^{2}$. Let $w_{\bar{C}_{i}}\left(H^{a}, M^{i}\right)$ be the winding number of $H^{a}$ with respect to $M^{i}$ along $C_{i}$ traversed in the negative direction. Then

$$
w_{\bar{C}_{i}}\left(H^{a}, M^{i}\right)=w_{\bar{C}_{i}}\left(H^{a}, R^{i}\right)+w_{\bar{C}_{i}}\left(R^{i}, M^{i}\right)=-w_{C_{i}}\left(H^{a}, R^{i}\right)+w_{\partial D^{2}}\left(R^{i}, M^{i}\right)=-w_{i}(g, h)+2,
$$

where we have used that $\bar{C}_{i}$ is identified with $\partial D^{2}$ and that the unoriented line tangent to $\partial D^{2}$ rotates twice with respect to the horizontal unoriented line. By the Extension Theorem in 6, p. 145], it is possible to construct an extension $\hat{h}$ of $h$ to $\hat{\Sigma}$ such that $\mathcal{C}(\hat{g}, \hat{h})=\mathcal{C}(g, h) \cup\left\{z_{1}, \ldots, z_{n}\right\}$, where $z_{1}, \ldots, z_{n}$ are the centers of the discs $D_{1}, \ldots, D_{n}$ and

$$
\begin{equation*}
\operatorname{ind}_{(\hat{g}, \hat{h})}\left(z_{i}\right)=w_{\bar{C}_{i}}\left(H^{a}, M^{i}\right)=2-w_{i}(g, h) \tag{2.1}
\end{equation*}
$$

Therefore,

$$
[\mathcal{C}(g, h)]=[\mathcal{C}(\hat{g}, \hat{h})]-\sum_{i=1}^{n} \operatorname{ind}_{(\hat{g}, \hat{h})}\left(z_{i}\right)=2 \chi(\hat{\Sigma})-2 n+\sum_{i=1}^{n} w_{i}(g, h)=2 \chi(\Sigma)+\sum_{i=1}^{n} w_{i}(g, h)
$$

where we used that $\chi(\Sigma)+n=\chi(\hat{\Sigma})$ as follows from the formula $\chi(A \cup B)=\chi(A)+\chi(B)-\chi(A \cap B)$. We have thus completed the proof of Theorem 1.2.(1).

Let us first prove Theorem 1.2 (2) when $\Sigma$ has no boundary. Let us consider an embedded closed disc $D$ containing $\mathcal{C}$ in its interior. There is a section $H^{\text {out }}$ of $E^{a}$ which is nowhere vanishing on $\Sigma \backslash D$ and there is a section $H^{\text {in }}$ which is nowhere vanishing over $D$. The winding number of $H^{\text {out }}$ with respect to $H^{\text {in }}$ along $\partial D$ is $w\left(H^{\text {out }}, H^{\text {in }}\right)=2 \chi(\Sigma)$. For each $z \in \mathcal{C}$ consider an embedded closed disc $D^{z}$ centered at $z$ and contained in $D$. After shrinking the discs $D^{z}$ we may assume that they are pairwise disjoint. Let $H^{z}$ be a section of $\left.E^{a}\right|_{D^{z}}$ which has just one zero at $z$ with index $\operatorname{ind}(z)=\imath(z)$. Thus the winding number of $H^{z}$ with respect to $H^{\text {in }}$ along $\partial D^{z}$ is $w\left(H^{z}, H^{\mathrm{in}}\right)=\imath(z)$. Since $2 \chi(\Sigma)=\sum_{z \in \mathcal{C}} \iota(z)$ by assumption, we get

$$
w\left(H^{\mathrm{out}}, H^{\mathrm{in}}\right)=\sum_{z \in \mathcal{C}} w\left(H^{z}, H^{\mathrm{in}}\right)
$$

Consider the surface

$$
\tilde{\Sigma}:=D \backslash \bigsqcup_{z \in \mathcal{C}} D^{z}
$$

It satisfies $\partial \tilde{\Sigma}=\partial D \sqcup\left(\sqcup_{z \in \mathcal{C}} \overline{\partial D^{z}}\right)$. Since $w\left(H^{\text {out }}, H^{\text {in }}\right)-\sum_{z \in \mathcal{C}} w\left(H^{z}, H^{\text {in }}\right)=0$, the Extension Theorem in [6, p. 145] implies that there is a nowhere vanishing section $\tilde{H}$ of $\left.E^{a}\right|_{\tilde{\Sigma}}$ coinciding
with $H^{\text {out }}$ on $\partial D$ and with $H^{z}$ on every $\partial D^{z}$. Thus, $H^{\text {out }}, \tilde{H}$, and all $H^{z}$ glue together to yield a section $H$ of $E^{a} \rightarrow \Sigma$ having the desired properties.

When $\Sigma$ has boundary, we construct the closed surface $\hat{\Sigma}$ as in the proof of Theorem 1.2.(1). We define $\hat{\mathcal{C}}:=\mathcal{C} \cup\left\{z_{1}, \ldots, z_{n}\right\}$ and $\hat{\imath}: \hat{\mathcal{C}} \rightarrow \mathbb{Z}$ as the extension of $\imath$ such that $\imath\left(z_{i}\right)=2-w_{i}$ for all $i=1, \ldots, n$. Applying Theorem [1.2)(2) for closed surfaces to $\hat{\Sigma}$ and $\hat{\imath}$ and using (2.1) yields Theorem 1.2, (2) for the case of surfaces with boundary, as well.

## 3. Proof of Theorem 1.4

Let $C_{i}$ be a component of $\partial \Sigma$ for some $i \in\{1, \ldots, n\}$. There is $j \in\{1, \ldots, n\}$ such that $F\left(C_{i}\right)=C_{j}$. Recall that $\left.\mathrm{d} F\right|_{C_{i}}$ is expressed by the matrix

$$
N_{i}=c_{i}\left(\begin{array}{ll}
a_{i} & 0 \\
b_{i} & 1
\end{array}\right)
$$

with respect to the positive orthonormal bases $\nu_{i}, \tau_{i}$ and $\nu_{j}, \tau_{j}$.
The metric $\left.F^{*} g\right|_{C_{i}}$ is represented by the endomorphism $\mathrm{d} F^{T} \cdot \mathrm{~d} F$ via (1.3). A computation shows that the matrix representing $\mathrm{d} F^{T} \cdot \mathrm{~d} F$ with respect to the basis $\nu_{i}, \tau_{i}$ is

$$
N_{i}^{T} N_{i}=c_{i}^{2} Q_{i}, \quad \text { with } \quad Q_{i}=\left(\begin{array}{cc}
a_{i}^{2}+b_{i}^{2} & b_{i} \\
b_{i} & 1
\end{array}\right) .
$$

We point out that the condition that $\left(a_{i}, b_{i}\right)$ is never equal to $(1,0)$ is equivalent to $Q_{i}$ having distinct eigenvalues, since $Q_{i}$ is symmetric. Let $q_{i}: C_{i} \rightarrow \mathbb{R} P^{1} \cong \mathbb{R} / \pi \mathbb{Z}$ be the eigendirection of $Q_{i}$ with larger eigenvalue. By (1.6), $w_{i}\left(g, F^{*} g\right)$ is the degree of the map $q_{i}: C_{i} \rightarrow \mathbb{R} / \pi \mathbb{Z}$. Therefore, our goal is to show that the degree of $q_{i}$ is equal to the winding number of $\left(a_{i}-1, b_{i}\right): C_{i} \rightarrow \mathbb{R}^{2}$ around the origin. To this purpose, let us parametrize $C_{i}$ in the positive direction by $\theta_{i} \in \mathbb{R} / 2 \pi \mathbb{Z}$ and, to ease notation, let us drop all the subscripts $i$ in what follows.

We may assume without loss of generality that the curve $(a-1, b): \mathbb{R} / 2 \pi \mathbb{Z} \rightarrow \mathbb{R}^{2}$ intersects the positive real axis transversely. In this case $w(a-1, b)$ counts the number of points $\theta_{0} \in \mathbb{R} / 2 \pi \mathbb{Z}$ such that $\left(a\left(\theta_{0}\right)-1, b\left(\theta_{0}\right)\right)$ lies on the positive real axis, namely $a\left(\theta_{0}\right)>1$ and $b\left(\theta_{0}\right)=0$, with sign: the intersection is counted positively if $b^{\prime}\left(\theta_{0}\right)>0$ and negatively if $b^{\prime}\left(\theta_{0}\right)<0$.

On the other hand, the degree of $q$ is computed using a regular value $\xi \in \mathbb{R} / \pi \mathbb{Z}$ of $q$. Being regular means that $q^{\prime}\left(\theta_{0}\right) \neq 0$ for all $\theta_{0} \in q^{-1}(\xi)$. In this case, the degree of $q$ counts number of points $\theta_{0} \in q^{-1}(\xi)$ with sign: the point $\theta_{0}$ is counted positively if $q^{\prime}\left(\theta_{0}\right)>0$ and negatively if $q^{\prime}\left(\theta_{0}\right)<0$.

Choosing $\xi=0$, we see that $\theta_{0} \in q^{-1}(0)$ if and only if $(1,0) \in \mathbb{R}^{2}$ is an eigenvector of $Q$ with eigenvalue larger than 1 . This happens exactly when $b\left(\theta_{0}\right)=0$ and $a\left(\theta_{0}\right)>1$, that is when $(a-1, b)$ intersects the positive real axis. Therefore, we prove that 0 is a regular value of $q$ and that $w(a-1, b)$ is the degree of $q$ if we can show that for every such $\theta_{0}$ the numbers $b^{\prime}\left(\theta_{0}\right)$ and $q^{\prime}\left(\theta_{0}\right)$ have the same sign.

For this purpose, let $v(\theta)=(x(\theta), y(\theta)) \in \mathbb{R}^{2}$ be a generator of the line $q(\theta)$ such that $v\left(\theta_{0}\right)=$ $(1,0)$ and write $\lambda(\theta)$ for the corresponding eigenvalue of $Q(\theta)$, so that $\lambda\left(\theta_{0}\right)=a\left(\theta_{0}\right)$. Then $q^{\prime}\left(\theta_{0}\right)=y^{\prime}\left(\theta_{0}\right)$. To compute $y^{\prime}\left(\theta_{0}\right)$ we differentiate the vector equation $(Q(\theta)-\lambda(\theta) I) v(\theta)=0$ at $\theta_{0}$ :

$$
\left(Q\left(\theta_{0}\right)-\lambda\left(\theta_{0}\right) I\right) v^{\prime}\left(\theta_{0}\right)+\left(Q^{\prime}\left(\theta_{0}\right)-\lambda^{\prime}\left(\theta_{0}\right) I\right) v\left(\theta_{0}\right)=0
$$

Therefore, substituting the values for $Q\left(\theta_{0}\right), \lambda\left(\theta_{0}\right)$ and $Q^{\prime}\left(\theta_{0}\right)$ and taking the $y$-component of the vector equation, we get

$$
\left(1-a\left(\theta_{0}\right)\right) y^{\prime}\left(\theta_{0}\right)+b^{\prime}\left(\theta_{0}\right)=0
$$

Thus,

$$
q^{\prime}\left(\theta_{0}\right)=y^{\prime}\left(\theta_{0}\right)=\frac{b^{\prime}\left(\theta_{0}\right)}{a\left(\theta_{0}\right)-1}
$$

from which we see that $q^{\prime}\left(\theta_{0}\right)$ and $b^{\prime}\left(\theta_{0}\right)$ have the same sign since $a\left(\theta_{0}\right)>1$. This completes the proof.

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